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Research Article

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Existence and uniqueness of solutions to the norm minimum problem on digraphs

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Abstract: In this article, based on the path homology theory of digraphs, which has been initiated and studied by Grigor'yan, Lin, Muranov, and Yau, we prove the existence and uniqueness of solutions to the problem

$$\|w\| = \min_{u\in\Omega_2(G),u\neq 0}\left\{\frac{1}{2}\|\partial u - w\|_2^2 + |u|_1\right\}$$

for $w \in H_1(G)$ and any digraph G generated by squares and triangles belonging to the same cluster.

Keywords: digraphs, cluster, path homology, norm

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1 Introduction

A digraph G is a pair (V, E), where V is a finite set known as the set of vertices and $E \subseteq V \times V\backslash\{\text{diag}\}$ is the set of directed edges. For vertices $a, b \in V$, the pair $(a, b) \in E$ will be denoted by $a \to b$. In particular, a *square* is a digraph with four distinct vertices a, b, c, and d such that $a \to b$, $b \to d$, $a \to c$, and $c \to d$. A *triangle* is a digraph with three distinct vertices a, b, and c such that $a \to b$, $b \to c$, and $a \to c$.

An elementary p-path (or p-path for short) on G is a sequence $\{i_k\}_{k=0}^{p}$ of $p + 1$ vertices. If all pairs (i_k, i_{k+1}) are edges, then the p-path is called *allowed*.

Let $\mathbb{K}$ be a field. Let $\Lambda_p(V)$ be the $\mathbb{K}$-linear space consisting of all the formal linear combinations of all elementary p-paths with the coefficients in $\mathbb{K}$. An elementary p-path $i_0 \cdots i_p$ as an element of Λ_p is denoted as $e_{i_0\cdots i_p}$. The boundary operator $\partial_p : \Lambda_p(V) \to \Lambda_{p-1}(V)$ is a $\mathbb{K}$-linear map such that for any elementary path $e_{i_0\cdots i_p}$,

$$\partial e_{i_0\cdots i_n} = \sum_{q=0}^{p}(-1)^q e_{i_0\cdots \widehat{i_q}\cdots i_n},$$

where $\widehat{i_q}$ means omission of the index i_q.

Let $\mathcal{A}_p$ be the subspace of $\Lambda_p(V)$, which consists of all the formal linear combinations of allowed paths on G, that is,

$$\mathcal{A}_p(G) = \text{span}\left\{e_{i_0\cdots i_p} : i_0 \cdots i_p \text{ is allowed}\right\}.$$

For an element $v = \sum v^{i_0\cdots i_p} e_{i_0\cdots i_p} \in \mathcal{A}_p(G)$, v is called a (a, b)-*cluster* if, for any $v^{i_0\cdots i_p} \neq 0$, $i_0 = a$ and $i_p = b$, where a and b are two fixed vertices in V.

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Note that the boundary of an allowed path may not be allowed. Nevertheless, $\mathcal{A}_p(G)$ has the following subspace:

$$\Omega_p(G) = \{x \in \mathcal{A}_p(G) : \partial x \in \mathcal{A}_{p-1}(G)\},$$

which satisfies $\partial_p \Omega_p(G) \subseteq \Omega_{p-1}(G)$ for all $p \geq -1$. The elements in $\Omega_p(G)$ are called ∂-invariant p-paths. The *path homology* of G referred to in this article is the homology of the chain complex $\{\Omega_p(G), \partial_p\}_{p\geq 0}$, denoted as $H_p(G, \mathbb{K})$ or $H_p(G)$ for short (cf. [1–6]).

In this article, our motivation is to prove the existence and uniqueness of solutions to the problem

$$\|w\| = \min_{u\in\Omega_2, u\neq 0} \left\{ \frac{1}{2} \|\partial u - w\|_2^2 + |u|_1 \right\}$$

for $w \in H_1(G)$ and study the "smallest" representative element in the path homology class of digraphs under the given norm. It should be noted that since $\Omega_p(G)$ has no unified form, we only consider the case of $w \in H_1(G)$ for digraphs that are generated by triangles or squares of the same cluster.

In information theory, signal processing, statistics, machine learning, and optimization theory, there is a lot of literature on analyzing, solving, and applying 1-norm minimization (cf. [7–13]). Our idea is to apply the existing results in signal theory, convex programming, and optimization theory to the study of path homology groups of digraphs. The main result of this article is as follows.

Theorem 1.1. *Suppose G is a finite digraph generated by squares or triangles that belong to the same cluster. Then, for any representative element w of the homology class in $H_1(G)$, the problem*

$$\|w\| = \min_{u\in\Omega_2(G), u\neq 0} \left\{ \frac{1}{2} \|\partial u - w\|_2^2 + |u|_1 \right\} \qquad (*)$$

has a unique solution u^* such that $(A^T)_I(w - \mathrm{Au}^*) = \mathrm{sign}(u_I^*)$, where A is the matrix of the boundary operator $\partial_2 : \Omega_2(G) \to \Omega_1(G)$ and $I := \mathrm{supp}(u^*)$.[1]

Finally, in Section 4, we illustrate the "smallest" representative element in the homology group $H_1(G)$ of some simple digraphs by examples.

2 Auxiliary results for the main theorem

In this section, before proving the main theorem, we give some auxiliary results. First,

Lemma 2.1. *Let $G = (V, E)$ be a digraph generated either by squares that belong to the same cluster or by triangles that belong to the same cluster. Then, the matrix A of the boundary operator $\partial_2 : \Omega_2(G) \to \Omega_1(G)$ is a full-column rank matrix.*

Proof. Case 1. G is generated by squares that belong to the same cluster (Figure 1). Then,

$$\begin{aligned}
&\Omega_1(G) = \mathcal{A}_1(G) = \mathrm{span}\{e_{12}, e_{13}, \ldots, e_{1(n-1)}, e_{2n}, e_{3n}, \ldots, e_{(n-1)n}\},\\
&\Omega_2(G) = \mathrm{span}\{e_{12n} - e_{13n}, e_{13n} - e_{14n}, \ldots, e_{1(n-2)n} - e_{1(n-1)n}\},\\
&\dim\Omega_1(G) = 2(n-2), \dim\Omega_2(G) = n-3
\end{aligned}$$

1 The subscript I in this article represents the support set of the unique solution u^*, which is a subset of $\{1, 2, \ldots, \dim u^*\}$. Hence, u_I^* is the vector made of all non-zero elements of u^*, and $\mathrm{sign}(u_I^*)$ is the vector determined by the signs of all elements of u_I^*. Meanwhile, $(A^T)_I$ is sub-matrix of A^T composed of the elements at the intersection of the rows determined by I and all columns of A^T (maintaining the relative order of rows and columns).

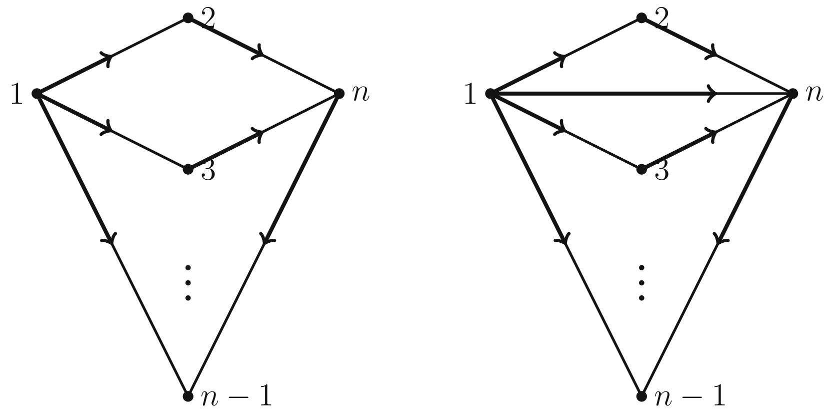

Figure 1: Case 1 and Case 2.

and

$$A_{2(n-2)\times(n-3)}=\begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0\\ -1 & 1 & 0 & 0 & \cdots & 0 & 0\\ 0 & -1 & 1 & 0 & \cdots & 0 & 0\\ 0 & 0 & -1 & 1 & \cdots & 0 & 0\\ & \cdots & & & \cdots & & \\ 0 & 0 & 0 & 0 & \cdots & -1 & 1\\ 0 & 0 & 0 & 0 & \cdots & 0 & -1\\ 1 & 0 & 0 & 0 & \cdots & 0 & 0\\ -1 & 1 & 0 & 0 & \cdots & 0 & 0\\ 0 & -1 & 1 & 0 & \cdots & 0 & 0\\ 0 & 0 & -1 & 1 & \cdots & 0 & 0\\ & \cdots & & & \cdots & & \\ 0 & 0 & 0 & 0 & \cdots & -1 & 1\\ 0 & 0 & 0 & 0 & \cdots & 0 & -1 \end{bmatrix}.$$

Hence, $R(A) = n - 3$, and A is a full-column rank matrix.

Case 2. G is generated by triangles that belong to the same cluster. Then,

$$\begin{aligned} &\Omega_1(G) = \mathcal{A}_1(G) = \operatorname{span}\{e_{12}, e_{13}, \ldots, e_{1(n-1)}, e_{1n}, e_{2n}, e_{3n}, \ldots, e_{(n-1)n}\},\\ &\Omega_2(G) = \operatorname{span}\{e_{12n}, e_{13n}, \ldots, e_{1(n-1)n}\},\\ &\dim\Omega_1(G) = 2(n-2)+1, \quad \dim\Omega_2(G) = n-2 \end{aligned}$$

and

$$A_{(2(n-2)+1)\times(n-2)}=\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0\\ 0 & 1 & 0 & \cdots & 0 & 0\\ 0 & 0 & 1 & \cdots & 0 & 0\\ & \cdots & & \cdots & & \\ 0 & 0 & 0 & \cdots & 0 & 1\\ -1 & -1 & -1 & \cdots & -1 & -1\\ 1 & 0 & 0 & \cdots & 0 & 0\\ 0 & 1 & 0 & \cdots & 0 & 0\\ 0 & 0 & 1 & \cdots & 0 & 0\\ & \cdots & & \cdots & & \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}.$$

Hence, $R(A) = n - 2$, and A is a full-column rank matrix. □

Remark 1. In fact, by [1], for digraph G discussed above, $\dim H_0(G) = 1$, $\dim H_1(G) = 0$, and $\dim H_p(G) = 0$ $(p \geq 2)$. Let f be a self-map on G (a digraph map which maps G to G). Then, the Lefschetz number $\Lambda(f) = \operatorname{trace} f|_{H_0} - \operatorname{trace} f|_{H_1} + \operatorname{trace} f|_{H_2} = 1 \neq 0$. Therefore, similar to [14], f has a fixed point.

Second, by [12], we have the following lemma.

Lemma 2.2. (cf. [12]). *Let f be a strictly convex function. If $f(Ax-b)+|x|_1$ is constant on a convex set S, then both $Ax-b$ and $|x|_1$ are constant on S.*

3 Proof of the main theorem

In this section, we prove the existence and uniqueness of solutions to the minimization problem.

Proof of Theorem 1.1. Step 1. Existence.

Case 1. G is generated by squares that belong to the same cluster. Then, for any given one-dimensional closed path $w \in H_1(G)$,

$$w = l_1e_{12} + l_2e_{13} + \cdots + l_{n-2}e_{1(n-1)} + l_1'e_{2n} + \cdots + l_{n-2}'e_{(n-1)n},$$

where $l_i, l_i' \in \mathbb{K}$, $1 \le i \le n-2$.

Since

$$0 = \partial w = -(l_1 + l_2 + \cdots + l_{n-2})e_1 + (l_1 - l_1')e_2 + \cdots + (l_{n-2} - l_{n-2}')e_{n-1} + (l_1' + \cdots + l_{n-2}')e_n,$$

it follows that

$$\begin{cases} l_1 + l_2 + \cdots + l_{n-2} = 0 \\ l_1 = l_1' \\ l_2 = l_2' \\ \cdots \\ l_{n-2} = l_{n-2}' \\ l_1' + l_2' + \cdots + l_{n-2}' = 0 \end{cases}$$

and

$$w = \begin{pmatrix} l_1 \\ l_2 \\ \vdots \\ -(l_1 + l_2 + \cdots + l_{n-3}) \\ l_1 \\ l_2 \\ \vdots \\ -(l_1 + l_2 + \cdots + l_{n-3}) \end{pmatrix}.$$

Let

$$0 \ne u = \operatorname{span}\{e_{12n} - e_{13n}, e_{13n} - e_{14n}, \ldots, e_{1(n-2)n} - e_{1(n-1)n}\} \in \Omega_2,$$
$$u = x_1(e_{12n} - e_{13n}) + x_2(e_{13n} - e_{14n}) + \cdots + x_{n-3}(e_{1(n-2)n} - e_{1(n-1)n}),\ x_i \in \mathbb{K}$$

and

$$\begin{aligned} f_1(x_1, x_2, \ldots, x_{n-3}) &= \frac{1}{2}\|\partial u - w\|_2^2 + |u|_1 \\ &= \frac{1}{2}\|\mathrm{Au} - w\|_2^2 + |u|_1 \\ &= (x_1 - l_1)^2 + (x_2 - x_1 - l_2)^2 + \cdots + (x_{n-3} - x_{n-4} - l_{n-3})^2 + (l_1 + l_2 + \cdots + l_{n-3} - x_{n-3})^2 + \sum_{i=1}^{n-3}|x_i| \end{aligned}$$

$$= \sum_{i=1}^{n-3} 2x_i^2 - 2\sum_{i=1}^{n-4} x_i x_{i+1} + \sum_{i=1}^{n-3} l_i^2 + \left(\sum_{i=1}^{n-3} l_i\right)^2 + \sum_{i=1}^{n-3} |x_i|$$
$$+ 2\sum_{i=1}^{n-4} (l_{i+1} - l_i)x_i - 2(l_1 + l_2 + \cdots + l_{n-4} + 2l_{n-3})x_{n-3}.$$

Then, the Hessian matrix of f_1 at any point $(x_1, x_2, \ldots, x_{n-3}) \in \mathbb{K}^{n-3}$ is given as follows:

$$(H_1)_{(n-3)\times(n-3)} = \begin{bmatrix} 4 & -2 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -2 & 4 & -2 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -2 & 4 & -2 & \cdots & 0 & 0 & 0 \\ & \cdots & & & \cdots & & \cdots & \\ 0 & 0 & 0 & 0 & \cdots & 4 & -2 & 0 \\ 0 & 0 & 0 & 0 & \cdots & -2 & 4 & -2 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -2 & 4 \end{bmatrix}.$$

Obviously, H_1 is a positive definite matrix and $(h_{11})_{H_1} > 0$. Thus, f_1 has the minimum points. That is, there exists a solution to the problem $(*)$ for Case 1.

CASE 2. G is generated by triangles that belong to the same cluster. Then, for any given one-dimensional closed path $w \in H_1(G)$,

$$w = l_1 e_{12} + l_2 e_{13} + \cdots + l_{n-2} e_{1(n-1)} + l_{n-1} e_{1n} + l_1' e_{2n} + \cdots + l_{n-2}' e_{(n-1)n},$$

where $l_i, l_i' \in \mathbb{K}$, $1 \le i \le n-2$.

Since $\partial w = 0$, it follows that

$$\begin{cases} l_i = l_i', \; i = 1, 2, \ldots, n-2 \\ l_1 + l_2 + \cdots + l_{n-1} = 0 \end{cases}$$

and

$$w_{(2(n-2)+1)\times 1} = \begin{pmatrix} l_1 \\ l_2 \\ \vdots \\ l_{n-2} \\ -(l_1 + l_2 + \cdots + l_{n-2}) \\ l_1 \\ l_2 \\ \vdots \\ l_{n-2} \end{pmatrix}.$$

Let

$$0 \ne u = \operatorname{span}\{e_{12n}, e_{13n}, \ldots, e_{1(n-1)n}\} \in \Omega_2(G),$$
$$u = x_1 e_{12n} + x_2 e_{13n} + \cdots + x_{n-2} e_{1(n-1)n}, \; x_i \in \mathbb{K}$$

and

$$f_2(x_1, x_2, \ldots, x_{n-2}) = \frac{1}{2}\|\partial u - w\|_2^2 + |u|_1$$
$$= \frac{1}{2}\|\mathrm{A}u - w\|_2^2 + |u|_1$$
$$= (x_1 - l_1)^2 + (x_2 - l_2)^2 + \cdots + (x_{n-2} - l_{n-2})^2 + \sum_{i=1}^{n-2} |x_i| + \frac{1}{2}[x_1 + \cdots + x_{n-2} - (l_1 + \cdots + l_{n-2})]^2$$

$$= \sum_{i=1}^{n-2} \frac{3}{2} x_i^2 - 2\sum_{i=1}^{n-2} l_i x_i - (l_1 + \cdots + l_{n-2}) \sum_{i=1}^{n-2} x_i + \sum_{i=1}^{n-2} |x_i| + \sum_{i=1}^{n-2} \frac{3}{2} l_i^2 + \sum_{1 \leqslant i \leqslant j \leqslant n-2} x_i x_j + \sum_{1 \leqslant i < j \leqslant n-2} l_i l_j.$$

Then, the Hessian matrix of f_2 is given as follows:

$$(H_2)_{(n-2)\times(n-2)} = \begin{bmatrix} 3 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 3 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 3 & 1 & \cdots & 1 \\ 1 & 1 & 1 & 3 & \cdots & 1 \\ & \cdots & & \cdots & & \\ 1 & 1 & 1 & 1 & \cdots & 3 \end{bmatrix}.$$

Similar to Case 1, we have that the matrix H_2 is also a positive definite matrix and $(h_{11})_{H_2} > 0$. Hence, f_2 has the minimum points, and there exists a solution to the problem $(*)$ for Case 2.

Step 2. Uniqueness. For any given w, by Step 1, the set of solutions of problem $(*)$ is not empty. Since $f = \frac{1}{2} \|\cdot\|_2^2$ is a strictly convex function, the problem $(*)$ is a convex problem. Hence, by Lemma 2.2, Au $- w =$ Constant.

On the other hand, by Lemma 2.1, the matrix A of $\partial_2 : \Omega_2(G) \to \Omega_1(G)$ is a full-column rank matrix. Thus, if $Au = Au'$, then $u = u'$. That is, the solution to the problem $(*)$ is unique.

Step 3. We will prove the property of the solution of $(*)$ in Theorem 1.1 by solving linear equations.

For Case 1 of Step 1, by the structural characteristics of f_1, it is sufficient to consider the following cases.

(1) Each $x_i \geqslant 0$ $(i = 1, 2, \ldots, n-3)$. Then,

$$\begin{cases} f_{x_1} = 4x_1 - 2x_2 + 2(l_2 - l_1) + 1 = 0 \\ f_{x_2} = 4x_2 - 2x_1 - 2x_3 + 2(l_3 - l_2) + 1 = 0 \\ f_{x_3} = 4x_3 - 2x_2 - 2x_4 + 2(l_4 - l_3) + 1 = 0 \\ \cdots\cdots \\ f_{x_{n-4}} = 4x_{n-4} - 2x_{n-5} - 2x_{n-3} + 2(l_{n-3} - l_{n-4}) + 1 = 0 \\ f_{x_{n-3}} = 4x_{n-3} - 2x_{n-4} - 2(l_1 + l_2 + \cdots + l_{n-4} + 2l_{n-3}) + 1 = 0. \end{cases}$$

By the first equation, $2x_2 = 4x_1 + 2(l_2 - l_1) + 1$. Substituting it into the second equation up to the $(n-3)$-th equation, we have that

$$\begin{cases} 2x_3 = 6x_1 - 4l_1 + 2l_2 + 2l_3 + 3, \\ 2x_4 = 8x_1 - 6l_1 + 2(l_2 + l_3 + l_4) + 6, \\ 2x_5 = 10x_1 - 8l_1 + 2(l_2 + \cdots + l_5) + 10, \\ \cdots \\ 2x_i = 2ix_1 - 2(i-1)l_1 + 2(l_2 + \cdots + l_i) + \dfrac{i(i-1)}{2}, \ 2 \leqslant i \leqslant n-3. \end{cases}$$

Hence,

$$\begin{cases} x_1 = l_1 - \dfrac{n-3}{4}, \\ x_2 = l_1 + l_2 - \dfrac{n-4}{2}, \\ x_3 = l_1 + l_2 + l_3 - \dfrac{3n-5}{4}, \\ x_4 = l_1 + l_2 + l_3 + l_4 - \dfrac{4n-6}{4}, \\ \cdots\cdots \\ x_i = \displaystyle\sum_{k=1}^{i} l_k - \dfrac{i(n-i-2)}{4}, \ i = 1, \ldots, n-3. \end{cases}$$

Therefore,

$$u=\begin{pmatrix} x_1\\ x_2\\ \vdots\\ x_i\\ \vdots\\ x_{n-3}\end{pmatrix}=\begin{pmatrix} l_1-\dfrac{n-3}{4}\\ l_1+l_2-\dfrac{n-4}{2}\\ \vdots\\ l_1+l_2+\cdots+l_i-\dfrac{i(n-i-2)}{4}\\ \vdots\\ l_1+l_2+\cdots+l_{n-3}-\dfrac{n-3}{4}\end{pmatrix}.$$

(2) $x_1\leqslant 0$ and $x_i\geqslant 0$ $(i=2,3,\ldots,n-3)$. Then,

$$u=\begin{pmatrix} x_1\\ x_2\\ \vdots\\ x_i\\ \vdots\\ x_{n-3}\end{pmatrix}=\begin{pmatrix} l_1-\dfrac{(n-3)(n-6)}{4(n-2)}\\ l_1+l_2-\dfrac{(n-4)^2}{2(n-2)}\\ \vdots\\ l_1+l_2+\cdots+l_i-\dfrac{(n-i-2)(in-2i-4)}{4(n-2)}\\ \vdots\\ l_1+l_2+\cdots+l_{n-3}-\dfrac{n^2-5n+2}{4(n-2)}\end{pmatrix}.$$

(3) $x_2\leqslant 0$ and $x_i\geqslant 0$ $(i=1,3,\ldots,n-3)$. Then,

$$u=\begin{pmatrix} x_1\\ x_2\\ x_3\\ \vdots\\ x_{n-4}\\ x_{n-3}\end{pmatrix}=\begin{pmatrix} l_1-\dfrac{n^2-9n+20}{2(2n-4)}\\ l_1+l_2-\dfrac{2n^2-20n+44}{2(2n-4)}\\ l_1+l_2+l_3-\dfrac{3n^2-29n+64}{2(2n-4)}\\ \vdots\\ l_1+l_2+\cdots+l_{n-4}-\dfrac{2n^2-14n+8}{2(2n-4)}\\ l_1+l_2+\cdots+l_{n-3}-\dfrac{n^2-7n+4}{2(2n-4)}\end{pmatrix},$$

$$x_j=\sum_{k=1}^{j}l_k-\frac{n^2j+(2-n)j^2-(4j+8)n+10j+16}{2(2n-4)},\quad j\geqslant 2.$$

(4) $x_{n-3}\leqslant 0$ and $x_i\geqslant 0$ $(i=1,2,\ldots,n-4)$. Then,

$$u=\begin{pmatrix} x_1\\ x_2\\ x_3\\ \vdots\\ x_{n-4}\\ x_{n-3}\end{pmatrix}=\begin{pmatrix} l_1-\dfrac{n^2-5n+2}{2(2n-4)}\\ l_1+l_2-\dfrac{2n^2-12n+8}{2(2n-4)}\\ l_1+l_2+l_3-\dfrac{3n^2-21n+18}{2(2n-4)}\\ \vdots\\ l_1+l_2+\cdots+l_{n-4}-\dfrac{2n^2-16n+32}{2(2n-4)}\\ l_1+l_2+\cdots+l_{n-3}-\dfrac{n^2-9n+18}{2(2n-4)}\end{pmatrix},$$

$$x_j = \sum_{k=1}^{j} l_k - \frac{j[n^2 - (j+4)n + 2j]}{2(2n-4)}, \quad j \geqslant 1.$$

Hence, by calculation, we have that $(A^T)_I(w - \mathrm{Au}) = \mathrm{sign}(u_I)$, where $I \coloneqq \mathrm{supp}(u)$ for all cases. Therefore, for any given w, by the uniqueness of solutions to the problem $(*)$, the unique solution u^* must be one of all possible cases satisfying $(A^T)_I(w - \mathrm{Au}^*) = \mathrm{sign}(u_I^*)$, where $I \coloneqq \mathrm{supp}(u^*)$.

For Case 2 of Step 1, consider the following cases.

(1) Each $x_i \geqslant 0 (i = 1, 2, \ldots, n-2)$. Then,

$$\begin{cases} f_{x_1} = 3x_1 - (3l_1 + l_2 + \cdots + l_{n-2}) + x_2 + x_3 + \cdots + x_{n-2} + 1 = 0 \\ f_{x_2} = 3x_2 - (l_1 + 3l_2 + l_3 + \cdots + l_{n-2}) + x_1 + x_3 + \cdots + x_{n-2} + 1 = 0 \\ f_{x_3} = 3x_3 - (l_1 + l_2 + 3l_3 + l_4 + \cdots + l_{n-2}) + x_1 + x_2 + x_4 + \cdots + x_{n-2} + 1 = 0 \\ \cdots\cdots \\ f_{x_{n-2}} = 3x_{n-2} - (l_1 + l_2 + \cdots + l_{n-3} + 3l_{n-2}) + x_1 + x_2 + \cdots + x_{n-3} + 1 = 0. \end{cases}$$

Hence,

$$u = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-2} \end{pmatrix} = \begin{pmatrix} l_1 - \frac{1}{n} \\ l_2 - \frac{1}{n} \\ \vdots \\ l_{n-2} - \frac{1}{n} \end{pmatrix}.$$

(2) Some $x_i \leqslant 0$. Without loss of generality, $x_1 \leqslant 0$ and $x_i \geqslant 0$ $(i = 2, 3, \ldots, n-3)$. Then,

$$u = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-2} \end{pmatrix} = \begin{pmatrix} l_1 + \frac{n-2}{n} \\ l_2 - \frac{2}{n} \\ l_3 - \frac{2}{n} \\ \vdots \\ l_{n-2} - \frac{2}{n} \end{pmatrix}.$$

Then, we also have that the unique solution u^* of the problem $(*)$ satisfies $(A^T)_I(w - \mathrm{Au}^*) = \mathrm{sign}(u_I^*)$, where $I \coloneqq \mathrm{supp}(u^*)$.

Therefore, Theorem 1.1 is proved. □

Remark 2. The conclusion of Theorem 1.1 is independent of the selection of the basis of $\Omega_2(G)$.

Furthermore, by the partitioned matrix, we have the following corollary.

Corollary 3.1. *Let $G = (V, E)$ be a digraph generated by clusters satisfying the following conditions*:

(1) *each cluster is composed of different squares or triangles*;

(2) *different clusters intersect at most at one vertex*.

Then the matrix A of the boundary operator $\partial_2 : \Omega_2(G) \to \Omega_1(G)$ is a full column rank matrix and there is a unique solution u^* to the problem $(*)$ satisfying $(A^T)_I(w - \mathrm{Au}^*) = \mathrm{sign}(u_I^*)$ where $I \coloneqq \mathrm{supp}(u^*)$.

4 Examples

In this section, we first show how coefficients and norms play an important role in the problem $(*)$.

Example 4.1. Let G be a digraph as follows (Figure 2). Then,

$$\Omega_2(G) = \text{span}\{e_{125} - e_{135}, e_{135} - e_{145}\}.$$

Let

$$w = \partial[4(e_{125} - e_{145})] = 4(e_{12} - e_{14} + e_{25} - e_{45})$$

be a one-dimensional closed path on G. Suppose

$$u = x_1(e_{125} - e_{135}) + x_2(e_{135} - e_{145}) \in \Omega_2(G).$$

Then,

$$\partial u - w = (x_1 - 4)e_{12} + (x_2 - x_1)e_{13} + (4 - x_2)e_{14} + (x_1 - 4)e_{25} + (x_2 - x_1)e_{35} + (4 - x_2)e_{45}.$$

Hence, $|\partial u - w|_0$ depends on whether the three formulas $(x_1 - 4)$, $(x_2 - x_1)$, and $(4 - x_2)$ are zero or not, and as long as two of the three formulas are zero, the third one must be zero. Thus, it is sufficient to consider the following cases:
(1) If $x_1 = 4, x_2 = 4, |\partial u - w|_0 + |u|_1 = 8$;
(2) If $x_1 = 4, x_2 \neq 4, |\partial u - w|_0 + |u|_1 = 4 + 4 + |x_2|$;
(3) If $x_2 = x_1, x_1 \neq 4, |\partial u - w|_0 + |u|_1 = 4 + 2|x_1|$;
(4) If $x_2 = 4, x_1 \neq 4, |\partial u - w|_0 + |u|_1 = 4 + 4 + |x_1|$;
(5) If $x_1 \neq 4, x_2 \neq 4, x_2 \neq x_1, |\partial u - w|_0 + |u|_1 = 6 + |x_1| + |x_2|$.

Therefore, there exists no non-zero solution of the problem

$$\min\{|\partial u - w|_0 + |u|_1\}$$

in $\mathbb{R}$ or $\mathbb{Q}$.

Consider another closed 1-path

$$w = \partial[\frac{1}{2}(e_{125} - e_{135}) + 4(e_{135} - e_{145})] = \frac{1}{2}(e_{12} + e_{25}) + \frac{7}{2}(e_{13} + e_{35}) - 4(e_{14} + e_{45}).$$

Then, the minimum points of $\min_{u\in\Omega_2(G),u\neq 0}\{|\partial u - w|_0 + |u|_1\}$ are not unique. Specifically,

$$\partial u - w = \left(x_1 - \frac{1}{2}\right)e_{12} + \left(x_2 - x_1 - \frac{7}{2}\right)e_{13} + (4 - x_2)e_{14} + \left(x_1 - \frac{1}{2}\right)e_{25} + \left(x_2 - x_1 - \frac{7}{2}\right)e_{35} + (4 - x_2)e_{45}.$$

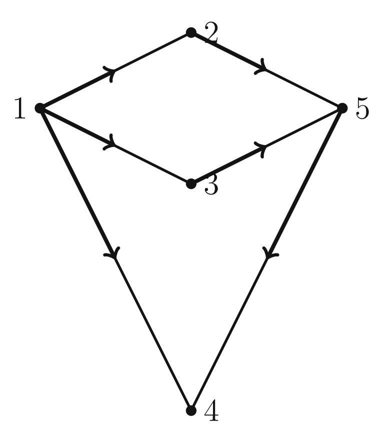

Figure 2: Example 4.1.

Similarly, consider the following cases:
(1) If $x_1 = \frac{1}{2}$, $x_2 = 4$, $|\partial u - w|_0 + |u|_1 = \frac{1}{2} + 4 = \frac{9}{2}$;
(2) If $x_1 = \frac{1}{2}$, $x_2 \neq 4$, $|\partial u - w|_0 + |u|_1 = 4 + \frac{1}{2} + |x_2| \geqslant \frac{9}{2}$;
(3) If $x_2 = 4$, $x_1 \neq \frac{1}{2}$, $|\partial u - w|_0 + |u|_1 = 4 + 4 + |x_1| \geqslant 8$;
(4) If $x_2 - x_1 = \frac{7}{2}$, $x_1 \neq \frac{1}{2}(x_2 \neq 4)$, $|\partial u - w|_0 + |u|_1 = 4 + |x_1| + \left|x_1 + \frac{7}{2}\right| \geqslant 4 + \frac{7}{2}$;
(5) If $x_1 \neq \frac{1}{2}$, $x_2 \neq 4$, $x_2 - x_1 \neq \frac{7}{2}$, $|\partial u - w|_0 + |u|_1 = 6 + |x_1| + |x_2| \geq 6$.

Hence, $\min f(x_1, x_2) = \frac{9}{2}$ when $u = \begin{pmatrix} \frac{1}{2} \\ 4 \end{pmatrix}$ or $\begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}$.

Next, for given digraphs, we try to find the "smallest" representative element in the homology class $H_1(G)$ with coefficients in any field $\mathbb{K}$ (in particular, $\mathbb{K} = \mathbb{Q}$ or $\mathbb{R}$).

Example 4.2. Let G be the digraph in Example 4.1. Then,

$$\Omega_2(G) = \text{pan}\{e_{125} - e_{135}, e_{135} - e_{145}\}.$$

For any element $u \in \Omega_2(G)$, it can be written as follows:

$$u = x_1(e_{125} - e_{135}) + x_2(e_{135} - e_{145}), \quad x_1, x_2 \in \mathbb{K}.$$

Since $\dim H_1(G) = 0$, it follows that

$$\begin{aligned} H_1(G) &= \{[0]\} \\ &= \{[(e_{12} + e_{25}) - (e_{14} + e_{45})]\} \\ &= \{[(e_{13} + e_{35}) - (e_{14} + e_{45})]\} \\ &= \{[(e_{12} + e_{25}) - (e_{13} + e_{35})]\}. \end{aligned}$$

Consider the following closed 1-paths.
(1) $w = (e_{12} + e_{25}) - (e_{13} + e_{35})$. Then,

$$f(x_1, x_2) = \frac{1}{2}\|\partial u - w\|_2^2 + |u|_1 = (x_1 - 1)^2 + (x_2 - x_1 + 1)^2 + x_2^2 + |x_1| + |x_2|.$$

Hence, when $u = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{3}{4} \\ 0 \end{pmatrix}$, $\|w\| = \min f(x_1, x_2) = \frac{7}{8}$.

(2) $w = (e_{13} + e_{35}) - (e_{14} + e_{45})$. Then,

$$f(x_1, x_2) = \frac{1}{2}\|\partial u - w\|_2^2 + |u|_1 = x_1^2 + (x_2 - x_1 - 1)^2 + (1 - x_2)^2 + |x_1| + |x_2|.$$

Hence, when $u = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{3}{4} \end{pmatrix}$, $\|w\| = \min f(x_1, x_2) = \frac{7}{8}$.

(3) $w = (e_{12} + e_{25}) - (e_{14} + e_{45})$. Then,

$$f(x_1, x_2) = \frac{1}{2}\|\partial u - w\|_2^2 + |u|_1 = (x_1 - 1)^2 + (x_2 - x_1)^2 + (1 - x_2)^2 + |x_1| + |x_2|.$$

Hence, when $u = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$, $\|w\| = \min f(x_1, x_2) = \frac{3}{2}$.

Remark 3. By Example 4.2, we know that in all intuitively visible one-dimensional closed paths

$$\begin{aligned} &(e_{12} + e_{25}) - (e_{14} + e_{45}) \\ &(e_{13} + e_{35}) - (e_{14} + e_{45}) \\ &(e_{12} + e_{25}) - (e_{13} + e_{35}), \end{aligned}$$

the paths that happen to be the boundaries of the elements in the basis of $\Omega_2(G)$ are the “smallest” representative elements for digraphs generated by squares that belong to the same cluster.

In the following example, we illustrate that for digraph G whose $\Omega_2(G)$ is generated by triangles that belong to the same cluster, Remark 3 still holds.

Example 4.3. Without loss of generality, take the following digraph G as an example (Figure 3). Let

$$\Omega_2(G) = \operatorname{span}\{e_{125}, e_{135}, e_{145}\}$$
$$u = x_1e_{125} + x_2e_{135} + x_3e_{145}, x_1, x_2, x_3 \in \mathbb{K}.$$

Consider the following cases.

(1) $w = e_{12} + e_{25} - e_{15} = \partial e_{125}$. Then,

$$\begin{aligned} f(x_1, x_2, x_3) &= \frac{1}{2}\|\partial u - w\|_2^2 + |u|_1 \\ &= \frac{3}{2}x_1^2 + \frac{3}{2}x_2^2 + \frac{3}{2}x_3^2 + x_1x_2 + x_1x_3 + x_2x_3 - 3x_1 - x_2 - x_3 + \frac{3}{2} + |x_1| + |x_2| + |x_3| \end{aligned}$$

and

$$u = \begin{pmatrix} \frac{2}{3} \\ 0 \\ 0 \end{pmatrix}, \quad \|w\| = \min f(x_1, x_2, x_3) = \frac{5}{6}.$$

(2) $w = (e_{12} + e_{25}) - (e_{13} + e_{35}) = \partial(e_{125} - e_{135})$. Then,

$$\begin{aligned} f(x_1, x_2, x_3) &= \frac{1}{2}\|\partial u - w\|_2^2 + |u|_1 \\ &= \frac{3}{2}x_1^2 + \frac{3}{2}x_2^2 + \frac{3}{2}x_3^2 + x_1x_2 + x_1x_3 + x_2x_3 - 2x_1 + 2x_2 + 2 + |x_1| + |x_2| + |x_3| \end{aligned}$$

and

$$u = \begin{pmatrix} \frac{1}{2} \\ \frac{-1}{2} \\ 0 \end{pmatrix}, \quad \|w\| = \min f(x_1, x_2, x_3) = \frac{3}{2}.$$

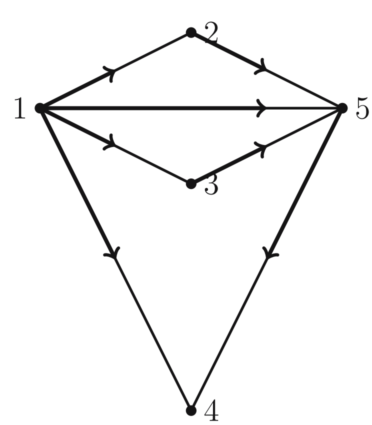

Figure 3: Example 4.3.

Then, we have that $w = e_{12} + e_{25} - e_{15} = \partial e_{125}$ is the boundary of one basis element of $\Omega_2(G)$, and its norm is the smallest in all intuitively visible one-dimensional elementary closed paths ($\|\partial e_{125}\| < \|\partial(e_{125} - e_{135})\|$).

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