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Research Article

Genhong Zhong, Xiaoyan Ma*, and Fei Wang

Approximations related to the complete p-elliptic integrals

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Abstract: In this paper, the authors present some monotonicity properties for certain functions involving the complete p-elliptic integrals of the first and second kinds, by showing the monotonicity and concavity-convexity properties of certain combinations defined in terms of $\mathscr{K}_p$, $\mathscr{E}_p$ and the inverse hyperbolic tangent arth_p, which is of importance in the computation of the generalized pi and in the elementary proof of Ramanujan's cubic transformation. By these results, several well-known results for the classical complete elliptic integrals including its bounds and logarithmic inequalities are remarkably improved.

Keywords: complete p-elliptic integrals, monotonicity, inequality, bounds inequalities, logarithmic inequalities

MSC 2020: 33C75, 33E05, 33F05

1 Introduction

Throughout this paper, we denote the set of positive integers (the inverse hyperbolic tangent) by $\mathbb{N}$ (arth, respectively), and let $r'^p + r^p = 1$ for each $r \in [0, 1]$ and $p \in (1, \infty)$. For $r \in (0, 1)$, *the complete elliptic integrals of the first and second kinds* are defined as follows:

$$\begin{cases} \mathscr{K} = \mathscr{K}(r) = \displaystyle\int_0^{\pi/2} (1 - r^2 \sin^2 t)^{-1/2}\,\mathrm{d}t, \\ \mathscr{K}' = \mathscr{K}'(r) = \mathscr{K}(r'), \\ \mathscr{K}(0) = \pi/2,\ \mathscr{K}(1) = \infty, \quad \text{and} \end{cases} \tag{1.1}$$

$$\begin{cases} \mathscr{E} = \mathscr{E}(r) = \displaystyle\int_0^{\pi/2} (1 - r^2 \sin^2 t)^{1/2}\,\mathrm{d}t, \\ \mathscr{E}' = \mathscr{E}'(r) = \mathscr{E}(r'), \\ \mathscr{E}(0) = \pi/2,\ \mathscr{E}(1) = 1, \end{cases} \tag{1.2}$$

respectively, which are the particular cases of the Gaussian hypergeometric function:

$$F(a, b, c, x) = {}_2F_1(a, b; c; x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} x^n \quad (|x| < 1) \tag{1.3}$$

* **Corresponding author: Xiaoyan Ma,** Department of Mathematics, Zhejiang Sci-Tech University, Hangzhou 310018, China, e-mail: mxy@zstu.edu.cn
Genhong Zhong: Keyi College of Zhejiang Sci-Tech University, Shaoxing 312300, China, e-mail: genhongz@126.com
Fei Wang: Zhejiang Institute of Mechanical and Electrical Engineering, Hangzhou 310053, China, e-mail: wf509529@163.com

for $a, b, c \in \mathbb{C}$ with $c \neq 0, -1, -2, \ldots$, where $(a)_n = a(a+1)(a+2)\cdots(a+n-1)$ for $n \in \mathbb{N}$, and $(a)_0 = 1$ for $a \neq 0$. As a matter of fact, it is well known that

$$\mathscr{K}(r) = \frac{\pi}{2}F\left(\frac{1}{2}, \frac{1}{2}; 1; r^2\right) \quad \text{and} \quad \mathscr{E}(r) = \frac{\pi}{2}F\left(-\frac{1}{2}, \frac{1}{2}; 1; r^2\right) \tag{1.4}$$

[1, 17.3.9–17.3.10]. For the properties of the complete elliptic integrals, the readers can refer to [2–21] and the bibliographies therein.

As far as the complete elliptic integrals of the first and second kinds are concerned, there are kinds of bounds for them in terms of the inverse hyperbolic tangent function [1, 15.1.4], that is,

$$\frac{\text{arth}r}{r} = F\left(\frac{1}{2}, 1; \frac{3}{2}; r^2\right) = \sum_{n=0}^{\infty} \frac{1}{2n+1} r^{2n}. \tag{1.5}$$

In 1992, Anderson et al. mentioned such bound, and they presented the double inequality [7, Theorem 3.10]

$$\left(\frac{\text{arth}r}{r}\right)^{1/2} < \frac{2}{\pi}\mathscr{K}(r) < \frac{\text{arth}r}{r} \tag{1.6}$$

for $r \in (0, 1)$. This was improved by Alzer and Qiu in [4, Theorem 18] as

$$\left(\frac{\text{arth}r}{r}\right)^{\alpha_1} < \frac{2}{\pi}\mathscr{K}(r) < \left(\frac{\text{arth}r}{r}\right)^{\beta_1} \tag{1.7}$$

for $r \in (0, 1)$ with the best possible constants $\alpha_1 = 3/4$ and $\beta_1 = 1$. In [15, Theorem 1.2], Wang et al. obtained the double inequality

$$\frac{\pi}{2} - \frac{3\pi}{16}\frac{r - (1-r^2)\text{arth}r}{r} < \mathscr{E}(r) < \frac{\pi}{2} - \left(\frac{\pi}{2} - 1\right)\frac{r - (1-r^2)\text{arth}r}{r} \tag{1.8}$$

holds for all $r \in (0, 1)$.

In 2014, Takeuchi [22] introduced a form of the generalized complete elliptic integrals as an application of generalized trigonometric functions. *The complete p-elliptic integrals of the first and second kinds* are, respectively, defined as follows: for $p \in (1, \infty)$ and $r \in [0, 1)$,

$$\mathscr{K}_p = \mathscr{K}_p(r) = \int_0^{\pi_p/2} \frac{\mathrm{d}\theta}{(1 - r^p \sin_p^p\theta)^{1-1/p}} = \frac{\pi_p}{2}F\left(\frac{1}{p}, 1 - \frac{1}{p}; 1; r^p\right), \mathscr{K}_p(0) = \frac{\pi_p}{2}, \mathscr{K}_p(1^-) = \infty \tag{1.9}$$

and

$$\mathscr{E}_p = \mathscr{E}_p(r) = \int_0^{\pi_p/2} (1 - r^p \sin_p^p\theta)^{1/p}\,\mathrm{d}\theta = \frac{\pi_p}{2}F\left(\frac{1}{p}, -\frac{1}{p}; 1; r^p\right), \mathscr{E}_p(0) = \frac{\pi_p}{2}, \mathscr{E}_p(1^-) = 1, \tag{1.10}$$

where $\sin_p\theta$ is *the generalized sine function*, defined by the inverse function of

$$\arcsin_p(\theta) = \int_0^{\theta} \frac{1}{(1-t^p)^{1/p}}\mathrm{d}t, \ 0 \le \theta \le 1,$$

$$\frac{\pi_p}{2} = \arcsin_p(1) = \int_0^1 \frac{1}{(1-t^p)^{1/p}}\mathrm{d}t = \frac{\pi/p}{\sin(\pi/p)}.$$

In [23], it was obtained that, for $p \in (1, \infty)$,

$$\frac{\text{arth}_p r}{r} < \mathscr{K}_p(r) < \frac{\pi_p}{2}\frac{\text{arth}_p r}{r}, \tag{1.11}$$

where

$$\frac{\mathrm{arth}_p r}{r} = F\left(\frac{1}{p}, 1; \frac{1}{p} + 1; r^p\right) = \sum_{n=0}^{\infty} \frac{1}{pn+1} r^{pn} \tag{1.12}$$

is *the generalized inverse hyperbolic tangent function.*

Recently, Wang and Qi [24, Theorem 1.3] obtained the bounds

$$\frac{\pi_{p,q}}{2}\frac{\mathrm{arth}_q r}{r}(1-\alpha_3 r^q) < \mathscr{K}_{p,q}(r) < \frac{\pi_{p,q}}{2}\frac{\mathrm{arth}_q r}{r}(1-\beta_3 r^q) \tag{1.13}$$

for $r \in (0,1)$, $p, q \in (1,\infty)$ with the best weights $\alpha_3 = 1 - 2/\pi_{p,q}$ and $\beta_3 = 1/[pq(q+1)]$, where

$$\pi_{p,q} = 2\int_0^1 \frac{1}{(1-t^q)^{1/p}}\mathrm{d}t.$$

When $p = q$, $\mathscr{K}_{p,q}(r) = \mathscr{K}(r)$, $\mathrm{arth}_q(r) = \mathrm{arth}_p r$, $\pi_{p,q} = \pi_p$. And as we know, for $p = 2$, these functions reduce to well-known special cases $\mathscr{K}_p(r) = \mathscr{K}(r)$, $\mathscr{E}_p(r) = \mathscr{E}(r)$, $\mathrm{arth}_p(r) = \mathrm{arth} r$, $\pi_p = \pi$, and numerous properties of the complete p-elliptic integrals have been obtained (cf. [22–29]) and bibliographies therein.

Inspired by the inequalities (1.7), (1.8), (1.11), and (1.13), the topic of studying the properties of the complete p-elliptic integrals is to approximate $\mathscr{K}_p(r)$ and $\mathscr{E}_p(r)$ by means of the function $r \mapsto (\mathrm{arth}_p r)/r$.

The main purpose of this paper is to present several new monotonicity properties of the complete p-elliptic integrals. By obtained results, some approximations of $\mathscr{K}_p(r)$ and $\mathscr{E}_p(r)$ by certain combinations in terms of $\mathrm{arth}_p(r)/r$ and polynomials are derived.

2 Preliminaries and proofs

In the sequel, we sometimes omit the variable r of the generalized and complete elliptic integrals when there is no confusion, and always let arth_p denote the generalized inverse hyperbolic tangent.

In order to prove our main results stated in Section 3, let us recall the following well-known formulas [22, Proposition 2.1]: For $r \in (0,1)$, $p \in (1,\infty)$,

$$\frac{\mathrm{d}\mathscr{K}_p}{\mathrm{d}r} = \frac{\mathscr{E}_p(r) - r'^p\mathscr{K}_p(r)}{rr'^p}, \quad \frac{\mathrm{d}\mathscr{E}_p}{\mathrm{d}r} = \frac{\mathscr{E}_p(r) - \mathscr{K}_p(r)}{r}, \tag{2.1}$$

$$\frac{\mathrm{d}}{\mathrm{d}r}(\mathscr{K}_p - \mathscr{E}_p) = \frac{r^{p-1}\mathscr{E}_p(r)}{r'^p}, \tag{2.2}$$

$$\frac{\mathrm{d}\,\mathrm{arth}_p}{\mathrm{d}r} = \frac{1}{1-r^p} = \frac{1}{r'^p}, \tag{2.3}$$

which will be frequently applied later.

Now we present several lemmas in this section.

Lemma 2.1. (See [8, Theorem 1.25]) *Let $-\infty < a < b < \infty$, $f, g : [a,b] \to R$ be continuous on $[a,b]$ and be differentiable on (a,b) such that $g'(x) \neq 0$ on (a,b). Then both the functions $[f(x) - f(a)]/[g(x) - g(a)]$ and $[f(x) - f(b)]/[g(x) - g(b)]$ are (strictly) increasing (decreasing) on (a,b) if $f'(x)/g'(x)$ is (strictly) increasing (decreasing) on (a,b).*

Lemma 2.2. (See [16, Theorem 2.1]) *Suppose that the power series $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and $g(x) = \sum_{n=0}^{\infty} b_n x^n$ have the radius of convergence $r > 0$ with $b_n > 0$ for all $n \in \{0,1,2,\ldots\}$. Let $h(x) = f(x)/g(x)$ and $H_{f,g} = (f'/g')g - f$, then the following statements are true:*

(1) *If the nonconstant sequence* $\{a_n/b_n\}|_{n=0}^{\infty}$ *is increasing (decreasing), then* $h(x)$ *is strictly increasing (decreasing) on* $(0, r)$;
(2) *If the nonconstant sequence* $\{a_n/b_n\}$ *is increasing (decreasing) for* $0 < n \le n_0$ *and decreasing (increasing) for* $n > n_0$, *then the function* h *is strictly increasing (decreasing) on* $(0, r)$ *if and only if* $H_{f,g}(r^-) \ge (\le) 0$. *While if* $H_{f,g}(r^-) < (>) 0$, *then exists* $\delta \in (0, r)$ *such that* $h(x)$ *is strictly increasing (decreasing) on* $(0, \delta)$ *and strictly decreasing (increasing) on* (δ, r).

Our first lemma presents several properties of the generalized inverse hyperbolic tangent function arth_p.

Lemma 2.3.
(1) *For* $p > 1$, *the function* $g_1(r) \equiv [r - r'^p \operatorname{arth}_p r]/r^{p+1}$ *is strictly increasing and convex from* $(0, 1)$ *onto* $(p/(p+1), 1)$.
(2) *For* $p > (3 + \sqrt{17})/4$, *the function* $g_2(r) \equiv [1 + (p-2)r^p]g_1(r)$ *is increasing and convex from* $(0, 1)$ *onto* $(p/(p+1), p-1)$.
(3) *For* $p \in ((3 + \sqrt{17})/4, 1 + \sqrt{2})$, *the function* $g_3(r) \equiv [r + (p-1)r'^p \operatorname{arth}_p r]/(rr')$ *is strictly increasing and convex from* $(0, 1)$ *onto* (p, ∞).
(4) *For* $p \in ((3 + \sqrt{17})/4, 1 + \sqrt{2})$, *the function* $g_4(r) \equiv (r - r'^p \operatorname{arth}_p r)/[r'(\operatorname{arth}_p r - r)]$ *is strictly increasing from* $(0, 1)$ *onto* (p, ∞).

Proof.
(1) By (1.12), $g_1(r)$ can be written as follows:

$$g_1(r) = \frac{1 - r'^p(\operatorname{arth}_p r/r)}{r^p} = p\sum_{n=0}^{\infty} \frac{r^{pn}}{(pn+1)(pn+p+1)}, \tag{2.4}$$

yielding the monotonicity and convexity for g_1 and the limiting value $g_1(0^+) = p/(p+1)$. Clearly, $g_1(1^-) = 1$.

(2) Clearly, $g_2(0) = p/(p+1)$, $g_2(1) = p - 1$. By (2.4), g_2 can be written as follows:

$$\begin{aligned} g_2(r) &= p\left[\sum_{n=0}^{\infty} \frac{r^{pn}}{(pn+1)(pn+p+1)} + (p-2)\sum_{n=0}^{\infty} \frac{r^{p(n+1)}}{(pn+1)(pn+p+1)}\right] \\ &= \frac{p}{p+1} + p\sum_{n=0}^{\infty} \frac{p(p-1)n + (2p^2 - 3p - 1)}{(pn+1)(pn+p+1)(pn+2p+n)} r^{p(n+1)} \\ &= \frac{p}{p+1} + p\sum_{n=0}^{\infty} \frac{f(n,p)}{(pn+1)(pn+p+1)(pn+2p+n)} r^{p(n+1)}, \end{aligned} \tag{2.5}$$

where $f(n, p) = p(p-1)n + (2p^2 - 3p - 1) = (n+2)p^2 - (n+3)p - 1$. It is easy to see that $f(n, p)$ is strictly increasing with respect to p. Then $f(n, p) > f(n, (3 + \sqrt{17})/4) = [(7 + \sqrt{17})n]/8 \ge 0$. Hence, the monotonicity and convexity properties of g_2 follow from (2.5).

(3) The limiting values of g_3 are clear. By (2.3) and by differentiation, we obtain

$$\begin{aligned} g_3'(r) &= \frac{[p - p(p-1)r^{p-1}\operatorname{arth}_p r]rr' - [r + (p-1)r'^p \operatorname{arth}_p r](1 - 2r^p)/r'^{(p-1)}}{(rr')^2} \\ &= \frac{r[pr'^p - 1 + 2r^p] - (p-1)r'^p \operatorname{arth}_p r[(p-2)r^p + 1]}{r^2 r'^{(p+1)}} \\ &= \frac{(p-1)(r - r'^p \operatorname{arth}_p r) + (2-p)r^p[r + (p-1)r'^p \operatorname{arth}_p r]}{r^2 r'^{(p+1)}} \\ &= \frac{r^{p-1}}{r'^{(p+1)}} \cdot \frac{r - r'^p \operatorname{arth}_p r}{r^{p+1}}\left[(p-1) + (2-p)r^p\left(1 + \frac{pr'^p \operatorname{arth}_p r}{r - r'^p \operatorname{arth}_p r}\right)\right] \end{aligned} \tag{2.6}$$

$$= \frac{r^{p-1}}{r'^{(p+1)}} \cdot \frac{r - r'^p \mathrm{arth}_p r}{r^{p+1}} \left[(p-1) + (2-p) r^p \left(1 - p + \frac{pr}{r - r'^p \mathrm{arth}_p r} \right) \right]$$
$$= \frac{r^{p-1}}{r'^{(p+1)}} \cdot \frac{r - r'^p \mathrm{arth}_p r}{r^{p+1}} \cdot \left\{ (p-1) + (2-p) \left[(1-p) r^p + \frac{p r^{p+1}}{r - r'^p \mathrm{arth}_p r} \right] \right\}$$
$$= \frac{r^{p-1}}{r'^{(p+1)}} \cdot \{(p-1)[1 + (p-2) r^p] g_1(r) + p(2-p)\}$$
$$= \frac{r^{p-1}}{r'^{(p+1)}} \cdot [(p-1) g_2(r) + p(2-p)]$$
$$= \frac{r^{p-1}}{r'^{(p+1)}} \cdot G_1(r),$$

where $G_1(r) = (p-1)g_2(r) + p(2-p)$. Hence, by the part (2), G_1 is increasing on (0,1) for $p > (3+\sqrt{17})/4$. On the other hand, it is easy to verify that

$$G_1(0) = \frac{p(-p^2 + 2p + 1)}{p+1} > 0$$

for $p \in ((3+\sqrt{17})/4, 1+\sqrt{2})$. Consequently, by (2.6), g_3' is strictly increasing on $(0,1)$ with $g_3'(0) = 0$, and hence, the monotonicity and convexity properties of g_3 follow from part (2).

(4) Let $G_2(r) = r/r' - r'^{(p-1)}\mathrm{arth}_p r$ and $G_3(r) = \mathrm{arth}_p r - r$. Then $G_2(0) = G_3(0) = 0$, $g_4(r) = G_2(r)/G_3(r)$ and

$$G_2'(r)/G_3'(r) = g_3(r). \tag{2.7}$$

Hence, the monotonicity of g_4 follows from Lemma 2.1 and part (3). Applying l'Hôpital's rule and (2.7), $g_4(0^+) = p$ and $g_4(1^-) = \infty$. □

Lemma 2.4. *Given $p \in [2, \infty)$, the function $h(r) \equiv \mathscr{E}_p/[r'^{(p-1)}\mathscr{K}_p] + (p-2)r'$ is strictly increasing and convex from $(0,1)$ onto $(p-1, \infty)$.*

Proof. Clearly, $h(0) = p-1$ and $h(1) = \infty$. Differentiation gives

$$h'(r) = \left\{ \frac{r'^{(p-1)}(\mathscr{E}_p - \mathscr{K}_p)\mathscr{K}_p}{r} - \mathscr{E}_p \left[\frac{\mathscr{E}_p - r'^p \mathscr{K}_p}{rr'} - (p-1)\frac{r^{p-1}}{r'}\mathscr{K}_p \right] \right\} \cdot [r'^{(p-1)}\mathscr{K}_p]^{-2} - (p-2)\left(\frac{r}{r'}\right)^{p-1}$$
$$= \left(\frac{r}{r'}\right)^{p-1} \cdot \frac{\mathscr{E}_p}{r'^p \mathscr{K}_p} \left\{ (p-1) - \left[\frac{(\mathscr{K}_p - \mathscr{E}_p) r'^p}{r^p \mathscr{E}_p} + \frac{\mathscr{E}_p - r'^p \mathscr{K}_p}{r^p \mathscr{K}_p} + (p-2)\frac{r'^p \mathscr{K}_p}{\mathscr{E}_p} \right] \right\} \tag{2.8}$$
$$= \left(\frac{r}{r'}\right)^{p-1} \cdot \frac{\mathscr{E}_p}{r'^p \mathscr{K}_p} [(p-1) - H_1(r)],$$

where

$$H_1(r) = \frac{(\mathscr{K}_p - \mathscr{E}_p) r'^p}{r^p \mathscr{E}_p} + \frac{\mathscr{E}_p - r'^p \mathscr{K}_p}{r^p \mathscr{K}_p} + (p-2)\frac{r'^p \mathscr{K}_p}{\mathscr{E}_p}.$$

By [23, Lemma 3.4(2),(3),(5)], for $p \ge 2$, we see that $H_1(r)$ is decreasing from (0,1) onto $(0, p-1)$. Hence, it follows from (2.8) that h' is the product of three positive and increasing functions. This yields the monotonicity and convexity properties of h. □

Lemma 2.5. *The function $j(x) = 1/\sin x - 1/x - 1/\pi$ is increasing from $(0, \pi)$ onto $(-1/\pi, \infty)$, and exists $x_0 = 1.451765874910260 \cdots \in (0, \pi)$ for $j(x_0) = 0$.*

Proof. By the Taylor expansion of $x/\sin x$ ([30, Equality (2.15)]), we obtain

$$j(x) = \sum_{k=1}^{\infty} \frac{2^{2k} - 2}{(2k)!} |B_{2k}| x^{2k-1} - 1/\pi = \frac{1}{6}x + \sum_{k=2}^{\infty} \frac{2^{2k} - 2}{(2k)!} |B_{2k}| x^{2k-1} - 1/\pi, \tag{2.9}$$

where B_n ($n \in \mathbb{N}_0, \mathbb{N}_0 = \mathbb{N} \cup \{0\}$) are the Bernoulli numbers, defined by

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}, \quad |t| < 2\pi.$$

By differentiation,

$$j'(x) = \frac{1}{6} + \sum_{k=2}^{\infty} \frac{2^{2k} - 2}{(2k)!} |B_{2k}|(2k-1)x^{2k-2}.$$

It is easy to see that for all $x \in (0, \pi)$, $j'(x) > 0$. Consequently, $j(x)$ is increasing on $(0, \pi)$. It follows from (2.9) that the limits $j(0) = -1/\pi$ and $j(\pi) = \infty$. By the mathematical software Maple, we compute that $j(x_0) = 0$. □

3 Main results

In this section, we reveal some monotonicity properties of $\mathscr{K}_p(r)$ and $\mathscr{E}_p(r)$, and some of which remarkably improve the related well-known results for them such as [4, Theorem 18] and [15, Theorem 1.2].

Theorem 3.1.

(1) *For $r \in (0, 1)$, $p \in [2, 1 + \sqrt{2}]$, the function*

$$f_1(r) \equiv \frac{\log(2\mathscr{K}_p(r)/\pi_p)}{\log(\mathrm{arth}_p r/r)}$$

is strictly increasing from $(0, 1)$ onto $(1 - 1/p^2, 1)$. In particular, for all $r \in (0, 1)$, $p \in [2, 1 + \sqrt{2}]$, the double inequality

$$\frac{\pi_p}{2}\left(\frac{\mathrm{arth}_p r}{r}\right)^{\alpha_1} < \mathscr{K}_p(r) < \frac{\pi_p}{2}\left(\frac{\mathrm{arth}_p r}{r}\right)^{\beta_1} \tag{3.1}$$

holds with the best possible constants $\alpha_1 = 1 - 1/p^2$ and $\beta_1 = 1$.

(2) *For $r \in (0, 1)$, $p \in (1, \infty)$, the function*

$$f_2(r) = \frac{r(2\mathscr{K}_p(r)/\pi_p - 1)}{\mathrm{arth}_p r - r}$$

is strictly decreasing from $(0, 1)$ onto $(2/\pi_p, 1 - 1/p^2)$. In particular, for all $r \in (0, 1)$, $p \in (1, \infty)$, the double inequality

$$1 - \alpha_2 + \alpha_2 \frac{\mathrm{arth}_p r}{r} < \frac{2}{\pi_p}\mathscr{K}_p < 1 - \beta_2 + \beta_2 \frac{\mathrm{arth}_p r}{r} \tag{3.2}$$

holds with the best possible constants $\alpha_2 = 2/\pi_p$ and $\beta_2 = 1 - 1/p^2$.

Proof. (1) Let

$$F_1(r) = \log\frac{2\mathscr{K}_p}{\pi_p}, \; F_2(r) = \log\frac{\mathrm{arth}_p r}{r},$$
$$F_3(r) = \mathscr{K}_p - \frac{\mathrm{arth}_p r}{r}\mathscr{E}_p, \; F_4(r) = \left(1 - r'^p \frac{\mathrm{arth}_p r}{r}\right)\mathscr{K}_p.$$

Then $F_1(0^+) = F_2(0^+) = F_3(0^+) = F_4(0^+) = 0$, $f_1(r) = F_1(r)/F_2(r)$, and by differentiation,

$$\frac{F_1'(r)}{F_2'(r)} = 1 - \frac{r\mathscr{K}_p - \mathscr{E}_p \mathrm{arth}_p r}{(r - r'^p \mathrm{arth}_p r)\mathscr{K}_p} = 1 - \frac{F_3(r)}{F_4(r)}, \tag{3.3}$$

$$\begin{aligned}\frac{F_3'(r)}{F_4'(r)} &= \left(\frac{\mathscr{E}_p - r'^p\mathscr{K}_p}{rr'^p} + \frac{\mathscr{K}_p - \mathscr{E}_p}{r^2}\mathrm{arth}_p r - \mathscr{E}_p\frac{r - r'^p\mathrm{arth}_p r}{r^2 r'^p}\right)\\
&\quad\times\left[\frac{(pr^p + 2r'^p)\mathrm{arth}_p r - 2r}{r^2}\mathscr{K}_p + \frac{(r - r'^p\mathrm{arth}_p r)\mathscr{E}_p}{r^2 r'^p}\right]^{-1}\\
&= [(\mathrm{arth}_p r - r)r'^p\mathscr{K}_p]\{(r - r'^p\mathrm{arth}_p r)\mathscr{E}_p + [(pr^p + 2r'^p)\mathrm{arth}_p r - 2r]r'^p\mathscr{K}_p\}^{-1}\\
&= \left\{p + \frac{r - r'^p\mathrm{arth}_p r}{\mathrm{arth}_p r - r}\cdot\frac{\mathscr{E}_p}{r'^p\mathscr{K}_p} + \frac{(pr^p + 2r'^p - p)\mathrm{arth}_p r + (p-2)r}{\mathrm{arth}_p r - r}\right\}^{-1}\\
&= \left\{p + \frac{r - r'^p\mathrm{arth}_p r}{\mathrm{arth}_p r - r}\cdot\frac{\mathscr{E}_p}{r'^p\mathscr{K}_p} + \frac{(2-p)r'^p\mathrm{arth}_p r + (p-2)r}{\mathrm{arth}_p r - r}\right\}^{-1}\\
&= [p + g_4(r)h(r)]^{-1},\end{aligned} \tag{3.4}$$

where g_4 and h are, respectively, defined as in Lemmas 2.3(4) and in 2.4. It follows from (3.4), Lemmas 2.3(4) and 2.4 that the function F_3'/F_4' is strictly decreasing on (0,1), and is F_3/F_4 by Lemma 2.1. Consequently, it follows from (3.3) and Lemma 2.1 that f_1 is strictly increasing on (0,1).

By Lemma 2.1 (l'Hôpital's rule), Lemma 2.3(4), Lemma 2.4, and by (3.3)–(3.4), we obtain the limiting values:

$$f_1(0^+) = 1 - \lim_{r\to 0^+}\frac{1}{p + g_4(r)h(r)} = 1 - \frac{1}{p^2} \quad\text{and}\quad f_1(1^-) = 1 - \lim_{r\to 1^-}\frac{1}{p + g_4(r)h(r)} = 1.$$

By the monotonicity of f_1, the double inequality (3.1) holds with $\alpha_1 = 1 - 1/p^2$ and $\beta_1 = 1$.

It is easy to see that for all $r \in (0,1)$ and $p \in [2, 1+\sqrt{2}]$,

$$\mathscr{K}_p(r) > \frac{\pi_p}{2}\left(\frac{\mathrm{arth}_p r}{r}\right)^{\alpha_1} \Leftrightarrow \alpha_1 \le \inf_{r\in(0,1)} f_1(r) = f_1(0^+) = 1 - 1/p^2$$

and

$$\mathscr{K}_p(r) < \frac{\pi_p}{2}\left(\frac{\mathrm{arth}_p r}{r}\right)^{\beta_1} \Leftrightarrow \beta_1 \ge \sup_{r\in(0,1)} f_1(r) = f_1(1^-) = 1.$$

Hence, the last assertion follows.

(2) Let $F_5(r) = 2\mathscr{K}_p(r)/\pi_p - 1, F_6(r) = \mathrm{arth}_p r/r - 1$. By using (1.9) and (1.12), we obtain

$$f_2(r) = \frac{F_5(r)}{F_6(r)} = \frac{\sum_{n=1}^{\infty}\frac{\left(\frac{1}{p},n\right)\left(1-\frac{1}{p},n\right)}{(n!)^2}r^{np}}{\sum_{n=1}^{\infty}\frac{1}{pn+1}r^{np}} = \frac{\sum_{n=1}^{\infty}a_n r^{np}}{\sum_{n=1}^{\infty}b_n r^{np}}, \tag{3.5}$$

where

$$a_n = \frac{\left(\frac{1}{p},n\right)\left(1-\frac{1}{p},n\right)}{(n!)^2}, \quad\text{and}\quad b_n = \frac{1}{pn+1}.$$

Let $c_n = a_n/b_n$, then

$$\frac{c_{n+1}}{c_n} = 1 - \frac{1}{p^2(n+1)^2} < 1.$$

We conclude from (3.5) that f_2 is strictly decreasing on (0,1).

It easy to see that $F_5(0) = 0$ and $F_6(0^+) = 0$. Applying (2.1), we have

$$\frac{F_5'(r)}{F_6'(r)} = \frac{2}{\pi_p}\frac{r(\mathscr{E}_p - r'^p\mathscr{K}_p)}{r - r'^p\mathrm{arth}_p r} = \frac{2}{\pi_p}\frac{(\mathscr{E}_p - r'^p\mathscr{K}_p)/r^p}{(r - r'^p\mathrm{arth}_p r)/r^{p+1}} = \frac{2}{\pi_p}\frac{F_7(r)}{g_1(r)},$$

where $F_7(r) = [\mathscr{E}_p - r'^p\mathscr{K}_p]/r^p$, $g_1(r)$ is in Lemma 2.3(1). By l'Hôpital's rule, Lemma 2.3(1), and [23, Lemma 3.4 (3)], we obtain the limiting values

$$f_2(0^+) = \lim_{r\to 0^+}\frac{F_5'(r)}{F_6'(r)} = \frac{2}{\pi_p}\lim_{r\to 0^+}\frac{F_7(r)}{g_1(r)} = 1 - \frac{1}{p^2} \text{ and}$$
$$f_2(1^-) = \lim_{r\to 1^-}\frac{F_5'(r)}{F_6'(r)} = \frac{2}{\pi_p}\lim_{r\to 1^-}\frac{F_7(r)}{g_1(r)} = \frac{2}{\pi_p}. \qquad \square$$

Theorem 3.2. *For $p \in (1, \infty)$, let*

$$p_0 = \frac{1}{6}(135 + 6\sqrt{249})^{1/3} + \frac{7}{2(135 + 6\sqrt{249})^{1/3}} + \frac{1}{2} \approx 2.092193586\cdots$$

be the unique solution to the equation $2p^3 - 3p^2 - 2p - 1 = 0$ and $p_* = 2.16398023116776\cdots$ be the unique solution to the equation $1/p + 1 - \pi/[p\sin(\pi/p)] = 0$, and the function f_3 is defined on $(0, 1)$ by

$$f_3(r) \equiv \frac{r(\pi_p/2 - \mathscr{E}_p(r))}{r - r'^p\operatorname{arth}_p r}.$$

Then the following statements are true:
(1) *If $p \in (1, p_0]$, then f_3 is strictly decreasing from $(0, 1)$ onto $(\pi_p/2 - 1, [(p+1)\pi_p]/(2p^3))$. In particular, for all $r \in (0, 1)$, $p \in (1, p_0]$, the double inequality*

$$\left(\frac{\pi_p}{2} - \alpha_3\right) + \alpha_3 r'^p\left(\frac{\operatorname{arth}_p r}{r}\right) < \mathscr{E}_p(r) < \left(\frac{\pi_p}{2} - \beta_3\right) + \beta_3 r'^p\left(\frac{\operatorname{arth}_p r}{r}\right) \tag{3.6}$$

holds with the best possible constants $\alpha_3 = [(p+1)\pi_p]/(2p^3)$ and $\beta_3 = \pi_p/2 - 1$.
(2) *If $p \in (p_0, p_*)$, then there exists unique $r_0 \in (0, 1)$ such that $f_3(r)$ is strictly increasing on $(0, r_0)$, and strictly decreasing on $(r_0, 1)$. Consequently, for $r \in (0, 1)$, one has*

$$\mathscr{E}_p(r) < \frac{\pi_p}{2} - \min\left\{\frac{\pi_p}{2} - 1, \frac{(p+1)\pi_p}{2p^3}\right\}\left[1 - r'^p\frac{\operatorname{arth}_p r}{r}\right]. \tag{3.7}$$

(3) *If $p \in (p_*, \infty)$, then f_3 is strictly increasing from $(0,1)$ onto $([(p+1)/\pi_p]/(2p^3), \pi_p/2 - 1)$, and the reverse inequality of (3.6) holds for all $r \in (0, 1)$.*

Proof. Clearly, $f_3(1) = \pi_p/2 - 1$. Let $F_8(r) = r[\pi_p/2 - \mathscr{E}_p(r)]$, $F_9(r) = r - r'^p\operatorname{arth}_p r$. By using the series expansion (1.10) and (1.12), we obtain

$$\begin{aligned} f_3(r) = \frac{F_8(r)}{F_9(r)} &= \frac{\pi_p/2 - \mathscr{E}_p(r)}{1 - r'^p\operatorname{arth}_p r/r} = -\frac{\pi_p}{2}\frac{\sum_{n=1}^{\infty}\frac{(1/p,n)(-1/p,n)}{(n!)^2}r^{np}}{1 - \sum_{n=0}^{\infty}\frac{1}{pn+1}r^{np} + \sum_{n=0}^{\infty}\frac{1}{pn+1}r^{(1+n)p}} \\ &= \frac{\pi_p}{2p}\frac{\sum_{n=1}^{\infty}\frac{(1/p,n)(1-1/p,n)}{(n!)^2(pn-1)}r^{np}}{\sum_{n=1}^{\infty}\frac{1}{(pn-p+1)(pn+1)}r^{np}} = \frac{\pi_p}{2p}\frac{\sum_{n=1}^{\infty}A_n r^{np}}{\sum_{n=1}^{\infty}B_n r^{np}} = \frac{\pi_p}{2p}\frac{\sum_{n=0}^{\infty}A_{n+1}r^{np}}{\sum_{n=0}^{\infty}B_{n+1}r^{np}}, \end{aligned} \tag{3.8}$$

where

$$A_n = \frac{(1/p,n)(1-1/p,n)}{(n!)^2(pn-1)}, \quad \text{and} \quad B_n = \frac{1}{(pn-p+1)(pn+1)}.$$

By (3.8), we obtain

$$f_3(0^+) = \lim_{r\to 0^+} f_3(r) = \frac{\pi_p}{2p}\frac{A_1}{B_1} = \frac{(p+1)\pi_p}{2p^3}. \tag{3.9}$$

Let $C_n = A_n/B_n$, then

$$\frac{C_{n+1}}{C_n}-1=\frac{p(p^2-2p-1)n+(p^3-p^2-p-1)}{p^2(n+1)^2(pn-p+1)}=\frac{pJ_1(p)n+J_2(p)}{p^2(n+1)^2(pn-p+1)}$$
$$=\frac{J(p,n)}{p^2(n+1)^2(pn-p+1)}, \tag{3.10}$$

where

$$J(p,n)=p(p^2-2p-1)n+(p^3-p^2-p-1),$$
$$J_1(p)=p^2-2p-1,\ J_2(p)=p^3-p^2-p-1.$$

Case 1. $p\in(1,p_0)$.

$$pJ_1(p)=p(p^2-2p-1)=p(p+\sqrt{2}-1)[p-(1+\sqrt{2})].$$

Clearly, $pJ_1(p)<0$ for $p<p_0<1+\sqrt{2}$, we obtain

$$J(p,n)<J(p,1)=pJ_1(p)+J_2(p)=2p^3-3p^2-2p-1. \tag{3.11}$$

By differentiation,

$$J'(p,1)=6(p^2-p-1/3)=6\left(p-\frac{3+\sqrt{21}}{6}\right)\left(p-\frac{3-\sqrt{21}}{6}\right),$$

then $J(p,1)$ is strictly decreasing from $(1,(3+\sqrt{21})/6)$ onto $(-4.28211277\cdots,-4)$ and strictly increasing from $((3+\sqrt{21})/6,p_0)$ onto $(-4.28211277\cdots,0)$, then $J(p,1)<0$. From (3.10) and (3.11), we obtain that $C_{n+1}/C_n<1$ for $n\geq1$, and thereby C_n is strictly decreasing with respect to n. With an application of Lemma 2.2(1) and (3.8), the monotonicity of f_3 on $(0,1)$ in this case is expressed as follows.

Case 2. $p\in(p_0,\infty)$.

By using (2.1) and (2.3), we obtain

$$\frac{F_8'(r)}{F_9'(r)}=\frac{(\pi_p/2-\mathscr{E}_p)+(\mathscr{K}_p-\mathscr{E}_p)}{pr^{p-1}\text{arth}_p r}, \tag{3.12}$$

and thus,

$$H_{F_8,F_9}(r)=\frac{F_8'}{F_9'}F_9-F_8=\frac{(\pi_p/2-\mathscr{E}_p)+(\mathscr{K}_p-\mathscr{E}_p)}{pr^{p-1}\text{arth}_p r}[r-r'^p\text{arth}_p r]-r\left(\frac{\pi_p}{2}-\mathscr{E}_P\right)$$
$$=\left[\frac{r^2(\pi_p/2-2\mathscr{E}_p)}{p\text{arth}_p r}+\frac{r}{p}\frac{r\mathscr{K}_p}{\text{arth}_p r}\right]\cdot\frac{r-r'^p\text{arth}_p r}{r^{p+1}}-r\left(\frac{\pi_p}{2}-\mathscr{E}_P\right) \tag{3.13}$$
$$=\left[\frac{r^2(\pi_p/2-2\mathscr{E}_p)}{p\text{arth}_p r}+\frac{r}{p}\frac{r\mathscr{K}_p}{\text{arth}_p r}\right]\cdot g_1(r)+r\left(\mathscr{E}_P-\frac{\pi_p}{2}\right).$$

It is not difficult to verify that

$$\lim_{r\to1}\frac{r\mathscr{K}_p(r)}{\text{arth}_p r}=1,$$

and thus

$$H_{F_8,F_9}(1^-)=\lim_{r\to1^-}H_{F_8,F_9}(r)=\frac{1}{p}+1-\frac{\pi_p}{2}=\frac{1}{p}+1-\frac{\pi}{p\sin(\pi/p)}=-\frac{\pi}{p}\cdot j\left(\frac{\pi}{p}\right), \tag{3.14}$$

where j is in Lemma 2.5.

Then there exists unique zero point $p_*=2.16398023116776\cdots\in(1,\infty)$ such that $H_{F_8,F_9}(1^-)=0$, $H_{F_8,F_9}(1^-)<0$ for (p_0,p_*) and $H_{F_8,F_9}(1^-)>0$ for (p_*,∞).

(i). $p \in (p_0, p_*)$.

Then (3.10) implies that there exists $n_0 > 1$ such that sequence A_n/B_n is increasing for $1 \le n \le n_0$ and decreasing for $n > n_0$, for the limiting value of $H_{F_8,F_9}(r)$ at 1. It follows from (3.14) and Lemma 2.2(2) that $H_{F_8,F_9}(1^-) < 0$ for $p \in (p_0, p_*)$, so $f_3(r)$ is piecewise monotone on (0,1), and therefore, the inequality (3.7) follows from

$$f_3(r) > \min\left\{\frac{\pi_p}{2} - 1, \frac{(p+1)\pi_p}{2p^3}\right\}.$$

(ii). $p \in (p_*, \infty)$

Applying Lemma 2.2 (2) and (3.14), $f_3(r)$ is strictly increasing from $(0, 1)$ onto $([(p+1)\pi_p]/(2p^3), \pi_p/2 - 1)$ if and only if $p \ge p_*$, yielding the third assertion. □

Remark 3.3.

(1) When $p = 2$, the inequality (3.1) in Theorem 3.1 reduces to the inequality (1.7).

(2) By using (1.12), we clearly see that

$$\frac{\mathrm{arth}_p r}{r} = 1 + \sum_{n=1}^{\infty} \frac{r^{pn}}{pn+1} > 1, \tag{3.15}$$

$$\frac{r'^p \mathrm{arth}_p r}{r} = (1 - r^p)\sum_{n=0}^{\infty} \frac{r^{pn}}{pn+1} = 1 + \sum_{n=1}^{\infty} \frac{r^{pn}}{pn+1} - \sum_{n=0}^{\infty} \frac{r^{p(n+1)}}{pn+1} = 1 - p\sum_{n=0}^{\infty} \frac{r^{p(n+1)}}{(pn+1)(pn+p+1)} < 1. \tag{3.16}$$

The inequality (3.15) implies that the upper bound for $\mathscr{K}_p(r)$ given in (3.2) is better than that given in (1.11) for all $r \in (0, 1)$. It is clear that $2/\pi_p(=[p\sin(\pi/p)]/\pi)$ is increasing from $(1, \infty)$ onto $(0, 1)$. This yields $1 - 2/\pi_p > 0$ for $p \in (1, \infty)$. Therefore, the lower bound for $\mathscr{K}_p(r)$ given in (3.2) is better than that given in (1.13) for $q = p$ by the inequality (3.16).

(3) When $p = 2$, inequality (3.6) in Theorem 3.2 reduces to inequality (1.8).

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