Research Article

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Hessian equations of Krylov type on compact Hermitian manifolds

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Abstract: In this article, we are concerned with the equations of Krylov type on compact Hermitian manifolds, which are in the form of the linear combinations of the elementary symmetric functions of a Hermitian matrix. Under the assumption of the $\mathcal{C}$-subsolution, we obtain *a priori* estimates in Γ_{k-1} cone. By using the method of continuity, we prove an existence theorem, which generalizes the relevant results. As an application, we give an alternative way to solve the deformed Hermitian Yang-Mills equation on compact Kähler threefold.

Keywords: Hessian equations, Hermitian manifolds, subsolution condition

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1 Introduction

Let (M, ω) be a compact Kähler manifold of complex dimension n. In 1978, Yau [1] proved the famous Calabi-Yau conjecture by solving the following complex Monge-Ampère equation on M

$$(\omega + \sqrt{-1}\partial\bar{\partial}u)^n = f\omega^n,$$

with positive function f. There have been many generalizations of Yau's work. One extension of Yau's Theorem is to the case of Hermitian manifolds, which is initiated by Cherrier [2] in 1987. The Monge-Ampère equation on compact Hermitian manifolds was solved by Tosatti and Weinkove [3], building on several earlier works. See [2,4–7] and the references therein.

The complex Hessian equation can be expressed as follows:

$$(\omega + \sqrt{-1}\partial\bar{\partial}u)^k \wedge \omega^{n-k} = f\omega^n, \quad 2 \le k \le n-1.$$

On compact Kähler manifolds (M, ω), Hou [8] proved the existence of a smooth admissible solution of the complex Hessian equation by assuming the nonnegativity of the holomorphic bisectional curvature. Later, Hou et al. [9] obtained the second-order estimate without any curvature assumption. Using Hou et al.'s estimate, Dinew and Kolodziej [10] applied a blow-up argument to prove the gradient estimate and solved the complex Hessian equation on compact Kähler manifolds. The corresponding problem on Hermitian manifolds was solved by Zhang [11] and Székelyhidi [12] independently.

The complex Hessian quotient equations include the complex Monge-Ampère equation and the complex Hessian equation. Let χ be a real (1, 1) form, the complex Hessian quotient equations can be expressed as follows:

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$$(\chi + \sqrt{-1}\partial\bar{\partial}u)^k \wedge \omega^{n-k} = f(z)(\chi + \sqrt{-1}\partial\bar{\partial}u)^l \wedge \omega^{n-l}, \quad 1 \le l < k \le n, z \in M.$$

When $f(z)$ is constant, one special case is the so-called Donaldson equation [13]:

$$(\chi + \sqrt{-1}\partial\bar{\partial}u)^n = c(\chi + \sqrt{-1}\partial\bar{\partial}u)^{n-1} \wedge \omega^k.$$

After some progresses made in [14–17], Song and Weinokove [18] solved the Donaldson equation on closed Kähler manifolds via J-flow. Fang et al. [19] extended the Donaldson equation to

$$(\chi + \sqrt{-1}\partial\bar{\partial}u)^n = c_k(\chi + \sqrt{-1}\partial\bar{\partial}u)^{n-k} \wedge \omega^k$$

and solved this equation on closed Kähler manifolds by assuming a cone condition. When $f(z)$ is not constant, analogous results were obtained by Sun [20,21] on compact Hermitian manifolds.

In this article, we are concerned with Hessian equations of Krylov type in the form of the linear combinations of the Hessian, which can be written as follows:

$$(\chi + \sqrt{-1}\partial\bar{\partial}u)^k \wedge \omega^{n-k} = \sum_{l=0}^{k-1} \alpha_l(\chi + \sqrt{-1}\partial\bar{\partial}u)^l \wedge \omega^{n-l}, \quad 2 \le k \le n. \tag{1.1}$$

The Dirichlet problem of (1.1) on $(k-1)$-convex domain Ω in $\mathbb{R}^n$ was first studied by Krylov [22] about 20 years ago. He observed that if $\alpha_l(x) \ge 0$ for $0 \le l \le k-1$, the natural admissible cone to make (1.1) elliptic is also the Γ_k-cone, which is the same as the k-Hessian equation case, where

$$\Gamma_k = \{\lambda \in \mathbb{R}^n | \sigma_1(\lambda) > 0, \ldots, \sigma_k(\lambda) > 0\}.$$

Guan and Zhang [23] solved the equation of Krylov type on the problem of prescribing convex combination of area measures. Pingali [24] proved *a priori* estimates to the following equation in Kähler case:

$$(\chi + \sqrt{-1}\partial\bar{\partial}u)^n = \sum_{l=0}^{n-1} C_n^l \alpha_l(\chi + \sqrt{-1}\partial\bar{\partial}u)^{n-k} \wedge \omega^l,$$

where $\alpha_l \ge 0$ are smooth real functions such that either $\alpha_l = 0$ or $\alpha_l > 0$, and $\sum_{l=0}^{n-2} \alpha_l > 0$. Recently, Phong and Tô [25] solved Hessian equations of Krylov type on compact Kähler manifolds, where α_l are non-negative constants for $0 \le l \le k-1$. When α_l are non-negative smooth functions for $0 \le l \le k-1$, analogous results on compact Kähler manifolds are obtained by Chen [26] and Zhou [27] independently.

Naturally, we want to extend this result to Hermitian manifolds. On the other hand, Zhou [27] believed that the condition on $\alpha_{k-1}(x) > 0$ is not necessary. In fact, Guan and Zhang [23] considered equation (1.1) without the sign requirement for the coefficient function $\alpha_{k-1}(x)$.

In this article, we mainly concern equation (1.1) on Hermitian manifold without any sign requirement for $\alpha_{k-1}(x)$. Let Γ_{k-1}^g be the set of all the real (1, 1) forms, eigenvalues of which belong to Γ_{k-1}. To ensure the ellipticity and non degeneracy of the equation in Γ_{k-1}, we require smooth real functions α_l to satisfy the conditions: for $0 \le l \le k-2$, either $\alpha_l > 0$ or $\alpha_l \equiv 0$, and $\sum_{l=0}^{k-2} \alpha_l > 0$. Let $\chi_u = \chi + \sqrt{-1}\partial\bar{\partial}u$ and $\chi_{\underline{u}} = \chi + \sqrt{-1}\partial\bar{\partial}\underline{u}$. To state our main results, we need also the following condition of C-subsolution, which is similar to C-subsolution introduced by Székelyhidi [12].

Definition 1.1. A smooth real function $\underline{u}$ is a C-subsolution to (1.1), if $\chi_{\underline{u}} \in \Gamma_{k-1}^g$, and at each point $x \in M$, the set

$$\left\{\lambda(\widetilde{\chi}) \in \Gamma_{k-1} | \widetilde{\chi}^k \wedge \omega^{n-k} = \sum_{l=0}^{k-1} \alpha_l(x)\widetilde{\chi}^l \wedge \omega^{n-l} \quad \text{and} \quad \widetilde{\chi} - \chi_{\underline{u}} \ge 0\right\}$$

is bounded.

Theorem 1.2. *Let* (M, g) *be a compact Hermitian manifold,* χ *a real* (1, 1) *form on* M. *Suppose that* $\underline{u}$ *is a* C*-subsolution of equation* (1.1) *and at each point* $x \in M$,

$$\chi_{\underline{u}(x)}^k \wedge \omega^{n-k} \leq \sum_{l=0}^{k-1} \alpha_l(x)\chi_{\underline{u}(x)}^l \wedge \omega^{n-l}. \tag{1.2}$$

Then there exists a smooth real function u on M and a unique constant b solving

$$\chi_u^k \wedge \omega^{n-k} = \sum_{l=0}^{k-2} \alpha_l(x)\chi_u^l \wedge \omega^{n-l} + (\alpha_{k-1} + b)\chi_u^{k-1} \wedge \omega^{n-k+1}, \tag{1.3}$$

with $\sup_M(u - \underline{u}) = 0$ *and* $\chi_u \in \Gamma_{k-1}^g$.

Corollary 1.3. *Let* (M, g) *be a compact Kähler manifold,* χ *a closed* $(1, 1)$*-form. Suppose that* $\underline{u}$ *is a* C*-subsolution of equation* (1.1) *and*

$$\int_M \chi^k \wedge \omega^{n-k} \leq \sum_{l=0}^{k-1} c_l \int_M \chi \wedge \omega^{n-l}, \tag{1.4}$$

where $c_l = \inf_M \alpha_l$, $0 \leq l \leq k-1$. *Then there exists a smooth real function u on M and a unique constant b solving*

$$\chi_u^k \wedge \omega^{n-k} = \sum_{l=0}^{k-2} \alpha_l(x)\chi_u^l \wedge \omega^{n-l} + (\alpha_{k-1} + b)\chi_u^{k-1} \wedge \omega^{n-k+1}, \tag{1.5}$$

with $\sup_M(u - \underline{u}) = 0$ *and* $\chi_u \in \Gamma_{k-1}^g$.

Lately, Pingali [28] proved an existence result of the deformed Hermitian Yang-Mills equation with phase angle $\hat{\theta} \in \left(\frac{1}{2}\pi, \frac{3}{2}\pi\right)$ on compact Kähler threefold. Let Ω be a closed $(1, 1)$ form, $\Omega_u = \Omega + \sqrt{-1}\partial\bar{\partial}u$. From [24], the deformed Hermitian Yang-Mills equation on compact Kähler threefold can be written as follows:

$$\Omega_u^3 = 3\sec^2(\hat{\theta})\Omega_u \wedge \omega^2 + 2\tan(\hat{\theta})\sec^2(\hat{\theta})\omega^3. \tag{1.6}$$

As an application of Corollary 1.3, we give an alternative way to solve the deformed Hermitian Yang-Mills equation on compact Kähler threefold.

Corollary 1.4. *Let* (M, g) *be a compact Kähler threefold, constant phase angle* $\hat{\theta} \in \left(\frac{1}{2}\pi, \frac{3}{2}\pi\right)$, *and* Ω *a positive definite closed* $(1, 1)$ *form, satisfying the following conditions*:

$$3\Omega^2 - 3\sec^2(\hat{\theta})\omega^2 > 0, \tag{1.7}$$

$$\int_M \Omega^3 = 3\sec^2(\hat{\theta})\int_M \Omega \wedge \omega^2 + 2\tan(\hat{\theta})\sec^2(\hat{\theta})\int_M \omega^3, \tag{1.8}$$

$$\Omega + \sec(\hat{\theta})\omega \in \Gamma_2^g. \tag{1.9}$$

Then there exists a smooth solution to equation (1.6) *with* $\sup_M u = 0$ *and* $\Omega_u \in \Gamma_3^g$.

The rest of this article is organized as follows. In Section 2, we set up some notations and provide some preliminary results. In Section 3, we give the C^0 estimate by the Alexandroff-Bakelman-Pucci maximum principle. In Section 4, we establish the C^2 estimate for equation (1.1) by the method of Hou et al. [9] and the C-subsolution condition. In Section 5, we give the gradient estimate. In Section 6, we give the proof of Theorem 1.2, Corollaries 1.3, and 1.4 by the method of continuity. Although the method is very standard in the study of elliptic PDEs, it is not easy to carry out this method on a compact Hermitian manifold. Since the essential C-subsolution condition depends on $\alpha_0, \ldots, \alpha_{k-1}$, we have to find a uniform C-subsolution condition for the solution flow of the continuity method.

2 Preliminaries

In this section, we set up the notation and establish some lemmas. Let $\sigma_k(\lambda)$ denote the kth elementary symmetric function

$$\sigma_k(\lambda)=\sum_{1\le i_1<\cdots<i_k\le n}\lambda_{i_1}\cdots\lambda_{i_k},\quad \text{for}\quad \lambda=(\lambda_1,\dots,\lambda_n)\in\mathbb{R}^n,\quad 1\le k\le n.$$

For completeness, we define $\sigma_0(\lambda)=1$ and $\sigma_{-1}(\lambda)=0$. Let $\sigma_k(\lambda|i)$ denote the symmetric function with $\lambda_i=0$ and $\sigma_k(\lambda|ij)$ the symmetric function with $\lambda_i=\lambda_j=0$. Also denote by $\sigma_k(A|i)$ the symmetric function with A deleting the ith row and ith column, and $\sigma_k(A|ij)$ the symmetric function with A deleting the ith, jth rows and ith, jth columns, for all $1\le i,j\le n$. In local coordinates,

$$X_{i\bar j}=X\left(\frac{\partial}{\partial z^i},\frac{\partial}{\partial \bar z^i}\right)=\chi_{i\bar j}+u_{i\bar j},\quad \underline{X}_{i\bar j}=\chi_{i\bar j}+\underline{u}_{i\bar j}.$$

Define $\lambda(\chi_u)$ as the eigenvalue set of $\{X_{i\bar j}\}$ with respect to $\{g_{i\bar j}\}$. In local coordinates, equation (1.1) can be written in the following form:

$$\sigma_k(\lambda(\chi_u))=\sum_{l=0}^{k-1}\beta_l(x)\sigma_l(\lambda(\chi_u)),\tag{2.1}$$

where

$$\frac{\sigma_l(\lambda(\chi_u))}{C_n^l}=\frac{\chi_u^l\wedge\omega^{n-l}}{\omega^n},\quad \beta_l(x)=\frac{C_n^k}{C_n^l}\alpha_l(x).$$

Equivalently, we can rewrite equation (2.1) as follows:

$$\frac{\sigma_k(\lambda(\chi_u))}{\sigma_{k-1}(\lambda(\chi_u))}-\sum_{l=0}^{k-2}\beta_l\frac{\sigma_l(\lambda(\chi_u))}{\sigma_{k-1}(\lambda(\chi_u))}=\beta_{k-1}(x).\tag{2.2}$$

Lemma 2.1. [29,30] *For $\lambda\in\Gamma_m$ and $m>l\ge0$, $r>s\ge0$, $m\ge r$, $l\ge s$, we have*

$$\left[\frac{\sigma_m(\lambda)/C_n^m}{\sigma_l(\lambda)/C_n^l}\right]^{\frac{1}{m-l}}\le\left[\frac{\sigma_r(\lambda)/C_n^r}{\sigma_s(\lambda)/C_n^s}\right]^{\frac{1}{r-s}}.$$

The following lemma is similar to Lemma 2.3 in [27], but we need to discuss it more widely, that is, $\lambda(\chi_u)\in\Gamma_{k-1}$ instead of $\lambda(\chi_u)\in\Gamma_k$.

Lemma 2.2. *If $u\in C^2(M)$ is a solution of (2.2), $\lambda(\chi_u)\in\Gamma_{k-1}$ and $\beta_l(x)>0$, $0\le l\le k-2$, then*

$$\frac{\sigma_l(\lambda)}{\sigma_{k-1}(\lambda)}\le C\left(n,k,\inf_{0\le l\le k-2}\beta_l,\sup|\beta_{k-1}|\right)\quad \textit{for}\ 0\le l\le k-2.\tag{2.3}$$

Proof. If $\frac{\sigma_k}{\sigma_{k-1}}\le1$, then we obtain from equation (2.2)

$$\beta_l(x)\frac{\sigma_l}{\sigma_{k-1}}\le\frac{\sigma_k}{\sigma_{k-1}}-\beta(x)\le1-\beta(x)\le C\left(\sup_M|\beta_{k-1}|\right),\quad \text{for}\ 0\le l\le k-2.$$

If $\frac{\sigma_k}{\sigma_{k-1}}>1$, i.e., $\frac{\sigma_{k-1}}{\sigma_k}<1$, we see from Lemma 2.1 that

$$\frac{\sigma_l}{\sigma_{k-1}}\le\frac{(C_n^k)^{k-1-l}C_n^l}{(C_n^{k-1})^{k-l}}\left(\frac{\sigma_{k-1}}{\sigma_k}\right)^{k-1-l}\le\frac{(C_n^k)^{k-1-l}C_n^l}{(C_n^{k-1})^{k-l}}\le C(n,k)$$

for $0\le l\le k-2$, which completes the proof of Lemma 2.2. □

For any point $x_0 \in M$, choose a local frame such that $X_{i\bar{j}} = \delta_{ij} X_{i\bar{i}}$. For the convenience of notations, we will write equation (2.2) as follows:

$$F(X) = F_k(X) + \sum_{l=0}^{k-2} \beta_l F_l(X) = \beta_{k-1}(x), \tag{2.4}$$

where $F_k(X) = \frac{\sigma_k(\lambda(X))}{\sigma_{k-1}(\lambda(X))}$ and $F_l(X) = -\frac{\sigma_l(\lambda(X))}{\sigma_{k-1}(\lambda(X))}$. Let

$$F^{i\bar{j}} := \frac{\partial F}{\partial X_{i\bar{j}}} = \frac{\partial F}{\partial \lambda_k} \frac{\partial \lambda_k}{\partial X_{i\bar{j}}},$$

then at x_0, we have

$$F^{i\bar{j}} = F^{i\bar{i}} \delta_{ij}.$$

Let

$$\mathcal{F} := \sum_i F^{i\bar{i}}.$$

Lemma 2.3. [23] *If $\lambda \in \Gamma_{k-1}$ and $\alpha_l(x) > 0$, $0 \le l \le k-2$, then the operator F is elliptic and concave in Γ_{k-1}.*

From Lemma 2.4 in [27] and Lemma 2.2, we have the following lemma.

Lemma 2.4. *If $u \in C^2(M)$ is a solution of* (2.4), *$\lambda(\chi_u) \in \Gamma_{k-1}$, then at x_0,*

$$\frac{n-k+1}{k} \le \mathcal{F} \le C\left(n, k, \inf_{0 \le l \le k-2} \alpha_l, \sup|\alpha_{k-1}|\right). \tag{2.5}$$

Lemma 2.5. *Under assumptions of* Theorem 1.2, *there is a constant $\theta > 0$ such that*

$$F^{i\bar{i}}\left(u_{i\bar{i}} - \underline{u}_{i\bar{i}}\right) \le -\theta(1 + \mathcal{F}), \tag{2.6}$$

or

$$F^{1\bar{1}} \ge \theta. \tag{2.7}$$

Proof. Without loss of generality, we may assume that $X_{1\bar{1}} \ge \cdots \ge X_{n\bar{n}}$. Thus,

$$F^{n\bar{n}} \ge \cdots \ge F^{1\bar{1}}.$$

Since $\underline{u}$ is a C-subsolution, if $\varepsilon > 0$ is small enough, $\chi_{\underline{u}} - \varepsilon\omega$ still satisfies Definition 1.1. Since M is compact, there are uniform constants $N > 0$ and $\delta > 0$ such that

$$F(\widetilde{X}) > \beta_{k-1} + \delta, \tag{2.8}$$

where

$$\widetilde{X} = \underline{X} - \varepsilon g + \begin{pmatrix} N & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \end{pmatrix}_{n \times n}.$$

Direct calculation yields

$$F^{i\bar{i}}\left(u_{i\bar{i}} - \underline{u}_{i\bar{i}}\right) = F^{i\bar{i}}\left(X_{i\bar{i}} - \underline{X}_{i\bar{i}}\right) = F^{i\bar{i}}(X_{i\bar{i}} - \widetilde{X}_{i\bar{i}}) + F^{1\bar{1}}N - \varepsilon\mathcal{F}. \tag{2.9}$$

Since F is concave in Γ_{k-1}, from (2.8), we obtain

$$\sum_{i=1}^{n} F^{i\bar{i}}\left(X_{i\bar{i}} - \widetilde{X}_{i\bar{i}}\right) \le F(X) - F(\widetilde{X}) \le -\delta. \tag{2.10}$$

By substituting (2.10) into (2.9), we obtain

$$F^{i\bar{i}}\left(u_{i\bar{i}} - \underline{u}_{i\bar{i}}\right) \leq -\delta + F^{1\bar{1}}N - \varepsilon\mathcal{F}.$$

Set $\theta = \min\left\{\frac{\delta}{2}, \varepsilon, \frac{\delta}{2N}\right\}$. If $F^{1\bar{1}}N \leq \frac{\delta}{2}$, we have (2.6); otherwise, (2.7) must be true. □

3 C^0 estimate

In this section, we obtain the C^0 estimate by using the Alexandroff-Bakelman-Pucci maximum principle and prove the following Proposition 3.1, which is similar to the approach of Székelyhidi [12].

Proposition 3.1. *Let $\alpha_l(x) > 0$ for $0 \leq l \leq k-2$ and χ be a smooth real $(1,\ 1)$ form on (M, g). Assume that u and $\underline{u}$ are solution and C-subsolution to* (1.1) *with $\lambda(\chi_u) \in \Gamma_{k-1}$, $\lambda(\chi_{\underline{u}}) \in \Gamma_{k-1}$, respectively. We normalize u such that $\sup_M(u - \underline{u}) = 0$. There is a constant C depending on the given data, such that*

$$\sup_M |u| < C. \tag{3.1}$$

Proof. To simplify notation, we can assume $\underline{u} = 0$, otherwise we modify the background form χ. Therefore, $\sup_M u = 0$. The following goal is to prove that $L = \inf_M u$ has a uniform lower bound. Notice that $\lambda(\chi_u) \in \Gamma_{k-1}$, so $\lambda(\chi_u) \in \Gamma_1$, that is,

$$\Delta u = g^{i\bar{p}}u_{i\bar{p}} > -g^{i\bar{p}}\chi_{i\bar{p}} \geq -\widehat{C}.$$

Let $G : M \times M \to \mathbb{R}$ be the Green's function of a Gauduchon metric conformal to g. From [1], there is a uniform constant K such that

$$G(x, y) + K \geq 0, \quad \forall (x, y) \in M \times M, \quad \text{and} \quad \int_{y\in M} G(x, y)\omega^n(y) = 0.$$

Since $\sup_M u = 0$, there is a point $x_0 \in M$ such that $u(x_0) = 0$. Hence,

$$\begin{aligned} u(x_0) &= \int_M u\,\mathrm{d}\mu - \int_{y\in M} G(x_0, y)\Delta u(y)\omega^n(y) \\ &= \int_M u\,\mathrm{d}\mu - \int_{y\in M} (G(x_0, y) + K)\Delta u(y)\omega^n(y) \\ &\leq \int_M u\,\mathrm{d}\mu + \widehat{C}K, \end{aligned}$$

which yields

$$\int_M |u|\,\mathrm{d}\mu \leq \widehat{C}K.$$

Next, we choose local coordinates at the minimum point of u and $L = \inf_M u = u(0)$. Let $B(1) = \{z : |z| < 1\}$ and $v = u + \varepsilon|z|^2$ for a small $\varepsilon > 0$. From the Alexandroff-Bakelman-Pucci maximum principle, we obtain

$$c_0\varepsilon^{2n} \leq \int_\Omega \det(D^2 v), \tag{3.2}$$

where

$$\Omega = \left\{\begin{matrix} x \in B(1) : |Dv(x)| < \dfrac{\varepsilon}{2}, \\ v(y) \geq v(x) + Dv(x)\cdot(y - x), \quad \forall y \in B(1) \end{matrix}\right\}.$$

Let $\widetilde{\lambda} - \lambda(\chi_{\underline{u}}) \geq 0$ and

$$\frac{\sigma_k(\widetilde{\lambda})}{\sigma_{k-1}(\widetilde{\lambda})} - \sum_{l=0}^{k-2} \beta_l \frac{\sigma_l(\widetilde{\lambda})}{\sigma_{k-1}(\widetilde{\lambda})} = \beta_{k-1}(x).$$

$\underline{u}$ is a C subsolution, which means that $|\widetilde{\lambda}|$ is bounded. Since M is compact, there is uniform constant $\eta > 0$ such that $\lambda(\chi_{\underline{u}}) - \eta\mathbf{1}$ satisfies Definition 1.1. Since Ω is a contact set, we have $D^2 v(x) \geq 0$, for $x \in \Omega$, which implies $u_{i\bar{j}}(x) + \varepsilon\delta_{ij} \geq 0$. Choosing ε such that $0 < \varepsilon \leq \eta$, on Ω, we have

$$\lambda(\chi_u) - (\lambda\big(\chi_{\underline{u}}\big) - \eta\mathbf{1}) \geq \lambda(\chi_u) - (\lambda\big(\chi_{\underline{u}}\big) - \varepsilon\mathbf{1}) = \lambda\big(u_{i\bar{j}}\big) + \varepsilon\mathbf{1} \geq 0.$$

Since

$$\frac{\sigma_k(\lambda(\chi_u))}{\sigma_{k-1}(\lambda(\chi_u))} - \sum_{l=0}^{k-2} \beta_l \frac{\sigma_l(\lambda(\chi_u))}{\sigma_{k-1}(\lambda(\chi_u))} = \beta_{k-1}(x),$$

we obtain that $|\lambda(\chi_u)|$ is bounded, which yields $|u_{i\bar{j}}| \leq C$. Then

$$\det(D^2 v(x)) \leq 2^{2n} \det\big(v_{i\bar{j}}\big)^2 \leq C.$$

From this and (3.2), we obtain

$$c_0 \varepsilon^{2n} \leq \int_{\Omega} \det(D^2 v) \leq C \cdot \mathrm{vol}(\Omega). \tag{3.3}$$

On the other hand, we have for $x \in \Omega$

$$v(0) \geq v(x) - Dv(x) \cdot x > v(x) - \frac{\varepsilon}{2},$$

so

$$|v(x)| > |L + \frac{\varepsilon}{2}|.$$

Then,

$$\int_M |v(x)| \geq \int_{\Omega} |v(x)| \geq |L + \frac{\varepsilon}{2}| \cdot \mathrm{vol}(\Omega).$$

Since $\int_M |v(x)|$ is uniformly bounded, this inequality contradicts (3.3) if L is very large. □

4 C^2 estimate

In this section, we establish the C^2 estimate to equation (1.1). Our calculation is similar to that in [27], but on Hermitian manifolds, equation (1.1) are much more difficult to treat due to the torsion terms.

4.1 Notations and lemma

In local coordinates $z = (z_1, \ldots, z_n)$, the Chern connection ∇ and torsion are given, respectively, by

$$\nabla_{\frac{\partial}{\partial z_i}} \frac{\partial}{\partial z_j} = \Gamma_{ij}^k \frac{\partial}{\partial z_k}, \quad \Gamma_{ij}^k = g^{k\bar{l}} \frac{\partial g_{j\bar{l}}}{\partial z_i}, \quad T_{ij}^k = \Gamma_{ij}^k - \Gamma_{ji}^k,$$

while the curvature tensor $R_{i\bar{j}k\bar{l}}$ by

$$R_{i\bar{j}k\bar{l}} = g_{p\bar{l}} \frac{\partial \Gamma_{ik}^{p}}{\partial \bar{z}_j}.$$

For $u \in C^4(M)$, we denote

$$u_{ij} = \nabla_j \nabla_i u, \quad u_{i\bar{j}} = \nabla_{\bar{j}} \nabla_i u.$$

We have (see [4,11,31])

$$\begin{cases} u_{i\bar{j}l} = u_{l\bar{j}i} + T_{il}^{p} u_{p\bar{j}}, \\ u_{i\bar{j}k} = u_{ik\bar{j}} - g^{l\bar{m}} R_{k\bar{j}i\bar{m}} u_l, \\ u_{i\bar{j}\bar{k}} = u_{i\bar{k}\bar{j}} + \overline{T_{jk}^{l}} u_{i\bar{l}}, \\ u_{i\bar{j}k} = u_{ki\bar{j}} - g^{l\bar{m}} R_{i\bar{j}k\bar{m}} u_l + T_{ik}^{l} u_{l\bar{j}}, \end{cases} \tag{4.1}$$

and

$$u_{i\bar{j}k\bar{l}} = u_{k\bar{l}i\bar{j}} + g^{p\bar{q}}\left(R_{k\bar{l}i\bar{q}} u_{p\bar{j}} - R_{i\bar{j}k\bar{q}} u_{p\bar{l}}\right) + T_{ik}^{p} u_{p\bar{j}\bar{l}} + \overline{T_{jl}^{q}} u_{i\bar{q}k} - T_{ik}^{p} \overline{T_{jl}^{q}} u_{p\bar{q}}. \tag{4.2}$$

Let $A_{i\bar{j}} = g^{j\bar{p}} X_{i\bar{p}}$, $\lambda(A) = (\lambda_1, \ldots, \lambda_n)$ and $\lambda_1 \geq \cdots \geq \lambda_n$. For a fixed point $x_0 \in M$, choose a local coordinates such that $A_{i\bar{j}} = A_{i\bar{i}} \delta_{ij}$. Since $\lambda_1, \ldots, \lambda_n$ need not be distinct at x_0, we will perturb χ_u slightly such that $\lambda_1, \ldots, \lambda_n$ become smooth functions near x_0. Let D be a diagonal matrix such that $D^{11} = 0$ and $0 < D^{22} < \cdots < D^{nn}$ are small, satisfying $D^{nn} < 2D^{22}$. Define the matrix $\widetilde{A} = A - D$. At x_0, $\widetilde{A}$ has eigenvalues

$$\widetilde{\lambda}_1 = \lambda_1, \quad \widetilde{\lambda}_i = \lambda_i - D^{ii}, \quad i \geq 2.$$

Lemma 4.1.

$$\widetilde{\lambda}_{1,i\bar{i}} \geq X_{i\bar{i}1\bar{1}} + 2\mathrm{Re}\left(X_{1\bar{1}i} \overline{T_{1i}^{1}}\right) - C_0 \lambda_1 - C_0. \tag{4.3}$$

Proof. Commuting derivative of $\widetilde{\lambda}_1$ gives

$$\widetilde{\lambda}_{1,i} = \frac{\partial \widetilde{\lambda}_1}{\partial \widetilde{A}_{p\bar{q}}} \frac{\partial \widetilde{A}_{p\bar{q}}}{\partial z_i} = X_{1\bar{1}i} - (D^{11})_i,$$

$$\begin{aligned} \widetilde{\lambda}_{1,i\bar{i}} &= \frac{\partial^2 \widetilde{\lambda}_1}{\partial \widetilde{A}_{r\bar{s}} \partial \widetilde{A}_{p\bar{q}}} \frac{\partial \widetilde{A}_{p\bar{q}}}{\partial z_i} \frac{\partial \widetilde{A}_{r\bar{s}}}{\partial \bar{z}_i} + \frac{\partial \widetilde{\lambda}_1}{\partial \widetilde{A}_{p\bar{q}}} \frac{\partial^2 \widetilde{A}_{p\bar{q}}}{\partial \bar{z}_i \partial z_i} \\ &= \sum_{p \geq 2} \frac{|X_{1\bar{p}i}|^2 + |X_{p\bar{1}i}|^2}{\lambda_1 - \widetilde{\lambda}_p} - 2 \sum_{p \geq 2} \frac{\mathrm{Re}((D^{1p})_{\bar{i}} X_{p\bar{1}i}) + \mathrm{Re}((D^{p1})_{\bar{i}} X_{1\bar{p}i})}{\lambda_1 - \widetilde{\lambda}_p} \\ &\quad + \sum_{p \geq 2} \frac{(D^{1p})_i (D^{p1})_{\bar{i}} + (D^{p1})_i (D^{1p})_{\bar{i}}}{\lambda_1 - \widetilde{\lambda}_p} + X_{1\bar{1}i\bar{i}} + (D^{11})_{i\bar{i}}. \end{aligned} \tag{4.4}$$

$\lambda(A) \in \Gamma_1$ implies that $|\lambda_p| < (n-1)\lambda_1$, $p \geq 2$. If the matrix D is sufficiently small, then $|\widetilde{\lambda}_p| < (n-1)\lambda_1$, $p \geq 2$, which means that

$$\frac{1}{n\lambda_1} \leq \frac{1}{\lambda_1 - \widetilde{\lambda}_p} \leq \frac{1}{D^{pp}}.$$

We are trying to bound λ_1 from mentioned earlier, so we can assume $\lambda_1 > 1$. Hence,

$$\sum_{p \geq 2} \frac{(D^{1p})_i (D^{p1})_{\bar{i}} + (D^{p1})_i (D^{1p})_{\bar{i}}}{\lambda_1 - \widetilde{\lambda}_p} + (D^{11})_{i\bar{i}} \geq -C_0. \tag{4.5}$$

From here on, C_0 will always denote such a constant, which depends on the given data and may vary from line to line. Using

$$2\mathrm{Re}((D^{1p})_{\bar{i}}X_{p\bar{1}i}) \leq \frac{1}{2}|X_{p\bar{1}i}|^2 + C_0,$$

we have

$$\sum_{p\geq 2}\frac{|X_{1\bar{p}i}|^2 + |X_{p\bar{1}i}|^2}{\lambda_1 - \widetilde{\lambda}_p} - 2\sum_{p\geq 2}\frac{\mathrm{Re}((D^{1p})_{\bar{i}}X_{p\bar{1}i}) + \mathrm{Re}((D^{p1})_{\bar{i}}X_{1\bar{p}i})}{\lambda_1 - \widetilde{\lambda}_p} \geq \frac{1}{2n\lambda_1}\sum_{p\geq 2}(|X_{1\bar{p}i}|^2 + |X_{p\bar{1}i}|^2) - C_0. \tag{4.6}$$

From (4.2), we obtain

$$u_{1\bar{1}i\bar{i}} = u_{i\bar{i}1\bar{1}} + R_{i\bar{i}1\bar{p}}u_{p\bar{1}} - R_{1\bar{1}i\bar{p}}u_{p\bar{i}} + T^p_{1i}u_{p\bar{1}\bar{i}} + \overline{T^p_{1i}}u_{1\bar{p}i} - T^p_{1i}\overline{T^q_{1i}}u_{p\bar{q}}. \tag{4.7}$$

From this, we have

$$\begin{aligned} X_{1\bar{1}i\bar{i}} &= X_{i\bar{i}1\bar{1}} + \chi_{1\bar{1}i\bar{i}} - \chi_{i\bar{i}1\bar{1}} + R_{i\bar{i}1\bar{p}}u_{p\bar{1}} - R_{1\bar{1}i\bar{p}}u_{p\bar{i}} + T^p_{1i}u_{p\bar{1}\bar{i}} + \overline{T^p_{1i}}u_{1\bar{p}i} - T^p_{1i}\overline{T^q_{1i}}u_{p\bar{q}} \\ &\geq X_{i\bar{i}1\bar{1}} + \lambda_1 R_{i\bar{i}1\bar{1}} - \lambda_i R_{1\bar{1}i\bar{i}} + 2\mathrm{Re}(X_{1\bar{p}i}\overline{T^p_{1i}}) - \lambda_p|T^p_{1i}|^2 - C_0 \\ &\geq X_{i\bar{i}1\bar{1}} + 2\mathrm{Re}(X_{1\bar{1}i}\overline{T^1_{1i}}) - \frac{1}{2n\lambda_1}\sum_{p\geq 2}|X_{1\bar{p}i}|^2 - C_0\lambda_1 - C_0. \end{aligned} \tag{4.8}$$

Substituting (4.5), (4.6), and (4.8) into (4.4) gives (4.3). □

4.2 C^2 estimate

Proposition 4.2. *Let $\alpha_l(x) > 0$ for $0 \leq l \leq k-2$ and χ be a smooth real $(1,1)$ form on (M,g). Assume that u and $\underline{u}$ are solution and C-subsolution to* (1.1) *with $\lambda(\chi_u) \in \Gamma_{k-1}$, $\lambda(\chi_{\underline{u}}) \in \Gamma_{k-1}$, respectively. Then there is an estimate as follows*:

$$\sup_M|\partial\bar{\partial}u| \leq C\Big(\sup_M|\nabla u|^2 + 1\Big),$$

where C is a uniform constant.

Proof. We assume that the C subsolution $\underline{u} = 0$, since otherwise we modify the background form χ. We normalize u so that $\sup_M u = 0$. Consider the function

$$W = \log\widetilde{\lambda}_1 + \varphi(|\nabla u|^2) + \psi(u). \tag{4.9}$$

Here,

$$\begin{aligned} \varphi(t) &= -\frac{1}{2}\log\Big(1 - \frac{t}{2K}\Big), \quad 0 \leq t \leq K-1, \\ \psi(t) &= -E\log\Big(1 + \frac{t}{2L}\Big), \quad -L+1 \leq t \leq 0, \end{aligned}$$

where

$$K = \sup_M|\nabla u|^2 + 1, \quad L = \sup_M|u| + 1, \quad E = 2L(C_1 + 1),$$

and C_1 is to be determined later. Direct calculation gives

$$0 < \frac{1}{4K} \leq \varphi' \leq \frac{1}{2K}, \quad \varphi'' = 2(\varphi')^2, \tag{4.10}$$

and

$$C_1 + 1 \leq -\psi' \leq 2(C_1 + 1), \quad \psi'' \geq \frac{4\varepsilon}{1-\varepsilon}(\psi')^2, \quad \text{for all } \varepsilon \leq \frac{1}{4E+1}. \tag{4.11}$$

Since M is compact, W attains its maximum at some point $x_0 \in M$. From now on, all the calculations will be carried out at the point x_0 and the Einstein summation convention will be used. Calculating covariant derivatives, we obtain

$$0 = W_i = \frac{X_{1\bar{1}i}}{\lambda_1} + \varphi'(|\nabla u|^2)_i + \psi' u_i - \frac{(D^{11})_i}{\lambda_1}, \quad 1 \le i \le n, \tag{4.12}$$

$$0 \ge W_{i\bar{i}} = \frac{\widetilde{\lambda}_{1,i\bar{i}}}{\lambda_1} - \frac{\widetilde{\lambda}_{1,i}\widetilde{\lambda}_{1,\bar{i}}}{\lambda_1^2} + \psi' u_{i\bar{i}} + \psi''|u_i|^2 + \varphi'(|\nabla u|^2)_{i\bar{i}} + \varphi''|(|\nabla u|^2)_i|^2. \tag{4.13}$$

Multiplying (4.13) by $F^{i\bar{i}}$ and summing it over index i yield

$$0 \ge F^{i\bar{i}}\frac{\widetilde{\lambda}_{1,i\bar{i}}}{\lambda_1} - F^{i\bar{i}}\frac{|\widetilde{\lambda}_{1,i}|^2}{\lambda_1^2} + \psi' F^{i\bar{i}} u_{i\bar{i}} + \psi'' F^{i\bar{i}}|u_i|^2 + \varphi' F^{i\bar{i}}(|\nabla u|^2)_{i\bar{i}} + \varphi'' F^{i\bar{i}}|(|\nabla u\,|^2)_i|^2. \tag{4.14}$$

We will control some terms in (4.14). Covariant differentiating equation (2.4) twice in the $\frac{\partial}{\partial z^1}$ direction and the $\frac{\partial}{\partial \bar{z}^1}$ direction, we have

$$F^{i\bar{i}}X_{i\bar{i}1} + \sum_{l=0}^{k-2}(\beta_l)_1 F_l = (\beta_{k-1})_1 \tag{4.15}$$

and

$$F^{i\bar{j},p\bar{q}}X_{i\bar{j}1}X_{p\bar{q}\bar{1}} + F^{i\bar{i}}X_{i\bar{i}1\bar{1}} + 2\mathrm{Re}\left(\sum_{l=0}^{k-2}(\beta_l)_{\bar{1}}F_l^{i\bar{i}}X_{i\bar{i}1}\right) + \sum_{l=0}^{k-2}(\beta_l)_{1\bar{1}}F_l = (\beta_{k-1})_{1\bar{1}}. \tag{4.16}$$

Direct calculation deduces that

$$F^{i\bar{i}}X_{i\bar{i}} = F^{i\bar{i}}\lambda_i = F_k^{i\bar{i}}\lambda_i + \sum_{l=0}^{k-2}\beta_l F_l^{i\bar{i}}\lambda_i = \beta_{k-1} - \sum_{l=0}^{k-2}(k-l)\beta_l F_l. \tag{4.17}$$

From Lemmas 4.1 and (4.16), we can estimate the first term in (4.14)

$$\begin{aligned} F^{i\bar{i}}\frac{\widetilde{\lambda}_{1,i\bar{i}}}{\lambda_1} &\ge \frac{1}{\lambda_1}F^{i\bar{i}}X_{i\bar{i}1\bar{1}} + \frac{2}{\lambda_1}F^{i\bar{i}}\mathrm{Re}\left(X_{1\bar{1}i}\overline{T_{1i}^1}\right) - C_0\mathcal{F} \\ &= -\frac{1}{\lambda_1}F^{i\bar{j},p\bar{q}}X_{i\bar{j}1}X_{p\bar{q}\bar{1}} - \frac{2}{\lambda_1}\mathrm{Re}\left(\sum_{l=0}^{k-2}(\beta_l)_{\bar{1}}F_l^{i\bar{i}}X_{i\bar{i}1}\right) - \frac{1}{\lambda_1}\sum_{l=0}^{k-2}(\beta_l)_{1\bar{1}}F_l \\ &\quad + \frac{(\beta_{k-1})_{1\bar{1}}}{\lambda_1} + \frac{2}{\lambda_1}F^{i\bar{i}}\mathrm{Re}\left(X_{1\bar{1}i}\overline{T_{1i}^1}\right) - C_0\mathcal{F}. \end{aligned} \tag{4.18}$$

It is shown by Krylov in [22] that the $\left(\frac{\sigma_{k-1}}{\sigma_l}\right)^{\frac{1}{k-l-1}}$ is concave in Γ_{k-1} for $0 \le l \le k-2$, which means that

$$\left((-F_l)^{-\frac{1}{k-l-1}}\right)^{i\bar{i},j\bar{j}}X_{i\bar{i}1}X_{j\bar{j}\bar{1}} \le 0.$$

Direct computation gives

$$-F_l^{i\bar{i},j\bar{j}}X_{i\bar{i}1}X_{j\bar{j}\bar{1}} \ge \frac{k-l}{k-l-1}(-F_l)^{-1}|F_l^{i\bar{i}}X_{i\bar{i}1}|^2,$$

which yields

$$\begin{aligned} &-\frac{F^{i\bar{i},j\bar{j}}X_{i\bar{i}1}X_{j\bar{j}\bar{1}}}{\lambda_1} - \frac{2}{\lambda_1}\mathrm{Re}\left(\sum_{l=0}^{k-2}(\beta_l)_{\bar{1}}F_l^{i\bar{i}}X_{i\bar{i}1}\right) \\ &\ge \sum_{l=0}^{k-2}\frac{k-l}{k-l-1}\frac{\beta_l}{\lambda_1}(-F_l)^{-1}\left|F_l^{i\bar{i}}X_{i\bar{i}1} + \frac{k-l-1}{k-l}\frac{(\beta_l)_{\bar{1}}}{\beta_l}F_l\right|^2 + \sum_{l=0}^{k-2}\frac{k-l-1}{k-l}\frac{(\beta_l)_{\bar{1}}^2}{\beta_l\lambda_1}F_l \end{aligned}$$

$$\geq \sum_{l=0}^{k-2} \frac{k-l-1}{k-l} \frac{(\beta_l)_1^2}{\beta_l \lambda_1} F_l$$
$$\geq -C_0,$$

where the last inequality is given by Lemma 2.2. Noting that

$$-F^{i\bar{j},p\bar{q}} X_{i\bar{j}1} X_{p\bar{q}\bar{1}} \geq -F^{i\bar{i},j\bar{j}} X_{i\bar{i}1} X_{j\bar{j}\bar{1}} - F^{i\bar{1},1\bar{i}} |X_{i\bar{1}1}|^2,$$

we have

$$-\frac{1}{\lambda_1} F^{i\bar{j},p\bar{q}} X_{i\bar{j}1} X_{p\bar{q}\bar{1}} - \frac{2}{\lambda_1} \mathrm{Re}\left(\sum_{l=0}^{k-2} (\beta_l)_{\bar{1}} F_l^{i\bar{i}} X_{i\bar{i}1} \right) \geq -F^{i\bar{1},1\bar{i}} |X_{i\bar{1}1}|^2 - C_0. \tag{4.19}$$

Substituting (4.19) into (4.18) and by Lemma 2.2,

$$F^{i\bar{i}} \frac{\widetilde{\lambda}_{1,i\bar{i}}}{\lambda_1} \geq -\frac{F^{i\bar{1},1\bar{i}} |X_{i\bar{1}1}|^2}{\lambda_1} + \frac{2}{\lambda_1} F^{i\bar{i}} \mathrm{Re}\left(X_{1\bar{1}i} \overline{T^1_{1i}} \right) - C_0 \mathcal{F} - C_0. \tag{4.20}$$

Since

$$X_{1\bar{1}i} = \chi_{1\bar{1}i} + u_{1\bar{1}i} = \left(\chi_{1\bar{1}i} - \chi_{i\bar{1}1} + T^p_{i1} \chi_{p\bar{1}} \right) + X_{i\bar{1}1} - T^1_{i1} \lambda_1, \tag{4.21}$$

we have

$$|X_{1\bar{1}i}|^2 \leq |X_{i\bar{1}1}|^2 - 2\lambda_1 \mathrm{Re}\left(X_{i\bar{1}1} \overline{T^1_{i1}} \right) + C_0\left(\lambda_1^2 + |X_{1\bar{1}i}| \right). \tag{4.22}$$

From

$$\widetilde{\lambda}_{1,i} = X_{1\bar{1}i} - (D^{11})_i,$$

we estimate the second term in (4.14)

$$\begin{aligned} -F^{i\bar{i}} \frac{|\widetilde{\lambda}_{1,i}|^2}{\lambda_1^2} &= -F^{i\bar{i}} \frac{|X_{1\bar{1}i}|^2}{\lambda_1^2} + \frac{2}{\lambda_1^2} F^{i\bar{i}} \mathrm{Re}(X_{1\bar{1}i} (D^{11})_{\bar{i}}) - \frac{F^{i\bar{i}} |(D^{11})_i|^2}{\lambda_i^2} \\ &\geq -F^{i\bar{i}} \frac{|X_{1\bar{1}i}|^2}{\lambda_1^2} - \frac{C_0}{\lambda_1^2} F^{i\bar{i}} |X_{1\bar{1}i}| - C_0 \mathcal{F} \\ &\geq -F^{i\bar{i}} \frac{|X_{i\bar{1}1}|^2}{\lambda_1^2} - \frac{2}{\lambda_1} F^{i\bar{i}} \mathrm{Re}\left(X_{1\bar{1}i} \overline{T^1_{1i}} \right) - \frac{C_0}{\lambda_1^2} F^{i\bar{i}} |X_{1\bar{1}i}| - C_0 \mathcal{F}, \end{aligned} \tag{4.23}$$

where the last inequality is given by (4.21) and (4.22). By (4.1), we have the identities

$$u_{pi\bar{i}} = u_{i\bar{i}p} - T^i_{ip} \lambda_i + T^q_{ip} \chi_{q\bar{i}} + R_{i\bar{i}p\bar{q}} u_q, \tag{4.24}$$

$$u_{\bar{p}i\bar{i}} = u_{i\bar{i}\bar{p}} - \overline{T^i_{ip}} \lambda_i + \overline{T^q_{ip}} \chi_{i\bar{q}}. \tag{4.25}$$

It follows from (4.24) and (4.25) that

$$\begin{aligned} F^{i\bar{i}} u_{pi\bar{i}} u_{\bar{p}} &= F^{i\bar{i}} u_{i\bar{i}p} u_{\bar{p}} - F^{i\bar{i}} T^i_{ip} \lambda_i u_{\bar{p}} + F^{i\bar{i}} T^q_{ip} \chi_{q\bar{i}} u_{\bar{p}} + F^{i\bar{i}} R_{i\bar{i}p\bar{q}} u_q u_{\bar{p}} \\ &= -F^{i\bar{i}} \chi_{i\bar{i}p} u_{\bar{p}} - \sum_{l=0}^{k-2} (\beta_l)_p F_l u_{\bar{p}} + (\beta_{k-1})_p u_{\bar{p}} - F^{i\bar{i}} T^i_{ip} \lambda_i u_{\bar{p}} + F^{i\bar{i}} T^q_{ip} \chi_{q\bar{i}} u_{\bar{p}} + F^{i\bar{i}} R_{i\bar{i}p\bar{q}} u_q u_{\bar{p}} \\ &\geq -C_0\left(K^{\frac{1}{2}} \mathcal{F} + K^{\frac{1}{2}} + K^{\frac{1}{2}} + K^{\frac{1}{2}} \mathcal{F} + K\mathcal{F} \right) - C_0 K^{\frac{1}{2}} F^{i\bar{i}} \lambda_i, \end{aligned} \tag{4.26}$$

where the second equality is given by (4.15) and the last inequality given by Lemma 2.2. From (4.16) and $K^{\frac{1}{2}} \leq \frac{1}{4} + K$, we obtain

$$-C_0 K^{\frac{1}{2}} F^{i\bar{i}} \lambda_i = -C_0 K^{\frac{1}{2}} \left(\beta_{k-1} - \sum_{l=0}^{k-2} (k-l) \beta_l F_l \right) \geq -C_0 - C_0 K. \tag{4.27}$$

Noting $\varphi' \geq \frac{1}{4K}$, we substitute (4.27) into (4.26),

$$\varphi' F^{i\bar{i}} u_{pi\bar{i}} u_{\bar{p}} \geq -C_0\mathcal{F} - \frac{C_0}{K}\mathcal{F} - C_0 - \frac{C_0}{K}. \tag{4.28}$$

The same estimate also holds for $\varphi' F^{i\bar{i}} u_{\bar{p}i\bar{i}} u_p$. From (4.28), we can estimate the fifth term in (4.14)

$$\begin{aligned}\varphi' F^{i\bar{i}}(|\nabla u|^2)_{i\bar{i}} &= \varphi' F^{i\bar{i}}\Big(u_{pi\bar{i}}u_{\bar{p}} + u_{\bar{p}i\bar{i}}u_p\Big) + \varphi' F^{i\bar{i}}\sum_p\Big(|u_{pi}|^2 + |u_{\bar{p}i}|^2\Big)\\ &\geq -C_0\mathcal{F} - \frac{C_0}{K}\mathcal{F} - C_0 - \frac{C_0}{K} + \frac{1}{4K}F^{i\bar{i}}\sum_p\Big(|u_{pi}|^2 + |u_{\bar{p}i}|^2\Big).\end{aligned} \tag{4.29}$$

Substituting (4.20), (4.23), and (4.29) into (4.14)

$$\begin{aligned}0 \geq{} & \frac{-F^{i\bar{1},1\bar{i}}|X_{i\bar{1}1}|^2}{\lambda_1} - F^{i\bar{i}}\frac{|X_{i\bar{1}1}|^2}{\lambda_1^2} - \frac{C_0}{\lambda_1}F^{i\bar{i}}\frac{|X_{1\bar{1}i}|}{\lambda_1} + \frac{1}{4K}F^{i\bar{i}}\sum_p\Big(|u_{pi}|^2 + |u_{\bar{p}i}|^2\Big)\\ &+ \psi' F^{i\bar{i}}u_{i\bar{i}} + \psi'' F^{i\bar{i}}|u_i|^2 + \varphi'' F^{i\bar{i}}|(|\nabla u|^2)_i|^2 - C_0\mathcal{F} - C_0 - \frac{C_0}{K}\mathcal{F} - \frac{C_0}{K}.\end{aligned} \tag{4.30}$$

We set

$$\delta = \min\left\{\frac{1}{1+4E}, \frac{1}{2}\right\}, \tag{4.31}$$

where

$$\frac{1}{1+4E} = \frac{1}{1+8L(C_1+1)}, \quad C_1 = \left(1+\frac{1}{K}\right)\frac{C_0}{\theta},$$

with θ in Lemma 2.6. Then we have two cases to consider.

Case 1 $\lambda_n < -\delta\lambda_1$. By using the critical point condition (4.12), we obtain

$$\begin{aligned}-\frac{F^{i\bar{i}}|X_{1\bar{1}i}|^2}{\lambda_1^2} &= -F^{i\bar{i}}|\varphi'(|\nabla u|^2)_i + \psi' u_i - \frac{(D^{11})_i}{\lambda_1}|^2\\ &\geq -2(\varphi')^2F^{i\bar{i}}|(|\nabla u|^2)_i|^2 - 2F^{i\bar{i}}|\psi' u_i - \frac{(D^{11})_i}{\lambda_1}|^2\\ &\geq -2(\varphi')^2F^{i\bar{i}}|(|\nabla u|^2)_i|^2 - 4|\psi'|^2K\mathcal{F} - C_0\mathcal{F}.\end{aligned} \tag{4.33}$$

It follows from (4.17) that

$$\psi' F^{i\bar{i}}u_{i\bar{i}} = \psi' F^{i\bar{i}}(X_{i\bar{i}} - \chi_{i\bar{i}}) \geq \psi'\left(\beta_{k-1} - \sum_{l=0}^{k-2}(k-l)\beta_l F_l - C_0\mathcal{F}\right) \geq \psi'(1+\mathcal{F})C_0. \tag{4.34}$$

By the fact that

$$\frac{|X_{1\bar{1}i}|}{\lambda_1} = -\varphi'\Big(u_{pi}u_{\bar{p}} + u_p u_{\bar{p}i}\Big) - \psi' u_i + \frac{(D^{11})_i}{\lambda_1},$$

we have

$$-\frac{C_0}{\lambda_1}F^{i\bar{i}}\frac{|X_{1\bar{1}i}|}{\lambda_1} \geq -\frac{C_0}{\lambda_1}K^{-\frac{1}{2}}F^{i\bar{i}}\Big(|u_{pi}| + |u_{\bar{p}i}|\Big) + \frac{C_0}{\lambda_1}\psi' K^{\frac{1}{2}}\mathcal{F} - C_0\mathcal{F}. \tag{4.35}$$

Since

$$-F^{i\bar{1},\bar{1}i} = \frac{F^{i\bar{i}} - F^{1\bar{1}}}{\lambda_1 - \lambda_i} \quad \text{and} \quad \lambda_1 \geq \cdots \geq \lambda_n,$$

we have

$$\frac{-F^{i\bar{1},1\bar{i}}|X_{i\bar{1}1}|^2}{\lambda_1} \geq 0. \tag{4.36}$$

Since $\varphi'' = 2(\varphi')^2$ and $\psi'' > 0$, substituting (4.32), (4.33), (4.34), and (4.35) into (4.30), we obtain

$$\begin{aligned} 0 \geq & \frac{1}{4K}F^{i\bar{i}}\sum_p\Big(|u_{pi}|^2 + |u_{\bar{p}i}|^2\Big) - \frac{C_0}{\lambda_1}K^{\frac{1}{2}}F^{i\bar{i}}\sum_p\Big(|u_{pi}| + |u_{\bar{p}i}|\Big) + \frac{C_0}{\lambda_1}K^{\frac{1}{2}}\psi'\mathcal{F} \\ & + \psi'C_0(1+\mathcal{F}) - 4(\psi')^2K\mathcal{F} - C_0(1+\mathcal{F})\left(1+\frac{1}{K}\right) \\ \geq & \frac{1}{8K}F^{i\bar{i}}\sum_p\Big(|u_{pi}|^2 + |u_{\bar{p}i}|^2\Big) + \frac{C_0}{\lambda_1}K^{\frac{1}{2}}\psi'\mathcal{F} + \psi'C_0(1+\mathcal{F}) - 4(\psi')^2K\mathcal{F} - C_0(1+\mathcal{F})\left(1+\frac{1}{K}\right), \end{aligned} \tag{4.36}$$

where the last inequality is obtained by using the first term absorbing the $|u_{pi}|$, $|u_{\bar{p}i}|$ terms.

From Lemma 2.4, we know that $\mathcal{F}$ is controlled by the uniform positive constant, which means that $\mathcal{F}$ can be absorbed by C_0. Noticing that

$$F^{i\bar{i}}|u_{i\bar{i}}|^2 = F^{i\bar{i}}\Big(\lambda_i - \chi_{i\bar{i}}\Big)^2 \geq \frac{1}{2}F^{i\bar{i}}\lambda_i^2 - C_0\mathcal{F}, \quad \text{and} \quad F^{n\bar{n}} \geq \frac{\mathcal{F}}{n} \geq \frac{n-k+1}{nk},$$

we have

$$\begin{aligned} 0 \geq & \frac{1}{16K}F^{i\bar{i}}\lambda_i^2 - C_0K^{\frac{1}{2}}(1+C_1) - C_0(1+C_1) - C_0(1+C_1)^2K - C_0\left(1+\frac{1}{K}\right) \\ \geq & \frac{(n-k+1)\delta^2}{16nkK}\lambda_1^2 - C_0(1+C_1)^2K - C_0\left(1+\frac{1}{K}\right). \end{aligned}$$

This inequality implies $\lambda_1 \leq CK$.

Case 2 $\lambda_n \geq -\delta\lambda_1$. Let

$$I = \{i \in \{1,\dots,n\} | F^{i\bar{i}} > \delta^{-1}F^{1\bar{1}}\}.$$

For those indices, which are not in I, we have

$$\begin{aligned} -\sum_{i\notin I}\frac{F^{i\bar{i}}|X_{1\bar{1}i}|^2}{\lambda_1^2} &= -\sum_{i\notin I}F^{i\bar{i}}\left|\varphi'(|\nabla u|^2)_i + \psi'u_i - \frac{(D^{11})_i}{\lambda_1}\right|^2 \\ &\geq -2(\varphi')^2\sum_{i\notin I}F^{i\bar{i}}|(|\nabla u|^2)_i|^2 - 2\sum_{i\notin I}F^{i\bar{i}}|\psi'u_i - \frac{(D^{11})_i}{\lambda_1}|^2 \\ &\geq -2(\varphi')^2\sum_{i\notin I}F^{i\bar{i}}|(|\nabla u|^2)_i|^2 - \frac{4K}{\delta}|\psi'|^2F^{1\bar{1}} - C_0\mathcal{F}. \end{aligned} \tag{4.37}$$

From (4.12), (4.37), and

$$\frac{|X_{i\bar{1}1}|^2}{\lambda_1^2} \leq \frac{|X_{1\bar{1}i}|^2}{\lambda_1^2} + C_0\left(1 + \frac{|X_{1\bar{1}i}|}{\lambda_1}\right),$$

we obtain

$$\begin{aligned} -\sum_{i\notin I}\frac{F^{i\bar{i}}|X_{i\bar{1}1}|^2}{\lambda_1^2} \geq & -2(\varphi')^2\sum_{i\notin I}F^{i\bar{i}}|(|\nabla u|^2)_i|^2 - \frac{4K}{\delta}|\psi'|^2F^{1\bar{1}} + C_0\varphi'\sum_{i\notin I}F^{i\bar{i}}(|\nabla u|^2)_i + C_0\psi'\sum_{i\notin I}F^{i\bar{i}}u_i - C_0\mathcal{F} \\ \geq & -2(\varphi')^2\sum_{i\notin I}F^{i\bar{i}}|(|\nabla u|^2)_i|^2 - \frac{4K}{\delta}|\psi'|^2F^{1\bar{1}} - C_0\varphi'K^{\frac{1}{2}}F^{i\bar{i}}\sum_p\Big(|u_{pi}| + |u_{\bar{p}i}|\Big) + C_0K^{\frac{1}{2}}\psi'\delta^{-1}F^{1\bar{1}} - C_0\mathcal{F}. \end{aligned} \tag{4.38}$$

Since

$$-F^{i\bar{1},\bar{1}i} = \frac{F^{i\bar{i}} - F^{1\bar{1}}}{X_{1\bar{1}} - X_{i\bar{i}}} \quad \text{and} \quad \lambda_i \geq \lambda_n \geq -\delta\lambda_1,$$

we have

$$-\sum_{i\in I}F^{i\bar{1},\bar{1}i} \geq \frac{1-\delta}{1+\delta}\frac{1}{\lambda_1}\sum_{i\in I}F^{i\bar{i}}, \tag{4.39}$$

which yields

$$-\frac{F^{i\bar{1},1\bar{i}}|X_{i\bar{1}1}|^2}{\lambda_1} \geq \frac{1-\delta}{1+\delta}\sum_{i\in I}F^{i\bar{i}}\frac{|X_{i\bar{1}1}|^2}{\lambda_1^2}. \tag{4.40}$$

Recalling that $\varphi'' = 2(\varphi')^2$ and $0 < \delta \leq \frac{1}{2}$, we obtain from (4.12)

$$\begin{aligned}\sum_{i\in I}\varphi'' F^{i\bar{i}}|(|\nabla u|^2)_i|^2 &= 2\sum_{i\in I}F^{i\bar{i}}\left|\frac{X_{i\bar{1}1}}{\lambda_1} + \psi' u_i - \frac{(D^{11})_i}{\lambda_1} - T^1_{i1} + \frac{\chi_{11i} - \chi_{i11} + T^p_{i1}\chi_{p\bar{1}}}{\lambda_1}\right|^2\\ &\geq 2\sum_{i\in I}F^{i\bar{i}}\left(\delta\left|\frac{X_{i\bar{1}1}}{\lambda_1}\right|^2 - \frac{2\delta}{1-\delta}(\psi')^2|u_i|^2 - C_0\right)\\ &\geq 2\delta\sum_{i\in I}F^{i\bar{i}}\left|\frac{X_{i\bar{1}1}}{\lambda_1}\right|^2 - \frac{4\delta}{1-\delta}(\psi')^2F^{i\bar{i}}|u_i|^2 - C_0\mathcal{F}.\end{aligned} \tag{4.41}$$

Noticing the fact that $\psi'' \geq \frac{4\varepsilon}{1-\varepsilon}(\psi')^2$, for all $\varepsilon \leq \frac{1}{4E+1} = \delta$, we have

$$\psi'' F^{i\bar{i}}|u_i|^2 - \frac{4\delta}{1-\delta}(\psi')^2F^{i\bar{i}}|u_i|^2 \geq 0. \tag{4.42}$$

Inserting (4.38), (4.40), (4.41), and (4.42) into (4.30), we deduce that

$$\begin{aligned}0 &\geq \frac{1}{4K}\sum_p F^{i\bar{i}}(|u_{pi}|^2 + |u_{\bar{p}i}|^2) + \left(\frac{1-\delta}{1+\delta} + 2\delta - 1\right)\sum_{i\in I}F^{i\bar{i}}\frac{|X_{i\bar{1}1}|^2}{\lambda_1^2} + \psi' F^{i\bar{i}}u_{i\bar{i}}\\ &\quad - \frac{4K}{\delta}(\psi')^2F^{1\bar{1}} + C_0K^{\frac{1}{2}}\psi'\delta^{-1}F^{1\bar{1}} - \frac{C_0}{\lambda_1}F^{i\bar{i}}\frac{|X_{1\bar{1}i}|}{\lambda_1} - \left(1+\frac{1}{K}\right)C_0\mathcal{F}\\ &\quad - C_0 - \frac{C_0}{K} - C_0K^{-\frac{1}{2}}F^{i\bar{i}}\sum_p\left(|u_{pi}| + |u_{\bar{p}i}|\right)\\ &\geq \frac{1}{8K}\sum_p F^{i\bar{i}}(|u_{pi}|^2 + |u_{\bar{p}i}|^2) + \psi' F^{i\bar{i}}u_{i\bar{i}} - \frac{C_0}{\lambda_1}F^{i\bar{i}}\frac{|X_{1\bar{1}i}|}{\lambda_1} - \frac{4K}{\delta}(\psi')^2F^{1\bar{1}}\\ &\quad + C_0K^{\frac{1}{2}}\psi'\delta^{-1}F^{1\bar{1}} - \left(1+\frac{1}{K}\right)C_0\mathcal{F} - C_0 - \frac{C_0}{K},\end{aligned} \tag{4.43}$$

where the last inequality is given by using the first term absorbing the $|u_{pi}|$, $|u_{\bar{p}i}|$ terms. Combining (4.34) with (4.43), we obtain

$$\begin{aligned}0 &\geq \frac{1}{16K}\sum_p F^{i\bar{i}}(|u_{pi}|^2 + |u_{\bar{p}i}|^2) + \psi' F^{i\bar{i}}u_{i\bar{i}} - \frac{4K}{\delta}(\psi')^2F^{1\bar{1}}\\ &\quad + C_0K^{\frac{1}{2}}\psi'\delta^{-1}F^{1\bar{1}} + \frac{C_0}{\lambda_1}\psi' K^{\frac{1}{2}}\mathcal{F} - \left(1+\frac{1}{K}\right)C_0\mathcal{F} - C_0 - \frac{C_0}{K}.\end{aligned} \tag{4.44}$$

From Lemma 2.4, we know that $\mathcal{F}$ is controlled by the uniform positive constant, which means that $\mathcal{F}$ can be absorbed by C_0. Noticing that

$$F^{i\bar{i}}|u_{i\bar{i}}|^2 = F^{i\bar{i}}\left(\lambda_i - \chi_{i\bar{i}}\right)^2 \geq \frac{1}{2}F^{i\bar{i}}\lambda_i^2 - C_0\mathcal{F},$$

we obtain

$$0 \geq \frac{1}{32K}\sum_p F^{i\bar{i}}\lambda_i^2 + \psi' F^{i\bar{i}}u_{i\bar{i}} - \frac{4K}{\delta}(\psi')^2F^{1\bar{1}} + C_0K^{\frac{1}{2}}\psi'\delta^{-1}F^{1\bar{1}} + \frac{C_0}{\lambda_1}\psi' K^{\frac{1}{2}} - \left(1+\frac{1}{K}\right)C_0. \tag{4.45}$$

We may assume $\lambda_1 \geq 2(C_1 + 1)K$, then

$$\frac{C_0}{\lambda_1}\psi' K^{\frac{1}{2}} \geq -C_0 K^{-\frac{1}{2}} \geq -\left(1 + \frac{1}{K}\right)C_0.$$

There are two cases to consider from Lemma 2.5.

If (2.6) holds, then we obtain $\psi' F^{i\bar{i}} u_{i\bar{i}} \geq (C_1 + 1)\theta(1 + \mathcal{F}) \geq (C_1 + 1)\theta$. Substituting this into (4.45) yields

$$0 \geq \frac{1}{32K} F^{1\bar{1}}\lambda_1^2 + (C_1 + 1)\theta - \frac{16(C_1 + 1)^2 K}{\delta} F^{1\bar{1}} - \frac{C_0(C_1 + 1)K^{\frac{1}{2}}}{\delta} F^{1\bar{1}} - \left(1 + \frac{1}{K}\right)C_0.$$

Recall that

$$C_1 = \left(1 + \frac{1}{K}\right)\frac{C_0}{\theta}.$$

We then obtain

$$0 \geq \frac{1}{32K}\lambda_1^2 - \frac{16(C_1 + 1)^2 K}{\delta} - \frac{C_0(C_1 + 1)K^{\frac{1}{2}}}{\delta},$$

which implies $\lambda_1 \leq CK$.

If (2.7) holds, then, by (4.17),

$$0 \geq \frac{1}{32K}\lambda_1^2 - \frac{16(C_1 + 1)^2 K}{\delta} - \frac{C_0(C_1 + 1)K^{\frac{1}{2}}}{\delta} - \theta\left(1 + \frac{1}{K}\right)C_0 - \theta(C_1 + 1)C_0.$$

This inequality again implies $\lambda_1 \leq CK$. □

5 C^1 estimates

In this section, we obtain the following gradient estimate by the blowup method and the Liouville theorem as suggested by Dinew and Kolodziej [10]. The argument follows closely Proposition 5.1 in [27], so we omit the proof.

Proposition 5.1. *Let* $\alpha_l(x) > 0$ *for* $0 \leq l \leq k - 2$ *and* χ *be a smooth real* $(1, 1)$*-form on* (M, g). *Assume that* u *and* $\underline{u}$ *are solution and* C*-subsolution to* (1.1) *with* $\lambda(\chi_u) \in \Gamma_{k-1}$, $\lambda(\chi_{\underline{u}}) \in \Gamma_{k-1}$, *respectively. We normalize* u *such that* $\sup_M(u - \underline{u}) = 0$. *Then there is an estimate*

$$\sup_M |\nabla u| \leq C,$$

where C *is a uniform constant.*

6 Proof of main theorem

From the standard regularity theory of uniformly elliptic partial differential equations, we can obtain the high order regularity. We refer the readers to Tosatti et al. [32]. In this section, we prove Theorem 1.2 and Corollaries 1.3 and 1.4 by the method of continuity. As explicitly shown in the proofs of the estimates up to second-order, we have to find a uniform C-subsolution condition for all the solution flow of the continuity method.

Proof of Theorem 1.2

Proof. Define $\underline{\phi}$ by

$$\frac{\sigma_k(\lambda(\chi_{\underline{u}}))}{\sigma_{k-1}(\lambda(\chi_{\underline{u}}))} - \sum_{l=0}^{k-2}\beta_l \frac{\sigma_l(\lambda(\chi_{\underline{u}}))}{\sigma_{k-1}(\lambda(\chi_{\underline{u}}))} = \underline{\phi}(x),$$

where $\lambda(\chi_{\underline{u}}) \in \Gamma_{k-1}$. It is easy to see that

$$\lim_{t\to\infty}\left(\frac{\sigma_k(\lambda(\chi_{\underline{u}}) + te_i)}{\sigma_{k-1}(\lambda(\chi_{\underline{u}}) + te_i)} - \sum_{l=0}^{k-2}\beta_l\frac{\sigma_l(\lambda(\chi_{\underline{u}}) + te_i)}{\sigma_{k-1}(\lambda(\chi_{\underline{u}}) + te_i)}\right) > \underline{\phi}(x), \tag{6.1}$$

where e_i is the ith standard basis vector in $\mathbb{R}^n$. Since $\underline{u}$ is a C-subsolution of equation (1.1),

$$\lim_{t\to\infty}\left(\frac{\sigma_k(\lambda(\chi_{\underline{u}}) + te_i)}{\sigma_{k-1}(\lambda(\chi_{\underline{u}}) + te_i)} - \sum_{l=0}^{k-2}\beta_l\frac{\sigma_l(\lambda(\chi_{\underline{u}}) + te_i)}{\sigma_{k-1}(\lambda(\chi_{\underline{u}}) + te_i)}\right) > \beta_{k-1}. \tag{6.2}$$

We consider

$$\frac{\sigma_k(\lambda(\chi_u))}{\sigma_{k-1}(\lambda(\chi_u))} - \sum_{l=0}^{k-2}\beta_l\frac{\sigma_l(\lambda(\chi_u))}{\sigma_{k-1}(\lambda(\chi_u))} = (1-t)\underline{\phi}(x) + t\beta_{k-1}(x) + b_t, \tag{6.3}$$

where $\lambda(\chi_u) \in \Gamma_{k-1}$ and b_t is a constant for each t. Set

$$T := \{t' \in [0,1] | \exists u \in C^{2,\alpha}(M) \quad \text{and} \quad b_t \quad \text{solving (6.3) for } t \in [0,t']\}.$$

As shown in [20], the continuity method works if we can guarantee (1) $0 \in T$ and (2) uniform C^∞ estimates for all u. When $t = 0$, $b_0 = 0$ by the uniqueness, the first requirement is naturally met. For the second requirement, we only need to show a uniform C-subsolution for all the solution flow. The condition (1.2) yields

$$\underline{\phi}(x) \le \beta_{k-1}(x).$$

At the maximum point of $u - \underline{u}$,

$$\frac{\sigma_k(\lambda(\chi_u))}{\sigma_{k-1}(\lambda(\chi_u))} - \sum_{l=0}^{k-2}\beta_l\frac{\sigma_l(\lambda(\chi_u))}{\sigma_{k-1}(\lambda(\chi_u))} \le \frac{\sigma_k(\lambda(\chi_{\underline{u}}))}{\sigma_{k-1}(\lambda(\chi_{\underline{u}}))} - \sum_{l=0}^{k-2}\alpha_l\frac{\beta_l(\lambda(\chi_{\underline{u}}))}{\sigma_{k-1}(\lambda(\chi_{\underline{u}}))} = \underline{\phi}(x),$$

which means that

$$(1-t)\underline{\phi}(x) + t\beta_{k-1}(x) + b_t \le \underline{\phi}(x),$$

so

$$b_t \le 0.$$

This means that

$$(1-t)\underline{\phi}(x) + t\beta_{k-1}(x) + b_t \le \beta_{k-1}(x).$$

Then C-subsolution condition is uniform for all the solution flow. As a result, we have uniform C^∞ estimates of u. □

Proof of Corollary 1.3

Proof. We can find a smooth real function h satisfying that all $x \in M$

$$h(x) \ge \max\{\underline{\phi}(x), \beta_{k-1}(x)\}$$

and

$$\lim_{t\to\infty}\left(\frac{\sigma_k(\lambda(\chi_{\underline{u}}) + te_i)}{\sigma_{k-1}(\lambda(\chi_{\underline{u}}) + te_i)} - \sum_{l=0}^{k-2}\beta_l\frac{\sigma_l(\lambda(\chi_{\underline{u}}) + te_i)}{\sigma_{k-1}(\lambda(\chi_{\underline{u}}) + te_i)}\right) > h(x), \tag{6.4}$$

which means that the set

$$\left\{\widetilde{\chi} \in \Gamma_{k-1}^g | \widetilde{\chi}^k \wedge \omega^{n-k} \le \sum_{l=0}^{k-2}\alpha_l(x)\widetilde{\chi}^l \wedge \omega^{n-l} + \frac{C_n^{k-1}}{C_n^k}h(x)\widetilde{\chi}^{k-1} \wedge \omega^{n-k+1} \quad \text{and} \quad \widetilde{\chi} - \chi_{\underline{u}} \ge 0\right\}$$

is bounded. First, we consider

$$\frac{\sigma_k(\lambda(\chi_u))}{\sigma_{k-1}(\lambda(\chi_u))} - \sum_{l=0}^{k-2}\beta_l \frac{\sigma_l(\lambda(\chi_u))}{\sigma_{k-1}(\lambda(\chi_u))} = (1-t)\underline{\phi}(x) + th(x) + a_t, \tag{6.5}$$

where $\lambda(\chi_u) \in \Gamma_{k-1}$ and a_t is a constant for each t. Set

$$T_1 := \{t' \in [0,1] | \exists u \in C^{2,\alpha}(M) \quad \text{and} \quad a_t \text{ solving (6.5) for } t \in [0,t']\}.$$

When $a_0 = 0$, $0 \in T_1$. At the maximum point of $u - \underline{u}$, we obtain

$$\frac{\sigma_k(\lambda(\chi_u))}{\sigma_{k-1}(\lambda(\chi_u))} - \sum_{l=0}^{k-2}\beta_l \frac{\sigma_l(\lambda(\chi_u))}{\sigma_{k-1}(\lambda(\chi_u))} \le \frac{\sigma_k(\lambda(\chi_{\underline{u}}))}{\sigma_{k-1}(\lambda(\chi_{\underline{u}}))} - \sum_{l=0}^{k-2}\alpha_l \frac{\beta_l(\lambda(\chi_{\underline{u}}))}{\sigma_{k-1}(\lambda(\chi_{\underline{u}}))} = \underline{\phi}(x),$$

which means that

$$(1-t)\underline{\phi}(x) + th(x) + a_t \le \underline{\phi}(x),$$

so

$$a_t \le 0.$$

Obviously,

$$(1-t)\underline{\phi}(x) + th(x) + a_t \le h(x).$$

Second, we consider the family of equations:

$$\frac{\sigma_k(\lambda(\chi_u))}{\sigma_{k-1}(\lambda(\chi_u))} - \sum_{l=0}^{k-2}\beta_l \frac{\sigma_l(\lambda(\chi_u))}{\sigma_{k-1}(\lambda(\chi_u))} = (1-t)h(x) + t\beta_{k-1}(x) + b_t, \tag{6.6}$$

where $\lambda(\chi_u) \in \Gamma_{k-1}$ and b_t is a constant for each t. Set

$$T_2 := \{t' \in [0,1] | \exists u \in C^{2,\alpha}(M) \quad \text{and} \quad b_t \text{ solving (6.6) for } t \in [0,t']\}.$$

Clearly, $0 \in T_2$ with $b_0 = a_1$. Integrating (6.6) on M, we have

$$\begin{aligned}
\int_M \chi^k \wedge \omega^{n-k} &= \sum_{l=0}^{k-2}\int_M \alpha_l \chi_u^l \wedge \omega^{n-l} + \int_M \left((1-t)\frac{C_n^{k-1}}{C_n^k}h + t\alpha_{k-1} + \frac{C_n^{k-1}}{C_n^k}b_t\right)\chi_u^{k-1} \wedge \omega^{n-k+1} \\
&\ge \sum_{l=0}^{k-2}\int_M c\chi_u^l \wedge \omega^{n-l} + \int_M \left(\alpha_{k-1} + \frac{C_n^{k-1}}{C_n^k}b_t\right)\chi_u^{k-1} \wedge \omega^{n-k+1} \\
&\ge \sum_{l=0}^{k-1} c_l \int_M \chi^l \wedge \omega^{n-l} + \frac{C_n^{k-1}}{C_n^k}b_t \int_M \chi^{k-1} \wedge \omega^{n-k+1} \\
&\ge \int_M \chi^k \wedge \omega^{n-k} + \frac{C_n^{k-1}}{C_n^k}b_t \int_M \chi^{k-1} \wedge \omega^{n-k+1}.
\end{aligned}$$

The last inequality is given by the condition (1.4). Hence,

$$b_t \le 0.$$

This means that

$$(1-t)h(x) + t\beta_{k-1}(x) + b_t \le h(x).$$

Then C-subsolution condition is uniform for all the solution flow. As a result, we have uniform C^∞ estimates of u. □

Proof of Corollary 1.4

Proof. The cone condition (1.7) is equivalent to C-subsolution of equation (1.6) satisfying $\underline{u} \equiv 0$. To solve equation (1.6), we consider two cases.

Case 1 $\tan(\hat{\theta}) \geq 0$.

Since $2\tan(\hat{\theta})\sec^2(\hat{\theta}) \geq 0$, equation (1.6) is a special case of equation (1.5) and all conditions in Corollary 1.3 are satisfied. Then there exists a smooth function to solve equation (1.6) and $\Omega_u \in \Gamma_2^g$. Denote the eigenvalue of $g^{ik}(\Omega_{k\bar{j}} + u_{k\bar{j}})$ as $\lambda = (\lambda_1, \lambda_2, \lambda_3)$, and $\lambda_1 \geq \lambda_2 \geq \lambda_3$. Obviously, $\lambda_1 \geq \lambda_2 > 0$. From equation (1.6), we have

$$\lambda_1\lambda_2\lambda_3 = \sec^2(\hat{\theta})(\lambda_1 + \lambda_2 + \lambda_3) + 2\tan(\hat{\theta})\sec^2(\hat{\theta}) > 0.$$

Hence, $\lambda_3 > 0$, which implies $\Omega_u \in \Gamma_3^g$.

Case 2 $\tan(\hat{\theta}) < 0$.

In this case, $\hat{\theta} \in (\frac{\pi}{2}, \pi)$. The sign of $2\tan(\hat{\theta})\sec^2(\hat{\theta})$ does not satisfy our requirement. Let $\Omega_u = \widetilde{\Omega}_u - \sec(\hat{\theta})\omega$. Substituting Ω_u into equation (1.6), we obtain

$$\widetilde{\Omega}_u^3 = 3\sec(\hat{\theta})\widetilde{\Omega}_u^2 \wedge \omega + 2\sec^2(\hat{\theta})(\sec(\hat{\theta}) - \tan(\hat{\theta}))\omega^3, \tag{6.7}$$

where $2\sec^2(\hat{\theta})(\sec(\hat{\theta}) - \tan(\hat{\theta})) > 0$. From the cone condition (1.7), we have

$$3(\Omega + \sec(\hat{\theta})\omega)^2 - 6\sec(\hat{\theta})(\Omega + \sec(\hat{\theta})\omega) \wedge \omega > 0,$$

which means that equation (6.7) also satisfies the cone condition. Hence, equation (6.7) satisfies all the conditions in Corollary 1.3. Then there exists a smooth function to solve equation (6.7) and $\widetilde{\Omega}_u \in \Gamma_2^g$. Denote the eigenvalue of $g^{ik}(\Omega_{k\bar{j}} + \sec(\hat{\theta})g_{k\bar{j}} + u_{k\bar{j}})$ as $\widetilde{\lambda} = (\widetilde{\lambda}_1, \widetilde{\lambda}_2, \widetilde{\lambda}_3)$, and $\widetilde{\lambda}_1 \geq \widetilde{\lambda}_2 \geq \widetilde{\lambda}_3$. Obviously, $\widetilde{\lambda}_i = \lambda_i + \sec(\hat{\theta})$, for any $1 \leq i \leq 3$. Since $\widetilde{\lambda} \in \Gamma_2$, we obtain

$$\lambda_1 + \lambda_2 + \lambda_3 > -3\sec(\hat{\theta}).$$

Therefore,

$$\lambda_1\lambda_2\lambda_3 = \sec^2(\hat{\theta})(\lambda_1 + \lambda_2 + \lambda_3) + 2\tan(\hat{\theta})\sec^2(\hat{\theta}) \geq -3\sec^3(\hat{\theta}) + 2\tan(\hat{\theta})\sec^2(\hat{\theta}) > 0,$$

which implies $\Omega_u \in \Gamma_3^g$. □

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