Research Article

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Some new fixed point theorems for nonexpansive-type mappings in geodesic spaces

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Abstract: In this article, we present some new fixed point existence results for nonexpansive-type mappings in geodesic spaces. We also give a number of illustrative examples to settle our claims. We study the asymptotic behavior of Picard iterates generated by these class of mappings under different conditions. Finally, we approximate the solutions of the constrained minimization problem in the setting of Cartan, Alexandrov, and Toponogov (CAT(0)) spaces.

Keywords: hyperbolic metric space, nonexpansive mapping, minimization problem

MSC 2020: 47H10, 54H25, 47H09

1 Introduction and preliminaries

Let $(\mathcal{H}, \langle .,. \rangle)$ be a Hilbert space and $\mathcal{K}$ be a closed-convex subset of $\mathcal{H}$. Browder [1] introduced the following mapping known as firmly contractive if for all $x, y \in \mathcal{K}$

$$\|G(x) - G(y)\|^2 \le \langle x - y, G(x) - G(y) \rangle, \tag{1.1}$$

where $G : \mathcal{K} \to \mathcal{K}$ is a mapping. This class of mapping has significance in the study of convergence of sequences generated by nonlinear operators. Bruck [2] defined the following important class of mappings (firmly nonexpansive) in the setting of Banach spaces if for all $x, y \in \mathcal{K}$

$$\|G(x) - G(y)\| \le \|(1 - \lambda)(G(x) - G(y)) + \lambda(x - y)\|, \tag{1.2}$$

where $\lambda > 0$. The class of mapping satisfying (1.2) is also known as λ-firmly nonexpansive mappings. In Hilbert spaces, the class of mappings satisfying (1.1) coincides with the class of mappings satisfying (1.2). Firmly nonexpansive mappings have fruitful importance in nonlinear analysis due to the connection with monotone operators. Monotone operators were introduced by Minty [3] in the setting of Hilbert spaces. These operators have significant importance in modeling many problems arising in convex analysis and in the theory of partial differential equations.

In regard to fixed point theory, firmly nonexpansive mappings show similar behavior to the nonexpansive mappings on closed-convex subsets. However, they behave differently on nonconnected subsets. In fact, Smarzewski [4] proved the following interesting result.

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Theorem 1.1. *Let $\mathcal{E}$ be a uniformly convex (UC) Banach space and $\mathcal{K}$ be a union of nonempty bounded closed-convex subsets $\mathcal{K}_i$ (for $i = 1, 2, \ldots, m$) of $\mathcal{E}$, that is, $\mathcal{K} = \cup_{i=1}^{m} \mathcal{K}_i$. Assume that $G : \mathcal{K} \to \mathcal{K}$ is λ-firmly nonexpansive mapping for some $\lambda \in (0, 1)$. Then, G admits a fixed point in $\mathcal{K}$.*

In [4], it is noted that Theorem 1.1 is not true if G is a nonexpansive mapping even in the real line. For instance, if $\mathcal{K} = [-2, -1] \cup [1, 2]$, the mapping $G : \mathcal{K} \to \mathcal{K}$ defined by $G(x) = -x$ is a fixed point free nonexpansive.

In recent years, a number of articles have appeared dealing with the extension of well-established techniques and results from linear spaces to nonlinear spaces (or from normed spaces to metric spaces, cf. [5–16]). In this direction, Ariza-Ruiz et al. [17] extended Theorem 1.1 in more general setting of spaces (geodesic spaces). For some applications of this class of mappings, see [18,19]. A number of extensions and generalizations of λ-firmly nonexpansive mapping have appeared in the literature, see [20,21].

Motivated by Smarzewski [4], Ariza-Ruiz et al. [17], and others, we study some fixed point theorems similar to Theorem 1.1 for nonexpansive-type mappings in the setting of geodesic spaces. We provide some suitable examples that ensure extensions of the results presented herein over those that appeared in the literature. We obtain results concerning the exhibition of Picard iterates generated by these classes of mappings under different conditions on spaces as well as on mappings. Finally, we utilize our results to find the solutions of constrained minimization problem. This way, some results in [4,17, 22,23] are extended, generalized, and complemented.

Now, we recall some notations, definitions, and results from the literature. Let $(\mathcal{E}, \rho)$ be a metric space. Given a pair of points $x, y \in \mathcal{E}$, we say that a path $\zeta : [0, 1] \to \mathcal{E}$ joins x and y if $\zeta(0) = x$ and $\zeta(1) = y$. A path ζ is called a geodesic if $\rho(\zeta(s), \zeta(t)) = \rho(\zeta(0), \zeta(1))|s - t|$ for every $s, t \in [0, 1]$. A metric space $(\mathcal{E}, \rho)$ is said to be a geodesic space if every two points $x, y \in \mathcal{E}$ are connected by a geodesic. If geodesics are unique, Ω-hyperbolic spaces are precisely the Busemann spaces [24]. Some well-known spaces are special cases of these spaces. For example, all normed spaces, the Cartan, Alexandrov, and Toponogov (CAT(0))-spaces, Hadamard manifolds, and Hilbert open unit balls are equipped with the hyperbolic metric (cf. [17,25]). The following precise formulation of hyperbolic spaces was introduced by Kohlenbach [25].

Definition 1.2. A triplet $(\mathcal{E}, \rho, \Omega)$ is called a hyperbolic metric space if $(\mathcal{E}, \rho)$ is a metric space and $\Omega : \mathcal{E} \times \mathcal{E} \times [0, 1] \to \mathcal{E}$ is a function satisfying

(i) $\rho(\xi, \Omega(x, y, \Theta)) \leq (1 - \Theta)\rho(\xi, x) + \Theta\rho(\xi, y)$;
(ii) $\rho(\Omega(x, y, \Theta), \Omega(x, y, \theta)) = |\Theta - \theta|\rho(x, y)$;
(iii) $\Omega(x, y, \Theta) = \Omega(y, x, 1 - \Theta)$;
(iv) $\rho(\Omega(x, \xi, \Theta), \Omega(y, \gamma, \Theta)) \leq (1 - \Theta)\rho(x, y) + \Theta\rho(\xi, \gamma)$

for all $x, y, \xi, \gamma \in \mathcal{E}$ and $\Theta, \theta \in [0, 1]$.

Remark 1.3. By taking $\Omega(x, y, \Theta) = (1 - \Theta)x + \Theta y$ for all $x, y \in \mathcal{E}$, $\Theta \in [0, 1]$, it is clear that all normed linear spaces $\mathcal{E}$ are included in these spaces.

Remark 1.4. A triplet $(\mathcal{E}, \rho, \Omega)$ is a convex metric space in the sense of Takahashi [26] if only condition (i) is satisfied. Goebel and Kirk [5] considered the class of hyperbolic-type spaces by assuming conditions (i)-(iii). Reich and Shafrir [27] considered the class of hyperbolic metric spaces that contain a family of metric lines, such that, for each pair of distinct points $x, y \in \mathcal{E}$, there is a unique metric line (an isometric image of the real line) that passes through x and y. Therefore, hyperbolic conventions considered in Definition 1.2 are less restrictive than those considered in [27]. Condition (iii) implies that $\mathrm{seg}[x, y]$ is an isometric image of the real-line segment $[0, \rho(x, y)]$.

We adopt the customary notations

$$\Omega(x, y, \Theta) := (1 - \Theta)x \oplus \Theta y$$

to indicate the point $\Omega(x, y, \Theta)$ in a given hyperbolic metric space. To indicate geodesic segments, we use the following notation:

for $x, y \in \mathcal{E}$,

$$[x, y] = \{(1-\Theta)x \oplus \Theta y : \Theta \in [0,1]\}.$$

A subset $\mathcal{K}$ of $(\mathcal{E}, \rho, \Omega)$ is said to be convex if $[x, y] \subset \mathcal{K}$ whenever $x, y \in \mathcal{K}$. When there is no incertitude, we adopt $(\mathcal{E}, \rho)$ for $(\mathcal{E}, \rho, \Omega)$.

Definition 1.5. [28] A hyperbolic space $(\mathcal{E}, \rho)$ is said to be UC if for any $t > 0$ and $\varepsilon \in (0, 2]$, there exists a $\delta \in (0, 1]$ such that

$$\left.\begin{array}{l}\rho(x,\gamma) \le t\\ \rho(y,\gamma) \le t\\ \rho(x,y) \ge \varepsilon t\end{array}\right\} \Rightarrow \rho\left(\frac{1}{2}x \oplus \frac{1}{2}y, \gamma\right) \le (1-\delta)t$$

for all $x, y, \gamma \in \mathcal{E}$.

Remark 1.6. Leuştean [6] showed that the complete CAT(0) spaces are complete UC hyperbolic spaces.

A map $\upsilon : [a, b] \to \mathcal{E}$ is an affinely reparametrized geodesic if υ is a constant path or there exists an interval $[c, d]$ and a geodesic $\upsilon' : [c, d] \to \mathcal{E}$ such that $\upsilon = \upsilon' \circ \psi$, where $\psi : [a, b] \to [c, d]$ is the unique affine homeomorphism between the intervals $[a, b]$ and $[c, d]$. A geodesic space $(\mathcal{E}, \rho)$ is a Busemann space if for any two affinely reparametrized geodesics $\upsilon : [a, b] \to \mathcal{E}$ and $\upsilon' : [c, d] \to \mathcal{E}$, the map $D_{\upsilon,\upsilon'} : [a, b] \times [c, d] \to \mathbb{R}$ defined as

$$D_{\upsilon,\upsilon'}(s, t) = d(\upsilon(s), \upsilon'(t))$$

is convex, see [24,29]. If $(\mathcal{E}, \rho)$ is a Busemann space, then there exists a unique convexity mapping Ω such that $(\mathcal{E}, \rho, \Omega)$ is a uniquely geodesic Ω-hyperbolic space. In other words, for any $x \ne \gamma \in \mathcal{E}$ and any $\Theta \in [0, 1]$, there exists a unique element $y \in \mathcal{E}$ (namely $y = \Omega(x, \gamma, \Theta)$) such that

$$\rho(x, y) = \Theta\rho(x, \gamma) \quad \text{and} \quad \rho(y, \gamma) = (1-\Theta)\rho(x, \gamma).$$

Let x, y, and γ be three points in metric space $(\mathcal{E}, \rho)$, the point y is said to lie between x and γ if these points are pairwise distinct and

$$\rho(x, \gamma) = \rho(x, y) + \rho(y, \gamma).$$

Clearly, if y lies between x and γ, then y lies between γ and x. Moreover, the relation of betweenness satisfies a transitivity property.

Definition 1.7. [17] A metric space $(\mathcal{E}, \rho)$ satisfies the betweenness property if the following condition holds:

$$\begin{array}{l}\text{if } y \text{ lies between } x \text{ and } \gamma \text{ and, } \gamma \text{ lies between } y \text{ and } \xi,\\ \text{then } y \text{ and } \gamma \text{ both lie between } x \text{ and } \xi \text{ for all } x, y, \gamma, \xi \in \mathcal{E}.\end{array}$$

In general metric spaces, the betweenness property is not true.

Lemma 1.8. [17] *Let $(\mathcal{E}, \rho)$ be a metric space with betweenness property. For all $n \ge 2$, and all $x_0, x_1, \ldots, x_n \in \mathcal{E}$, we have the following*:

$$\begin{array}{l}\text{if } x_k \text{ lies between } x_{k-1} \text{ and } x_{k+1}, \text{ for all } k = 1, \ldots, n-1,\\ \text{then } x_k \text{ lies between } x_0 \text{ and } x_{k+1} \text{ for all } k = 1, \ldots, n-1.\end{array}$$

Lemma 1.9. [17] *Every Busemann space satisfies the betweenness property. Therefore*, Lemma 1.8 *is true for Busemann spaces.*

Lemma 1.10. [17] *Let $(\mathcal{E}, \rho)$ be a geodesic space. Let $x, y, \gamma, \xi \in \mathcal{E}$ and $[x, y]$ be a geodesic segment. If $\gamma, \xi \in [x, y]$, then either $\rho(x, \gamma) + \rho(\gamma, \xi) = \rho(x, \xi)$ or $\rho(\xi, \gamma) + \rho(\gamma, y) = \rho(\xi, y)$.*

Let $\{x_n\}$ be a bounded sequence in a metric space $(\mathcal{E}, \rho)$ and $\mathcal{K}$ be a nonempty subset of $\mathcal{E}$. A functional $r(\cdot,\{x_n\}) : \mathcal{E} \to [0, +\infty)$ can be defined as follows:

$$r(y, \{x_n\}) = \limsup_{n\to+\infty} \rho(y, x_n).$$

The asymptotic radius of $\{x_n\}$ with respect to $\mathcal{K}$ is defined as

$$r(\mathcal{K}, \{x_n\}) = \inf\{r(y, \{x_n\}) | y \in \mathcal{K}\}.$$

A point x in $\mathcal{K}$ is called an asymptotic center of $\{x_n\}$ with respect to $\mathcal{K}$ if

$$r(x, \{x_n\}) = r(\mathcal{K}, \{x_n\}).$$

$A(\mathcal{K}, \{x_n\})$ is denoted as set of all asymptotic centers of $\{x_n\}$ with respect to $\mathcal{K}$. A bounded sequence $\{x_n\}$ in a metric space $(\mathcal{E}, \rho)$ is said to Δ-converge to x if x is the unique asymptotic center for every subsequence $\{u_n\}$ of $\{x_n\}$. Let C be a nonempty subset of metric space $(\mathcal{E}, \rho)$ and $\{x_n\}$ be a sequence in $\mathcal{E}$. A sequence $\{x_n\}$ is said to be Fejér monotone with respect to C if for all $x^\dagger \in C$,

$$\rho(x^\dagger, x_{n+1}) \leq \rho(x^\dagger, x_n)$$

for all $n \geq 0$.

Proposition 1.11. [6] *Let $(\mathcal{E}, \rho)$ be a complete UC-hyperbolic space, $\mathcal{K}$ be a nonempty closed-convex subset of $\mathcal{E}$, and $\{x_n\}$ a bounded sequence in $\mathcal{E}$. Then $\{x_n\}$ has a unique asymptotic center with respect to $\mathcal{K}$.*

Lemma 1.12. [6] *Let $(\mathcal{E}, \rho)$ be a metric space and $\mathcal{K}$ be a nonempty subset of $\mathcal{E}$. Let $\{x_n\}$ be a bounded sequence in $\mathcal{E}$ and $A(\mathcal{K}, \{x_n\}) = \{y\}$. Let $\{a_n\}$ and $\{b_n\}$ be two sequences in $\mathbb{R}$ such that $a_n \in [0, +\infty)$ for all $n \in \mathbb{N}$, $\limsup a_n \leq 1$ and $\limsup b_n \leq 0$. Suppose that $\gamma \in \mathcal{K}$ and there exists $m, N \in \mathbb{N}$ such that*

$$\rho(\gamma, x_{n+m}) \leq a_n \rho(\gamma, x_n) + b_n \quad \textit{for all } n \geq N.$$

Then, $y = \gamma$.

Lemma 1.13. [17] *Let $(\mathcal{E}, \rho)$ be a metric space, C be a nonempty subset of $\mathcal{E}$. If $\{x_n\}$ is Fejér monotone with respect to C, then $A(C, \{x_n\}) = \{x\}$ and $A(\mathcal{E}, \{u_n\}) \subseteq C$ for every subsequence $\{u_n\}$* of $\{x_n\}$. *Then, the sequence $\{x_n\}$ Δ-converges to $x \in C$.*

Let $\mathcal{K}$ be a nonempty subset of a metric space $(\mathcal{E}, \rho)$. A mapping $G : \mathcal{K} \to \mathcal{K}$ is said to be compact if $G(\mathcal{K})$ has a compact closure.

Definition 1.14. [30] Let $G : \mathcal{K} \to \mathcal{K}$ with $F(G) \neq \emptyset$, where $F(G)$ is a set of fixed points of G, that is, $F(G) = \{x \in \mathcal{K} | G(x) = x\}$ The mapping G satisfies condition (I) if there is a nondecreasing function $f : [0, +\infty) \to [0, +\infty)$ with $f(0) = 0, f(r) > 0$ for $r \in (0, +\infty)$ such that $\rho(x, G(x)) \geq f(\rho(x, F(G)))$ for all $x \in \mathcal{K}$, where $\rho(x, F(G)) = \inf\{\rho(x, y) : y \in F(G)\}$.

2 Nonexpansive-type mappings

Definition 2.1. Let $(\mathcal{E}, \rho)$ be a hyperbolic space and $\mathcal{K}$ be a nonempty subset of $\mathcal{E}$. Let $G : \mathcal{K} \to \mathcal{E}$ is said to be λ-firmly nonexpansive if for given $\lambda \in (0, 1)$, the following condition holds:

$$\rho(G(x), G(y)) \leq \rho((1-\lambda)x \oplus \lambda G(x), (1-\lambda)y \oplus \lambda G(y))$$

for all $x, y \in \mathcal{K}$.

Remark 2.2.

– In view of Definition 1.2 (iv), it follows that

$$\rho(G(x), G(y)) \le \rho((1-\lambda)x \oplus \lambda G(x), (1-\lambda)y \oplus \lambda G(y)) \le (1-\lambda)\rho(x, y) + \lambda\rho(G(x), G(y)). \tag{2.1}$$

Thus, $\rho(G(x), G(y)) \le \rho(x, y)$. Therefore, every λ-firmly nonexpansive mapping is nonexpansive.

– Let $(\mathcal{E}, \rho)$ be a CAT(0) space and $\mathcal{K}$ be a nonempty closed-convex subset of $\mathcal{E}$. The metric projection $P_{\mathcal{K}} : \mathcal{E} \to \mathcal{K}$ is a firmly nonexpansive mapping [17].

– Let $(\mathcal{E}, \rho)$ be a CAT(0) space and $g : \mathcal{E} \to (-\infty, +\infty]$ be a proper, lower semicontinuous, and convex function. Then, its resolvent J_r (for any $r > 0$) is a firmly nonexpansive mapping [17].

Definition 2.3. Let $(\mathcal{E}, \rho)$ be a metric space and $\mathcal{K}$ be a nonempty subset of $\mathcal{E}$. Let $G : \mathcal{K} \to \mathcal{E}$ be a generalized nonexpansive if for all $x, y \in \mathcal{K}$,

$$\rho(G(x), G(y)) \le a\rho(x, y) + b\{\rho(x, G(x)) + \rho(y, G(y))\} + c\{\rho(x, G(y)) + \rho(y, G(x))\}, \tag{2.2}$$

where $a, b, c \ge 0$ with $a + 2b + 2c = 1$.

If $b = 0$ and $c > 0$, then (2.2) reduced into the following condition:

$$\rho(G(x), G(y)) \le a\rho(x, y) + c\{\rho(x, G(y)) + \rho(y, G(x))\} \tag{2.3}$$

for all $x, y \in \mathcal{K}$, where $a \ge 0$ with $a + 2c = 1$. The class of mapping satisfying (2.3) has been studied and investigated to obtain my fruitful fixed point theorems by many authors, see [22,23,31,32].

Proposition 2.4. *Let $(\mathcal{E}, \rho)$ be a hyperbolic space and $\mathcal{K}$ be a nonempty subset of $\mathcal{E}$. Let $G : \mathcal{K} \to \mathcal{K}$ be a λ-firmly nonexpansive for some $\lambda \in (0, 1)$, then G is a mapping satisfying* (2.3).

Proof. By the definition of mapping G and Definition 1.2 (i), we have

$$\begin{aligned}\rho(G(x), G(y)) &\le \rho((1-\lambda)x \oplus \lambda G(x), (1-\lambda)y \oplus \lambda G(y))\\ &\le (1-\lambda)\rho((1-\lambda)x \oplus \lambda G(x), y) + \lambda\rho((1-\lambda)x \oplus \lambda G(x), G(y))\\ &\le (1-\lambda)\{(1-\lambda)\rho(x, y) + \lambda\rho(G(x), y)\} + \lambda\{(1-\lambda)\rho(x, G(y)) + \lambda\rho(G(x), G(y))\}.\end{aligned}$$

This implies that

$$(1-\lambda^2)\rho(G(x), G(y)) \le (1-\lambda)^2\rho(x, y) + \lambda(1-\lambda)\{\rho(G(x), y) + \rho(y, G(x))\}$$

and

$$\rho(G(x), G(y)) \le \frac{(1-\lambda)}{(1+\lambda)}\rho(x, y) + \frac{\lambda}{(1+\lambda)}\{\rho(G(x), y) + \rho(y, G(x))\}.$$

Take $c = \frac{\lambda}{(1+\lambda)}$, since $\lambda \in (0, 1)$, $c = \frac{\lambda}{(1+\lambda)} > 0$, and take $a = \frac{(1-\lambda)}{(1+\lambda)}$, then the above inequality becomes:

$$\rho(G(x), G(y)) \le a\rho(x, y) + c\{\rho(G(x), y) + \rho(y, G(x))\},$$

where $a + 2c = 1$. □

We note that in the above proposition, the inclusion is strict as the following example shows.

Example 2.5. Let $\mathcal{E} = \mathbb{R}$ be a metric space equipped with the usual metric and $\mathcal{K} = [0, 1] \subset \mathbb{R}$. Define $G : \mathcal{K} \to \mathcal{K}$ by

$$G(x) = \begin{cases} \dfrac{2x}{3}, & \text{if } x \in \left[0, \dfrac{1}{2}\right] \\ \dfrac{7x}{10}, & \text{if } x \in \left(\dfrac{1}{2}, 1\right]. \end{cases}$$

First, we show that G is a mapping satisfying (2.3) for $a = \frac{2}{3}$ and $c = \frac{1}{6}$. To show this, we distinguish three cases as follows.

Case 1. $x, y \in \left[0, \frac{1}{2}\right]$. Then,

$$\frac{2}{3}\rho(x, y) + \frac{1}{6}\rho(G(x), y) + \frac{1}{6}\rho(G(y), x) \geq \frac{2}{3}\rho(x, y) = \frac{2}{3}|x - y| = \rho(G(x), G(y)).$$

Case 2. $x, y \in \left(\frac{1}{2}, 1\right]$ and $x < y$. Then,

$$\begin{aligned}\frac{2}{3}\rho(x, y) + \frac{1}{6}\rho(G(x), y) + \frac{1}{6}\rho(G(y), x) &\geq \frac{1}{6}\left|\frac{7x}{10} - y\right| + \frac{2}{3}|x - y| \\ &\geq \frac{1}{6}|x - y| + \frac{2}{3}|x - y| \\ &\geq \frac{5}{6}|x - y| > \frac{7}{10}|x - y| \\ &= \rho(G(x), G(y)).\end{aligned}$$

Case 3. $x \in \left[0, \frac{1}{2}\right]$ and $y \in \left(\frac{1}{2}, 1\right]$ and $\frac{7y}{10} \leq x$. Then,

$$\begin{aligned}\frac{2}{3}\rho(x, y) + \frac{1}{6}\rho(G(x), y) + \frac{1}{6}\rho(G(y), x) &= \frac{2}{3}|x - y| + \frac{1}{6}\left|\frac{2x}{3} - y\right| + \frac{1}{6}\left|\frac{7y}{10} - x\right| \\ &= \frac{2}{3}(y - x) + \frac{1}{6}\left(y - \frac{2x}{3}\right) + \frac{1}{6}\left(x - \frac{7y}{10}\right) \\ &= \frac{43y}{60} - \frac{11x}{18} > \frac{43y}{60} - \frac{12x}{18} > \frac{42y}{60} - \frac{12x}{18} \\ &= \rho(G(x), G(y)).\end{aligned}$$

Again, if $x < \frac{7y}{10}$,

$$\begin{aligned}\frac{2}{3}\rho(x, y) + \frac{1}{6}\rho(G(x), y) + \frac{1}{6}\rho(G(y), x) &= \frac{2}{3}|x - y| + \frac{1}{6}\left|\frac{2x}{3} - y\right| + \frac{1}{6}\left|\frac{7y}{10} - x\right| \\ &= \frac{2}{3}(y - x) + \frac{1}{6}\left(y - \frac{2x}{3}\right) + \frac{1}{6}\left(\frac{7y}{10} - x\right) \\ &= \frac{57y}{60} - \frac{17x}{18} = \frac{42y}{60} - \frac{12x}{18} + \frac{15y}{60} - \frac{5x}{18}.\end{aligned}$$

Since $\frac{7y}{10} > x$, $\frac{15y}{60} > \frac{5x}{14} > \frac{5x}{18}$. Thus,

$$\frac{2}{3}\rho(x, y) + \frac{1}{6}\rho(G(x), y) + \frac{1}{6}\rho(G(y), x) \geq \frac{42y}{60} - \frac{12x}{18} = \rho(G(x), G(y)).$$

Since G is not continuous, G is not firmly nonexpansive mapping.

Remark 2.6. The class of nonexpansive mappings and that of mappings satisfying condition (2.3) are independent in nature. It can be noticed that the class of mappings defined in Example 2.5 is not nonexpansive mapping but satisfying condition (2.3). We present the following example to complete our claim.

Example 2.7. Let $\mathcal{K} = [0, 1] \subset \mathbb{R}$ with usual metric and $G : \mathcal{K} \to \mathcal{K}$ be a mapping defined as

$$G(x) = 1 - x \quad \text{for all } x \in \mathcal{K}.$$

Then, $F(G) = \left\{\frac{1}{2}\right\}$ and G is a nonexpansive mapping. On the other hand, it can be seen that if $c > 0$, then $a < 1$; thus, for $x = 0.4$ and $y = 0.6$, we have

$$\rho(G(x), G(y)) = 0.2 > a \times 0.2 = a\rho(x, y) + c\{\rho(x, G(y)) + \rho(y, G(x))\}.$$

And G does not satisfy condition (2.3).

Lemma 2.8. [32] *Let $(\mathcal{E}, \rho)$ be a bounded metric space and $G : \mathcal{E} \to \mathcal{E}$ be a mapping satisfying* (2.3). *Then, G is asymptotically regular, that is, for any $x \in \mathcal{E}$,*

$$\lim_{n \to +\infty} \rho(G^{n+1}(x), G^n(x)) = 0.$$

3 Main results

First, we present the following useful lemma.

Lemma 3.1. *Let $(\mathcal{E}, \rho)$ be a metric space and $\mathcal{K}$ be a union of nonempty subsets $\mathcal{K}_i$ (for $i = 1, 2, \dots, m$) of $\mathcal{E}$, that is, $\mathcal{K} = \cup_{i=1}^m \mathcal{K}_i$. Let $G : \mathcal{K} \to \mathcal{K}$ be a mapping satisfying* (2.3), *suppose that G has bounded orbits and that for some $y \in \mathcal{K}$, the orbit $\{G^n(y)\}$ of G has a unique asymptotic center x_i with respect to each $\mathcal{K}_i$, $i = 1, 2, \dots, m$. Then, there exists p in $\{1, 2, \dots, m\}$ such that x_p is a periodic point of G.*

Proof. From the definition of G and triangle inequality, we have

$$\begin{aligned}\rho(G(x_i), G^{n+1}(y)) &\le a\rho(x_i, G^n(y)) + c\{\rho(x_i, G^{n+1}(y)) + \rho(G(x_i), G^n(y))\} \\ &\le a\rho(x_i, G^n(y)) + c\{\rho(x_i, G^n(y)) + \rho(G^n(y), G^{n+1}(y))\} \\ &\quad + c\{\rho(G(x_i), G^{n+1}(y)) + \rho(G^{n+1}(y), G^n(y))\}.\end{aligned}$$

This implies that

$$(1 - c)\rho(G(x_i), G^{n+1}(y)) \le (a + c)\rho(x_i, G^n(y)) + 2c\rho(G^n(y), G^{n+1}(y))$$

and

$$\rho(G(x_i), G^{n+1}(y)) \le \frac{(a + c)}{(1 - c)}\rho(x_i, G^n(y)) + \frac{2c}{(1 - c)}\rho(G^n(y), G^{n+1}(y)).$$

Since $a + 2c = 1$, $a + c = 1 - c$, and by triangle inequality,

$$\rho(G(x_i), G^n(y)) \le \rho(G(x_i), G^{n+1}(y)) + \rho(G^n(y), G^{n+1}(y)) \le \rho(x_i, G^n(y)) + \frac{1 + c}{(1 - c)}\rho(G^n(y), G^{n+1}(y)).$$

Since the mapping G has bounded orbits, by Lemma (2.8), G is asymptotically regular,

$$\limsup_{n \to +\infty} \rho(G(x_i), G^n(y)) \le \limsup_{n \to +\infty} \rho(x_i, G^n(y))$$

for all $i = 1, 2, \dots, m$. Thus,

$$r(G(x_i), \{G^n(y)\}) \le r(x_i, \{G^n(y)\}). \tag{3.1}$$

If there exists i_0 in $\{1, 2, \dots, m\}$ such that $G(x_{i_0}) \in \mathcal{K}_{i_0}$, then by Lemma 1.12 ($y = x_{i_0}$, $y = G(x_{i_0})$, $a_n = 1$, $b_n = 0$, $m = 1$, and $x_n = G^n(y)$), it follows that $G(x_{i_0}) = x_{i_0}$ and x_{i_0} is a fixed point of G, in fact x_{i_0} is a periodic point of G. Otherwise, suppose that $G(x_i) \notin \mathcal{K}_i$ for all $i \in \{1, 2, \dots, m\}$, then there exist integers $\{m_1, m_2, \dots, m_j\} \subseteq \{1, 2, \dots, m\}$, $j \ge 2$, such that $G(x_{m_i}) \in \mathcal{K}_{m_{i+1}}$ for all $i \in \{1, 2, \dots, j - 1\}$ and $G(x_{m_j}) \in \mathcal{K}_{m_1}$. Using the fact that x_{m_i} is the unique asymptotic center of $\{G^n(y)\}$ with respect to $\mathcal{K}_{m_i}$ and from (3.1), we have

$$r\left(x_{m_1}, \{G^n(y)\}\right) \le r\left(G\left(x_{m_j}\right), \{G^n(y)\}\right) \le r\left(x_{m_j}, \{G^n(y)\}\right) \le \dots \le r\left(G\left(x_{m_1}\right), \{G^n(y)\}\right) \le r\left(x_{m_1}, \{G^n(y)\}\right).$$

Therefore,

$$r\left(x_{m_1}, \{G^n(y)\}\right) = r\left(G\left(x_{m_j}\right), \{G^n(y)\}\right) \quad \text{and} \quad r\left(G\left(x_{m_i}\right), \{G^n(y)\}\right) = r\left(x_{m_{i+1}}, \{G^n(y)\}\right)$$

for all $i \in \{1, 2, \dots, j - 1\}$. By the uniqueness of the asymptotic centers,

$$x_{m_1} = G\big(x_{m_j}\big) \quad \text{and} \quad G\big(x_{m_i}\big) = x_{m_{i+1}} \text{ for all } i \in \{1, 2, \ldots, j-1\}. \tag{3.2}$$

Hence, $G^j(x_{m_1}) = x_{m_1}$, and x_{m_1} is a periodic point of G. □

The above lemma is generalization of [17, Lemma 4.4] for more general class of mappings.

Now, we present the following proposition, which is a generalization of [17, Proposition 4.3].

Proposition 3.2. *Let $(\mathcal{E}, \rho)$ be a Busemann space, $\mathcal{K}$ be a nonempty subset of $\mathcal{E}$, and $G : \mathcal{K} \to \mathcal{K}$ be a mapping satisfying* (2.3). *Then, any periodic point of G is a fixed point of G.*

Proof. Let x be a periodic point of G, then there is an $m \in \mathbb{N} \cup \{0\}$ such that $G^{m+1}(x) = x$. If $m = 0$, then obviously x is a fixed point of G, thus we suppose that $m \geq 1$. By the definition of mapping G and triangle inequality,

$$\begin{aligned}\rho(G^{m+1}(x), G^m(x)) &\leq a\rho(G^m(x), G^{m-1}(x)) + c\{\rho(G^{m+1}(x), G^{m-1}(x)) + \rho(G^m(x), G^m(x))\}\\ &\leq a\rho(G^m(x), G^{m-1}(x)) + c\{\rho(G^{m+1}(x), G^m(x)) + \rho(G^m(x), G^{m-1}(x))\}\end{aligned}$$

and

$$\rho(G^{m+1}(x), G^m(x)) \leq \frac{(a+c)}{(1-c)}\rho(G^m(x), G^{m-1}(x)) = \rho(G^m(x), G^{m-1}(x)).$$

Thus,

$$\rho(x, G^m(x)) = \rho(G^{m+1}(x), G^m(x)) \leq \rho(G^m(x), G^{m-1}(x)) \leq \ldots \rho(G(x), x) = \rho(G(x), G^{m+1}(x)). \tag{3.3}$$

Again, by the definition of mapping G,

$$\begin{aligned}\rho(G(x), G^{m+1}(x)) &\leq a\rho(x, G^m(x)) + c\{\rho(x, G^{m+1}(x)) + \rho(G(x), G^m(x))\}\\ &\leq a\rho(x, G^m(x)) + c\{\rho(x, G(x)) + \rho(x, G^m(x))\}\end{aligned}$$

and

$$\rho(G(x), G^{m+1}(x)) = \rho(x, G(x)) \leq \frac{(a+c)}{(1-c)}\rho(x, G^m(x)) = \rho(x, G^m(x)). \tag{3.4}$$

From (3.3) and (3.4),

$$\rho(x, G^m(x)) = \rho(G^{m+1}(x), G^m(x)) \leq \rho(G^m(x), G^{m-1}(x)) \leq \ldots \rho(G(x), x) = \rho(G(x), G^{m+1}(x)) \leq \rho(x, G^m(x)).$$

Thus, we must have

$$\rho(G(x), x) = \rho(G^2(x), G(x)) = \cdots = \rho(G^m(x), G^{m-1}(x)) = \rho(x, G^m(x)) = L. \tag{3.5}$$

Since $G^i(x) \neq G^{i+1}(x)$ for any $i = 1, 2, \ldots, m$, by the property of Busemann space $\mathcal{E}$, for given $\mu \in (0, 1)$, there exists a unique element $u_i \in \mathcal{E}$ (namely $u_i = W(G^i(x), G^{i+1}(x), \mu)$) such that

$$\rho(G^i(x), u_i) = \mu\rho(G^i(x), G^{i+1}(x)) \tag{3.6}$$

and

$$\rho(G^{i+1}(x), u_i) = (1-\mu)\rho(G^i(x), G^{i+1}(x)). \tag{3.7}$$

Again, $G^{i-1}(x) \neq G^i(x)$ for any $i = 1, 2, \ldots, m$, by the property of Busemann space $\mathcal{E}$, for given $\mu \in (0, 1)$, there exists a unique element $v_i \in \mathcal{E}$ (namely $v_i = W(G^{i-1}(x), G^i(x), \mu)$) such that

$$\rho(G^{i-1}(x), v_i) = \mu\rho(G^{i-1}(x), G^i(x)) \tag{3.8}$$

and

$$\rho(G^i(x), v_i) = (1-\mu)\rho(G^{i-1}(x), G^i(x)). \tag{3.9}$$

Now, we show that

$$L = \rho(u_i, v_i) = \rho(v_i, G^i(x)) + \rho(G^i(x), u_i).$$

From the triangle inequality, (3.6) and (3.9), we obtain

$$\rho(u_i, v_i) \le \rho(v_i, G^i(x)) + \rho(G^i(x), u_i) = (1-\mu)\rho(G^{i-1}(x), G^i(x)) + \mu\rho(G^i(x), G^{i+1}(x)) = (1-\mu)L + \mu L = L. \tag{3.10}$$

Furthermore, by the definition of mapping G,

$$\begin{aligned} L = \rho(G^{i+1}(x), G^i(x)) &\le a\rho(G^i(x), G^{i-1}(x)) + c\{\rho(G^{i+1}(x), G^{i-1}(x)) + \rho(G^i(x), G^i(x))\} \\ &\le aL + +c\{\rho(G^{i+1}(x), u_i) + \rho(u_i, v_i) + \rho(v_i, G^{i-1}(x))\}. \end{aligned}$$

From (3.7) and (3.8),

$$\begin{aligned} L &\le aL + +c\{(1-\mu)\rho(G^i(x), G^{i+1}(x)) + \rho(u_i, v_i) + \mu\rho(G^{i-1}(x), G^i(x))\} \\ &\le aL + +c\{(1-\mu)L + \rho(u_i, v_i) + \mu L\} \\ &\le (a+c)L + c\rho(u_i, v_i). \end{aligned}$$

This implies that

$$L \le \frac{c}{1-a-c}\rho(u_i, v_i) = \rho(u_i, v_i). \tag{3.11}$$

Combining (3.10) and (3.12), we obtain

$$L = \rho(u_i, v_i) = \rho(v_i, G^i(x)) + \rho(G^i(x), u_i). \tag{3.12}$$

Now, we distinguish the following cases:
Case 1. If $m = 1$, hence $i = 1$. Then, $G^{m-1}(x) = x$ and $G^2(x) = x$,

$$u_1 = W(G(x), x, \mu) = W(x, G(x), 1-\mu)$$

and

$$v_1 = W(x, G(x), \mu).$$

From Definition 1.2 (ii), we have

$$L = \rho(u_1, v_1) = \rho(W(x, G(x), 1-\mu), W(x, G(x), \mu)) = |1-\mu-\mu|\rho(x, G(x)) = |1-2\mu|L.$$

Therefore, $|1-2\mu| = 1$, a contradiction, since $\mu \in (0,1)$.
Case 2. If $m \ge 2$, then $m-1 \ge 1$. From (3.12), the point $G^i(x)$ lies between two points v_i and u_i for each $i = 1, 2, \dots, m$, further u_i lies between $G^i(x)$ and $G^{i+1}(x)$, by Lemma 1.9, we obtain that $G^i(x)$ and u_i both lies between v_i and $G^{i+1}(x)$. Moreover, v_i lies between $G^{i-1}(x)$ and $G^i(x)$, that $G^i(x)$ lies between $G^{i-1}(x)$ and $G^{i+1}(x)$ for all $i = 1, 2, \dots, m$. In view of Lemma 1.8, $G^{m-1}(x)$ lies between x and $G^m(x)$. Hence,

$$L = \rho(x, G^m(x)) = \rho(x, G^{m-1}(x)) + \rho(G^{m-1}(x), G^m(x)) = L + \rho(x, G^{m-1}(x)) > L$$

a contradiction, since $G^{m-1}(x) \ne x$. Therefore, $L = 0$. This completes the proof. □

Proposition 3.3. *Let $(\mathcal{E}, \rho)$ be a complete UC-hyperbolic space and $\mathcal{K}$ be a union of nonempty closed-convex subsets $\mathcal{K}_i$ (for $i = 1, 2, \dots, m$) of $\mathcal{E}$, that is, $\mathcal{K} = \cup_{i=1}^m \mathcal{K}_i$. Let $G : \mathcal{K} \to \mathcal{K}$ be a mapping satisfying* (2.3) *with bounded orbits. Then, G has periodic point.*

Proof. For all $y \in \mathcal{K}$ and for all $i = 1, \dots, m$, in view of Proposition 1.11, the orbit $\{G^n(y)\}$ has a unique asymptotic center x_i with respect to $\mathcal{K}_i$. From Lemma 3.1, it follows that one of the asymptotic centers x_i, $i = 1, \dots, m$, is a periodic point of G. □

The above proposition is generalization of [17, Proposition 4.5] for more general class of mappings. In this, we present the following fixed point theorem.

Theorem 3.4. *Let $(\mathcal{E}, \rho)$ be a complete UC-hyperbolic space and $\mathcal{K}$ be a union of nonempty closed-convex subsets $\mathcal{K}_i$ (for $i = 1, 2, \ldots, m$) of $\mathcal{E}$, that is, $\mathcal{K} = \cup_{i=1}^{m} \mathcal{K}_i$. Let $G : \mathcal{K} \to \mathcal{K}$ be a mapping satisfying* (2.3). *Then the following are equivalent*:
- *G has bounded orbits.*
- *G has fixed points.*

Proof. In view of Propositions 3.3 and 3.2, one can complete the proof. □

The above theorem is a generalization of [17, Theorem 4.1] for the more general class of mappings.

Remark 3.5. It can be seen that Theorem 3.4 is not true for nonexpansive mappings. The following illustrative example settles this claim:

Take $x \neq y \in \mathcal{E}$, $\mathcal{K}_1 = \{x\}$, $\mathcal{K}_2 = \{y\}$, and $\mathcal{K} = \mathcal{K}_1 \cup \mathcal{K}_2$ and define $G : \mathcal{K} \to \mathcal{K}$ as follows:

$$G(x) = y, G(y) = x.$$

Then, G is fixed point-free nonexpansive mapping. On the other hand, if G is a mapping satisfying (2.3), then we obtain the following contradiction:

$$\begin{aligned} 0 < \rho(x, y) = \rho(G(x), G(y)) &\leq a\rho(x, y) + c\{\rho(x, G(y)) + \rho(y, G(x))\} \\ &\leq a\rho(x, y) + c\{\rho(x, x) + \rho(y, y)\} = a\rho(x, y) < \rho(x, y) \end{aligned}$$

since $c > 0$, $a < 1$.

Lemma 3.6. *Let $(\mathcal{E}, \rho)$ be a uniquely geodesic space, $\mathcal{K}$ be a nonempty closed-convex subset of $\mathcal{E}$, and $G : \mathcal{K} \to \mathcal{K}$ be a mapping satisfying* (2.3). *Then, $F(G)$ is closed and convex.*

Proof. First, we show that $F(G)$ is closed. Let $\{x_n\}$ be a sequence in $F(G)$ such that $\{x_n\}$ strongly converges to $x \in \mathcal{K}$.

$$\rho(G(x), x_n) \leq \rho(G(x), G(x_n)) \leq a\rho(x, x_n) + c\{\rho(x, G(x_n)) + \rho(x_n, G(x))\}$$

and

$$\rho(G(x), x_n) \leq \rho(x, x_n) \to 0 \text{ as } n \to +\infty.$$

Thus, $G(x) = x \in F(G)$. Now, we show that $F(G)$ is convex. Let $x \neq y \in F(G)$ and $y \in [x, y]$. It can be seen that

$$\rho(x, G(y)) = \rho(G(x), G(y)) \leq \rho(x, y) \tag{3.13}$$

and

$$\rho(y, G(y)) = \rho(G(y), G(y)) \leq \rho(y, y). \tag{3.14}$$

Then, from (3.13) and (3.14),

$$\rho(x, y) \leq \rho(x, G(y)) + \rho(G(y), y) \leq \rho(x, y) + \rho(y, y) = \rho(x, y).$$

Therefore,

$$\rho(x, G(y)) + \rho(G(y), y) = \rho(x, y)$$

and $G(y) \in [x, y]$. From Lemma (1.10), we obtain the following:

$$\rho(x, y) + \rho(y, G(y)) = \rho(x, G(y)) = \rho(G(x), G(y)) \leq \rho(x, y),$$

or

$$\rho(y, y) + \rho(y, G(y)) = \rho(y, G(y)) = \rho(G(y), G(y)) \leq \rho(y, y).$$

In both occasions, we obtain $y = G(y)$. □

Theorem 3.7. *Let $(\mathcal{E}, \rho)$ be a complete UC-hyperbolic space, $\mathcal{K}$ be a nonempty closed-convex subset of $\mathcal{E}$, and $G : \mathcal{K} \to \mathcal{K}$ be a mapping satisfying (2.3) with $F(G) \neq \emptyset$. Then, the Picard iterate $\{G^n(x)\}$ (for any $x \in \mathcal{K}$) Δ-converges to a point in $F(G)$.*

Proof. Take $C := F(G)$, then from Lemma 3.6, C is closed and convex. Moreover, for all $x^\dagger \in C$

$$\rho(G^{n+1}(x), x^\dagger) \leq \rho(G^n(x), x^\dagger) \text{ for all } n \geq 0.$$

Thus, $\{G^n(x)\}$ is Fejér monotone with respect to C and bounded. In view of Proposition 1.11, sequence $\{G^n(x)\}$ has a unique asymptotic center with respect to C. Suppose $\{u_n\}$ is a subsequence of $\{G^n(x)\}$, and y is its unique asymptotic center. By the triangle inequality, we have

$$\begin{aligned}\rho(G(y), u_n) &\leq \rho(G(y), G(u_n)) + \rho(u_n, G(u_n))\\ &\leq a\rho(y, u_n) + c\{\rho(y, G(u_n)) + \rho(G(y), u_n)\} + \rho(u_n, G(u_n))\\ &\leq a\rho(y, u_n) + c\{\rho(y, u_n) + \rho(u_n, G(u_n))\} + c\rho(G(y), u_n) + \rho(u_n, G(u_n)).\end{aligned}$$

This implies that

$$\rho(G(y), u_n) \leq \frac{(a + c)}{(1 - c)}\rho(y, u_n) + \frac{(1 + c)}{(1 - c)}\rho(u_n, G(u_n))$$

and

$$\rho(G(y), u_n) \leq \rho(y, u_n) + \frac{(1 + c)}{(1 - c)}\rho(u_n, G(u_n)).$$

Since G is asymptotically regular at $x \in \mathcal{K}$, $\lim_{n\to+\infty}\rho(G^n(x), G^{n+1}(x)) = 0$ and $\lim_{n\to+\infty}\rho(u_n, G(u_n)) = 0$. Thus, all the assumptions of Lemma 1.12 are fulfilled, and it follows that $G(y) = y$, that is, $y \in C$. In view of Lemma (1.13), we can conclude that the sequence $\{x_n\}$ Δ-converges to a point in $F(G)$. □

Theorem 3.8. *Let $(\mathcal{E}, \rho)$ be a complete UC-hyperbolic space, $\mathcal{K}$ be a nonempty closed-convex subset of $\mathcal{E}$, and $G : \mathcal{K} \to \mathcal{K}$ be a mapping satisfying (2.3) such that G has bounded orbits. Then, the Picard iterate $\{G^n(x)\}$ (for any $x \in \mathcal{K}$) Δ-converges to a point in $F(G)$.*

Theorem 3.9. *Let $(\mathcal{E}, \rho)$ be a Busemann space and $\mathcal{K}$ be a nonempty bounded closed-convex subset of $\mathcal{E}$. Let $G : \mathcal{K} \to \mathcal{K}$ be a mapping satisfying (2.3), G satisfies condition (I) and $F(G) \neq \emptyset$. Then, the Picard iterate $\{G^n(x)\}$ (for any $x \in K$) strongly converges to a point in $F(G)$.*

Proof. From Theorem 3.7, it can be seen that for all $x^\dagger \in F(G)$,

$$\rho(G^{n+1}(x), x^\dagger) \leq \rho(G^n(x), x^\dagger) \text{ for all } n \geq 0.$$

Thus, the sequence $\{\rho(G^n(x), x^\dagger)\}$ is non-increasing and $\lim_{n\to+\infty}\rho(G^n(x), x^\dagger)$ exists. Since, $\lim_{n\to+\infty}\rho(G^n(x), x^\dagger)$ exists for all $x^\dagger \in F(G)$, $\lim_{n\to+\infty}\rho(G^n(x), F(G))$ exists. Since $\mathcal{K}$ is bounded, by Lemma (2.8), G is asymptotically regular, that is,

$$\lim_{n\to+\infty} \rho(G^{n+1}(x), G^n(x)) = 0. \tag{3.15}$$

Take $x_n = G^n(x)$. Since G satisfies condition (I) and (3.15), we obtain

$$\rho(x_n, G(x_n)) \geq f(\rho(x_n, F(G))).$$

Thus,

$$\lim_{n\to+\infty} \rho(x_n, F(G)) = 0. \tag{3.16}$$

Now, it turns out that the sequence $\{x_n\}$ is Cauchy. For the sake of completeness, we include the argument. For given $\varepsilon > 0$, in view of (4.1), there exists a $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$,

$$\rho(x_n, F(G)) < \frac{\varepsilon}{4}.$$

In particular,

$$\inf\left\{\rho\left(x_{n_0}, y\right) : y \in F(G)\right\} < \frac{\varepsilon}{4},$$

and there exists $y \in F(G)$ such that

$$\rho\left(x_{n_0}, y\right) < \frac{\varepsilon}{2}.$$

Therefore, for all $m, n \geq n_0$,

$$\rho(x_{n+m}, x_n) \leq \rho(x_{n+m}, y) + \rho(y, x_n) \leq \rho(x_n, y) < 2\frac{\varepsilon}{2} = \varepsilon,$$

and the sequence $\{x_n\}$ is Cauchy. Since $\mathcal{K}$ is a closed subset of $\mathcal{E}$, so $\{x_n\}$ converges to a point $x^\dagger \in C$. From the definition of mapping G,

$$\begin{aligned}\rho(x_{n+1}, G(x^\dagger)) = \rho(G(x_n), G(x^\dagger)) &\leq a\rho(x_n, x^\dagger) + c\{\rho(G(x_n), x^\dagger) + \rho(x_n, G(x^\dagger))\}\\ &\leq a\rho(x_n, x^\dagger) + c\rho(G(x_n), x^\dagger) + c\rho(x_n, G(x_n)) + c\rho(G(x_n), G(x^\dagger))\end{aligned}$$

and

$$\rho(x_{n+1}, G(x^\dagger)) \leq \frac{a}{(1-c)}\rho(x_n, x^\dagger) + \frac{c}{(1-c)}\rho(x_{n+1}, x^\dagger) + \frac{c}{(1-c)}\rho(x_n, G(x_n))$$

from (3.15), $x^\dagger = G(x^\dagger)$. Therefore, the sequence $\{x_n\}$ converges strongly to a point in $F(G)$. □

Theorem 3.10. *Let $(\mathcal{E}, \rho)$ be a Busemann space and $\mathcal{K}$ be a nonempty bounded closed-convex subset of $\mathcal{E}$. Let $G : \mathcal{K} \to \mathcal{K}$ be a compact mapping satisfying* (2.3) *and $F(G) \neq \emptyset$. Then, the Picard iterate $\{G^n(x)\}$ (for any $x \in \mathcal{K}$) strongly converges to a point in $F(G)$.*

4 An application to a constrained minimization problem

Let $(\mathcal{E}, \rho)$ be a complete CAT(0) space and $\Psi : \mathcal{E} \to (-\infty, +\infty]$ be a proper, lower semicontinuous and convex function. We employ Theorem 3.7 to find the minimizers of Ψ, that is, the solutions of the following minimization problem:

$$\min_{x \in \mathcal{E}} \Psi(x). \tag{4.1}$$

Take

$$\operatorname*{argmin}_{y \in \mathcal{E}} \Psi(x) = \{x \in \mathcal{E} | \Psi(x) \leq \Psi(y) \text{ for all } y \in \mathcal{E}\},$$

the set of minimizers of Ψ.

Proposition 4.1. [17] *Let $r > 0$ and J_r be a resolvent associated with Ψ. Then, $F(J_r) = \operatorname{argmin}_{y \in \mathcal{E}} \Psi(x)$.*

Theorem 4.2. *Suppose that the function Ψ has a minimizer. Then, for all $r > 0$ and all $x \in \mathcal{E}$, the Picard iterate $\{J_r^n(x)\}$ Δ-converges to some point in $\mathcal{E}$ which is a minimizer of Ψ.*

Proof. It can be easily seen that J_r (a resolvent associated with Ψ) satisfies (2.3). Therefore, conclusion directly follows from Theorem 3.7. □

5 Examples

In this section, we present couple of examples to illustrate facts.

Example 5.1. Let $\mathcal{E} = \{(x^{(1)}, x^{(2)}) \in \mathbb{R}^2 : x^{(1)}, x^{(2)} > 0\}$. Define $\rho : \mathcal{E} \times \mathcal{E} \to [0, +\infty)$ by

$$\rho(x, y) = |x^{(1)} - y^{(1)}| + |x^{(1)}x^{(2)} - y^{(1)}y^{(2)}|$$

for all $x = (x^{(1)}, x^{(2)})$ and $y = (y^{(1)}, y^{(2)})$ in $\mathcal{E}$. Then, it can be easily seen that ρ is a metric on $\mathcal{E}$, and $(\mathcal{E}, \rho)$ is a metric space. Now, for $\Theta \in [0, 1]$, define a function $\Omega : \mathcal{E} \times \mathcal{E} \times [0, 1] \to \mathcal{E}$ by

$$\Omega(x, y, \Theta) = \left((1 - \Theta)x^{(1)} + \Theta y^{(1)}, \frac{(1 - \Theta)x^{(1)}x^{(2)} + \Theta y^{(1)}y^{(2)}}{(1 - \Theta)x^{(1)} + \Theta y^{(1)}}\right).$$

It is shown in [7] that $(\mathcal{E}, \rho, \Omega)$ is a hyperbolic metric space but not a normed linear space.

Take $\mathcal{K}_1 = [1, 2] \times [1, 2]$, $\mathcal{K}_2 = [1, 2] \times [2, 4]$, $\mathcal{K}_3 = [2, 4] \times [1, 2]$, and $\mathcal{K}_4 = [2, 4] \times [2, 4]$. Then,

$$\mathcal{K} := \bigcup_{i=1}^{4} \mathcal{K}_i = [1, 4] \times [1, 4]$$

and $\mathcal{K}$ is a nonempty closed-convex subset of $\mathcal{E}$, and a mapping $G : \mathcal{K} \to \mathcal{K}$ is defined as follows:

$$G(x) = \begin{cases} (1, 1), & \text{if } (x^{(1)}, x^{(2)}) \neq (4, 4) \\ \left(\frac{5}{2}, \frac{5}{2}\right), & \text{if } (x^{(1)}, x^{(2)}) = (4, 4). \end{cases}$$

Now, we show that G is a mapping satisfying (2.3) for $a = 0$ and $c = \frac{1}{2}$ with $F(G) = (1, 1)$. We distinguish two cases:

Case 1. If $x = (x^{(1)}, x^{(2)})$, $y = (y^{(1)}, y^{(2)}) \neq (4, 4)$, then

$$\rho(G(x), G(y)) = \rho((1, 1), (1, 1)) = 0 \leq \frac{1}{2}\rho(G(x), y) + \frac{1}{2}\rho(x, G(y)).$$

Case 2. Again, if $x = (x^{(1)}, x^{(2)}) \neq (4, 4)$, and $y = (y^{(1)}, y^{(2)}) = (4, 4)$, then

$$\begin{aligned} \frac{1}{2}\rho(G(x), y) + \frac{1}{2}\rho(x, G(y)) &= \frac{1}{2}\rho((1, 1), (4, 4)) + \frac{1}{2}\rho\left((x^{(1)}, x^{(2)}), \left(\frac{5}{2}, \frac{5}{2}\right)\right) \\ &= 9 + \left|\frac{5}{2} - x^{(1)}\right| + \left|x^{(1)}x^{(2)} - \frac{25}{4}\right| \\ &> \frac{27}{4} = \rho\left((1, 1), \left(\frac{5}{2}, \frac{5}{2}\right)\right) = \rho(G(x), G(y)). \end{aligned}$$

However, if G is not a continuous mapping on $\mathcal{K}$, then G is neither nonexpansive nor firmly nonexpansive.

Now, we consider the well-known *river metric* ρ. A river metric space $(\mathbb{R}^2, \rho)$ is a $\mathbb{R}$-tree. Moreover, CAT(0) spaces include the $\mathbb{R}$-trees and CAT(0) are special cases of hyperbolic spaces (cf. [33]).

Example 5.2. Let $\mathcal{E} = \mathbb{R}^2$ be endowed with the *river metric* defined as

$$\rho(x, y) = \begin{cases} |y^{(2)} - x^{(2)}|, & \text{if } y^{(1)} = x^{(1)} \\ |x^{(2)}| + |y^{(2)}| + |y^{(1)} - x^{(1)}|, & \text{if } y^{(1)} \neq x^{(1)}, \end{cases}$$

for all $x = (x^{(1)}, x^{(2)})$, and $y = (y^{(1)}, y^{(2)})$ in $\mathbb{R}^2$. Take $\mathcal{K}_1 = [0, 2] \times [0, 2]$, $\mathcal{K}_2 = [0, 2] \times [2, 4]$, $\mathcal{K}_3 = [2, 4] \times [0, 2]$, and $\mathcal{K}_4 = [2, 4] \times [2, 4]$. Then,

$$\mathcal{K} := \bigcup_{i=1}^{4} \mathcal{K}_i = [0, 4] \times [0, 4]$$

and $\mathcal{K}$ is a nonempty closed-convex subset of $\mathcal{E}$, and $G:\mathcal{K}\to\mathcal{K}$ a mapping defined as follows:

$$G(x)=\begin{cases}\left(\dfrac{x^{(1)}}{4},\dfrac{x^{(2)}}{2}\right), & \text{if } (x^{(1)},x^{(2)})\neq(4,4)\\ (0,1), & \text{if } (x^{(1)},x^{(2)})=(4,4).\end{cases}$$

We consider the following cases:
Case (i) If $x=(x^{(1)},x^{(2)})$, $y=(y^{(1)},y^{(2)})\neq(4,4)$, then

$$\frac{1}{2}\rho(x,y)+\frac{1}{4}\rho(x,G(y))+\frac{1}{4}\rho(y,G(x))\geq\frac{1}{2}\rho(x,y)\geq\rho(G(x),G(y)).$$

Case (ii) If $x=(x^{(1)},x^{(2)})\neq(4,4)$, $x^{(1)}\neq4$, $y=(y^{(1)},y^{(2)})=(4,4)$, then $G(x)=\left(\frac{x^{(1)}}{4},\frac{x^{(2)}}{2}\right)$. If $x^{(1)}\neq0$, then

$$\frac{1}{2}\rho(x,y)+\frac{1}{4}\rho(x,G(y))+\frac{1}{4}\rho(y,G(x))\geq\frac{1}{2}\rho(x,y)=\frac{x^{(2)}}{2}+2+\frac{1}{2}(4-x^{(1)})>\frac{x^{(2)}}{2}+1+\frac{x^{(1)}}{4}=\rho(G(x),G(y))$$

since $\frac{|x^{(1)}|}{4}<1$. If $x^{(1)}=0$, then

$$\frac{1}{2}\rho(x,y)+\frac{1}{4}\rho(x,G(y))+\frac{1}{4}\rho(y,G(x))\geq\frac{1}{2}\rho(x,y)=4+\frac{x^{(2)}}{2}>\left|1-\frac{x^{(2)}}{2}\right|+1=\rho(G(x),G(y)).$$

Case (iii) If $x=(4,x^{(2)})$, $y=(y^{(1)},y^{(2)})=(4,4)$, then

$$\begin{aligned}\frac{1}{2}\rho(x,y)+\frac{1}{4}\rho(x,G(y))+\frac{1}{4}\rho(y,G(x))&\geq\frac{1}{4}\rho(x,G(y))+\frac{1}{4}\rho(y,G(x))\\&=\frac{x^{(2)}}{4}+\frac{5}{4}+\frac{x^{(2)}}{8}+\frac{7}{4}=\frac{3x^{(2)}}{8}+3\\&>\frac{3x^{(2)}}{8}+2+\frac{x^{(2)}}{8}=\frac{x^{(2)}}{2}+2=\rho(G(x),G(y)).\end{aligned}$$

Therefore, G is a mapping satisfying (2.3) for $a=\frac{1}{2}$ and $c=\frac{1}{4}$ with $F(G)=(0,0)$. On the other hand, G is not continuous, and G is not nonexpansive.

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