Research Article

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A note on commutators of strongly singular Calderón-Zygmund operators

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Abstract: In this article, the authors consider the commutators of strongly singular Calderón-Zygmund operator with Lipschitz functions. A sufficient condition is given for the boundedness of the commutators from Lebesgue spaces $L^p(\mathbb{R}^n)$ to certain Campanato spaces $C^{p,\beta}(\mathbb{R}^n)$.

Keywords: strongly singular Calderón-Zygmund operator, commutator, Lipschitz space, Campanato space

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1 Introduction and result

Let T be the classical singular integral operator, and the commutator T_b generated by T and a locally integrable function b is given by

$$T_b f = bT(f) - T(bf).$$

A well-known result by Coifman et al. [1] states that T_b is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$ when $b \in \mathrm{BMO}(\mathbb{R}^n)$. They also gave some characterization of $\mathrm{BMO}(\mathbb{R}^n)$ in virtue of the L^p boundedness of the aforementioned commutator (see also [2,3]).

In 1978, Janson [2] studied the boundedness of the commutator T_b when $b \in \dot{\Lambda}_\gamma(\mathbb{R}^n)$, the homogeneous Lipschitz space of order $0 < \gamma < 1$, which is the space of all functions b, such that

$$\|b\|_{\dot{\Lambda}_\gamma(\mathbb{R}^n)} = \sup_{\substack{x,y\in\mathbb{R}^n \\ x\neq y}} \frac{|b(x) - b(y)|}{|x - y|^\gamma} < \infty. \tag{1.1}$$

Janson proved that T_b is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ for $1 < p < q < \infty$ if and only if $b \in \dot{\Lambda}_\gamma(\mathbb{R}^n)$ with $\gamma = n(1/p - 1/q)$. In 1995, Paluszyński [4] made a further study of the problem and proved that T_b is bounded from $L^p(\mathbb{R}^n)$ to some Triebel-Lizorkin spaces $\dot{F}_p^{\gamma,\infty}(\mathbb{R}^n)$ if and only if $b \in \dot{\Lambda}_\gamma(\mathbb{R}^n)$, for $1 < p < \infty$ and $0 < \gamma < 1$.

In 2015, Zhang et al. [5] gave another kind of interesting results for T_b when b belongs to Lipschitz spaces. They proved that T_b is bounded from $L^p(\mathbb{R}^n)$ to $C^{p,\beta}(\mathbb{R}^n)$ if and only if $b \in \dot{\Lambda}_\gamma(\mathbb{R}^n)$, for $1 < p < \infty$, $-n/p \le \beta < 0$ and $0 < \gamma = \beta + n/p < 1$, where $C^{p,\beta}(\mathbb{R}^n)$ is Campanato space (see Definition 1.2).

On the other hand, motivated by the study of multiplier operator with symbol given by $\frac{e^{i|\xi|^\alpha}}{|\xi|^\beta}$ away from the origin ($0 < \alpha < 1, \beta > 0$), Alvarez and Milman [6] introduced the following strongly singular Calderón-Zygmund operator.

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Definition 1.1. [6] Let $T : \mathcal{S} \to \mathcal{S}'$ be a bounded linear operator. T is called a strongly singular Calderón-Zygmund operator if the following conditions are fulfilled.

(1) T can be extended to a continuous operator from L^2 into itself.

(2) There exists a continuous function $K(x, y)$ on $\{(x, y) : x \neq y\}$ such that

$$|K(x, y) - K(x, z)| + |K(y, x) - K(z, x)| \leqslant C\frac{|y - z|^{\delta}}{|x - z|^{n+\delta/\alpha}},$$

if $2|y - z|^{\alpha} \leq |x - z|$ for some $0 < \delta \leq 1$ and $0 < \alpha < 1$, and,

$$\langle Tf, g\rangle = \int K(x, y)f(y)g(x)\mathrm{d}y\mathrm{d}x,$$

for $f, g \in \mathcal{S}$ with disjoint supports.

(3) For some $(1 - \alpha)n/2 \leq \eta < n/2$, both T and its conjugate operator T^* can be extended to continuous operators from $L^q(\mathbb{R}^n)$ into $L^2(\mathbb{R}^n)$, where $1/q = 1/2 + \eta/n$.

In 1986, Alvarez and Milman studied the boundedness of strongly singular Calderón-Zygmund operator on Lebesgue spaces and Hardy spaces in [6,7]. Later on, there are many authors discussed the mapping properties of strongly singular Calderón-Zygmund operators in various spaces. See, for instance, [8–11]. We would like to note that, as stated in [6,7], the strongly singular Calderón-Zygmund operators include pseudo-differential operator with a symbol in the Hörmander class $S^{-\eta}_{\alpha,\delta}$, where $0 < \delta \leq \alpha < 1$, $(1 - \alpha)n/2 \leq \eta < n/2$.

Now, we define the commutator generated by strongly singular Calderón-Zygmund operator T and a locally integrable function b as follows:

$$T_b f(x) = b(x)T(f)(x) - T(bf)(x).$$

The main result of Alvarez et al. in [8] yields the boundedness of T_b on $L^p(\mathbb{R}^n)$, $1 < p < \infty$, when $b \in \mathrm{BMO}(\mathbb{R}^n)$. Afterward, the mapping properties of T_b, when b belongs to BMO space or Lipschitz space, on Lebesgue spaces, Morrey spaces, Herz type spaces, and Hardy spaces have been studied by several authors. See [10–15] for example.

In this article, we will continue the study of the commutator of strongly singular Calderón-Zygmund operator when the symbol b belongs to Lipschitz space. The aim is to extend some of the results in [5] to a strongly singular Calderón-Zygmund operator.

As usual, let $B = B(x_0, r)$ denote the ball centered at x_0 with radius r. For $a > 0$, aB stands for the ball concentric with B having a times its radius, that is, $aB = B(x_0, ar)$. Denote by $|B|$ the Lebesgue measure of B and by χ_B its characteristic function. For $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, we write

$$f_B = \frac{1}{|B|}\int_B f(x)\mathrm{d}x.$$

Definition 1.2. Let $1 \leq p < \infty$, $-n/p \leq \beta < 1$, the Campanato space $C^{p,\beta}(\mathbb{R}^n)$ is given by

$$C^{p,\beta}(\mathbb{R}^n) = \{f \in L^p_{\mathrm{loc}}(\mathbb{R}^n), \|f\|_{C^{p,\beta}(\mathbb{R}^n)} < \infty\},$$

where

$$\|f\|_{C^{p,\beta}(\mathbb{R}^n)} := \sup_B \frac{1}{|B|^{\beta/n}}\left(\frac{1}{|B|}\int_B |f(x) - f_B|^p\mathrm{d}x\right)^{1/p},$$

and the supremum is taken over all balls B in $\mathbb{R}^n$.

Our result can be stated as follows.

Theorem 1.1. *Let T be a strongly singular Calderón-Zygmund operator, and, α, η, and δ be as in Definition* 1.1. *Suppose that* $\frac{n(1-\alpha)+2\eta}{2\eta} < p < \infty$, $-n/p \le \beta < 0$, *and* $0 < \gamma = \beta + n/p < 1$. *If* $b \in \dot{\Lambda}_\gamma(\mathbb{R}^n)$, *then* T_b *is bounded from* $L^p(\mathbb{R}^n)$ *to* $C^{p,\beta}(\mathbb{R}^n)$, *that is, there exists a constant* $C > 0$ *such that for all* $f \in L^p(\mathbb{R}^n)$,

$$\|T_b(f)\|_{C^{p,\beta}(\mathbb{R}^n)} \le C\|b\|_{\dot{\Lambda}_\gamma(\mathbb{R}^n)}\|f\|_{L^p(\mathbb{R}^n)}.$$

Remark 1.1. Theorem 1.1 gives a new kind of boundedness for commutator T_b when b belongs to certain Lipschitz spaces, compared with the (L^p, L^q)-boundedness and the $(M^{p,\beta}, M^{q,\beta+\gamma})$-boundedness of T_b, when $\frac{n(1-\alpha)+2\eta}{2\eta} < p < q < \infty$ and $0 < \gamma = n\left(\frac{1}{p} - \frac{1}{q}\right) < 1$, obtained in [12, Corollary 1] and [10, Theorem 2.2], respectively.

2 Proof of Theorem 1.1

To prove Theorem 1.1, we need some known results. The first one is due to DeVore and Sharpley [16] and Janson et al. [17] (see also Paluszyński [4], Lemma 1.5).

Lemma 2.1. *Let* $0 < \gamma < 1$ *and* $b \in \dot{\Lambda}_\gamma(\mathbb{R}^n)$, *then for all* $1 \le p < \infty$,

$$\|b\|_{\dot{\Lambda}_\gamma(\mathbb{R}^n)} \approx \sup_B \frac{1}{|B|^{\gamma/n}}\left(\frac{1}{|B|}\int_B |b(x) - b_B|^p \mathrm{d}x\right)^{1/p} \approx \sup_B \frac{1}{|B|^{\gamma/n}}\|b - b_B\|_{L^\infty(B)}.$$

The next result is easy to check by using (1.1). See also DeVore and Sharpley [16], page 14.

Lemma 2.2. [16] *Let* $0 < \gamma < 1$, $b \in \dot{\Lambda}_\gamma(\mathbb{R}^n)$, *and B and B' be balls in* $\mathbb{R}^n$. *If* $B' \subset B$, *then*

$$|b_{B'} - b_B| \le C\|b\|_{\dot{\Lambda}_\gamma(\mathbb{R}^n)}|B|^{\gamma/n}.$$

Now we recall the boundedness of strongly singular Calderón-Zygmund operator T on Lebesgue spaces. Let us observe that T is bounded from L^∞ to BMO ([6], Theorem 2.1), from L^1 to $L^{1,\infty}$ ([7], Theorem 4.1), and from H^1 to L^1 ([12], Lemma 2), and note the assumption (3) in Definition 1.1, and by interpolation between these estimates, we achieve the following L^p-boundedness of T. We refer to [12] (page 1052) and [11] (pages 42 and 43), for details.

Lemma 2.3. *Let T be a strongly singular Calderón-Zygmund operator, and, α, η, and δ be the same as in Definition* 1.1.

(*i*) *If* $1 < p < \infty$, *then T is bounded from* $L^p(\mathbb{R}^n)$ *to itself.*

(*ii*) *If* $\frac{n(1-\alpha)+2\eta}{2\eta} \le u < \infty$, *then there is a positive number v satisfying* $0 < u/v \le \alpha$, *such that T is bounded from* $L^u(\mathbb{R}^n)$ to $L^v(\mathbb{R}^n)$.

Furthermore, the index v can be chosen as $v = \frac{uq'}{2q' - uq' + 2u - 2}$ *when* $\frac{n(1-\alpha)+2\eta}{2\eta} \le u \le 2$ *and* $v = \frac{uq'}{2}$ *when* $2 \le u < \infty$, *where q is given in Definition* 1.1 *and q' is its conjugate index.*

Now, let us prove Theorem 1.1.

Proof of Theorem 1.1. For any $f \in L^p(\mathbb{R}^n)$, it suffices to prove

$$\frac{1}{|B|^{\beta/n}}\left(\frac{1}{|B|}\int_B |T_b f(y)-(T_b f)_B|^p \mathrm{d}y\right)^{1/p} \leq C\|b\|_{\dot{\Lambda}_\gamma(\mathbb{R}^n)}\|f\|_{L^p(\mathbb{R}^n)}, \tag{2.1}$$

for all balls B in $\mathbb{R}^n$.

For any ball $B = B(x_0, r)$ centered at x_0 with radius r, we divide the proof into two cases.

Case 1. *The case when* $r > 1$. Denote by $B^* = 8B = B(x_0, 8r)$ the ball with the same center as B and 8 times the radius. Let $f_1 = f\chi_{B^*}$ and $f_2 = f - f_1$. For any real number c, by Minkowski's inequality and Hölder's inequality, we have

$$\begin{aligned}
&\frac{1}{|B|^{\beta/n}}\left(\frac{1}{|B|}\int_B |T_b f(y)-(T_b f)_B|^p \mathrm{d}y\right)^{1/p}\\
&\leq \frac{1}{|B|^{\beta/n}}\left(\frac{1}{|B|}\int_B |T_b f(y)-c|^p \mathrm{d}y\right)^{1/p} + \frac{1}{|B|^{\beta/n}}\left(\frac{1}{|B|}\int_B |(T_b f)_B-c|^p \mathrm{d}y\right)^{1/p}\\
&\leq \frac{1}{|B|^{\beta/n}}\left(\frac{1}{|B|}\int_B |T_b f(y)-c|^p \mathrm{d}y\right)^{1/p} + \frac{1}{|B|^{\beta/n}}|(T_b f)_B-c|\\
&\leq \frac{1}{|B|^{\beta/n}}\left(\frac{1}{|B|}\int_B |T_b f(y)-c|^p \mathrm{d}y\right)^{1/p} + \frac{1}{|B|^{\beta/n}}\frac{1}{|B|}\int_B |T_b f(z)-c|\mathrm{d}z\\
&\leq \frac{1}{|B|^{\beta/n}}\left(\frac{1}{|B|}\int_B |T_b f(y)-c|^p \mathrm{d}y\right)^{1/p} + \frac{1}{|B|^{\beta/n}}\left(\frac{1}{|B|}\int_B |T_b f(z)-c|^p \mathrm{d}z\right)^{1/p}\\
&\leq \frac{2}{|B|^{\beta/n}}\left(\frac{1}{|B|}\int_B |T_b f(y)-c|^p \mathrm{d}y\right)^{1/p}.
\end{aligned}$$

Let $c = -(T((b-b_B)f_2))_B$ and notice that $T_b f = T_{b-b_B} f$, one has

$$\begin{aligned}
&\frac{1}{|B|^{\beta/n}}\left(\frac{1}{|B|}\int_B |T_b f(y)-(T_b f)_B|^p \mathrm{d}y\right)^{1/p}\\
&\leq \frac{2}{|B|^{\beta/n}}\left(\frac{1}{|B|}\int_B |T_{b-b_B} f(y)+(T((b-b_B)f_2))_B|^p \mathrm{d}y\right)^{1/p}\\
&\leq \frac{2}{|B|^{\beta/n}}\left(\frac{1}{|B|}\int_B |(b(y)-b_B)Tf(y)|^p \mathrm{d}y\right)^{1/p} + \frac{2}{|B|^{\beta/n}}\left(\frac{1}{|B|}\int_B |T((b-b_B)f_1)(y)|^p \mathrm{d}y\right)^{1/p} \qquad (2.2)\\
&\quad + \frac{2}{|B|^{\beta/n}}\left(\frac{1}{|B|}\int_B |T((b-b_B)f_2)(y)-(T((b-b_B)f_2))_B|^p \mathrm{d}y\right)^{1/p}\\
&:= I_1 + I_2 + I_3.
\end{aligned}$$

For I_1, note that $1 < p < \infty$ and $0 < \gamma = \beta + n/p < 1$, it follows from Lemmas 2.1 and 2.3 that

$$\begin{aligned}
I_1 &= \frac{2}{|B|^{\gamma/n}}\left(\int_B |(b(y)-b_B)Tf(y)|^p \mathrm{d}y\right)^{1/p}\\
&\leq \frac{2}{|B|^{\gamma/n}}\|b-b_B\|_{L^\infty(B)}\left(\int_B |Tf(y)|^p \mathrm{d}y\right)^{1/p}
\end{aligned}$$

$$\leq C\|b\|_{\dot{\Lambda}_\gamma(\mathbb{R}^n)}\|Tf\|_{L^p(\mathbb{R}^n)}$$
$$\leq C\|b\|_{\dot{\Lambda}_\gamma(\mathbb{R}^n)}\|f\|_{L^p(\mathbb{R}^n)}.$$

Next we estimate I_2. Again we note that $1 < p < \infty$ and $0 < \gamma = \beta + n/p < 1$. By Lemmas 2.1, 2.2, and 2.3, we deduce

$$\begin{aligned}
I_2 &\leq \frac{2}{|B|^{\gamma/n}}\|T((b-b_B)f_1)\|_{L^p(\mathbb{R}^n)}\\
&\leq \frac{C}{|B|^{\gamma/n}}\|(b-b_B)f_1\|_{L^p(\mathbb{R}^n)}\\
&\leq \frac{C}{|B|^{\gamma/n}}\Big\{\|(b-b_{B^*})f\|_{L^p(B^*)} + \|(b_{B^*}-b_B)f\|_{L^p(B^*)}\Big\}\\
&\leq \frac{C}{|B|^{\gamma/n}}\Big\{\|b-b_{B^*}\|_{L^\infty(B^*)} + |b_{B^*}-b_B|\Big\}\|f\|_{L^p(\mathbb{R}^n)}\\
&\leq \frac{C}{|B|^{\gamma/n}}\Big\{C|B^*|^{\gamma/n}\|b\|_{\dot{\Lambda}_\gamma(\mathbb{R}^n)} + C\|b\|_{\dot{\Lambda}_\gamma(\mathbb{R}^n)}|B^*|^{\gamma/n}\Big\}\|f\|_{L^p(\mathbb{R}^n)}\\
&\leq C\|b\|_{\dot{\Lambda}_\gamma(\mathbb{R}^n)}\|f\|_{L^p(\mathbb{R}^n)}.
\end{aligned}$$

Now, let us consider I_3. Since for any $w, y \in B = B(x_0, r)$ and any $z \in (B^*)^c$ one has $2|y-w|^\alpha < |z-w|$, it follows from Definition 1.1 that

$$\begin{aligned}
|T((b-b_B)f_2)(y) - T((b-b_B)f_2)(w)| &\leq \int_{\mathbb{R}^n} |K(y,z)-K(w,z)||(b(z)-b_B)f_2(z)|\mathrm{d}z\\
&= \int_{(B^*)^c} |K(y,z)-K(w,z)||b(z)-b_B||f(z)|\mathrm{d}z\\
&\leq C\int_{(B^*)^c} \frac{|y-w|^\delta}{|z-w|^{n+\delta/\alpha}}|b(z)-b_B||f(z)|\mathrm{d}z\\
&\leq C\sum_{k=1}^\infty \int_{2^kB^*\setminus 2^{k-1}B^*} \frac{|y-w|^\delta}{|z-w|^{n+\delta/\alpha}}|b(z)-b_B||f(z)|\mathrm{d}z\\
&\leq Cr^{\delta-\delta/\alpha}\sum_{k=1}^\infty \frac{2^{-k\delta/\alpha}}{|2^kB^*|}\int_{2^kB^*}|b(z)-b_B||f(z)|\mathrm{d}z.
\end{aligned} \tag{2.3}$$

Observe that the last term of (2.3) is always independent of w and y, for any $w, y \in B$. Then we can write

$$\begin{aligned}
I_3 &= \frac{2}{|B|^{\beta/n}}\left(\frac{1}{|B|}\int_B |T((b-b_B)f_2)(y) - (T((b-b_B)f_2))_B|^p\mathrm{d}y\right)^{1/p}\\
&\leq \frac{2}{|B|^{\beta/n}}\left\{\frac{1}{|B|}\int_B\left[\frac{1}{|B|}\int_B |T((b-b_B)f_2)(y) - T((b-b_B)f_2)(w)|\mathrm{d}w\right]^p\mathrm{d}y\right\}^{1/p}\\
&\leq \frac{2}{|B|^{\beta/n}}\left\{\frac{1}{|B|}\int_B\left[\frac{1}{|B|}\int_B\left(Cr^{\delta-\delta/\alpha}\sum_{k=1}^\infty\frac{2^{-k\delta/\alpha}}{|2^kB^*|}\int_{2^kB^*}|b(z)-b_B||f(z)|\mathrm{d}z\right)\mathrm{d}w\right]^p\mathrm{d}y\right\}^{1/p}\\
&\leq \frac{Cr^{\delta(1-1/\alpha)}}{|B|^{\beta/n}}\sum_{k=1}^\infty\frac{2^{-k\delta/\alpha}}{|2^kB^*|}\int_{2^kB^*}|b(z)-b_B||f(z)|\mathrm{d}z\\
&\leq \frac{Cr^{\delta(1-1/\alpha)}}{|B|^{\beta/n}}\sum_{k=1}^\infty\frac{2^{-k\delta/\alpha}}{|2^kB^*|}\int_{2^kB^*}|b(z)-b_{2^kB^*}||f(z)|\mathrm{d}z + \frac{Cr^{\delta(1-1/\alpha)}}{|B|^{\beta/n}}\sum_{k=1}^\infty\frac{2^{-k\delta/\alpha}}{|2^kB^*|}\int_{2^kB^*}|b_{2^kB^*}-b_B||f(z)|\mathrm{d}z\\
&:= I_{3,1} + I_{3,2}.
\end{aligned} \tag{2.4}$$

Applying Hölder's inequality, Lemma 2.1, and noting that $0 < \gamma = \beta + n/p < 1$, we obtain

$$\begin{aligned} I_{3,1} &\le \frac{Cr^{\delta(1-1/\alpha)}}{|B|^{\beta/n}} \sum_{k=1}^{\infty} \frac{2^{-k\delta/\alpha}}{|2^k B^*|} \left(\int_{2^k B^*} |b(z) - b_{2^k B^*}|^{p'} \mathrm{d}z \right)^{1/p'} \left(\int_{2^k B^*} |f(z)|^p \mathrm{d}z \right)^{1/p} \\ &\le \frac{Cr^{\delta(1-1/\alpha)}}{|B|^{\beta/n}} \sum_{k=1}^{\infty} \frac{2^{-k\delta/\alpha}}{|2^k B^*|} |2^k B^*|^{\gamma/n + 1/p'} \|b\|_{\dot{\Lambda}_\gamma(\mathbb{R}^n)} \|f\|_{L^p(\mathbb{R}^n)} \\ &\le C \|b\|_{\dot{\Lambda}_\gamma(\mathbb{R}^n)} \|f\|_{L^p(\mathbb{R}^n)} \frac{r^{\delta(1-1/\alpha)}}{|B|^{\beta/n}} \sum_{k=1}^{\infty} 2^{-k\delta/\alpha} |2^k B^*|^{\beta/n} \\ &\le C \|b\|_{\dot{\Lambda}_\gamma(\mathbb{R}^n)} \|f\|_{L^p(\mathbb{R}^n)} r^{\delta(1-1/\alpha)} \sum_{k=1}^{\infty} 2^{k(\beta - \delta/\alpha)} \\ &\le C \|b\|_{\dot{\Lambda}_\gamma(\mathbb{R}^n)} \|f\|_{L^p(\mathbb{R}^n)}, \end{aligned}$$

where in the last step we made use of the fact that $r^{\delta(1-1/\alpha)} \le 1$ since $r > 1$ and $\delta(1 - 1/\alpha) < 0$ and the fact that the series $\sum_{k=1}^{\infty} 2^{k(\beta-\delta/\alpha)}$ is convergent since $\beta - \delta/\alpha < 0$.

For $I_{3,2}$, noting that $0 < \gamma = \beta + n/p < 1$ and applying Lemma 2.2 and Hölder's inequality, we have

$$\begin{aligned} I_{3,2} &= \frac{Cr^{\delta(1-1/\alpha)}}{|B|^{\beta/n}} \sum_{k=1}^{\infty} \frac{2^{-k\delta/\alpha}}{|2^k B^*|} \int_{2^k B^*} |b_{2^k B^*} - b_B| |f(z)| \mathrm{d}z \\ &\le \frac{Cr^{\delta(1-1/\alpha)}}{|B|^{\beta/n}} \sum_{k=1}^{\infty} \frac{2^{-k\delta/\alpha}}{|2^k B^*|} \|b\|_{\dot{\Lambda}_\gamma(\mathbb{R}^n)} |2^k B^*|^{\gamma/n} \int_{2^k B^*} |f(z)| \mathrm{d}z \\ &\le C \|b\|_{\dot{\Lambda}_\gamma(\mathbb{R}^n)} \frac{r^{\delta(1-1/\alpha)}}{|B|^{\beta/n}} \sum_{k=1}^{\infty} \frac{2^{-k\delta/\alpha}}{|2^k B^*|} |2^k B^*|^{\gamma/n} \left(\int_{2^k B^*} |f(z)|^p \mathrm{d}z \right)^{1/p} |2^k B^*|^{1-1/p} \\ &\le C \|b\|_{\dot{\Lambda}_\gamma(\mathbb{R}^n)} \|f\|_{L^p(\mathbb{R}^n)} \frac{r^{\delta(1-1/\alpha)}}{|B|^{\beta/n}} \sum_{k=1}^{\infty} 2^{-k\delta/\alpha} |2^k B^*|^{\beta/n} \\ &\le C \|b\|_{\dot{\Lambda}_\gamma(\mathbb{R}^n)} \|f\|_{L^p(\mathbb{R}^n)} r^{\delta(1-1/\alpha)} \sum_{k=1}^{\infty} 2^{k(\beta - \delta/\alpha)} \\ &\le C \|b\|_{\dot{\Lambda}_\gamma(\mathbb{R}^n)} \|f\|_{L^p(\mathbb{R}^n)}, \end{aligned}$$

where in the last step we also made use of the fact that $r^{\delta(1-1/\alpha)} \le 1$ and $\sum_{k=1}^{\infty} 2^{k(\beta-\delta/\alpha)}$ is convergent. This, together with the estimates for $I_{3,1}$, yields

$$I_3 \le C \|b\|_{\dot{\Lambda}_\gamma(\mathbb{R}^n)} \|f\|_{L^p(\mathbb{R}^n)}.$$

Combining the estimates for I_1, I_2, and I_3 leads to (2.1) for the case $r > 1$.

Case 2. *The case* $0 < r \le 1$. Set $\tilde{B} = B(x_0, r^\alpha)$ and denote $\tilde{B}^* = 8\tilde{B} = B(x_0, 8r^\alpha)$. Let $f_3 = f\chi_{\tilde{B}^*}$ and $f_4 = f - f_3$. For the same reason as that in (2.2), we deduce that

$$\begin{aligned} &\frac{1}{|B|^{\beta/n}} \left(\frac{1}{|B|} \int_B |T_b f(y) - (T_b f)_B|^p \mathrm{d}y \right)^{1/p} \\ &\le \frac{2}{|B|^{\beta/n+1/p}} \left(\int_B |(b(y) - b_B) T f(y)|^p \mathrm{d}y \right)^{1/p} + \frac{2}{|B|^{\beta/n+1/p}} \left(\int_B |T((b - b_B) f_3)(y)|^p \mathrm{d}y \right)^{1/p} \\ &\quad + \frac{2}{|B|^{\beta/n+1/p}} \left(\int_B |T((b - b_B) f_4)(y) - (T((b - b_B) f_4))_B|^p \mathrm{d}y \right)^{1/p} \\ &\coloneqq J_1 + J_2 + J_3. \end{aligned}$$

Similar to J_1, we have

$$J_1 = \frac{2}{|B|^{\gamma/n}}\left(\int_B |(b(y) - b_B)Tf(y)|^p \mathrm{d}y\right)^{1/p} \le C\|b\|_{\dot{\Lambda}_\gamma(\mathbb{R}^n)}\|f\|_{L^p(\mathbb{R}^n)}.$$

To estimate J_2, we first observe that, by Lemma 2.3, there is an s satisfying $0 < p/s \le \alpha$ such that T is bounded from L^p to L^s since $\frac{n(1-\alpha)+2\eta}{2\eta} < p < \infty$. This, together with Hölder's inequality, Lemmas 2.1 and 2.2, gives

$$\begin{aligned}
J_2 &= \frac{2}{|B|^{\beta/n+1/p}}\left(\int_B |T((b-b_B)f_3)(y)|^p \mathrm{d}y\right)^{1/p}\\
&\le \frac{2}{|B|^{\beta/n+1/s}}\|T((b-b_B)f_3)\|_{L^s(\mathbb{R}^n)}\\
&\le \frac{C}{|B|^{\beta/n+1/s}}\|(b-b_B)f_3\|_{L^p(\mathbb{R}^n)}\\
&\le \frac{C}{|B|^{\beta/n+1/s}}\left\{\|(b-b_{\tilde{B}^*})f\|_{L^p(\tilde{B}^*)} + \|(b_{\tilde{B}^*}-b_B)f\|_{L^p(\tilde{B}^*)}\right\}\\
&\le \frac{C}{|B|^{\beta/n+1/s}}\left\{\|b-b_{\tilde{B}^*}\|_{L^\infty(\tilde{B}^*)} + |b_{\tilde{B}^*}-b_B|\right\}\|f\|_{L^p(\mathbb{R}^n)}\\
&\le \frac{C}{|B|^{\beta/n+1/s}}|\tilde{B}^*|^{\gamma/n}\|b\|_{\dot{\Lambda}_\gamma(\mathbb{R}^n)}\|f\|_{L^p(\mathbb{R}^n)}\\
&\le Cr^{(\alpha-1)\beta+(\alpha/p-1/s)n}\|b\|_{\dot{\Lambda}_\gamma(\mathbb{R}^n)}\|f\|_{L^p(\mathbb{R}^n)}\\
&\le C\|b\|_{\dot{\Lambda}_\gamma(\mathbb{R}^n)}\|f\|_{L^p(\mathbb{R}^n)},
\end{aligned}$$

where the last two steps follow from the condition $0 < \gamma = \beta + n/p < 1$ and the fact that $r^{(\alpha-1)\beta+(\alpha/p-1/s)n} \le 1$ since $0 < r \le 1$ and $(\alpha-1)\beta + (\alpha/p - 1/s)n > 0$.

Finally, let us consider J_3. Since $0 < r \le 1$ and $2|y-w|^\alpha < |z-w|$ for any $w, y \in B = B(x_0, r)$ and $z \in (\tilde{B}^*)^c$, similar to (2.3), we have

$$\begin{aligned}
|T((b-b_B)f_4)(y) - T((b-b_B)f_4)(w)| &\le \int_{\mathbb{R}^n} |K(y,z)-K(w,z)||(b(z)-b_B)f_4(z)|\mathrm{d}z\\
&\le C\int_{(\tilde{B}^*)^c} \frac{|y-w|^\delta}{|z-w|^{n+\delta/\alpha}}|b(z)-b_B||f(z)|\mathrm{d}z\\
&\le C\sum_{k=1}^\infty \int_{2^k\tilde{B}^*\setminus 2^{k-1}\tilde{B}^*} \frac{|y-w|^\delta}{|z-w|^{n+\delta/\alpha}}|b(z)-b_B||f(z)|\mathrm{d}z\\
&\le C\sum_{k=1}^\infty \frac{2^{-k\delta/\alpha}}{|2^k\tilde{B}^*|}\int_{2^k\tilde{B}^*}|b(z)-b_B||f(z)|\mathrm{d}z.
\end{aligned}$$

Reasoning as in (2.4), we can also write

$$\begin{aligned}
J_3 &= \frac{2}{|B|^{\beta/n}}\left(\frac{1}{|B|}\int_B |T((b-b_B)f_4)(y) - (T((b-b_B)f_4))_B|^p \mathrm{d}y\right)^{1/p}\\
&\le \frac{C}{|B|^{\beta/n}}\sum_{k=1}^\infty \frac{2^{-k\delta/\alpha}}{|2^k\tilde{B}^*|}\int_{2^k\tilde{B}^*}|b(z)-b_B||f(z)|\mathrm{d}z\\
&\le \frac{C}{|B|^{\beta/n}}\sum_{k=1}^\infty \frac{2^{-k\delta/\alpha}}{|2^k\tilde{B}^*|}\int_{2^k\tilde{B}^*}|b(z)-b_{2^k\tilde{B}^*}||f(z)|\mathrm{d}z + \frac{C}{|B|^{\beta/n}}\sum_{k=1}^\infty \frac{2^{-k\delta/\alpha}}{|2^k\tilde{B}^*|}\int_{2^k\tilde{B}^*}|b_{2^k\tilde{B}^*}-b_B||f(z)|\mathrm{d}z\\
&:= J_{3,1} + J_{3,2}.
\end{aligned}$$

To estimate $J_{3,1}$, we first observe the fact that $r^{\beta(\alpha-1)} \le 1$ since $0 < r \le 1$ and $\beta(\alpha-1) > 0$ and the series $\sum_{k=1}^{\infty} 2^{k(\beta-\delta/\alpha)}$ is convergent since $\beta - \delta/\alpha < 0$. Then by Hölder's inequality and Lemma 2.1 and noticing that $0 < \gamma = \beta + n/p < 1$, we obtain

$$\begin{aligned}
J_{3,1} &= \frac{C}{|B|^{\beta/n}} \sum_{k=1}^{\infty} \frac{2^{-k\delta/\alpha}}{|2^k\tilde{B}^*|} \int_{2^k\tilde{B}^*} |b(z) - b_{2^k\tilde{B}^*}||f(z)|\mathrm{d}z \\
&\le \frac{C}{|B|^{\beta/n}} \sum_{k=1}^{\infty} \frac{2^{-k\delta/\alpha}}{|2^k\tilde{B}^*|} \Bigg(\int_{2^k\tilde{B}^*} |b(z) - b_{2^k\tilde{B}^*}|^{p'}\mathrm{d}z \Bigg)^{1/p'} \|f\|_{L^p(\mathbb{R}^n)} \\
&\le \frac{C}{|B|^{\beta/n}} \sum_{k=1}^{\infty} \frac{2^{-k\delta/\alpha}}{|2^k\tilde{B}^*|} |2^k\tilde{B}^*|^{\gamma/n+1/p'} \|b\|_{\dot{\Lambda}_\gamma(\mathbb{R}^n)} \|f\|_{L^p(\mathbb{R}^n)} \\
&\le \frac{C}{|B|^{\beta/n}} \sum_{k=1}^{\infty} 2^{-k\delta/\alpha} |2^k\tilde{B}^*|^{\beta/n} \|b\|_{\dot{\Lambda}_\gamma(\mathbb{R}^n)} \|f\|_{L^p(\mathbb{R}^n)} \\
&\le C\|b\|_{\dot{\Lambda}_\gamma(\mathbb{R}^n)} \|f\|_{L^p(\mathbb{R}^n)} r^{\beta(\alpha-1)} \sum_{k=1}^{\infty} 2^{k(\beta-\delta/\alpha)} \\
&\le C\|b\|_{\dot{\Lambda}_\gamma(\mathbb{R}^n)} \|f\|_{L^p(\mathbb{R}^n)}.
\end{aligned}$$

For $J_{3,2}$, applying Lemma 2.2 and Hölder's inequality, and observing again the fact that $r^{\beta(\alpha-1)} \le 1$, the series $\sum_{k=1}^{\infty} 2^{k(\beta-\delta/\alpha)}$ is convergent and $0 < \gamma = \beta + n/p < 1$, and we have

$$\begin{aligned}
J_{3,2} &= \frac{C}{|B|^{\beta/n}} \sum_{k=1}^{\infty} \frac{2^{-k\delta/\alpha}}{|2^k\tilde{B}^*|} \int_{2^k\tilde{B}^*} |b_{2^k\tilde{B}^*} - b_B||f(z)|\mathrm{d}z \\
&\le \frac{C}{|B|^{\beta/n}} \sum_{k=1}^{\infty} \frac{2^{-k\delta/\alpha}}{|2^k\tilde{B}^*|} |2^k\tilde{B}^*|^{\gamma/n} \|b\|_{\dot{\Lambda}_\gamma(\mathbb{R}^n)} \int_{2^k\tilde{B}^*} |f(z)|\mathrm{d}z \\
&\le \frac{C}{|B|^{\beta/n}} \sum_{k=1}^{\infty} \frac{2^{-k\delta/\alpha}}{|2^k\tilde{B}^*|} |2^k\tilde{B}^*|^{\gamma/n} \|b\|_{\dot{\Lambda}_\gamma(\mathbb{R}^n)} \|f\|_{L^p(2^k\tilde{B}^*)} |2^k\tilde{B}^*|^{1/p'} \\
&\le \frac{C}{|B|^{\beta/n}} \sum_{k=1}^{\infty} 2^{-k\delta/\alpha} |2^k\tilde{B}^*|^{\beta/n} \|b\|_{\dot{\Lambda}_\gamma(\mathbb{R}^n)} \|f\|_{L^p(\mathbb{R}^n)} \\
&\le C\|b\|_{\dot{\Lambda}_\gamma(\mathbb{R}^n)} \|f\|_{L^p(\mathbb{R}^n)} r^{\beta(\alpha-1)} \sum_{k=1}^{\infty} 2^{k(\beta-\delta/\alpha)} \\
&\le C\|b\|_{\dot{\Lambda}_\gamma(\mathbb{R}^n)} \|f\|_{L^p(\mathbb{R}^n)}.
\end{aligned}$$

The estimate for $J_{3,2}$, together with the ones for $J_{3,1}$, yields

$$J_3 \le C\|b\|_{\dot{\Lambda}_\gamma(\mathbb{R}^n)} \|f\|_{L^p(\mathbb{R}^n)}.$$

Combining the estimates for J_1, J_2, and J_3, we deduce (2.1) for the case $0 < r \le 1$.

Now, we finish the proof of Theorem 1.1. □

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