

## Research Article

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# Stein-Weiss inequality for local mixed radial-angular Morrey spaces

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**Abstract:** In this article, a generalization of the well-known Stein-Weiss inequality for the fractional integral operator on functions with different integrability properties in the radial and the angular direction in local Morrey spaces is established. We find that some conditions can be relaxed for the Stein-Weiss inequality for local mixed radial-angular Morrey spaces.

**Keywords:** Stein-Weiss inequality, fractional integral operator, mixed radial-angular space, local Morrey space

**MSC 2020:** 42B20, 42B25, 42B35

## 1 Introduction

In this article, we devote to extending the celebrated Stein-Weiss inequality to local mixed radial-angular Morrey spaces.

The Stein-Weiss inequality [1] gives the two power-weighted norm inequalities for fractional integral operator. After that, the two-weighted inequalities for fractional integral operator were extended to general weight functions. For instance, the weighted norm inequalities with Muckenhoupt weights for fractional integral operator were considered in [2–5]. Note that, if we consider the power weights, De Nápoli et al. [6] and Hidano and Kurokawa [7] proved that some conditions in [1] can be relaxed. And more generally, if the functions under consideration have different integrability properties in the radial and the angular direction, i.e., mixed radial-angular spaces, D’Ancona and Luca’ [8] pointed out that the conditions in [1,6,7] can be extended to a more general setting. Obviously, the functions with different integrability properties in the radial and the angular direction may not be radial functions, and the result in [8, Theorem 1.3] essentially improves the results of [6,7]. Note that recently the mixed radial-angular spaces have been successfully used to study Strichartz estimates and partial differential equations to improve the corresponding results (see [9–13], etc.).

As we know, Morrey spaces, initially introduced by Morrey in [14], are natural generalizations of Lebesgue spaces. Many important results in harmonic analysis, such as the mapping properties of some important integral operators, have been extended to Morrey spaces. Particularly, the boundedness of the fractional integral operator on Morrey-type spaces was established in [15–19]. Nowadays, the Stein-Weiss inequality has been successfully generalized to weighted Morrey spaces by Ho [20]. Recently, based on the results of [6, Theorem 1.2] and [7, Theorem 2.1], Ho [21] considered the Stein-Weiss inequality for radial functions in local Morrey spaces and showed that some conditions can be relaxed. Inspired by [8,21],

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we will consider the Stein-Weiss inequality for functions with different integrability properties in the radial and the angular direction in local Morrey spaces.

In view of [8, Theorem 1.3], it is expectable that when we consider functions with different integrability properties in local Morrey spaces, the conditions of [21, Theorem 3.1] can also be relaxed. As the central versions of the classical Morrey spaces, the local Morrey spaces are also important function spaces to study the mapping properties of integral operators. We refer the readers to [22–25] for more studies of local Morrey spaces. See also [26] for the extrapolation theory on local Morrey spaces with variable exponents.

The main result of this article can be seen as a complement of the mapping properties of fractional integral operator acting on local Morrey spaces. As applications of the main result, we obtain the Poincaré and Sobolev inequalities for local mixed radial-angular Morrey spaces.

The organization of the remainder of this article is as follows. Section 2 contains the definitions of local Morrey spaces and local mixed radial-angular Morrey spaces used in this article. The celebrated Stein-Weiss inequality and some extensions of it to radial functions will also be presented in this section. The main result of this article, Stein-Weiss inequality for local mixed radial-angular Morrey spaces, is proved in Section 3. As applications, we give the Poincaré and Sobolev inequalities for functions with different integrability properties in the radial and the angular direction in local Morrey spaces in Section 4.

## 2 Definitions and preliminaries

Throughout the article, we use the following notations.

For any  $r > 0$  and  $x \in \mathbb{R}^n$ , let  $B(x, r) = \{y : |y - x| < r\}$  be the ball centered at  $x$  with radius  $r$ . Let  $\mathcal{B} = \{B(x, r) : x \in \mathbb{R}^n, r > 0\}$  be the set of all such balls. We use  $\chi_E$  and  $|E|$  to denote the characteristic function and the Lebesgue measure of a measurable set  $E$ . Let  $\mathcal{M}(E)$  be the class of Lebesgue measurable functions on  $E$ . For a nonnegative function  $w \in L^1_{\text{loc}}(\mathbb{R}^n)$  and  $1 \leq p < \infty$ , the weighted Lebesgue space  $L^p_w(\mathbb{R}^n)$  consists of all  $f \in \mathcal{M}(\mathbb{R}^n)$  such that  $\|f\|_{L^p_w} := \left( \int_{\mathbb{R}^n} |f(x)|^p w(x) dx \right)^{1/p} < \infty$ . In particular, if  $w(x) = |x|^{-\alpha p}$ , then we write  $L^p_w(\mathbb{R}^n) = L^p_\alpha(\mathbb{R}^n)$ . By  $A \lesssim B$ , we mean that  $A \leq CB$  for some constant  $C > 0$ .

Let  $0 < \gamma < n$ , the fractional integral operator  $I_\gamma$  is defined by

$$I_\gamma f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x - y|^\gamma} dy, \quad x \in \mathbb{R}^n,$$

where  $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ .

We first recall the famous Stein-Weiss inequality for the fractional integral operator.

**Theorem 2.1.** Let  $n \geq 1$ ,  $0 < \gamma < n$ ,  $1 < p < \infty$ ,  $\alpha < \frac{n}{p'}$ ,  $\beta < \frac{n}{q}$ ,  $\alpha + \beta \geq 0$ , and  $\frac{1}{q} = \frac{1}{p} + \frac{\gamma + \alpha + \beta}{n} - 1$ . If  $p \leq q < \infty$ , then for all  $f \in L^p_\alpha(\mathbb{R}^n)$ , we have

$$\|I_\gamma f\|_{L^q_{-\beta}} \lesssim \|f\|_{L^p_\alpha}.$$

The reader is referred to [1, Theorem B\*], for the proof of the above theorem.

Define by

$$\mathcal{R}(\mathbb{R}^n) = \{f \in \mathcal{M}(\mathbb{R}^n) : f(x) = f(y), \quad \text{if } |x| = |y|\}.$$

For any  $1 \leq p < \infty$  and  $\alpha \in \mathbb{R}$ , define  $L^p_{\alpha, \text{rad}}(\mathbb{R}^n) = L^p_\alpha(\mathbb{R}^n) \cap \mathcal{R}(\mathbb{R}^n)$ . Then the Stein-Weiss inequality for the radial functions can be stated as follows.

**Theorem 2.2.** Let  $n \geq 1$ ,  $0 < \gamma < n$ ,  $1 < p < \infty$ ,  $\alpha < \frac{n}{p'}$ ,  $\beta < \frac{n}{q}$ ,  $\alpha + \beta \geq (n - 1)\left(\frac{1}{q} - \frac{1}{p}\right)$ , and  $\frac{1}{q} = \frac{1}{p} + \frac{\gamma + \alpha + \beta}{n} - 1$ . If  $p \leq q < \infty$ , then for all  $f \in L^p_{\alpha, \text{rad}}(\mathbb{R}^n)$ , we have

$$\|I_\gamma f\|_{L_{-\beta}^q} \leq \|f\|_{L_\alpha^p}.$$

The proof of Theorem 2.2 can be found in [6, Theorem 1.2] and [7, Theorem 2.1].

Next, we give the definition of local Morrey spaces.

**Definition 2.1.** Let  $1 \leq p < \infty$ ,  $w : \mathbb{R}^n \rightarrow [0, \infty)$ , and  $u : (0, \infty) \rightarrow (0, \infty)$  be Lebesgue measurable functions. The local Morrey spaces  $LM_w^{p,u}(\mathbb{R}^n)$  consist of all  $f \in \mathcal{M}(\mathbb{R}^n)$  such that

$$\|f\|_{LM_w^{p,u}} = \sup_{r>0} \frac{1}{u(r)} \|f\chi_{B(0,r)}\|_{L_w^p} < \infty.$$

In particular, if  $w(x) = |x|^{\alpha p}$ , then we write  $LM_w^{p,u}(\mathbb{R}^n) = LM_\alpha^{p,u}(\mathbb{R}^n)$ .

We refer the readers to [22–25] for the mapping properties for various integral operators on local Morrey spaces. For  $1 \leq p < \infty$ ,  $\alpha \in \mathbb{R}$ , and  $u : (0, \infty) \rightarrow (0, \infty)$ , the radial local Morrey spaces  $LM_{\alpha,\text{rad}}^{p,u}(\mathbb{R}^n)$  consist of all radial functions  $f \in LM_\alpha^{p,u}(\mathbb{R}^n)$ , i.e.,

$$LM_{\alpha,\text{rad}}^{p,u}(\mathbb{R}^n) = LM_\alpha^{p,u}(\mathbb{R}^n) \cap \mathcal{R}(\mathbb{R}^n).$$

The Stein-Weiss inequality for radial local Morrey spaces is as follows, see [21, Theorem 3.1] for the proof.

**Theorem 2.3.** Let  $n \geq 1$ ,  $0 < \gamma < n$ ,  $1 < p < \infty$ ,  $\alpha < \frac{n}{p}$ ,  $\beta < \frac{n}{q}$ ,  $\alpha + \beta \geq (n-1)\left(\frac{1}{q} - \frac{1}{p}\right)$ , and  $\frac{1}{q} = \frac{1}{p} + \frac{\gamma + \alpha + \beta}{n} - 1$  and  $u : (0, \infty) \rightarrow (0, \infty)$ . If  $p \leq q < \infty$  and there exists a constant  $C > 0$  such that for any  $r > 0$ ,  $u$  satisfies

$$u(2r) \leq Cu(r), \quad (2.1)$$

$$\sum_{j=0}^{\infty} 2^{-(j+1)(\frac{n}{q}-\beta)} u(2^{j+1}r) \leq Cu(r), \quad (2.2)$$

then for any  $f \in LM_{\alpha,\text{rad}}^{p,u}(\mathbb{R}^n)$ , we have

$$\|I_\gamma f\|_{LM_{-\beta}^{q,u}} \leq \|f\|_{LM_\alpha^{p,u}}.$$

Note that when  $n = 1$ , Theorem 2.2 coincides with Theorem 2.1. And in this case, the conditions imposed on  $p, q, \alpha, \beta$  of Theorems 2.1, 2.2, and 2.3 are the same. However, when  $n \geq 2$ , the condition  $\alpha + \beta \geq (n-1)\left(\frac{1}{q} - \frac{1}{p}\right)$  in Theorems 2.2 and 2.3 implies that  $\alpha + \beta$  may be negative, while Theorem 2.1 requires  $\alpha + \beta \geq 0$ . This observation yields that the Stein-Weiss inequality imposed on radial functions has better performance.

The mixed radial-angular spaces, including the functions with different integrability properties in the radial and the angular direction may not be radial functions, are extensions of Lebesgue spaces. Now we recall their definitions.

**Definition 2.2.** For  $n \geq 1$ ,  $1 \leq p, \bar{p} \leq \infty$ , the mixed radial-angular space  $L_{|x|}^p L_\theta^{\bar{p}}(\mathbb{R}^n)$  consist of all functions  $f \in \mathcal{M}(\mathbb{R}^n)$  for which

$$\|f\|_{L_{|x|}^p L_\theta^{\bar{p}}} := \left( \int_0^\infty \left( \int_{\mathbb{S}^{n-1}} |f(r, \theta)|^{\bar{p}} d\theta \right)^{p/\bar{p}} r^{n-1} dr \right)^{1/p} < \infty,$$

where  $\mathbb{S}^{n-1}$  denotes the unit sphere in  $\mathbb{R}^n$ . If  $p = \infty$  or  $\bar{p} = \infty$ , then we have to make appropriate modifications.

Similar to the power-weighted Lebesgue spaces, we can define the power-weighted mixed radial-angular spaces.

**Definition 2.3.** For  $n \geq 1$ ,  $\alpha \in \mathbb{R}$ ,  $1 \leq p, \bar{p} \leq \infty$ , the power-weighted mixed radial-angular spaces  $L_{|x|}^{p,\alpha} L_{\theta}^{\bar{p}}(\mathbb{R}^n)$  consists of all  $f \in \mathcal{M}(\mathbb{R}^n)$  such that

$$\|f\|_{L_{|x|}^{p,\alpha} L_{\theta}^{\bar{p}}} := \|f v\|_{L_{|x|}^p L_{\theta}^{\bar{p}}} < \infty,$$

where  $v(x) = |x|^{\alpha}$ .

The mixed radial-angular spaces were initially introduced to improve some classical results in PDE, see [9–12]. Later, the mixed radial-angular spaces were frequently used in harmonic analysis. For instance, Liu et al. [27–29] considered the mapping properties of various operators with rough kernels on mixed radial-angular spaces. See also [30] for the extrapolation theorems on mixed radial-angular spaces. One can see that the mixed radial-angular spaces are particular cases of mixed-norm Lebesgue spaces studied by Benedek and Panzone [31]. The readers are referred to [32–38] for more studies on mixed-norm Lebesgue spaces.

When we consider function with different integrability properties in the radial and the angular direction, D’Ancona and Luca’ [8, Theorem 1.3] obtained the following extensions of Stein-Weiss inequality for mixed radial-angular spaces.

**Theorem 2.4.** Let  $n \geq 1$ ,  $0 < \gamma < n$ ,  $1 < p < \infty$ ,  $\alpha < \frac{n}{p}$ ,  $\beta < \frac{n}{q}$ ,  $\alpha + \beta \geq (n-1)\left(\frac{1}{q} - \frac{1}{p} + \frac{1}{\bar{p}} - \frac{1}{\bar{q}}\right)$ , and  $\frac{1}{q} = \frac{1}{p} + \frac{\gamma + \alpha + \beta}{n} - 1$ . If  $p \leq q < \infty$  and  $1 \leq \bar{p} \leq \bar{q} \leq \infty$ , then for all  $f \in L_{|x|}^{p,\alpha} L_{\theta}^{\bar{p}}(\mathbb{R}^n)$ , we have

$$\|I_{\gamma} f\|_{L_{|x|}^{q,-\beta} L_{\theta}^{\bar{q}}} \lesssim \|f\|_{L_{|x|}^{p,\alpha} L_{\theta}^{\bar{p}}}.$$

By combining the definitions of local Morrey spaces and mixed radial-angular spaces, we can define local mixed radial-angular Morrey spaces as follows:

**Definition 2.4.** Let  $1 \leq p, \bar{p} \leq \infty$ , and  $u : (0, \infty) \rightarrow (0, \infty)$  be a Lebesgue measurable function. The local mixed radial-angular Morrey spaces  $LM_u^{p,\bar{p}}(\mathbb{R}^n)$  consist of all  $f \in \mathcal{M}(\mathbb{R}^n)$  such that

$$\|f\|_{LM_u^{p,\bar{p}}} = \sup_{r>0} \frac{1}{u(r)} \|f \chi_{B(0,r)}\|_{L_{|x|}^p L_{\theta}^{\bar{p}}} < \infty.$$

Roughly speaking, the local mixed radial-angular Morrey spaces are just the functions in local Morrey spaces which have different integrability properties in the radial and the angular direction. Therefore, local mixed radial-angular Morrey spaces are extensions of radial local Morrey spaces, since the integrability in the angular direction holds automatically for radial functions. Consequently, it is meaningful to consider Stein-Weiss inequality for local mixed radial-angular Morrey spaces.

### 3 Main result

This section establishes the Stein-Weiss inequality on local mixed radial-angular Morrey spaces. To do this, we first give the definition of power-weighted local mixed radial-angular Morrey spaces, which are combinations of power-weighted mixed radial-angular spaces and local Morrey spaces.

**Definition 3.1.** Let  $\alpha \in \mathbb{R}$ ,  $1 \leq p, \bar{p} \leq \infty$ , and  $u : (0, \infty) \rightarrow (0, \infty)$  be a Lebesgue measurable function. The power-weighted local mixed radial-angular Morrey space  $LM_{\alpha,u}^{p,\bar{p}}(\mathbb{R}^n)$  consists of all  $f \in \mathcal{M}(\mathbb{R}^n)$  such that

$$\|f\|_{LM_{\alpha,u}^{p,\bar{p}}} = \sup_{r>0} \frac{1}{u(r)} \|f \chi_{B(0,r)}\|_{L_{|x|}^{p,\alpha} L_{\theta}^{\bar{p}}} < \infty.$$

Now we are in a position to state our main result in this article.

**Theorem 3.1.** Let  $n \geq 1$ ,  $0 < \gamma < n$ ,  $1 < p < \infty$ ,  $\alpha < \frac{n}{p}$ ,  $\beta < \frac{n}{q}$ ,  $\alpha + \beta \geq (n-1)\left(\frac{1}{q} - \frac{1}{p} + \frac{1}{\bar{p}} - \frac{1}{\bar{q}}\right)$ ,  $\frac{1}{q} = \frac{1}{p} + \frac{\gamma + \alpha + \beta}{n} - 1$ , and  $u : (0, \infty) \rightarrow (0, \infty)$  be a Lebesgue measurable function. If  $p \leq q < \infty$  and  $1 \leq \bar{p} \leq \bar{q} \leq \infty$  and there is a constant  $C > 0$  such that for any  $r > 0$ ,  $u$  satisfies (2.1) and (2.2) in Theorem 2.3, then for any  $f \in LM_{\alpha, u}^{p, \bar{p}}(\mathbb{R}^n)$ , we have

$$\|I_\gamma f\|_{LM_{-\beta, u}^{q, \bar{q}}} \lesssim \|f\|_{LM_{\alpha, u}^{p, \bar{p}}}.$$

**Proof.** Let  $f \in LM_{\alpha, u}^{p, \bar{p}}(\mathbb{R}^n)$ . For any  $r > 0$ , define

$$D_k = B(0, 2^{k+1}) \setminus B(0, 2^k), \quad k \in \mathbb{N} \setminus \{0\}.$$

Denote by  $f = \sum_{k=0}^{\infty} f_k$ , where  $f_k = f \chi_{D_k}$ ,  $k \geq 1$ , and  $f_0 = f \chi_{B(0, 2r)}$ .

Noting that  $I_\gamma$  is a sublinear operator, there holds

$$|I_\gamma f| \leq \sum_{k=0}^{\infty} |I_\gamma f_k|. \quad (3.1)$$

Since  $f \in LM_{\alpha, u}^{p, \bar{p}}(\mathbb{R}^n)$ , we have  $|f| \in LM_{\alpha, u}^{p, \bar{p}}(\mathbb{R}^n)$ . As a consequence, Theorem 2.4 guarantees

$$\frac{1}{u(r)} \|\chi_{B(0, r)} I_\gamma |f_0|\|_{L_{|\cdot|}^{q, -\beta} L_\theta^{\bar{q}}} \leq \frac{1}{u(r)} \|I_\gamma |f_0|\|_{L_{|\cdot|}^{q, -\beta} L_\theta^{\bar{q}}} \lesssim \frac{1}{u(r)} \|f_0\|_{L_{|\cdot|}^{p, \alpha} L_\theta^{\bar{p}}}.$$

Consequently, (2.1) implies

$$\frac{1}{u(r)} \|\chi_{B(0, r)} I_\gamma |f_0|\|_{L_{|\cdot|}^{q, -\beta} L_\theta^{\bar{q}}} \lesssim \frac{1}{u(2r)} \|f \chi_{B(0, 2r)}\|_{L_{|\cdot|}^{p, \alpha} L_\theta^{\bar{p}}} \lesssim \|f\|_{LM_{\alpha, u}^{p, \bar{p}}}. \quad (3.2)$$

Next we consider the terms  $I_\gamma f_k$ , where  $k \geq 1$ . From the definitions of  $I_\gamma$ , for any  $x \in B(0, r)$ , we have

$$I_\gamma |f_k|(x) \lesssim \frac{1}{2^{ky} r^\gamma} \int_{D_k} |f_k(y)| dy.$$

By using Hölder's inequality on mixed Lebesgue spaces (see [31]), we obtain

$$I_\gamma |f_k|(x) \lesssim \frac{1}{2^{ky} r^\gamma} \|f_k\|_{L_{|\cdot|}^{p, \alpha} L_\theta^{\bar{p}}} \|\chi_{B(0, 2^{k+1}r)}\|_{L_{|\cdot|}^{p', -\alpha} L_\theta^{\bar{p}'}}. \quad (3.3)$$

A direct calculation yields

$$\begin{aligned} \|\chi_{B(0, 2^{k+1}r)}\|_{L_{|\cdot|}^{p', -\alpha} L_\theta^{\bar{p}'}} &= w_n^{1/p'} \left( \int_0^{2^{k+1}r} r^{-p'\alpha + n-1} dr \right)^{1/p'} \\ &= w_n^{1/p'} \frac{(2^{k+1}r)^{-\alpha + n/p'}}{-p'\alpha + n} \\ &= C 2^{(k+1)(-\alpha + n/p')} r^{-\alpha + n/p'} \end{aligned} \quad (3.4)$$

for some constant  $C > 0$ , where  $w_n$  is the induced Lebesgue measure of  $\mathbb{S}^{n-1}$ .

From the assumption  $\frac{1}{q} = \frac{1}{p} + \frac{\gamma + \alpha + \beta}{n} - 1$ , we obtain

$$-\alpha + \frac{n}{p'} - \gamma = -\alpha + n \left( 1 - \frac{1}{p} \right) - \gamma = \beta - \frac{n}{q}. \quad (3.5)$$

Inequality (3.3), together with (3.4) and (3.5), shows that

$$\begin{aligned} I_\gamma |f_k|(x) &\lesssim 2^{(k+1)(-\alpha + n/p' - \gamma)} r^{-\alpha + n/p' - \gamma} \|f_k\|_{L_{|\cdot|}^{p, \alpha} L_\theta^{\bar{p}}} \\ &= 2^{(k+1)(\beta - n/q)} r^{\beta - n/q} \|f_k\|_{L_{|\cdot|}^{p, \alpha} L_\theta^{\bar{p}}}. \end{aligned} \quad (3.6)$$

By multiplying  $\chi_{B(0,r)}$  on both sides of inequality (3.3), and then using (3.4) and (3.6), we arrive at

$$\begin{aligned}\|\chi_{B(0,r)} I_Y |f_k| \|_{L_{|x|}^{q,-\beta} L_\theta^{\bar{q}}} &\lesssim 2^{(k+1)(\beta-n/q)} r^{\beta-n/q} \|f_k\|_{L_{|x|}^{p,\alpha} L_\theta^{\bar{p}}} \|\chi_{B(0,r)}\|_{L_{|x|}^{q,-\beta} L_\theta^{\bar{q}}} \\ &\lesssim 2^{(k+1)(\beta-n/q)} r^{\beta-n/q} \|f_k\|_{L_{|x|}^{p,\alpha} L_\theta^{\bar{p}}} \times r^{-\beta+n/q} \\ &\lesssim 2^{(k+1)(\beta-n/q)} \|f_k\|_{L_{|x|}^{p,\alpha} L_\theta^{\bar{p}}}.\end{aligned}$$

As a consequence,

$$\begin{aligned}\frac{1}{u(r)} \|\chi_{B(0,r)} I_Y |f_k| \|_{L_{|x|}^{q,-\beta} L_\theta^{\bar{q}}} &\leq \frac{u(2^{k+1}r)}{u(r)} 2^{(k+1)(\beta-n/q)} \frac{1}{u(2^{k+1}r)} \|\chi_{B(0,2^{k+1}r)}\|_{L_{|x|}^{p,\alpha} L_\theta^{\bar{p}}} \\ &\leq \frac{u(2^{k+1}r)}{u(r)} 2^{(k+1)(\beta-n/q)} \|f\|_{LM_{a,u}^{p,\bar{p}}}.\end{aligned}\quad (3.7)$$

As a result of (2.2), (3.1), (3.2), and (3.7), there holds

$$\sum_{k=0}^{\infty} \frac{1}{u(r)} \|\chi_{B(0,r)} I_Y |f_k| \|_{L_{|x|}^{q,-\beta} L_\theta^{\bar{q}}} \leq \left(1 + \sum_{k=0}^{\infty} \frac{u(2^{k+1}r)}{u(r)} 2^{(k+1)(\beta-n/q)}\right) \|f\|_{LM_{a,u}^{p,\bar{p}}} \leq \|f\|_{LM_{a,u}^{p,\bar{p}}},$$

since  $\beta < n/q$ .

By taking the supremum over  $r > 0$  on the above inequalities, one obtains

$$\begin{aligned}\|I_Y f\|_{LM_{a,u}^{q,\bar{q}}} &= \sup_{r>0} \frac{1}{u(r)} \|\chi_{B(0,r)} I_Y |f| \|_{L_{|x|}^{q,-\beta} L_\theta^{\bar{q}}} \\ &\leq \sup_{r>0} \sum_{k=0}^{\infty} \frac{1}{u(r)} \|\chi_{B(0,r)} I_Y |f_k| \|_{L_{|x|}^{q,-\beta} L_\theta^{\bar{q}}} \\ &\leq \|f\|_{LM_{a,u}^{p,\bar{p}}},\end{aligned}$$

which finishes the proof.  $\square$

### Remark 3.1.

- (i) When the functions under consideration in Theorem 3.1 are radial, we can choose  $\bar{p} = \bar{q} = s$  for some  $1 < s < \infty$ . In this sense, we obtain Theorem 2.3 as a consequence of Theorem 3.1.
- (ii) Conditions (2.1) and (2.2) are satisfied for many functions  $u$ . For instance, if we take  $u(r) = r^\sigma$  for some  $0 < \sigma < \frac{n}{q} - \beta$ , then  $u$  fulfills (2.1) and (2.2).

## 4 Applications

By applying Theorem 3.1, we will establish the Poincaré and Sobolev inequalities for local mixed radial-angular Morrey spaces.

The Poincaré inequality is closely related to the Harnack's inequalities, see [39]. Now we give the first result of this section.

**Theorem 4.1.** *Let  $n > 1$ ,  $1 < p < \infty$ ,  $\alpha < \frac{n}{p}$ ,  $\beta < \frac{n}{q}$ ,  $\alpha + \beta \geq (n-1)\left(\frac{1}{q} - \frac{1}{p} + \frac{1}{p} - \frac{1}{q}\right)$ ,  $\frac{1}{q} = \frac{1}{p} + \frac{\alpha+\beta-1}{n}$ , and  $u : (0, \infty) \rightarrow (0, \infty)$  be a Lebesgue measurable function. If  $p \leq q < \infty$  and  $1 \leq \bar{p} \leq \bar{q} \leq \infty$  and there is a constant  $C > 0$  such that for any  $r > 0$ ,  $u$  satisfies (2.1) and (2.2) in Theorem 2.3, then for any  $D \in \mathbb{B}$  and any continuous differentiable function  $f$  satisfying either  $\int_D f(x) dx = 0$  or  $\text{supp} f \subseteq D$ , there holds*

$$\sup_{r>0} \frac{1}{u(r)} \|\chi_{D \cap B(0,r)}\|_{L_{|x|}^{q,-\beta} L_\theta^{\bar{q}}} \leq \sup_{r>0} \frac{1}{u(r)} \|\nabla f \chi_{B(0,r)}\|_{L_{|x|}^{p,\alpha} L_\theta^{\bar{p}}}.$$

where  $\nabla$  is the gradient operator.

**Proof.** By virtue of [40, (4.34) and (4.35)], we know that for any  $x \in D$ ,

$$|f(x)| \leq I_{n-1}(|\nabla f(x)|\chi_D(x)).$$

By using Theorem 3.1 with  $\gamma = n - 1$ , we obtain the desired result.  $\square$

The second application is connected with the Sobolev inequality. As we know, the Sobolev inequality gives a two-weighted norm inequality for the Laplacian operator  $\Delta$ . Next we present the Sobolev inequality for local mixed radial-angular Morrey spaces.

**Theorem 4.2.** Let  $n > 2$ ,  $1 < p < \infty$ ,  $\alpha < \frac{n}{p}$ ,  $\beta < \frac{n}{q}$ ,  $\alpha + \beta \geq (n - 1)\left(\frac{1}{q} - \frac{1}{p} + \frac{1}{\bar{p}} - \frac{1}{\bar{q}}\right)$ ,  $\frac{1}{q} = \frac{1}{p} + \frac{\alpha + \beta - 2}{n}$ , and  $u : (0, \infty) \rightarrow (0, \infty)$  be a Lebesgue measurable function. If  $p \leq q < \infty$  and  $1 \leq \bar{p} \leq \bar{q} < \infty$  and there is a constant  $C > 0$  such that for any  $r > 0$ ,  $u$  satisfies (2.1) and (2.2) in Theorem 2.3, then for any  $D \in \mathcal{B}$  and any twice continuous differentiable function  $f$  satisfying  $\text{supp } f \subseteq D$ , there holds

$$\sup_{r>0} \frac{1}{u(r)} \|f\chi_{D \cap B(0,r)}\|_{L_{|x|}^{q,-\beta} L_{\theta}^{\bar{q}}} \leq \sup_{r>0} \frac{1}{u(r)} \|\Delta f\chi_{B(0,r)}\|_{L_{|x|}^{p,\alpha} L_{\theta}^{\bar{p}}}. \quad (4.1)$$

**Proof.** Noting that  $f = I_{n-2}(\Delta f)$ , (4.1) is a consequence of Theorem 3.1 with  $\gamma = n - 2$ .  $\square$

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