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Research Article

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Unique solvability for an inverse problem of a nonlinear parabolic PDE with nonlocal integral overdetermination condition

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Abstract: In this work, the solvability for an inverse problem of a nonlinear parabolic equation with nonlocal integral overdetermination supplementary condition is examined. The proof of the existence and uniqueness of the solution of the inverse nonlinear parabolic problem upon the data is established by using the fixed-point technique. In addition, the inverse problem is investigated by using the cubic B-spline collocation technique together with the Tikhonov regularization. The resulting nonlinear system of parabolic equation is approximated using the MATLAB subroutine *lsqnonlin*. The obtained results demonstrate the accuracy and efficiency of the technique, and the stability of the approximate solutions even in the existence of noisy data. The stability analysis is also conducted for the discretized system of the direct problem.

Keywords: nonlinear parabolic equation, collocation technique, Tikhonov regularization, nonlinear optimization

MSC 2020: 65Mxx, 35R30, 47H10

1 Introduction

The diffusion equation is a partial differential equation that describes many problems in many fields, such as physics, mechanics, biology, and technology. Indeed, thanks to the modeling of these phenomena through partial differential equations, we have understood the role of this or that parameter and above all, obtain forecasts that are sometimes extremely precise. Various natural phenomena can be modeled by partial differential equations with different boundary conditions like nonlocal conditions and other types [1–6]. Cannon was the first who attracted researchers to these problems with an integral condition [7,8].

Motivated by all the aforementioned facts, the authors of [9,10] studied the inverse problems in the same aforesaid domain for a hyperbolic equation and for a class of fractional reaction-diffusion equations, respectively. They established the unique solvability upon the data and showed continuous dependency.

This new research is considered as the development of previous research from the hyperbolic problem [9,10] to the parabolic problem, which we will study here and also the development of the inverse problem

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with the nonlocal condition. So, we find difficulties because of the studied equation, which creates a complication in the proof of the unique solvability. Therefore, we investigate the inverse problem for the existence and uniqueness to determine a pair $\{u, f\}$ satisfying the nonlinear equation:

$$u_t - a\Delta u + bu + u^q = f(t)h(x, t), \quad x \in \Omega, \ t \in [0, T], \tag{1}$$

with the initial conditon

$$u(x, 0) = \varphi(x), \quad x \in \Omega, \tag{2}$$

the boundary condition

$$u(x, t) = 0, \quad (x, t) \in \partial\Omega \times [0, T], \tag{3}$$

and the nonlocal condition

$$\int_{\Omega} v(x)u(x, t)\mathrm{d}x = \theta(t), \quad t \in [0, T], \tag{4}$$

where $q > 1$, Ω is a bounded domain in $\mathbb{R}^n$ and $\partial\Omega$ is its smooth boundary. The h, φ, and θ are known functions.

Inverse problems for the parabolic equations (see [11,12], and references therein) arise naturally in many reality phenomena, where supplementary or additional information comes in the form of the integral condition (4). The integral condition plays an important role and tool of modelization in the theory of partial differential equations in engineering and physics [7,8,13–16].

The theory of the existence and uniqueness of inverse problems has been studied by many authors [7,8, 17–24]. In [7,17–19], the authors established the existence, uniqueness, and continuous dependency in a class of hyperbolic equations. In [8], the authors established these theories in parabolic equations. In [20], the authors studied unique solvability for the inverse problem to identify the unknown coefficient in a nondivergence parabolic equation. Kanca [22] studied an inverse problem of a heat equation for recovering the time-wise coefficient with integral conditions and established the conditions for the unique solvability. Based on these works and to develop these theories and works, a new study is presented here for the inverse problem for a heat equation with the integral condition by reducing the problem to a fixed-point principle.

In addition, the inverse problem for the pseudo-parabolic equation has been scarcely examined, numerically, to determine the time or/and space-dependent coefficients. For instance, the inverse problems of determining the time-dependent coefficients have been studied in [25–29], while [30,31] examined it for the space-dependent coefficients in the pseudo-parabolic equations. Ramazanova et al. [32] determined the time-dependent coefficient, theoretically. Furthermore, [33–36] proved the unique solvability. Huntul and Tamsir [37,38] investigated an inverse problem for the diffusion equation to reform the time-dependent coefficient from the additional measurements.

Recently, Huntul and Oussaeif [39] proved the solvability of the nonlocal inverse parabolic problem and examined it numerically. Here, the existence and uniqueness of the solutions in inverse problem (1)–(4) is established using the fixed-point theorem. Moreover, it is investigated using the cubic B-spline collocation technique and the Tikhonov regularization for identifying a stable and accurate approximate solution. We also discuss the stability for the discretized form of the direct problem.

The rest of the manuscript is coordinated as follows. Section 2 illustrates the preliminaries. The unique solvability of the inverse problem is given in Section 3. The cubic B-spline collocation technique is given in Section 4. Section 5 analyses the stability. Section 6 gives a minimization technique of the Tikhonov objective functional. Section 7 presents the numerical investigations. Finally, Section 8 states the conclusions.

2 Preliminaries

We begin with certain notations and definition as similar notions and definition had studied by Oussaeif and Bouziani (see [9] in section 2) when the problem is hyperbolic.

$$g^*(t) = \int_\Omega v(x)h(x,t)\mathrm{d}x, \quad Q_T = \Omega \times [0,T]. \tag{5}$$

Notation 1. Let $C((0,T), L_2(\Omega))$ be the space of all continuous functions on $(0,T)$ with values in $L_2(\Omega)$ defined by:

$$\|u\|_{C((0,T),L_2(\Omega))} = \max_{(0,T)} \|u\|_{L_2(\Omega)} < \infty.$$

Notation 2. We have the space $L_2(\Omega)$ by $\|u\| \equiv \|u\|_{L_2(\Omega)}$ and the inequality (Cauchy's ε-inequality):

$$2|ab| \leqslant \varepsilon a^2 + \frac{1}{\varepsilon} b^2, \quad \text{for } \varepsilon > 0.$$

3 Solvability of the solution of direct nonlinear parabolic problems by the method of energy inequality

This section studies the solutions of parabolic problems with Dirichlet boundary conditions. The existence and uniqueness of strong solutions for nonlinear problems are established by the method of energy inequality, where difficulty in the choice of the multiplier is found, and the uniqueness which is emanating from an *a priori* estimate.

Let $T > 0$, $\Omega \subset \mathbb{R}^n$, Γ is smooth boundary, and

$$Q = \Omega \times (0,T) = \{(x,t) \in \mathbb{R}^{n+1} : x \in \Omega, 0 < t < T\}.$$

Consider

$$\begin{cases} u_t - a\Delta u + b(x,t)u + c(x,t)u^q = y(x,t), \\ u(x,0) = \varphi(x), \\ u(x,t)|_\Gamma = 0, \end{cases} \tag{6}$$

whose nonlinear parabolic equation is

$$\mathcal{L}u = u_t - a\Delta u + b(x,t)u + c(x,t)u^q = y(x,t), \tag{7}$$

subject to

$$lu = u(x,0) = \varphi(x), \tag{8}$$

with the boundary conditions

$$u(x,t)|_\Gamma = 0, \quad \forall t \in (0,T), \tag{9}$$

where q is positive odd integers, the functions $\varphi(x)$, $y(x,t)$ are known, and $b(x,t)$ and $c(x,t)$ hold the assumptions:
$(A1)$ $b_1 \le b(x,t) \le b_0$, $c_1 \le c(x,t) \le c_0$, $(x,t) \in \bar{Q}$.

A priori bound is derived, and the unique solvability to the problem (7)–(9) is estabilihsed. Next, we construct exact solutions by using the tanh function method for equation (7) with $y(x,t) = 0$, $b(x,t) = 1$, and $c(x,t) = 1$. Suppose that $Lu = F$, where $L = (\mathcal{L}, l)$ is an operator, related to (7)–(9). The L acts from Banach space E to Hilbert space F described as follows. The E consists of all $u(x,t)$ with

$$\|u\|_E^2 = \sup_{0 \le \tau \le T} \|u(x,\tau)\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(Q)} + \|u\|^{q+1}_{L^{q+1}(Q)}, \tag{10}$$

and $F = (f, u_0)$ with

$$\|\mathcal{F}\|_F^2 = \|f\|^2_{L^2(Q)} + \|\varphi\|^2_{L^2(\Omega)}. \tag{11}$$

The associated inner product is

$$F : (\mathcal{F}, G)_F = (y_1, y_2)_{L^2(Q)} + (\varphi_1, \varphi_2)_{L^2(\Omega)}, \quad \forall \mathcal{F}, G \quad \text{in } F, \tag{12}$$

where $\mathcal{F} = (y_1, \varphi_1)$ and $G = (y_2, \varphi_2)$. Suppose that the data function u_0 satisfies (9):

$$u_0|_\Gamma = 0.$$

First, *a priori* estimate is derived.

3.1 *A priori* bound

Theorem 1. *If all A1 are fulfilled then for $u \in D(L)$, $\exists$ constant $c > 0$ s.t.*

$$\sup_{0\le\tau\le T} \|u(x,\tau)\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(Q)} + \|u\|^2_E + \|u\|^{q+1}_{L^{q+1}(Q)} \le c(\|f\|^2_{L^2(Q)} + \|\varphi\|^2_{L^2(\Omega)}), \tag{13}$$

and the domain $D(L)$ of L is defined as follows:

$$D(L) = \{u \in L^2(Q)/u_t, \nabla u, \Delta u \in L^2(Q), u \in L^{q+1}(Q)\},$$

satisfying (9).

Proof. We take the scalar product in $L^2(Q)$ of equation (7) and $Mu = u$ as follows:

$$(\mathcal{L}u, Mu)_{L^2(Q^\tau)} = (u_t, u)_{L^2(Q^\tau)} - a(\Delta u, u)_{L^2(Q^\tau)} + (bu, u)_{L^2(Q^\tau)} + (cu^q, u)_{L^2(Q^\tau)} = (f, u)_{L^2(Q^\tau)}, \tag{14}$$

where $Q^\tau = \Omega \times (0, T)$. The successive integration the right-hand side (RHS) of (14), yields

$$(u_t, u)_{L^2(Q^\tau)} = \int_{Q^\tau} u_t u = \frac{1}{2}\int_0^l u^2 - \frac{1}{2}\int_0^l \varphi^2 = \frac{1}{2}\|u(x,\tau)\|^2_{L^2(\Omega)} - \frac{1}{2}\|\varphi\|^2_{L^2(\Omega)}, \tag{15}$$

$$-a(\Delta u, u)_{L^2(Q^\tau)} = -a\int_{Q^\tau} \Delta u \cdot u = a\int_{Q^\tau} \nabla u^2 = a\|\nabla u\|^2_{L^2(Q)}, \tag{16}$$

$$(bu, u)_{L^2(Q^\tau)} = \int_{Q^\tau} b(x,t)u^2 \mathrm{d}x\mathrm{d}t, \tag{17}$$

and

$$(cu^q, u)_{L^2(Q^\tau)} = \int_{Q^\tau} c(x,t)u^{q+1}\mathrm{d}x\mathrm{d}t. \tag{18}$$

Substituting (15)–(18) into (14), we obtain

$$\frac{1}{2}\|u(x,\tau)\|^2_{L^2(\Omega)} - \frac{1}{2}\|\varphi\|^2_{L^2(\Omega)} + a\|\nabla u\|^2_{L^2(Q)} + \int_{Q^\tau} b(x,t)u^2\mathrm{d}x\mathrm{d}t + \int_{Q^\tau} c(x,t)u^{q+1}\mathrm{d}x\mathrm{d}t = (f, u). \tag{19}$$

Estimating the last term on RHS of (19) by using Cauchy's inequality with ε, ($|ab| \le \frac{a^2}{2\varepsilon} + \frac{\varepsilon b^2}{2}$), we obtain

$$\begin{aligned} &\frac{1}{2}\|u(x,\tau)\|^2_{L^2(\Omega)} + a\|\nabla u\|^2_{L^2(Q)} + \int_{Q^\tau} b(x,t)u^2\mathrm{d}x\mathrm{d}t + \int_{Q^\tau} c(x,t)u^{q+1}\mathrm{d}x\mathrm{d}t \\ &= (f, u) + \frac{1}{2}\|\varphi\|^2_{L^2(\Omega)} \\ &\le \frac{1}{2}\|\varphi\|^2_{L^2(\Omega)} + \|f\|_{L^2(Q)}\|u\|_{L^2(Q)} \\ &\le \frac{1}{2}\|\varphi\|^2_{L^2(\Omega)} + \frac{1}{2\varepsilon}\|f\|^2_{L^2(Q)} + \frac{\varepsilon}{2}\|u\|^2_{L^2(Q)}. \end{aligned} \tag{20}$$

By using $(A1)$, (20) becomes

$$\frac{1}{2}\|u(x,\tau)\|^2_{L^2(\Omega)} + a\|\nabla u\|^2_{L^2(\Omega)} + b_1\int\limits_{Q^\tau} u^2 \mathrm{d}x\mathrm{d}t + c_1\int\limits_{Q^\tau} u^{q+1}\mathrm{d}x\mathrm{d}t \le \frac{1}{2}\|\varphi\|^2_{L^2(\Omega)} + \frac{1}{2\varepsilon}\|f\|^2_{L^2(Q)} + \frac{\varepsilon}{2}\|u\|^2_{L^2(Q)}. \tag{21}$$

Then,

$$\frac{1}{2}\|u(x,\tau)\|^2_{L^2(\Omega)} + a\|\nabla u\|^2_{L^2(\Omega)} + \left(b_1 - \frac{\varepsilon}{2}\right)\|u\|^2_{L^2(Q)} + c_1\int\limits_{Q^\tau} u^{q+1}\mathrm{d}x\mathrm{d}t \le \frac{1}{2}\|\varphi\|^2_{L^2(\Omega)} + \frac{1}{2\varepsilon}\|f\|^2_{L^2(Q)}. \tag{22}$$

$\Rightarrow$

$$\sup_{0\le\tau\le T}\|u(x,\tau)\|^2_{L^2(\Omega)} + 2a\|\nabla u\|^2_{L^2(\Omega)} + (2b_1-\varepsilon)\|u\|^2_{L^2(Q)} + 2c_1\int\limits_{Q^\tau} u^{q+1}\mathrm{d}x\mathrm{d}t \le \|\varphi\|^2_{L^2(\Omega)} + \frac{1}{\varepsilon}\|f\|^2_{L^2(Q)}. \tag{23}$$

So, it comes

$$\sup_{0\le\tau\le T}\|u(x,\tau)\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(Q)} + \int\limits_{Q^\tau} u^{q+1}\mathrm{d}x\mathrm{d}t \le \frac{\max(1,\frac{1}{\varepsilon})}{\min(2a, 2c_1, (2b_1-\varepsilon), 1)}(\|\varphi\|^2_{L^2(\Omega)} + \|f\|^2_{L^2(Q)}), \tag{24}$$

$\Rightarrow$

$$\sup_{0\le\tau\le T}\|u(x,\tau)\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(Q)} + \int\limits_{Q^\tau} u^{q+1}\mathrm{d}x\mathrm{d}t \le c(\|\varphi\|^2_{L^2(\Omega)} + \|f\|^2_{L^2(Q)}), \tag{25}$$

where

$$c = \frac{\max\left(1,\frac{1}{\varepsilon}\right)}{\min(2a, 2c_1, (2b_1-\varepsilon), 1)}.$$

So, we have

$$\|u\|_E \le \sqrt{c}\;\|Lu\|_F. \tag{26}$$

Let the range of L is $R(L)$. As there is no knowledge about $R(L)$, except that $R(L) \subset F$, it is essential to extend L, so that (12) satisfies for the extended L and the range of extended L is the whole space F. First, we define the following proposition: □

Proposition 1. *The operator $L : E \longrightarrow F$ has a closure*

Proof. Let $(u_n)_{n\in\mathbb{N}} \subset D(L)$ a sequence, where

$$u_n \longrightarrow 0 \quad \text{in } E,$$

and

$$Lu_n \longrightarrow (f;\varphi) \quad \text{in } F. \tag{27}$$

We must prove that

$$f \equiv 0 \quad \text{and} \quad \varphi \equiv 0.$$

The convergence of u_n to 0 in E drives

$$u_n \longrightarrow 0 \quad \text{in } D'(Q). \tag{28}$$

According to the continuity of the derivation of $C_0^\infty(Q)'$ in $D'(Q)$, and the continuity of the distribution of the function u^q, the relation (28) involves

$$\mathcal{L}u_n \longrightarrow 0 \quad \text{in } D'(Q). \tag{29}$$

Moreover, the convergence of $\mathcal{L}u_n$ to f in $L^2(Q)$ gives

$$\mathcal{L}u_n \longrightarrow f \quad \text{in } D'(Q). \tag{30}$$

Due to the uniqueness of the limit in $D'(Q)$, we deduce from (29) and (30)

$$f = 0.$$

Then, it is generated from (27) that

$$lu_n \longrightarrow \varphi \quad \text{in } L^2(\Omega).$$

Now

$$\|u\|_E^2 = \sup_{0\le\tau\le T} \|u(x,\tau)\|_{L^2(\Omega)}^2 + \|u_x\|_{L^2(Q)}^2 + \|u\|_{L^2(Q)}^2 + \|u\|_{L^{q+1}(Q)}^{q+1} \ge \|u(x,0)\|_{L^2(\Omega)}^2 \ge \|\varphi\|_{L^2(\Omega)}^2.$$

Since

$$u_n \longrightarrow 0 \quad \text{in } E,$$

$$\|u\|_E^2 \longrightarrow 0 \quad \text{in } \mathbb{R},$$

which gives

$$0 \ge \|\varphi\|_{L^2(\Omega)}^2.$$

Hence,

$$\varphi = 0.$$

Let the closure of L with $D(L)$ is $\bar{L}$. □

Definition 1. A solution of

$$\bar{L}u = \mathcal{F}$$

is called as a strong solution of (7)–(9). Then, we can extend estimate (13) to strong solutions as follows:

$$\sup_{0\le\tau\le T} \|u(x,\tau)\|_{L^2(\Omega)}^2 + \|\nabla u\|_{L^2(\Omega)}^2 + \|u\|_{L^2(Q)}^2 + \int_{Q^\tau} u^{q+1}\mathrm{d}x\mathrm{d}t \le c(\|\varphi\|_{L^2(\Omega)}^2 + \|f\|_{L^2(Q)}^2), \quad \forall u \in D(\bar{L}). \tag{31}$$

From the estimate (30), we deduce

Corollary 1. *The $R(\bar{L})$ is closed in F and $R(\bar{L}) = \overline{R(L)}$, where $R(\bar{L})$ of $\bar{L}$ and $\overline{R(L)}$ is closure of $R(L)$.*

Proof. Let $z \in \overline{R(L)}$, so there is a Cauchy sequence $(z_n)_{n\in\mathbb{N}}$ in F constituted of the elements of the set $R(L)$ such as

$$\lim_{n\longrightarrow+\infty} z_n = z.$$

Then, there is a corresponding sequence $u_n \in D(L)$ such as

$$z_n = Lu_n.$$

The estimate (26), we obtain

$$\|u_p - u_q\|_E \le C\|Lu_p - Lu_q\|_F \to 0,$$

where p, q tend toward infinity. We can deduce that $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in E, so like E is a Banach space, $\exists\ u \in E$ such as

$$\lim_{n\longrightarrow+\infty} u_n = u \quad \text{in } E.$$

By virtue of the definition of $\bar{L}$ ($\lim_{n\longrightarrow+\infty} u_n = u$ in E; if $\lim_{n\longrightarrow+\infty} Lu_n = \lim_{n\longrightarrow+\infty} z_n = z$, then $\lim_{n\longrightarrow+\infty} \bar{L}u_n = z$ as like $\bar{L}$ and is closed, so $\bar{L}u = z$), the function u check:

$$u \in D(\bar{L}), \quad \bar{L}u = z.$$

Then, $z \in R(\bar{L})$, and so

$$\overline{R(L)} \subset R(\bar{L}).$$

Also, we conclude here that $R(\bar{L})$ is closed because it is Banach (any complete subspace of a metric space (not necessarily complete) is closed). It is left to show the reverse inclusion.

If $z \in R(\bar{L})$, then $\exists$ is a Cauchy sequence $(z_n)_{n\in\mathbb{N}}$ in F constituted of the elements of the set $R(\bar{L})$ such that

$$\lim_{n\longrightarrow+\infty} z_n = z,$$

or $z \in R(\bar{L})$, because $R(\bar{L})$ is a closed subset a completed F, so $R(\bar{L})$ is complete. Then there is a corresponding sequence $u_n \in D(\bar{L})$ such that

$$\bar{L}u_n = z_n.$$

Once again, there is a corresponding sequel $(Lu_n)_{n\in\mathbb{N}} \subset R(L)$ such as

$$\bar{L}u_n = Lu_n \text{ on } R(L), \forall n \in \mathbb{N}.$$

So,

$$\lim_{n\longrightarrow+\infty} Lu_n = z,$$

consequently $z \in \overline{R(L)}$, and then we conclude that

$$R(\bar{L}) \subset \overline{R(L)}.$$

□

3.2 Existence of the solution

Theorem 2. *Let A_1 be fulfilled. Then, $\forall F = (f, \varphi)$, $\exists$ a unique strong solution $u = \bar{L}^{-1}\mathcal{F} = \overline{L^{-1}}\mathcal{F}$ to problems* (7)–(9).

Proof. Consider

$$(Lu, W)_F = \int_Q \mathcal{L}u \cdot w\mathrm{d}x\mathrm{d}t + \int_\Omega lu \cdot w_0 \mathrm{d}x, \tag{32}$$

where

$$W = (w, w_0).$$

So, for $w \in L^2(Q)$ and $\forall\ u \in D_0(L) = \{u, u \in D(L) : \ell u = 0\}$, we obtain

$$\int_Q \mathcal{L}u \cdot w\mathrm{d}x\mathrm{d}t = 0.$$

By putting $w = u$, we obtain

$$\int_{Q^\tau} u_t u + \int_{Q^\tau} b(x,t)u^2 \mathrm{d}x\mathrm{d}t + \int_{Q^\tau} c(x,t)u^{q+1}\mathrm{d}x\mathrm{d}t = a\int_{Q^\tau} \Delta u\cdot u,$$
$$\int_{Q^\tau} u_t u + \int_{Q^\tau} b(x,t)u^2 \mathrm{d}x\mathrm{d}t + \int_{Q^\tau} c(x,t)u^{q+1}\mathrm{d}x\mathrm{d}t = -a\int_{Q^\tau} (\nabla u)^2,$$
$$\int_{Q^\tau} u_t u + \int_{Q^\tau} b(x,t)u^2 \mathrm{d}x\mathrm{d}t + \int_{Q^\tau} c(x,t)u^{q+1}\mathrm{d}x\mathrm{d}t \le 0,$$
$$\int_{Q^\tau} u_t u + b_1\int_{Q^\tau} u^2 \mathrm{d}x\mathrm{d}t + c_1\int_{Q^\tau} u^{q+1}\mathrm{d}x\mathrm{d}t \le 0.$$

So, we obtain $u = w = 0$.

Since the range of ℓ is dense everywhere in F with $\|\varphi\|_{L^2(\Omega)}$, equation (32) $\Rightarrow \omega_0 = 0$. Hence, $W = 0$ implies $(\overline{R(L)} = F)$. ☐

Corollary 2. *If, for any $u \in D(L)$, the estimate is*

$$\|u\|_E \le C\,\|\mathcal{F}\|_F.$$

Then, P1 has a unique solution if it exists.

Proof. Let u_1 and u_2 are two solutions to P_1

$$\begin{cases} Lu_1 = \mathcal{F} \\ Lu_2 = \mathcal{F} \end{cases} \Rightarrow Lu_1 - Lu_2 = 0,$$

and as L is linear, so we obtain

$$L(u_1 - u_2) = 0,$$

and according to (26),

$$\|u_1 - u_2\|_E^2 \le c\,\|0\|_F^2 = 0,$$

which gives

$$u_1 = u_2.$$ ☐

4 Unique solvability of the inverse problem

Using same process in Section 3 [9], suppose that the functions emerging in problem data are measurable and fulfill conditions:

$$\begin{cases} h \in C((0,T), L_2(\Omega)),\, v \in V = \{v, \nabla v \in L_2(\Omega),\, v \in L_{p+1}(\Omega)\},\, E \in W_2^2(0,T), \\ \|h(x,t)\| \leqslant m; \quad |g^*(t)| \geqslant r > 0 \quad \text{for } r \in \mathbb{R},\, (x,t) \in Q_T, \quad \varphi(x) \in W_2^1(\Omega). \end{cases} \tag{H}$$

The relation between f and u is given by the following linear operator:

$$A : L_2(0,T) \to L_2(0,T). \tag{33}$$

with the values

$$(Af)(t) = \frac{1}{g^*}\left\{a\int_\Omega \nabla u \nabla v \mathrm{d}x + \int_\Omega u^p v \mathrm{d}x\right\}. \tag{34}$$

In light of this, the previous relation between f and u is shaped in the form of the second kind linear equation f over $L_2(0, T)$:

$$f = Af + W, \tag{35}$$

where

$$W = \frac{E' + bE}{g^*} \quad \text{and} \quad E(0) = 0. \tag{36}$$

Theorem 3. *In this theorem as* Theorem 3.2 *in* [9], *suppose that the data function of the inverse problem* (1)–(4) *satisfies the conditions* (H). *Then we have the equivalent between the following assertions*:
(*i*) *If the inverse problems* (1)–(4) *have a unique solution, then so is* (17).
(*ii*) *If* (17) *has a solution and verify the compatibility condition*

$$E(0) = 0, \tag{37}$$

then ∃ *a solution of inverse problems* (1)–(4).

Proof. (*i*) Using same idea and steps for [9], the problems (1)–(4) are solvable. Let $\{z, f\}$ be the solution of inverse problems (1)–(4). Multiplying equation (33) by the function v and integrating over Ω, and using (4) and (34), it follows from (36) that

$$f = Af + \frac{E' + bE}{g^*}.$$

This leads that f solves equation (35).

(*ii*) Again from [9], equation (35) has a solution in $L_2(0, T)$, and so be f.

Problems (1)–(3) can be considered as a direct problem with a unique solution $u \in E$.

It is left to show that u verifies the condition (4). By using equation (1), it yields that

$$\frac{d}{dt}\int_\Omega uv dx + a\int_\Omega \nabla u\nabla v dx + b\int_\Omega uv dx + \int_\Omega u^p v dx = f(t)\int_\Omega v(x)h(x, t) dx. \tag{38}$$

So, we obtain

$$\frac{d}{dt}\int_\Omega uv dx + a\int_\Omega \nabla u\nabla v dx + b\int_\Omega uv dx + \int_\Omega u^p v dx = f(t)g^*(t). \tag{39}$$

Differently, being a solution to (35), u satisfies the following relation

$$E' + bE + a\int_\Omega \nabla u\nabla v dx + b\int_\Omega uv dx + \int_\Omega u^p v dx = f(t)g^*(t). \tag{40}$$

By subtracting (39) from (40), we obtain

$$\frac{d}{dt}\int_\Omega uv dx + b\int_\Omega uv dx = E' + bE. \tag{41}$$

Integrating the previous equation and taking (36) into account, we conclude that u satisfies (4), and finally, we find that the pair $\{u, f\}$ is a solution of the original inverse problems (1)–(4). This achieves the proof. □

Now we are trying to take care of some of the properties of the operator A.

Lemma 1. *Suppose that the condition* (H) *fulfilled. Then, for which* A *is a contracting operator in* $L_2(0, T)$ (*see* Lemma 3.3 *in* [9]).

Proof. Definitely (34) gives

$$\|Af\|_{L_2(0,T)} \leq \frac{k}{p}\left(\left(\int_0^T \|\nabla u(\cdot,\tau)\|^2_{L_2(\Omega)}\mathrm{d}\tau\right)^{\frac{1}{2}} + \int_0^T \|u(\cdot,\tau)\|^p_{L_p(\Omega)}\mathrm{d}\tau\right), \tag{42}$$

where

$$k = \max_{t\in(0,T)}(\|\nabla v\|_{L_2(\Omega)}, \|v\|_{L_{p+1}(\Omega)}).$$

By multiplying equation (1) by u scalarly in $L_2(Q_T)$ and integrating the resulting expressions with the use of (2) and (3), we obtain the identity

$$\frac{1}{2}\|u(\cdot,t)\|^2_{L_2(\Omega)} + a\|\nabla u\|^2_{L_2(Q_\tau)} + b\|u\|^2_{L_2(Q_\tau)} + \|u\|^{p+1}_{L_{p+1}(0,\tau,L_{p+1}(\Omega))} = \int_0^\tau\left(f(t)\int_\Omega h(x,t)u\mathrm{d}x\right)\mathrm{d}t + \frac{1}{2}\|\varphi\|^2_{L_2(\Omega)}. \tag{43}$$

Now, employing the Cauchy's inequality, we have

$$\frac{1}{2}\|u(\cdot,t)\|^2_{L_2(\Omega)} + a\|\nabla u\|^2_{L_2(Q_\tau)} + b\|u\|^2_{L_2(Q_\tau)} + \|u\|^{p}_{L_{p+1}(0,\tau,L_{p+1}(\Omega))} \leq \frac{m^2}{2\varepsilon}\int_0^\tau |f(t)|^2\mathrm{d}t + \frac{1}{2}\|\varphi\|^2_{L_2(Q_\tau)} + \frac{1}{2}\|u\|^2_{L_2(Q_\tau)}. \tag{44}$$

By using Gronwall lemma, we obtain

$$\frac{1}{2}\|u(\cdot,t)\|^2_{L_2(\Omega)} + a\|\nabla u\|^2_{L_2(Q_\tau)} + b\|u\|^2_{L_2(Q_\tau)} + \|u\|^{p}_{L_{p+1}(0,\tau,L_{p+1}(\Omega))} \leq \left(\frac{m^2}{2}\int_0^T |f(\tau)|^2\mathrm{d}t + \frac{1}{2}\|\varphi\|^2_{L_2(\Omega)}\right)e^{\frac{1}{2}T}. \tag{45}$$

Passing to the maximum in the left-hand side (LHS) of the last inequality leads to

$$\frac{1}{2}\max_{t\in(0,T)}\|u(\cdot,t)\|^2_{L_2(\Omega)} + a\|\nabla u\|^2_{L_2(Q_\tau)} + b\|u\|^2_{L_2(Q_\tau)} + \|u\|^{p}_{L_{p+1}(0,\tau,L_{p+1}(\Omega))} \leq \left(\frac{m^2}{2}\int_0^T |f(\tau)|^2\mathrm{d}t + \frac{1}{2}\|\varphi\|^2_{L_2(\Omega)}\right)e^{\frac{1}{2}T}. \tag{46}$$

Since $\max_{t\in(0,T)}\|u(\cdot,t)\|^2_{L_2(\Omega)} > \|u(x,0)\|^2_{L_2(\Omega)}$, and omitting some terms, we obtain the following:

$$\|\nabla u\|^2_{L_2(Q_T)} + \|u\|^p_{L_{p+1}(0,T,L_{p+1}(\Omega))} \leq \frac{m^2e^{\frac{1}{2}T}}{2\min\{1,a\}}\left(\int_0^T |f(\tau)|^2\mathrm{d}t\right). \tag{47}$$

Thus, according to (42) and (47), we obtain the following estimate:

$$\|Af\|_{L_2(0,T)} \leq \delta\int_0^T |f(\tau)|^2\mathrm{d}t, \quad 0 \leqslant t \leqslant T, \tag{48}$$

where

$$\delta = \frac{k}{r}\sqrt{\frac{m^2e^{\frac{1}{2}T}}{2\min\{1,a\}}}.$$

So, we obtain

$$\|Af\|_{L_2(0,T)} \leq \delta\|f\|_{L_2(0,T)}. \tag{49}$$

The inequality (49) proves Lemma 1. □

In regard to the results of the unique solvability of main inverse problem, the below result could be useful.

Theorem 4. *Let the conditions* (H) *and* (37) *hold. Then the statements*:
(*i*) *a solution* $\{z, f\}$ *of the inverse problem* (36)–(47) *exists and is unique, and*
(*ii*) *with any initial iteration* $f_0 \in L_2(0, T)$,

$$f_{n+1} = \widetilde{A} f_n, \tag{50}$$

converge to f *the* $L_2(0, T)$*-norm is valid.*

See reference [9] for proof.

Corollary 3. *Let the assumptions of* Theorem 2 *fulfilled, then the solution f depends continuously with respect to the data W of the equation* (17). *See* [9] *for proof.*

5 The cubic B-spline collocation technique

We consider the problems (1)–(3) when a, b, $f(t)$, and $h(x, t)$ are given and $u(x, t)$ is unknown. First, we divide [0, 1] into equal length mesh $\Delta x = x_{i+1} - x_i$, $i = 0, 1, \ldots, M$. For the discrete form of the problem, we use $u(x_i, t_j) = u_i^j$, $f(t_j) = f^j$ and $h(x_i, t_j) = h_i^j$, where $x_i = ih$, $t_j = jk$, $\Delta x = \frac{l}{M}$, and $\Delta t = \frac{T}{N}$ for $i = 0, 1, \ldots, M$ and $j = 0, 1, \ldots, N$. The cubic B-spline functions $\mathrm{CB}_i(x)$, $i = -1, 0, \ldots, M, M+1$ are given by [40–43]:

$$\mathrm{CB}_i(x) = \frac{1}{(\Delta x)^3}\begin{cases} (x - x_{i-2})^3, & x \in [x_{i-2}, x_{i-1}), \\ (x - x_{i-2})^3 - 4(x - x_{i-1})^3, & x \in [x_{i-1}, x_i), \\ (x_{i+2} - x)^3 - 4(x_{i+1} - x)^3, & x \in [x_i, x_{i+1}), \\ (x_{i+2} - x)^3, & x \in [x_{i+1}, x_{i+2}), \\ 0, & \text{else}, \end{cases} \tag{51}$$

where the set of cubic B-spline functions $\{\mathrm{CB}_{-1}, \mathrm{CB}_0, \ldots, \mathrm{CB}_M, \mathrm{CB}_{M+1}\}$ form a basis over [0, 1]. Table 1 presents $\mathrm{CB}_i(x)$, $\mathrm{CB}_i'(x)$, and $\mathrm{CB}_i''(x)$ at x_i.

At the point (x, t_j), we suppose that $u(x, t)$ is expressed as follows:

$$u(x, t_j) = \sum_{k=-1}^{M+1} C_k(t_j)\mathrm{CB}_k(x), \tag{52}$$

where $C_k(t)$ are the time-dependent coefficients to be estimated from the initial condition, boundary values of the initial condition, and the collocation method. The variation of the $u(x, t)$, over the element is given as follows:

$$u(x, t_j) = \sum_{k=i-1}^{i+1} C_k(t_j)\mathrm{CB}_k(x). \tag{53}$$

Now, the u, u_x, and u_{xx} are defined as follows:

Table 1: The values of cubic B-spline functions and its derivatives at the knots

	x_{i-2}	x_{i-1}	x_i	x_{i+1}	x_{i+2}
$\mathrm{CB}_i(x)$	0	1	4	1	0
$\mathrm{CB}_i'(x)$	0	$\frac{3}{\Delta x}$	0	$-\frac{3}{\Delta x}$	0
$\mathrm{CB}_i''(x)$	0	$\frac{6}{(\Delta x)^2}$	$\frac{-12}{(\Delta x)^2}$	$\frac{6}{(\Delta x)^2}$	0

$$u_i^j = C_{i-1}^j + 4C_i^j + C_{i+1}^j, \tag{54}$$

$$(u_x)_i^j = -\frac{3}{\Delta x}C_{i-1}^j + \frac{3}{\Delta x}C_{i+1}^j, \tag{55}$$

$$(u_{xx})_i^j = \frac{6}{(\Delta x)^2}C_{i-1}^j - \frac{12}{(\Delta x)^2}C_i^j + \frac{6}{(\Delta x)^2}C_{i+1}^j, \tag{56}$$

where $C_i^j = C_i(t_j)$. Next, we take $q = 2$ in problem (1) and discretize using collocation technique over Crank-Nicolson, where the forward difference is employed to discretize the time derivative as follows:

$$\begin{aligned} &\frac{u_i^{j+1} - u_i^j}{\Delta t} - a\left(\frac{(u_{xx})_i^{j+1} + (u_{xx})_i^j}{2}\right) + b\left(\frac{u_i^{j+1} + u_i^j}{2}\right) + u_i^j u_i^{j+1} + u_i^{j+1}u_i^j - u_i^j u_i^j \\ &= \frac{1}{2}(f^{j+1}h_i^{j+1} + f^j h_i^j), \quad i = 0, 1, \ldots, M, \; j = 0, 1, \ldots, N, \end{aligned} \tag{57}$$

where, the nonlinear term u^2 is discretized as follows:

$$(u^2)_i^j = u_i^j u_i^{j+1} + u_i^{j+1}u_i^j - u_i^j u_i^j. \tag{58}$$

Re-arranging, we have

$$A_i^j u_i^{j+1} - \frac{\Delta t}{2}a(u_{xx})_i^{j+1} = B_i^j u_i^j + \frac{\Delta t}{2}a(u_{xx})_i^j + \frac{\Delta t}{2}(f^{j+1}h_i^{j+1} + f^j h_i^j), \quad i = 0, 1, \ldots, M, \; j = 0, 1, \ldots, N, \tag{59}$$

where

$$A_i^j = 1 + \frac{\Delta t}{2}b + 2\Delta t u_i^j, \quad B_i^j = 1 - \frac{\Delta t}{2}b + \Delta t u_i^j.$$

Now, by using u and u_{xx} from equations (54), we have

$$\begin{aligned} &\left(A_i^j - 3\frac{\Delta t}{(\Delta x)^2}a\right)C_{i-1}^{j+1} + \left(4A_i^j + 6\frac{\Delta t}{(\Delta x)^2}a\right)C_i^{j+1} + \left(A_i^j - 3\frac{\Delta t}{(\Delta x)^2}a\right)C_{i+1}^{j+1} \\ &= \left(B_i^j + 3\frac{\Delta t}{(\Delta x)^2}a\right)C_{i-1}^j + \left(4B_i^j - 6\frac{\Delta t}{(\Delta x)^2}a\right)C_i^j + \left(B_i^j + 3\frac{\Delta t}{(\Delta x)^2}a\right)C_{i+1}^j \\ &+ \frac{\Delta t}{2}(f^{j+1}h_i^{j+1} + f^j h_i^j), \quad i = 1, 2, \ldots, M-1, \quad j = 0, 1, \ldots, N. \end{aligned} \tag{60}$$

Now, we discretize $u(0, t) = 0$ and $u(1, t) = 0$ as follows:

$$C_{-1}^j + 4C_0^j + C_1^j = 0, \quad j = 1, 2, \ldots, N, \tag{61}$$

and

$$C_{M-1}^j + 4C_M^j + C_{M+1}^j = 0, \quad j = 0, 1, \ldots, N. \tag{62}$$

From equations (61) and (63), we have

$$C_{-1}^j = -4C_0^j - C_1^j, \quad C_{M+1}^j = -C_{M-1}^j - 4C_M^j, \quad j = 0, 1, \ldots, N. \tag{63}$$

For $i = 0$ and $i = M$, substituting equation (63) in equation (60), we obtain

$$18a\frac{\Delta t}{(\Delta x)^2}C_0^{j+1} = -18a\frac{\Delta t}{(\Delta x)^2}C_0^j + \frac{\Delta t}{2}(f^{j+1}h_0^{j+1} + f^j h_0^j), \quad j = 0, 1, \ldots, N, \tag{64}$$

and

$$18a\frac{\Delta t}{(\Delta x)^2}C_M^{j+1} = -18a\frac{\Delta t}{(\Delta x)^2}C_M^j + \frac{\Delta t}{2}(f^{j+1}h_M^{j+1} + f^j h_M^j), \quad j = 0, 1, \ldots, N. \tag{65}$$

At t_{j+1}, $j = 0, 1, \dots, N$, (64), (60), and (65) can be put into $(M+1)\times(M+1)$ order system as follows:

$$\begin{pmatrix} 18a\frac{\Delta t}{(\Delta x)^2} & 0 & 0 & 0 & \cdots & 0 & 0 \\ \hat{p}_1^j & \hat{q}_1^j & \hat{p}_1^j & 0 & \cdots & 0 & 0 \\ 0 & \hat{p}_2^j & \hat{q}_2^j & \hat{p}_2^j & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & \hat{p}_{M-2}^j & \hat{q}_{M-2}^j & \hat{p}_{M-2}^j & 0 \\ 0 & \cdots & 0 & 0 & \hat{p}_{M-1}^j & \hat{q}_{M-1}^j & \hat{p}_{M-1}^j \\ 0 & \cdots & 0 & 0 & 0 & 0 & 18a\frac{\Delta t}{(\Delta x)^2} \end{pmatrix} \begin{pmatrix} C_0^{j+1} \\ C_1^{j+1} \\ C_2^{j+1} \\ \vdots \\ C_{M-2}^{j+1} \\ C_{M-1}^{j+1} \\ C_M^{j+1} \end{pmatrix} = \begin{pmatrix} R_0^j \\ R_1^j \\ R_2^j \\ \vdots \\ R_{M-2}^j \\ R_{M-1}^j \\ R_M^j \end{pmatrix},$$

where

$$\begin{aligned} &\hat{p}_i^j = A_i^j - 3\frac{\Delta t}{(\Delta x)^2}a, \quad \hat{q}_i^j = 4A_i^j + 6\frac{\Delta t}{(\Delta x)^2}a, \quad i = 0, 1, \dots, M, \ j = 0, 1, \dots, N, \\ &R_0^j = -18a\frac{\Delta t}{(\Delta x)^2}C_0^j + \frac{\Delta t}{2}(f^{j+1}h_0^{j+1} + f^j h_0^j), \quad j = 0, 1, \dots, N, \\ &R_i^j = \left(B_i^j + 3\frac{\Delta t}{(\Delta x)^2}a\right)C_{i-1}^j + \left(4B_i^j - 6\frac{\Delta t}{(\Delta x)^2}a\right)C_i^j + \left(B_i^j + 3\frac{\Delta t}{(\Delta x)^2}a\right)C_{i+1}^j + \frac{\Delta t}{2}(f^{j+1}h_i^{j+1} + f^j h_i^j), \\ &\qquad i = 1, 2, \dots, M-1, \ j = 0, 1, \dots, N, \\ &R_M^j = -18a\frac{\Delta t}{(\Delta x)^2}C_M^j + \frac{\Delta t}{2}(f^{j+1}h_M^{j+1} + f^j h_M^j), \quad j = 0, 1, \dots, N. \end{aligned}$$

Now, we determine the initial vector $(C_{-1}^0, C_0^0, \dots, C_M^0, C_{M+1}^0)$ as follows:

$$u(x,0) = \varphi(x) \ \Rightarrow \ C_{i-1}^0 + 4C_i^0 + C_{i+1}^0 = \varphi(x_i), \quad i = 0, 1, \dots, M, \tag{66}$$

which gives $M+1$ equation in $M+3$ unknowns. For wiping out C_{-1}^0 and C_{M+1}^0, we use the derivative of the initial conditions at the boundaries as follows:

$$u_x(0,0) = \varphi_x(x_0) \ \Rightarrow \ C_{-1}^0 = C_1^0 - \frac{\Delta x}{5}\varphi_x(x_0), \tag{67}$$

and

$$u_x(1,0) = \varphi_x(x_M) \ \Rightarrow \ C_{M+1}^0 = C_{M-1}^0 + \frac{\Delta x}{5}\varphi_x(x_M). \tag{68}$$

Removing C_{-1}^0 and C_{M+1}^0 from (66), we obtain $(M+1)\times(M+1)$ order system as follows:

$$\begin{pmatrix} 4 & 2 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 4 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 4 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 4 & 1 & 0 \\ 0 & \cdots & 0 & 0 & 1 & 4 & 1 \\ 0 & \cdots & 0 & 0 & 0 & 2 & 4 \end{pmatrix} \begin{pmatrix} C_0^0 \\ C_1^0 \\ C_2^0 \\ \vdots \\ C_{M-2}^0 \\ C_{M-1}^0 \\ C_M^0 \end{pmatrix} = \begin{pmatrix} \varphi(x_0) + \frac{\Delta x}{5}\varphi_x(x_0) \\ \varphi(x_1) \\ \varphi(x_2) \\ \vdots \\ \varphi(x_{M-2}) \\ \varphi(x_{M-1}) \\ \varphi(x_M) - \frac{\Delta x}{5}\varphi_x(x_M) \end{pmatrix}.$$

6 Stability analysis

The von Neumann stability [45,44] is analyzed for the discretized form of the nonlinear parabolic equation (1). Taking $f(t)h(x,t) = 0$ and discretizing (1) as follows:

$$\begin{aligned}&\left(\hat{A}-3\frac{\Delta t}{(\Delta x)^2}a\right)C_{i-1}^{j+1}+\left(4\hat{A}+6\frac{\Delta t}{(\Delta x)^2}a\right)C_i^{j+1}+\left(\hat{A}-3\frac{\Delta t}{(\Delta x)^2}a\right)C_{i+1}^{j+1}\\&=\left(\hat{B}+3\frac{\Delta t}{(\Delta x)^2}a\right)C_{i-1}^{j}+\left(4\hat{B}-6\frac{\Delta t}{(\Delta x)^2}a\right)C_i^{j}+\left(\hat{B}+3\frac{\Delta t}{(\Delta x)^2}a\right)C_{i+1}^{j},\end{aligned}\tag{69}$$

where $\hat{A} = 1 + \frac{\Delta t}{2}b + 2\Delta tU$, $\hat{B} = 1 - \frac{\Delta t}{2}b + \Delta tU$, by supposing $u_i^j = U$ as a local constant. Now we consider $C_i^j = \delta^j e^{ki\phi}$ at the point x_i, where $k = \sqrt{-1}$. Substituting it in (69), we obtain

$$\left(\left(2\hat{A}-6\frac{\Delta t}{(\Delta x)^2}a\right)\cos(\phi)+4\hat{A}+6\frac{\Delta t}{(\Delta x)^2}a\right)\tilde{\delta}=\left(2\hat{B}+6\frac{\Delta t}{(\Delta x)^2}a\right)\cos(\phi)+4\hat{B}-6\frac{\Delta t}{(\Delta x)^2}a.\tag{70}$$

Simplifying and rearranging terms, we obtain

$$\tilde{\delta}=\frac{\lambda_1\cos^2\left(\frac{\phi}{2}\right)+\lambda_2}{\lambda_3\cos^2\left(\frac{\phi}{2}\right)+\lambda_4},\tag{71}$$

where

$$\lambda_1=4\hat{B}+12\frac{\Delta t}{(\Delta x)^2}a,\quad \lambda_2=2\hat{B}-6\frac{\Delta t}{(\Delta x)^2}a,$$

$$\lambda_3=4\hat{A}-12\frac{\Delta t}{(\Delta x)^2}a,\quad \lambda_4=2\hat{A}+12\frac{\Delta t}{(\Delta x)^2}a.$$

From equation (71), we have $|\tilde{\delta}| \le 1$, and so, the proposed collocation technique is unconditionally stable.

7 Numerical solution of the inverse problem

To identify the stable and accurate solution for terms $f(t)$ along with $u(x,t)$ satisfying (1)–(4), we formulate the inverse problem as the regularized nonlinear Tikhonov minimize function:

$$J(f)=\left\|\int_\Omega v(x)u(x,t)\mathrm{d}x-\theta(t)\right\|^2+\lambda\|f(t)\|^2,\tag{72}$$

where u solves (1)–(3) for known $f(t)$, and $\lambda \ge 0$ is of regularization that is introduced to stabilize the approximation solutions. For discrete form, (72) turns into

$$J(\underline{f})=\sum_{j=1}^{N}\left[\int_\Omega v(x)u(x,t_j)\mathrm{d}x-\theta(t_j)\right]^2+\lambda\sum_{j=1}^{N}f_j^2.\tag{73}$$

The MATLAB subroutine [46] is utilized to minimize the cost function (73). This routine attempts to find the minimum of a sum of squares by starting from a given initial guess. Simple bounds on the variable are allowed and the explicit calculation (analytical or numerical) of the gradient is not required to be supplied by the user. Furthermore, within *lsqnonlin*, we use the Trust Region Reflective algorithm [47], which is based on the interior-reflective Newton method.

To measure the errors in this data, $\theta(t_j)$, in (73), is replaced by perturbed (noisy) data $\theta^\varepsilon(t_j)$, as follows:

$$\theta^\varepsilon(t_j)=\theta(t_j)+\varepsilon_j,\quad j=0,1,\dots,N,\tag{74}$$

where ε_j are r.v.'s with mean zero and with S.D.

$$\sigma = p \times \max_{t\in[0,T]} |\theta(t)|, \tag{75}$$

where p represents the % noise.

8 Numerical experiments

The solutions for $f(t)$ and $u(x,t)$ are constructed in this section for the case of noisy (74) and exact data. We use

$$\mathrm{rmse}(f) = \left[\frac{T}{N}\sum_{j=1}^{N}(f^{\text{numerical}}(t_j) - f^{\text{exact}}(t_j))^2\right]^{1/2}, \tag{76}$$

for measuring the accuracy. Now, we choose $T = 1$, for simplicity. The lower bound for $f(t)$ is taken as -10^2 while 10^2 for the upper bound.

8.1 Example 1

First, problems (1)–(4) are considered with smooth heat source term:

$$f(t) = 1 + t, \quad t \in [0,1], \tag{77}$$

together with

$$u(x,t) = x^2(1-x)^2 e^{-2t}, \quad (x,t) \in D_T, \tag{78}$$

$$u(0,t) = u(1,t) = 0, \quad t \in [0,1], \quad \varphi(x) = u(x,0) = (1-x)^2x^2, \quad x \in [0,1], \tag{79}$$

and the nonlocal integral condition

$$\theta(t) = \int_0^1 v(x)u(x,t)\mathrm{d}x = \frac{e^{-2t}}{30} \quad t \in [0,1], \tag{80}$$

where $v(x) = 1$ and the rest of the data are as follows:

$$\begin{aligned} &a = 1, \quad b = 1, \quad \%p = 2, \quad \Omega = (0,1), \\ &h(x,t) = \frac{1}{1+t}\{e^{-4t}((-1+x)^4x^4 - e^{2t}(2 - 12x + 13x^2 - 2x^3 + x^4))\}. \end{aligned} \tag{81}$$

First, the accuracy of (1)–(3) is measured with the data (79) and (81) when $f(t)$ is given by (77). The exact (78) and approximate $u(x,t)$, and absolute errors with various grid sizes are illustrated in Figure 1. It can be noticed that there is a closed agreement between exact and approximate $u(x,t)$. Figure 2 depicts the approximate nonlocal integral measurement in (4) in comparison of the analytical solution (80) obtained by using the cubic B-spline collocation method with $M = N \in \{10, 20, 40\}$. A good agreement was observed between the analytical (80) and the approximate $\theta(t)$ as the mesh size decreases (see Table 2).

In the inverse problems (1)–(4), the initial guess for $\underline{f}$ is taken as follows:

$$f^0(t_j) = f(0) = 1, \quad j = 1, 2, \dots, N. \tag{82}$$

We take $M = N = 40$ and start the analysis for recovering $f(t)$ and u, when $p = 0$ in (75). Figure 3(a) depicts J(73) without and with regularization, where a monotonically decreasing convergence is acquired in 17 to 10 iterations for getting a stipulated tolerance of $O(10^{-21})$ to $O(10^{-8})$. Figures 3(b) and (c) illustrate the exact (77) and approximate $f(t)$ with $\lambda \in \{0, 10^{-10}, 10^{-9}\}$. From these figures, one can see that when $\lambda = 0$, we obtain inaccurate and unstable $f(t)$ with $\mathrm{rmse}(f) = 0.0675$, which was expected due to ill-posed

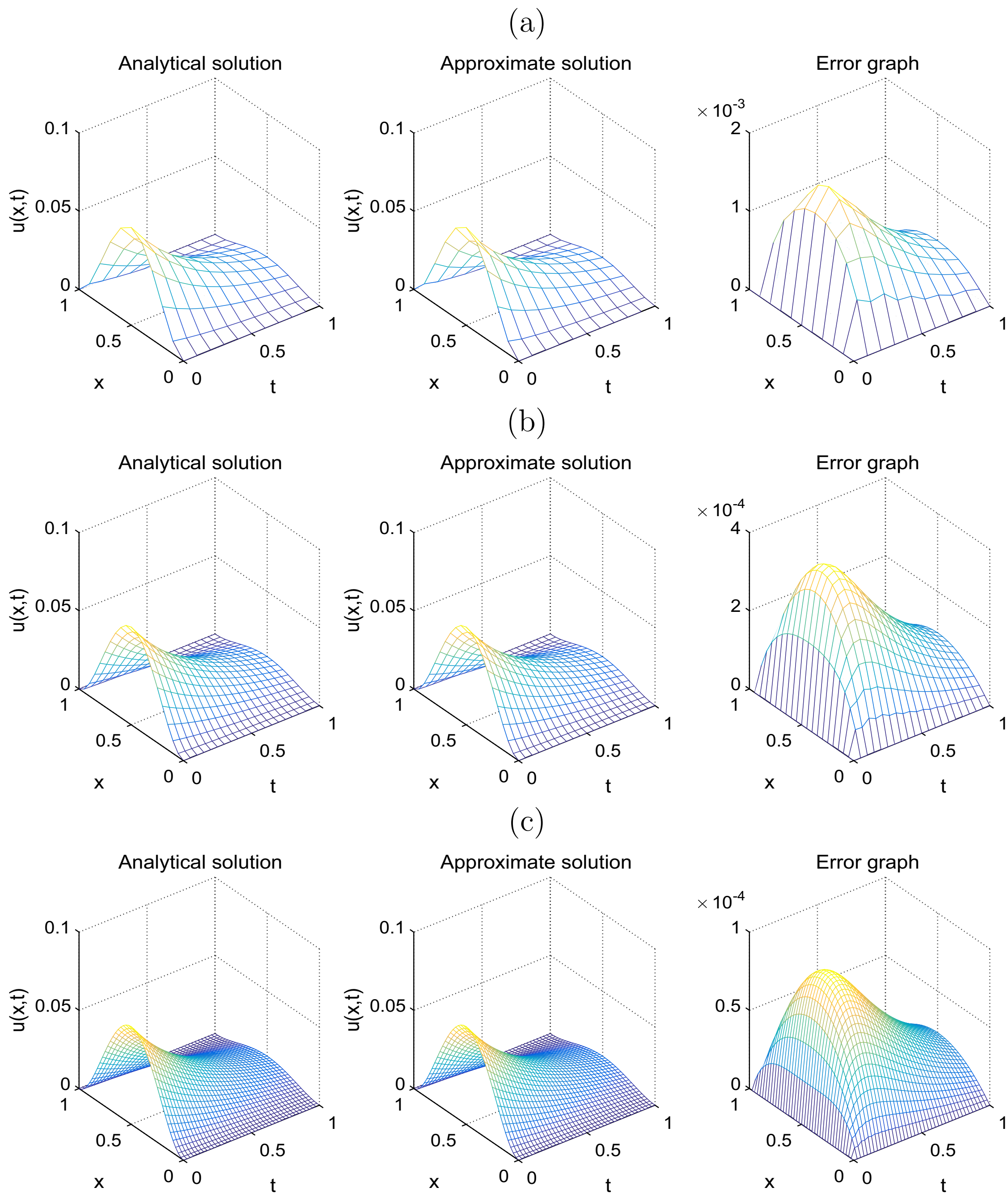

Figure 1: The exact (78) and approximate $u(x, t)$ with absolute errors for $M = N$: (a) 10, (b) 20, and (c) 40, for the direct problem.

problem. So, regularization is employed for stabilizing the solution. From all chosen λ, it is deduced that $\lambda \in \{10^{-10}, 10^{-9}\}$ gives a reasonable and stable accurate approximations for $f(t)$ obtaining $\text{rmse}(f) \in \{0.0052, 0.0063\}$.

Now, we add $p \in \{0.1\%, 1\%\}$ to the nonlocal integral $\theta(t)$, as shown in equation (4) via (75). The heat source $f(t)$ is depicted in Figures 4 and 5. From Figures 4(a) and 5(a), it can be observed that as noise p is increased, the approximate results start to build up oscillations with $\text{rmse}(f) \in \{0.6498, 4.5890\}$. Figures 4(b) and 5(b) show the reconstructed potential coefficient for numerous λ, and one can see that the most accurate solution is obtained for $\lambda \in \{10^{-9}, 10^{-8}\}$, obtaining $\text{rmse}(f) \in \{0.0123, 0.0501\}$, see Table 3 for more information. Figure 6 demonstrates the absolute errors between the exact (78) and approximate solutions u,

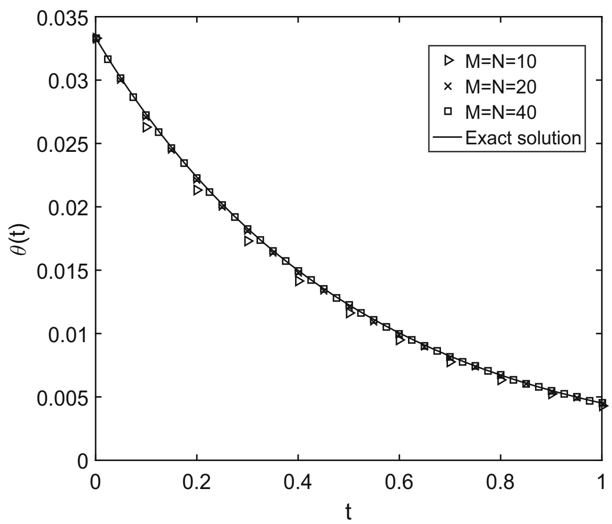

Figure 2: The exact (80) approximate $\theta(t)$, for Example 1.

Table 2: The rmse error norm for $\theta(t)$, for direct problem

M* = *N	**10**	**20**	**40**
rmse(θ)	6.9×10^{-4}	1.6×10^{-4}	5.1×10^{-5}

where the impact of $\lambda > 0$ in decreasing the unstable behavior of the reconstructed u can be identified. The numerically obtained results are stable and accurate. The main difficulty in regularization when we solve the nonlinear inverse problem is how to choose an appropriate regularization parameter λ, which compromises between accuracy and stability. However, one can use techniques such as the L-curve method [48] or Morozov's discrepancy principle [49] to find such a parameter, but in our work, we have used trial and error. As mentioned in [50], the regularization parameter λ is selected based on experience by first choosing a small value and gradually increasing it until any numerical oscillations in the unknown coefficient disappear.

8.2 Example 2

Now, we deal with the problem (1)–(4), with a nonsmooth heat source coefficient $f(t)$, and therefore, it is a critical test for the proposed technique of regularization, from the governing equation

$$u_t - a\Delta u + bu + u^q = f(t)h(x,t), \quad (x,t) \in [0,1] \times [0,1], \tag{83}$$

subject to

$$\varphi(x) = u(x,0) = \sin^2(\pi x), \quad x \in [0,1], \tag{84}$$

boundary condition:

$$u(0,t) = u(1,t) = 0, \quad t \in [0,1], \tag{85}$$

and nonlocal integral condition:

$$\theta(t) = \int_0^1 v(x)u(x,t)\mathrm{d}x = \frac{e^{-t}}{2}, \quad t \in [0,1], \tag{86}$$

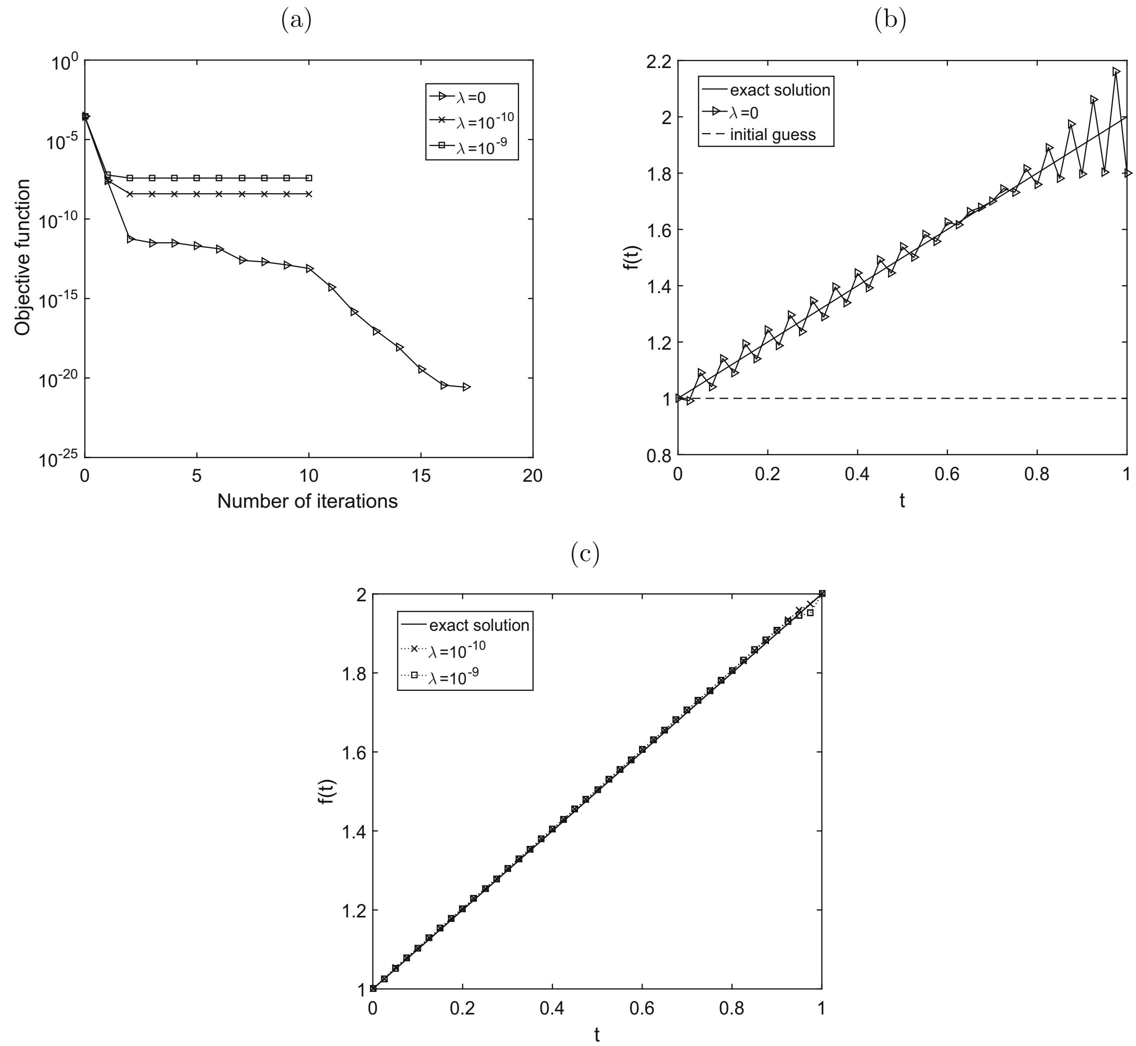

Figure 3: (a) The $J(73)$ and the exact (77) and approximate $f(t)$, for $p = 0$ with λ: (b) 0 and (c) 10^{-10}, 10^{-9}, for Example 1.

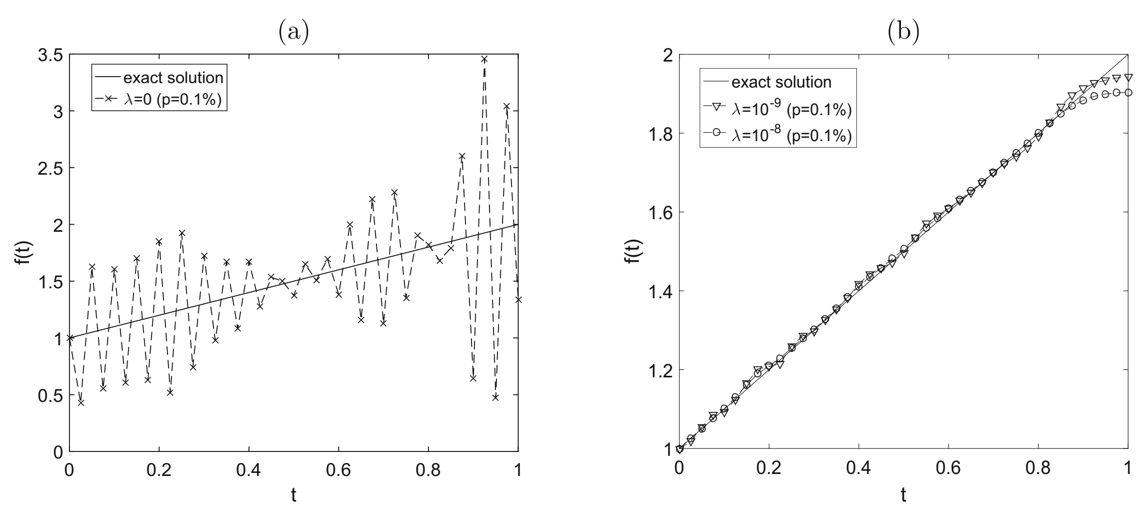

Figure 4: The exact (77) and approximate $f(t)$, for $p = 0.1\%$ with λ: (a) 0 and (b) 10^{-9}, 10^{-8}, for Example 1.

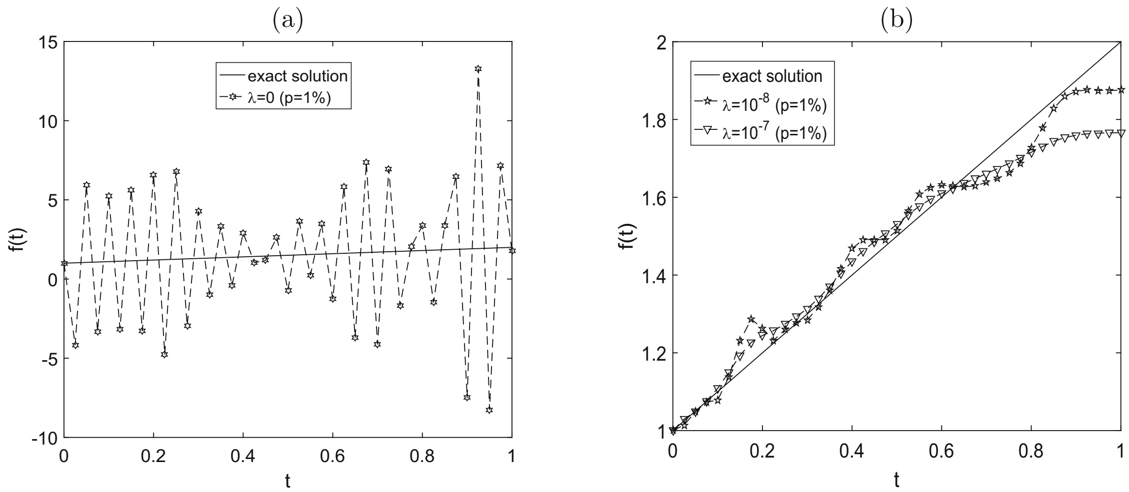

Figure 5: The exact (77) and approximate $f(t)$, for $p = 1\%$ with λ: (a) 0 and (b) 10^{-8}, 10^{-7}, for Example 1.

Table 3: The rmse and the least value of (73) for $p \in \{0, 0.1\%, 1\%\}$, with $\lambda = 10^{-10}$, 10^{-9}, 10^{-8}, and 10^{-7} at last iteration, for Example 1

p	λ	rmse(f)	Minimum values of (73)
0	0	0.0675	1.2×10^{-22}
	10^{-10}	0.0052	3.2×10^{-9}
	10^{-9}	0.0063	3.9×10^{-7}
0.1%	0	0.6498	8.9×10^{-22}
	10^{-10}	0.0228	3.5×10^{-5}
	10^{-8}	0.0123	1.1×10^{-4}
	10^{-7}	0.0162	7.5×10^{-4}
1%	0	4.589	9.7×10^{-26}
	10^{-9}	0.0971	3.1×10^{-3}
	10^{-8}	0.0501	2.1×10^{-2}
	10^{-7}	0.0706	6.9×10^{-2}

where $v(x) = 1$ and

$$a = 1,\ \ b = 1,\ \ p = 2,\ \ q = 2,\ \ h(x,t) = \frac{1}{8\left(\left|t^2 - \frac{1}{2}\right| - 1\right)}\{e^{-2t}(3 - 4(1 + 4e^t\pi^2)\cos(2\pi x) + \cos(4\pi x))\}. \tag{87}$$

The exact solution is considered as follows:

$$u(x,t) = e^{-t}\sin^2(\pi x), \tag{88}$$

$$f(t) = \left|t^2 - \frac{1}{2}\right| - 1. \tag{89}$$

Then, with this input data, the conditions of Theorems 1 and 2 are fulfilled, and so, the solution is unique. The initial guess for $\underline{f}$ for this example has been chosen as follows:

$$f^0(t_j) = f(0) = -\frac{1}{2}, \quad j = 1, 2, \ldots, N. \tag{90}$$

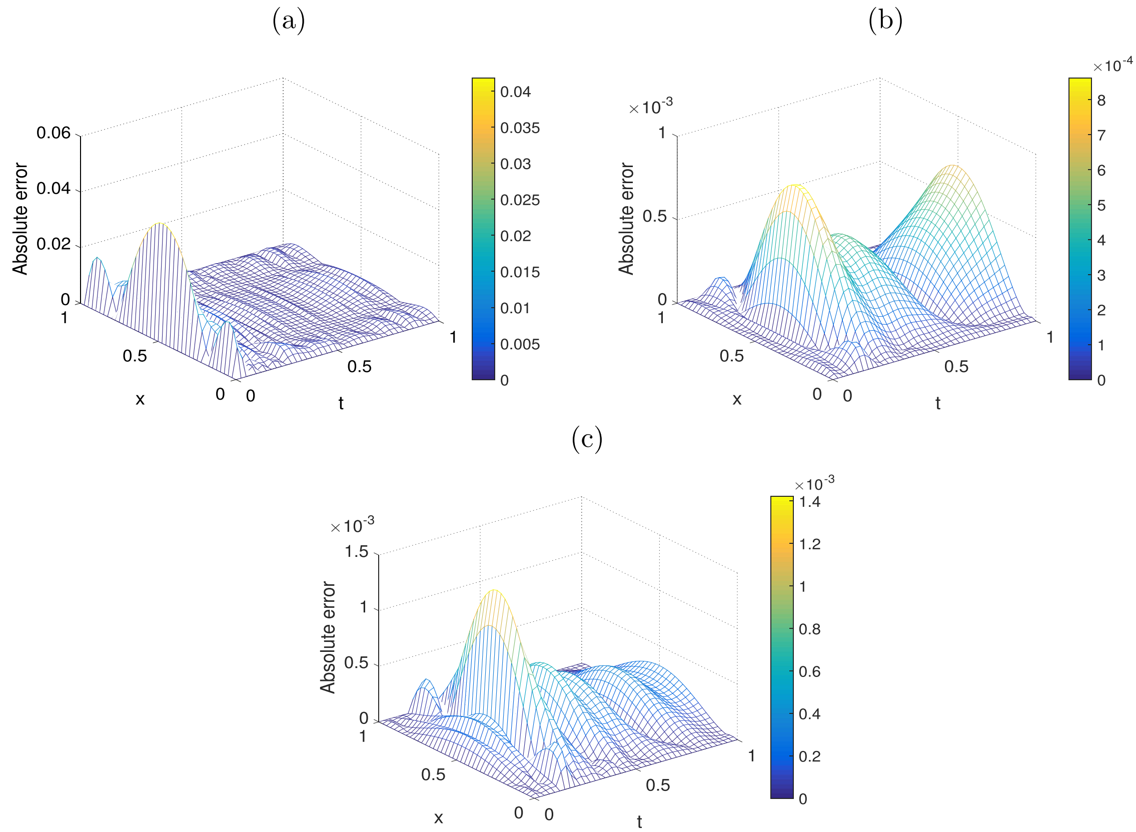

Figure 6: The errors between the exact (78) and approximate $u(x, t)$ with λ: (a) 0, (b) 10^{-8} and (c) 10^{-7}, with $p = 1\%$, for Example 1.

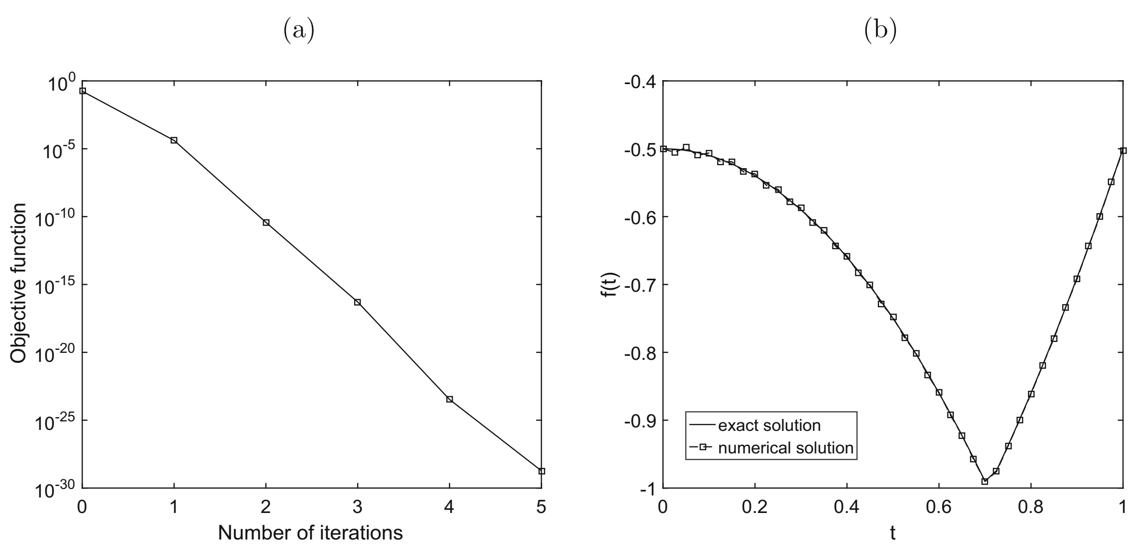

Figure 7: (a) J(73) and (b) the exact (89) and approximate $f(t)$, with $p = 0$ and without regularzation parameter, for Example 2.

We fix $\Delta x = \Delta t = 0.025$, and first choose the case when $p = 0$ in $\theta(t)$, as in (75). The J(73) is demonstrated in Figure 7(a), where a monotonically decreasing convergence is obtained in five iterations for getting a stipulated tolerance of $O(10^{-29})$. Figure 7(b) illustrates the computed heat source $f(t)$ without regularization, obtaining rmse$(f) = 0.0024$, and see Table 4.

Table 4: The rmse and the least value of (73) for $p \in \{0, 0.1\%, 1\%\}$, with $\lambda = 10^{-8}, 10^{-7}, 10^{-6}, 10^{-5}, 10^{-4}$, and 10^{-3} at last iteration, for Example 2

p	λ	rmse(f)	Minimum values of (73)
0	0	0.0024	1.8×10^{-29}
0.1%	0	0.1372	2.1×10^{29}
	10^{-8}	0.0232	3.1×10^{6}
	10^{-7}	0.0095	1.1×10^{5}
	10^{-6}	0.0044	5.7×10^{5}
	10^{-5}	0.0086	4.3×10^{4}
1%	0	1.5812	9.1×10^{30}
	10^{-6}	0.037	1.1×10^{3}
	10^{-5}	0.0194	1.7×10^{3}
	10^{-4}	0.0341	4.3×10^{3}
	10^{-3}	0.0532	46.9×10^{2}

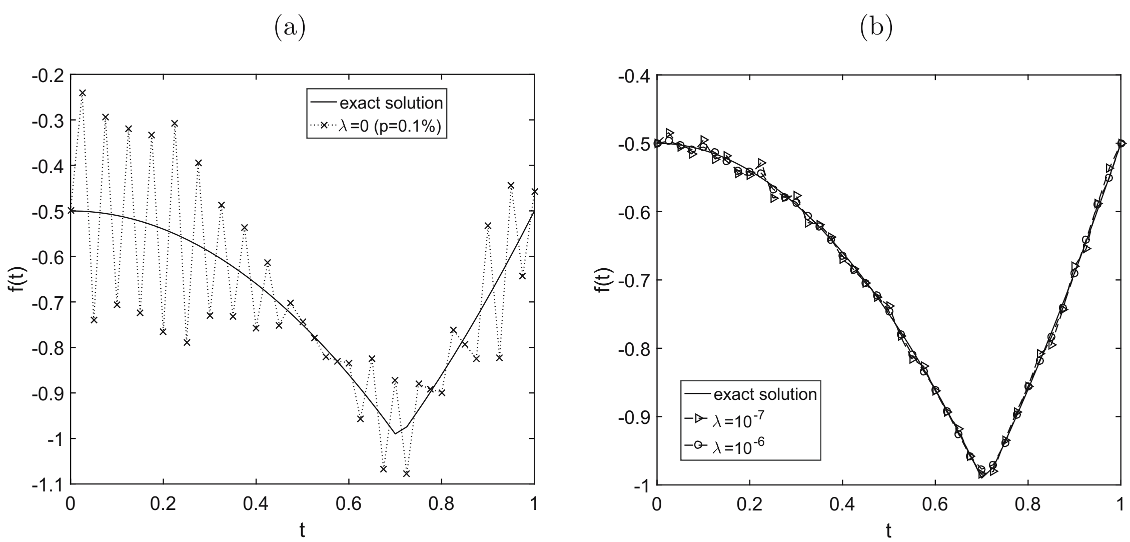

Figure 8: The exact (89) and approximate $f(t)$, for $p = 0.1\%$ with λ: (a) 0 and (b) 10^{-7}, 10^{-6}, for Example 2.

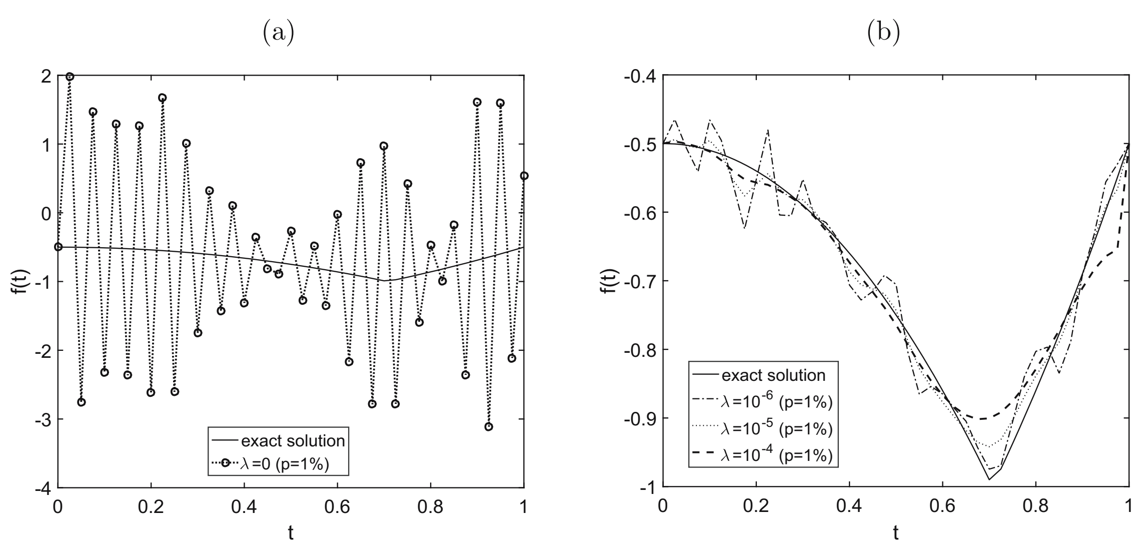

Figure 9: The exact (89) and approximate $f(t)$, for $p = 0.1\%$ with λ: (a) 0 and (b) 10^{-6}, 10^{-5}, 10^{-4}, for Example 2.

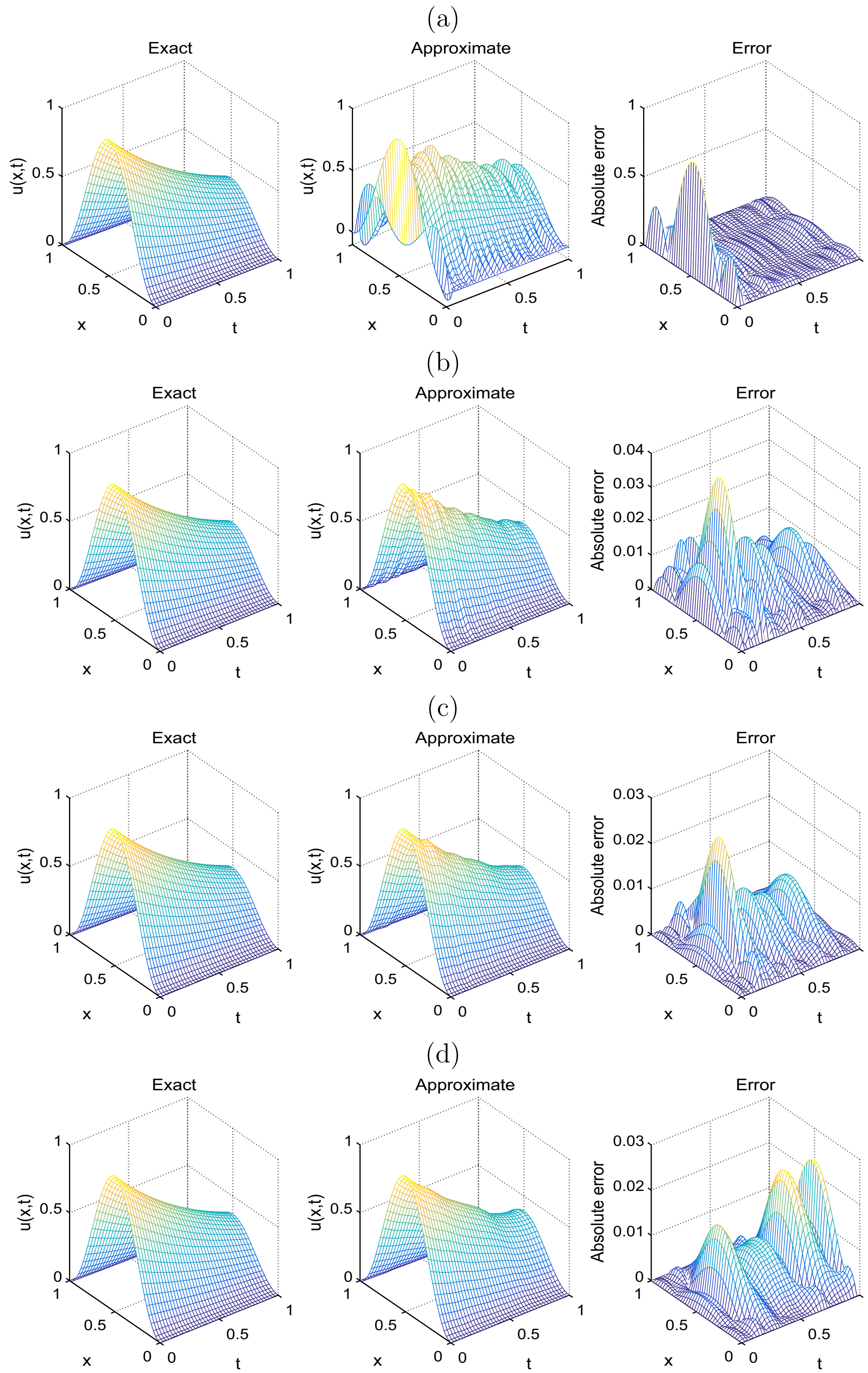

Figure 10: The exact (88) and approximate $u(x, t)$, and absolute errors with $p = 1\%$ and λ (a) 0, (b) 10^{-6}, (c) 10^{-5}, and (d) 10^{-4}, for Example 2.

Now, we examine the stability of the solution with noisy data. We include $p \in \{0.1\%, 1\%\}$ for simulating the input noisy data, via (75) for $\theta(t)$. We have also investigated higher amounts of noise p in (78), but the results obtained were less accurate and therefore, they are not presented. The determination of the heat source $f(t)$ is shown in Figures 8(a) and 9(a), where unstable results are obtained, if $\lambda = 0$, with $\mathrm{rmse}(f) = 0.1372$ and 1.5812. To stabilize $f(t)$, we employ regularization with $\lambda \in \{10^{-7}, 10^{-6}\}$ for $p = 0.1\%$, obtaining $\mathrm{rmse}(f) \in \{0.0095, 0.0044\}$ and $\lambda \in \{10^{-6}, 10^{-5}, 10^{-4}\}$ for $p = 1\%$ noise, obtaining $\mathrm{rmse}(f) \in \{0.0370, 0.0194, 0.0341\}$. Figures 8(b) and 9(b) demonstrate the reconstructed potential for different λ, and it is noticed that the most accurate solution is obtained for $\lambda = 10^{-6}$ and 10^{-5}. The exact (78) and approximate $u(x, t)$ with absolute error norms are shown in Figure 10, where the impact of $\lambda > 0$ in decreasing the unstable behavior of the reconstructed $u(x, t)$ can be identified. For more information about the rmse values (76) and the minimum value of J(73) at last iteration, we refer to Table 4. The similar conclusions can be drawn for the stable reconstruction of $f(t)$.

9 Conclusion

In this article, the inverse problem involving the determination of the time-dependent component and the temperature in the nonlinear parabolic equation from the nonlocal integral over-specification has been examined. The proof of existence and uniqueness of the solution upon the data has been proved by using the fixed-point technique. The inverse parabolic problem has also been investigated numerically by using the cubic B-spline collocation technique together with the Tikhonov regularization. From the obtained results, it has been deduced that stable accurate approximations for $f(t)$ has been obtained for $\lambda \in \{10^{-10}, 10^{-9}\}$, when noise $p = 0$, and for $\lambda \in \{10^{-9}, 10^{-8}\}$, when $p \in \{0.1\%, 1\%\}$. For nonsmooth heat source coefficient $f(t)$, it has been observed that stable accurate approximations for $f(t)$ has been obtained for $\lambda \in \{10^{-7}, 10^{-6}\}$, when perturbed data $p = 0.1\%$ and $\lambda \in \{10^{-6}, 10^{-5}\}$, when $p = 1\%$. The stability analysis shows that the present technique is unconditionally stable for the discretized system of (1). Finally, the proposed numerical method can be easily generalized for determining the unknown coefficient in a two-dimensional parabolic problem, which is an interesting field for future work.

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