Research Article

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The lattice of (2, 1)-congruences on a left restriction semigroup

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Abstract: All the (2, 1)-congruences on a left restriction semigroup become a complete sublattice of its lattice of congruences. The aim of this article is to study certain fundamental properties of this complete sublattice. We introduce k-, K-, t-, and T-operators on this sublattice and obtain some properties. As applications, the remarkable (2, 1)-congruences are characterized. These results extend the corresponding results on inverse semigroups and ample semigroups.

Keywords: left restriction semigroups, semilattices, projection kernel-traces, (2, 1)-congruences, complete lattices

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1 Introduction

Inverse semigroups play an important role in the theory of semigroups and appear in many branches of mathematics. So, many semigroup scholars have been trying to generalize inverse semigroups. One of the most successful generalizations is (left; right) ample semigroups, originated by Fountain (see, [1,2]). (Left; Right) restriction semigroups are generalizations of (left; right) ample semigroups (of course, extensions of inverse semigroups).

Left restriction semigroups (termed as weakly left E-ample semigroups in some context) arise from many sources (see [2,3]). They are a class of semigroups equipped with one additional unary operation $^+$ (i.e., unary semigroups), which satisfy certain identities. Such semigroups are isomorphic to unary subsemigroups of partial transformation semigroups $\mathcal{PT}_X$, where the unary operation $^+$ is of the form $\alpha \mapsto I_{\mathrm{dom}\alpha}$. The reader can consult [4,5] for history and more details. In particular, each inverse semigroup S can be shaped into a left restriction semigroup where the unary operation $^+$ is defined by the rule $a^+ = aa^{-1}$ for $a \in S$. Certainly, S also has another unary operation $^{-1}$, which will not have a crucial part to play at this stage. In this sense, left restriction semigroups are regarded as being natural extensions of inverse semigroups, obtained by weakening the condition of regularity, and they have many analogous properties. Consequently, such class of semigroups has been recently extensively researched by many semigroup experts from various perspectives (see [6–8]).

The theory of congruences plays a significant role in semigroup theory. Many researchers have been investigating such a theory (e.g., see [9–21] and references therein). The most useful tool in studying congruences is the kernel-trace approach, which is successfully used to research the congruences on inverse semigroups (e.g., see [9,10]). In 1986, Pastijn and Petrich [11] established the kernel-trace theory

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on general regular semigroups. Recently, El-Qallali in [12] developed the kernel-trace theory on ample semigroups, which is similar to that on inverse semigroups. He characterized the structural properties of the smallest and the greatest (admissible) congruences having the same kernel and the same trace as a given admissible congruence. From the universal algebraic point of view, a left restriction semigroup is an algebra of type (2, 1), so we will mainly concentrate on the (2, 1)-congruences. In the case of inverse semigroups, congruences must be admissible ones while the admissible congruences on ample semigroups are necessarily (2, 1)-congruences, even (2, 1, 1)-congruences (see [12, Lemma 2.2]). Meanwhile, the projections will take over the role of idempotents in the process of studying left restriction semigroups, correspondingly, the kernel and trace of congruences will be replaced by the projection kernel and trace, respectively. In this spirit, we shall analogously investigate the (2, 1)-congruences via the projection kernel-trace approach on left restriction semigroups, echoing the kernel-trace approach to (admissible) congruences on inverse semigroups and ample semigroups. Inspired by [9,12], the constructions of (2, 1)-congruences having the same projection kernel and the same projection trace will be explored on left restriction semigroups.

The kernel-trace approach can establish the two-dimensional "network" structure of the lattice of congruences, which may provide more structure information of the related semigroup. In [13], Petrich and Reilly set up a min network of congruences on inverse semigroups by means of the smallest group congruence, semilattice congruence, Clifford semigroup congruence, and E-unitary inverse semigroup congruence, which enlarges the lattice of congruences on such a semigroup, which is provided by Green [14]. More recently, El-Qallali [15] extended the approach of [13] to ample semigroups and depicted a network of admissible congruences on the basis of the smallest cancellative monoid, semilattice, Clifford ample semigroup, E-unitary ample semigroup admissible congruence, and so on. Along this way, we shall continue to research analogously remarkable (2, 1)-congruences and provide a similar min network of these congruences for left restriction semigroups.

The main aim of this article is to study the lattice of (2, 1)-congruences on a left restriction semigroup. The authors in [16] pointed out that all the (2, 1)-congruences on a left restriction semigroup become a complete sublattice. This result allows us to develop the kernel-trace theory on left restriction semigroups. The article will be organized as follows: after making some preparations, Section 3 is contributed to investigating the structure of (2, 1)-congruences whose projection traces coincide with that of a given (2, 1)-congruence on left restriction semigroups and determining the smallest as well as the greatest one of such (2, 1)-congruences. Moreover, it is proved that such (2, 1)-congruences constitute a complete sublattice of $C^{2,1}(S)$. However, the key to the kernel-trace theory on left restriction semigroups is how to define the $t-$ ($T-$)operator and the $k-$ ($K-$)operator on $C^{2,1}(S)$. So, in Section 3, we introduce t- and T-operators on $C^{2,1}(S)$. Similarly, we also probe the (2, 1)-congruences having the same projection kernel and consider k- and K-operators on $C^{2,1}(S)$ in Section 4. All of these conclusions generalize the corresponding ones on inverse semigroups and ample semigroups [9,10,12]. As applications, Section 5 is devoted to characterizing certain specific (2, 1)-congruences on a left restriction semigroup; for example, reduced, semilattice, Clifford left restriction semigroup, and P-unitary left restriction semigroup (2, 1)-congruences (Propositions 5.1, 5.3, Theorem 5.5, and Lemma 5.8). In particular, $\omega_t = \sigma$, $\omega_k = \eta$, $(\omega_k)_t = \nu$, $(\omega_t)_k = \pi$ are, respectively, the least reduced, semilattice, Clifford left restriction semigroup, and P-unitary left restriction semigroup (2, 1)-congruence on left restriction semigroups, where ω is the universal congruence (Corollaries 5.2, 5.4, 5.6, and Proposition 5.9). Moreover, such least congruences are utilized to build up a min network of (2, 1)-congruences as building bricks on left restriction semigroups. These results enrich and extend the related results concerning inverse semigroups [10,13] and ample semigroups [15].

2 Preliminaries

Throughout this article, we shall use notions and notations from the textbook of Howie [22]. We start by recalling some concepts and known results on left restriction semigroups in the sequel. For those not given in this article, the reader is referred to [3,4,6].

Definition 2.1. *A left restriction semigroup* is defined to be an algebra of type (2, 1), more precisely, an algebra $S = (S, \cdot, ^+)$ where $(S, \cdot)$ is a semigroup and $^+$ is a unary operation such that the following identities are satisfied:

$$x^+x = x, \quad (x^+y)^+ = x^+y^+, \quad (x^+y^+)^+ = x^+y^+ = y^+x^+, \quad xy^+ = (xy)^+x. \tag{2.1}$$

Dually, we can define *right restriction semigroups*. By a *restriction semigroup* we mean an algebra $(S, \cdot, ^+, ^*)$ of type (2, 1, 1) such that
(i) $(S, \cdot, ^+)$ is a left restriction semigroup;
(ii) $(S, \cdot, ^*)$ is a right restriction semigroup; and
(iii) for any $x \in S$, $(x^*)^+ = x^*$ and $(x^+)^* = x^+$.

The lemma gathers some easy but useful consequences; for details, see [6].

Lemma 2.2. *Let S be a left restriction semigroup and $x, y \in S$. Then*

$$x^+x^+ = x^+, \quad (x^+)^+ = x^+, \quad (xy)^+ = (xy^+)^+, \quad x^+(xy)^+ = (xy)^+ = (xy)^+x^+.$$

For a left restriction semigroup S, as usual, we call $P(S) = \{x^+ : x \in S\}$ the *set of projections* of S and its elements are called *projections* of S. By Lemma 2.2, every projection is necessarily an idempotent element of S, but, in general, not all of the idempotents are projections; for example, see [4]. So, by definition, it is easy to know that $P(S)$ is a sub-semilattice of S. Indeed, any semilattice Y can be regarded as being a left restriction semigroup with identity unary operator by defining $a^+ = a$ for every $a \in Y$. Therefore, such a left restriction semigroup is simply considered and called a semilattice.

In particular, we call a left restriction semigroup to be *reduced* if its set of projections is a singleton. It is easy to see that a reduced left restriction semigroup S must be a monoid with the unique projection as its identity 1. Indeed, any monoid M with identity 1 can be endowed with the structure of a left restriction semigroup by setting $s^+ = 1$ for all $s \in M$. Note that an inverse semigroup, regarded as a left restriction semigroup, is reduced if and only if it is a group while a left ample semigroup is reduced if and only if it is a right cancellative monoid. So, in what follows, we regard a monoid as a reduced left restriction semigroup with a unary operator $^+ : s \mapsto 1$.

A left restriction semigroup S is called *Clifford* if $P(S)$ is central, that is, if $e \in P(S)$, then $ea = ae$ holds for each $a \in S$. The known result [7, Proposition 2.3] tells us that S is a Clifford left restriction semigroup if and only if S is a semilattice of monoids M_α with Y, satisfying that $1_\alpha 1_\beta = 1_{\alpha\beta} = 1_\beta 1_\alpha$ for any $\alpha, \beta \in Y$.

Let S be a left restriction semigroup. Following [22], a subset $A \subseteq S$ is said to be *left unitary* if for all $a \in A$ and $s \in S$, whenever $as \in A$, we have $s \in A$. Dually, we may define the notion of a right unitary subset. If A is both left and right unitary, then we call it a *unitary subset* of S. In particular, a left restriction semigroup S is *P-unitary* if $P(S)$ is a unitary subset of S.

Proposition 2.3. *Let S be a left restriction semigroup. Then the following statements are equivalent*:
(1) *S is P-unitary*;
(2) *S is left P-unitary*;
(3) *If $x^+y = x^+$ for $x, y \in S$, then $y = y^+$.*

Proof. (1) ⇒ (2) ⇒ (3) are consequences of the definition.

(3) ⇒ (1). Let $a, b \in S$. If $ab^+ \in P(S)$, then $(ab)^+a = ab^+ = (ab)^+ \in P(S)$, so that $a = a^+ \in P(S)$, thereby $P(S)$ is a right unitary subset of S. If $a^+b \in P(S)$, then

$$a^+b^+b = a^+b = (a^+b)^+ = a^+b^+ \in P(S),$$

hence $b = b^+ \in P(S)$, thereby $P(S)$ is a left unitary subset of S. We complete the proof. □

Among left restriction semigroups, the notions of subalgebras, homomorphisms, congruences, and factor algebras are understood in type (2, 1). In order to emphasize this point, we use the expressions (2, 1)-subsemigroup, (2, 1)-morphism, (2, 1)-congruence, and (2, 1)-factor semigroup, respectively.

Let ρ be a (2, 1)-congruence on a left restriction semigroup. On the quotient S/ρ, define

$$^+ : S/\rho \to S/\rho;\ a\rho \mapsto a^+\rho.$$

Obviously, $^+$ is well defined. We use still S/ρ to denote the algebra $(S/\rho, \cdot, ^+)$ of type (2, 1). The following proposition is immediate and we here omit the details.

Proposition 2.4. *Let ρ be a (2, 1)-congruence on a left restriction semigroup S. Then so is S/ρ, and $P(S/\rho) = \{x^+\rho : x \in S\}$.*

Moreover, we have the following result.

Proposition 2.5. *Let α, β be (2, 1)-congruences on a left restriction semigroup S. If $\alpha \subseteq \beta$, then*
(*i*) *β/α is a (2, 1)-congruence on S/α;*
(*ii*) *S/β is (2, 1)-isomorphic to $(S/\alpha)/(\beta/\alpha)$.*

Proof.
(i) It is immediate from [16, Proposition 2.3] and [22, Theorem 1.5.4, p. 25].
(ii) Consider the mapping

$$\theta : (S/\alpha)/(\beta/\alpha) \to S/\beta;\ (x\alpha)(\beta/\alpha) \mapsto x\beta,$$

and note that by [22, Theorem 1.5.4, p. 25], θ is a semigroup isomorphism from $(S/\alpha)/(\beta/\alpha)$ onto S/β and

$$[((x\alpha)(\beta/\alpha))^+]\theta = [(x^+\alpha)(\beta/\alpha)]\theta = x^+\beta = (x\beta)^+ = [((x\alpha)(\beta/\alpha))\theta]^+.$$

Thus, we obtain that θ is indeed a (2, 1)-isomorphism. □

Remark 2.6. In view of Proposition 2.5, we can assert that the map $\beta \mapsto \beta/\alpha$ has actually established an isotone isomorphism from the lattice of those (2, 1)-congruences on S that contain α onto $C^{2,1}(S/\alpha)$, the lattice of all (2, 1)-congruences on S/α.

Let S be a semigroup with a subsemilattice E of $E(S)$ the set of idempotents. The relations $\widetilde{\mathcal{R}}_E$ and $\widetilde{\mathcal{L}}_E$ are efficient tools in the study of generalized regular semigroups (see, [3,4]). In [3], Gould provided an equivalent definition of a left restriction semigroup in terms of $\widetilde{\mathcal{R}}_E$, that is, S is weakly left (right) E-ample with respect to E if and only if $(S, \cdot, ^+)$ is a left (right) restriction semigroup with $P(S) = E$. Later on, when considering the relation $\widetilde{\mathcal{R}}_{P(S)}$ on a left restriction semigroup S, it is usual to drop the subscript and write $\widetilde{\mathcal{R}}_{P(S)}$ more simply as $\widetilde{\mathcal{R}}$ and make the same convention for $\widetilde{\mathcal{L}}_{P(S)}$. In the literature, the relation $\widetilde{\mathcal{R}}$ is used only for weakly left ample semigroups (a class of left restriction semigroups in which all the idempotents are projections see [23–25]), but here this will not cause any confusion. It is well known that for elements a, b of a left restriction semigroup, $(a, b) \in \widetilde{\mathcal{R}}$ if and only if $a^+ = b^+$, from which it follows that $\widetilde{\mathcal{R}}$ is indeed a left congruence on any left restriction semigroup; in particular, $a\widetilde{\mathcal{R}}a^+$. This observation will be repeatedly used in the sequel.

Kernel-trace approach is a useful tool to study congruences on an inverse semigroup. As (left; right) restriction semigroups are generalizations of inverse semigroups, the following problem is natural: Whether do (left; right) restriction semigroups have a kernel-trace approach similar to that on inverse semigroups? In order to solve it, we first start to define Pkernel and Ptrace on a left restriction semigroup as follows:

Definition 2.7. For an equivalence ϱ on S, the *projection kernel* of ϱ is

$$\mathrm{Pker}(\varrho) = \{a \in S : a\varrho e,\ e \in P(S)\}$$

and the *projection trace* of ϱ is

$$\mathrm{Ptr}(\varrho) = \varrho \cap (P(S) \times P(S)).$$

The following lemma is immediate.

Lemma 2.8. *Let ϱ_1, ϱ_2 be equivalences on S. If $\varrho_1 \subseteq \varrho_2$, then $\mathrm{Ptr}(\varrho_1) \subseteq \mathrm{Ptr}(\varrho_2)$ and $\mathrm{Pker}(\varrho_1) \subseteq \mathrm{Pker}(\varrho_2)$.*

Recall that a (2, 1)-congruence ρ on a left restriction semigroup S is called

(i) *projection-separating* if for any $a, b \in S$, $(a^+, b^+) \in \rho$ implies that $a^+ = b^+$.
(ii) *projection-pure* if and only if $e\rho \subseteq P(S)$ for any $e \in P(S)$.

By the definition above, it is easy to check that a (2, 1)-congruence ρ on S is projection-separating if and only if $\mathrm{Ptr}(\rho) = \varepsilon$, where ε is the identical congruence and that ρ is projection-pure if and only if $\mathrm{Pker}(\rho) = P(S)$. These two classes of (2, 1)-congruences play an important role in the study of left restriction semigroups. We shall denote by $\mathfrak{R}(S)$ the set of projection-pure (2, 1)-congruences on S and by $\mathfrak{P}(S)$ the set of projection-separating (2, 1)-congruences on S, respectively.

By the foregoing argument, the following lemma is straightforward.

Lemma 2.9. *Let S be a left restriction semigroup and ρ be a (2, 1)-congruence on S. Then $a\widetilde{\mathcal{R}}b \;\Rightarrow\; a\rho\widetilde{\mathcal{R}}(S/\rho)b\rho$, for any $a, b \in S$.*

Let $(Q, \leq)$ be a partially ordered set. Recall that an *interval* is the subset of the form:

$$[\alpha, \beta] = \{\gamma \in Q : \alpha \leq \gamma \leq \beta\},$$

for some $\alpha, \beta \in Q$, $\alpha \leq \beta$. It is worthy to point out when Q is a complete lattice, any interval is necessarily a complete sublattice of Q. This fact will be frequently used in the following sections.

3 *t*- and *T*-operators

In the remainder of this article, we always assume that S is a left restriction semigroup. We denote by $C(S)$ the set of congruences and by $C^{2,1}(S)$ the set of (2, 1)-congruences on S, respectively. It is easy to know that under set inclusion, both $C(S)$ and $C^{2,1}(S)$ are lattices, even $C(S)$ is complete. In [16, Theorem 2.1], the authors pointed out that $C^{2,1}(S)$ is a complete sublattice of $C(S)$, and the smallest and the greatest element of both $C(S)$ and $C^{2,1}(S)$ are ε and ω. We shall sometimes use $\varepsilon(S)$ and $\omega(S)$ to avoid confusion, and the similar notation for other relations. The main aim of this section is to study t- and T-operators on the complete lattice $C^{2,1}(S)$.

To begin with, we investigate the (2, 1)-congruences having the same projection trace on left restriction semigroups through the approach of [11], and the next lemma is necessary and used several times.

Lemma 3.1. *Let $\rho \in C^{2,1}(S)$ with $\mathrm{Ptr}(\rho) = \tau$. Then for $a, b \in S$*

(1) $(a^+\rho)\widetilde{\mathcal{R}}(S/\rho)(b^+\rho) \;\Leftrightarrow\; a^+\tau b^+$;
(2) $(a\rho)\widetilde{\mathcal{R}}(S/\rho)(b\rho) \;\Leftrightarrow\; a(\widetilde{\mathcal{R}}\tau\widetilde{\mathcal{R}})b$;
(3) $\rho \vee \widetilde{\mathcal{R}} = \widetilde{\mathcal{R}}\tau\widetilde{\mathcal{R}}$ *and* $\mathrm{Ptr}(\rho \vee \widetilde{\mathcal{R}}) = \tau$.

Proof.

(1) It is immediate from the definition.
(2) Suppose now that $(a\rho)\widetilde{\mathcal{R}}(S/\rho)(b\rho)$. Then $(a\rho)^+ = (b\rho)^+$, that is to mean $a^+\rho b^+$. Note that $a\widetilde{\mathcal{R}}a^+$ and $b^+\widetilde{\mathcal{R}}b$. Then $a\widetilde{\mathcal{R}}a^+\rho b^+\widetilde{\mathcal{R}}b$. Hence, $a(\widetilde{\mathcal{R}}\tau\widetilde{\mathcal{R}})b$.

Conversely, let $a(\widetilde{\mathcal{R}}\tau\widetilde{\mathcal{R}})b$. Then $a\widetilde{\mathcal{R}}x^+\tau y^+\widetilde{\mathcal{R}}b$ for $x, y \in S$. By using Lemma 2.9, $a\rho\widetilde{\mathcal{R}}(S/\rho)x^+\rho = y^+\rho\widetilde{\mathcal{R}}(S/\rho)b\rho$. Therefore, we have proved $(a\rho)\widetilde{\mathcal{R}}(S/\rho)(b\rho)$.

(3) Let $a(\rho \vee \widetilde{\mathcal{R}})b$. By [22, Proposition 1.5.11, p. 28], there exist $x_1, x_2, \cdots, x_n \in S$ such that

$$a\rho x_1\widetilde{\mathcal{R}}x_2\rho \cdots \rho x_n\widetilde{\mathcal{R}}b.$$

Since ρ is a (2, 1)-congruence on S, by Lemma 2.9 we obtain

$$a\rho = x_1\rho\widetilde{\mathcal{R}}x_2\rho = \cdots = x_n\rho\widetilde{\mathcal{R}}b\rho,$$

so that $a\rho\widetilde{\mathcal{R}}(S/\rho)b\rho$. By using (2), it follows $a(\widetilde{\mathcal{R}}\tau\widetilde{\mathcal{R}})b$, which implies $(\rho \vee \widetilde{\mathcal{R}}) \subseteq (\widetilde{\mathcal{R}}\tau\widetilde{\mathcal{R}})$. Clearly, $(\widetilde{\mathcal{R}}\tau\widetilde{\mathcal{R}}) \subseteq (\rho \vee \widetilde{\mathcal{R}})$. Again by (2), $\widetilde{\mathcal{R}}\tau\widetilde{\mathcal{R}}$ is an equivalence including ρ and $\widetilde{\mathcal{R}}$ on S. Then it yields $\rho \vee \widetilde{\mathcal{R}} = \widetilde{\mathcal{R}}\tau\widetilde{\mathcal{R}}$. Associating (1) with (2), we have $(\rho \vee \widetilde{\mathcal{R}})|_{P(S)} = \tau$. □

Let ϱ be a relation and denote by ϱ^* the least (2, 1)-congruence on S containing ϱ and by $\varrho^\flat$ the greatest congruence on S contained in ϱ. Let $\rho \in C^{2,1}(S)$ with $\mathrm{Ptr}(\rho) = \tau$. We put

$$\rho_t = \tau^*, \rho^T = (\rho \vee \widetilde{\mathcal{R}})^\flat = (\widetilde{\mathcal{R}}\tau\widetilde{\mathcal{R}})^\flat.$$

Theorem 3.2. *Let S be a left restriction semigroup. If $\rho \in C^{2,1}(S)$, then ρ_t and ρ^T are the least and the greatest (2, 1)-congruences with the same projection trace τ as ρ, respectively.*

Proof. Note that τ^* is the least (2, 1)-congruence including τ so that $\tau^* \subseteq \rho$. By hypothesis, we can obtain $\mathrm{Ptr}(\tau^*) = \tau$ since $\tau \subseteq \mathrm{Ptr}(\tau^*)$ and

$$\tau^* \subseteq \rho \;\Rightarrow\; \mathrm{Ptr}(\tau^*) \subseteq \mathrm{Ptr}(\rho) = \tau.$$

For the remainder, we first show that ρ^T is a (2, 1)-congruence with the projection trace τ. By [22, Proposition 1.5.10], we can see that

$$(\rho \vee \widetilde{\mathcal{R}})^\flat = \{(a, b) \in S \times S : (\forall\, x, y \in S^1)(xay, xby) \in \rho \vee \widetilde{\mathcal{R}}\}$$

is the greatest congruence contained in $\rho \vee \widetilde{\mathcal{R}}$. Furthermore, $(\rho \vee \widetilde{\mathcal{R}})^\flat$ respects the unary operation $^+$ since

$$\begin{aligned}(a, b) \in (\rho \vee \widetilde{\mathcal{R}})^\flat &\Rightarrow (a, b) \in \rho \vee \widetilde{\mathcal{R}}\\ &\Rightarrow (a, b) \in \widetilde{\mathcal{R}}\tau\widetilde{\mathcal{R}} \;\Rightarrow\; a^+\tau b^+ \;\Rightarrow\; a^+\rho b^+\\ &\Rightarrow xa^+y\rho xb^+y \;\Rightarrow\; (xa^+y)^+\tau(xb^+y)^+\\ &\Rightarrow xa^+y\widetilde{\mathcal{R}}(xa^+y)^+\tau(xb^+y)^+\widetilde{\mathcal{R}}xb^+y\\ &\Rightarrow (a^+, b^+) \in (\rho \vee \widetilde{\mathcal{R}})^\flat.\end{aligned}$$

Additionally, since $\rho \subseteq (\rho \vee \widetilde{\mathcal{R}})$ implies $\rho \subseteq (\rho \vee \widetilde{\mathcal{R}})^\flat \subseteq (\rho \vee \widetilde{\mathcal{R}})$, then

$$\mathrm{Ptr}(\rho) \subseteq \mathrm{Ptr}((\rho \vee \widetilde{\mathcal{R}})^\flat) \subseteq (\rho \vee \widetilde{\mathcal{R}})|_{P(S)} = \tau = \mathrm{Ptr}(\rho).$$

Hereby, we obtain $\mathrm{Ptr}((\rho \vee \widetilde{\mathcal{R}})^\flat) = \tau$.

In the end, we shall claim that $(\rho \vee \widetilde{\mathcal{R}})^\flat$ is the greatest (2, 1)-congruence with the projection trace τ, where $\tau = \rho\,|_{P(S)}$. In fact, if $\lambda \in C^{2,1}(S)$ with $\mathrm{Ptr}(\lambda) = \tau$, then by Lemma 3.1, we deduce

$$\lambda \subseteq (\lambda \vee \widetilde{\mathcal{R}}) = \widetilde{\mathcal{R}}\tau\widetilde{\mathcal{R}} = \rho \vee \widetilde{\mathcal{R}}.$$

This shows that $\lambda \subseteq (\rho \vee \widetilde{\mathcal{R}})^\flat$ since $(\rho \vee \widetilde{\mathcal{R}})^\flat$ is the greatest (2, 1)-congruence contained in $\rho \vee \widetilde{\mathcal{R}}$. Therefore, the proof is completed. □

Motivated by Lawson [20, Theorem 1.4.12, p. 25], we define $\sim$ on a left restriction semigroup S as follows: for $x, y \in S$,

$$x \sim y \quad \text{if and only if } y^+x = x^+y.$$

Lemma 3.3. The relation $\sim$ *is reflexive, symmetric, and compatible with the multiplication. Moreover,* $a^+ \sim b^+$ *for any* a, b *in* S.

Proof. By definition, it is not difficult to check that $\sim$ is reflexive and symmetric.

Let $a, b, c, d \in S$. If $a \sim b$ and $c \sim d$, then $b^+a = a^+b$, $d^+c = c^+d$, and so

$$\begin{aligned}
(bd)^+ac &= (bd)^+b^+ac = (bd)^+(a^+b)c && (\text{since by Lemma 2.2, } (bd)^+b^+ = (bd)^+)\\
&= a^+\cdot(bd)^+b\cdot c = a^+bd^+c && (\text{since by (2.1), } bd^+ = (bd)^+b)\\
&= b^+a\cdot c^+d = b^+\cdot ac^+\cdot d && (\text{since } b^+a = a^+b, d^+c = c^+d)\\
&= b^+(ac)^+ad = (ac)^+b^+ad && (\text{since by (2.1), } ac^+ = (ac)^+a)\\
&= (ac)^+a^+bd = (ac)^+bd && (\text{since } b^+a = a^+b).
\end{aligned}$$

Hence $ac \sim bd$, which implies that $\sim$ is compatible with the multiplication. We note that

$$(b^+)^+a^+ = b^+a^+ = a^+b^+ = (a^+)^+b^+,$$

and the result follows. □

Proposition 3.4. *If* $\rho \in C^{2,1}(S)$, *then*
(1) $\rho_t = \{(a, b) \in S \times S : (\exists c \in S)c^+a = c^+b, \quad c^+\rho a^+\rho b^+\} = (\rho \cap \sim)^*$.
(2) $\rho^T = \{(a, b) \in S \times S : (\forall y \in S^1)(ay)^+\rho(by)^+\}$.

Proof. (1). By [12, Corollary 3.3], the relation

$$\xi := \{(a, b) \in S \times S : \quad (\exists c \in S) \quad c^+a = c^+b, \quad c^+\rho a^+\rho b^+\}$$

is the least (2, 1)-congruence with the projection trace $\mathrm{Ptr}(\rho)$ on S. Note that by Theorem 3.2, ρ_t is also the least (2, 1)-congruence on S with projection trace $\mathrm{Ptr}(\rho)$, we can observe that $\rho_t = \xi$.

Let $a, b \in S$. If $a(\rho \cap \sim)b$, then $b^+a = a^+b$, $a^+\rho b^+$, and hence

$$b^+a^+\cdot a = b^+a^+\cdot b, \;\; b^+a^+\rho \quad a^+\rho \quad b^+,$$

thereby $(a, b) \in \xi = \rho_t$. It follows that $(\rho \cap \sim) \subseteq \rho_t$. Therefore, $(\rho \cap \sim)^* \subseteq \rho_t$. This shows that $\mathrm{Ptr}(\rho \cap \sim)^* \subseteq \mathrm{Ptr}(\rho_t)$. Conversely, if $(a, b) \in \mathrm{Ptr}(\rho_t) = \tau$, then by definition, $a = a^+$, $b = b^+$, and $a\rho b$. Obviously, $a^+b = a^+b^+ = b^+a^+ = b^+a$. In other words, $(a, b) \in \sim$. Therefore, $(a, b) \in (\rho \cap \sim)$ and whence $\mathrm{Ptr}(\rho_t) \subseteq \mathrm{Ptr}(\rho \cap \sim)^*$. We have now verified that $\mathrm{Ptr}(\rho_t) = \mathrm{Ptr}(\rho \cap \sim)^*$. But by Theorem 3.2, ρ_t is the least (2, 1)-congruence on S with projection trace $\mathrm{Ptr}(\rho) = \mathrm{Ptr}(\rho \cap \sim)^*$, now $\rho_t \subseteq (\rho \cap \sim)^*$. Consequently, $\rho_t = (\rho \cap \sim)^*$.

(2) We prove first that $\zeta = \{(a, b) \in S \times S : (\forall y \in S^1) \quad (ay)^+\rho(by)^+\}$ is a (2, 1)-congruence on S. Obviously, ζ is an equivalence on S. If $a\zeta b$ and $c \in S$, then by definition, for any $y \in S^1$, $(acy)^+\rho(bcy)^+$, which follows that $ac\zeta bc$. Therefore, ζ is a right congruence on S. Considering that ρ is a (2, 1)-congruence on S, we have $c(ay)^+\rho c(by)^+$ so that $(cay)^+ = (c(ay)^+)^+\rho(c(by)^+)^+ = (cby)^+$. So, $(ca, cb) \in \zeta$ and whence ζ is a left congruence on S. Now ζ is a congruence on S. Furthermore, we note $a\zeta b$ implies $a^+\rho b^+$ and so

$$(a^+y)^+ = a^+y^+\rho b^+y^+ = (b^+y)^+.$$

In other words,

$$a^+\rho b^+ \;\;\Rightarrow\;\; a^+\zeta b^+. \tag{3.1}$$

Therefore, ζ is a (2, 1)-congruence on S, and $\mathrm{Ptr}(\rho) \subseteq \mathrm{Ptr}(\zeta)$.

On the other hand, by definition, $a^+\zeta b^+$ can imply that $a^+\rho b^+$, so that $\mathrm{Ptr}(\zeta) \subseteq \mathrm{Ptr}(\rho)$. Now $\mathrm{Ptr}(\zeta) = \mathrm{Ptr}(\rho)$. So, by Theorem 3.2, we have $\zeta \subseteq \rho^T$. Conversely, if $a\rho^T b$, then $ay\rho^T by$. Since by Theorem 3.2, ρ^T is a (2, 1)-congruence on S, it follows that $(ay)^+\rho(by)^+$. Therefore, $(a, b) \in \zeta$ and whence $\rho^T \subseteq \zeta$. We complete the proof. □

We can now define t- and T-operators on $C^{2,1}(S)$.

Definition 3.5. On $C^{2,1}(S)$, we define the t-*operator* by

$$\bullet^t : C^{2,1}(S) \to C^{2,1}(S);\ \rho \mapsto \rho_t$$

and the T-*operator* by

$$\bullet^T : C^{2,1}(S) \to C^{2,1}(S);\ \rho \mapsto \rho^T.$$

We arrive now at the main result of this section.

Proposition 3.6. *Let S be a left restriction semigroup and $\rho, \theta \in C^{2,1}(S)$. Then the following statements hold*:
(1) *$[\rho_t, \rho^T]$ is a complete sublattice of $C^{2,1}(S)$ with the same projection trace as ρ.*
(2) *$t^2 = t = Tt$ and $T^2 = T = tT$.*
(3) *If $\rho \subseteq \theta$, then $\rho_t \subseteq \theta_t$ and $\rho^T \subseteq \theta^T$.*

Proof. (1) It is immediate from Theorem 3.2 and the fact that an arbitrary interval of a complete lattice is necessarily a complete sublattice.

(2) and (3) are the consequences of (1) and definitions of t- and T-operators. □

As already stated in Section 2, any (2, 1)-congruence ρ on S is projection-separating if and only if $\mathrm{Ptr}(\rho) = \varepsilon$. In this case, $\mathfrak{P}(S) = [\varepsilon_t, \varepsilon^T] = [\varepsilon, \varepsilon^T]$ is a complete sublattice of $C^{2,1}(S)$. Thus, we have the following proposition.

Proposition 3.7. *Let S be a left restriction semigroup and $\rho \in C^{2,1}(S)$. Then the mapping*

$$\phi : [\rho_t, \rho^T] \to \mathfrak{P}(S/\rho_t);\ \xi \mapsto \xi/\rho_t$$

is an isotone isomorphism of complete sublattices.

Proof. Suppose now $\xi \in [\rho_t, \rho^T]$. Then from Proposition 2.5 it follows that ξ/ρ_t is a (2, 1)-congruence on S/ρ_t. Observe that $\xi/\rho_t \in \mathfrak{P}(S/\rho_t)$ if and only if $\mathrm{Ptr}(\xi) = \mathrm{Ptr}(\rho_t)$. Considering Proposition 3.6, it is easy to see that ϕ is well-defined. Accordingly, we have that ϕ is an isotone isomorphism of complete sublattice from $[\rho_t, \rho^T]$ onto $[\rho_t/\rho_t, \rho^T/\rho_t]$ by Remark 2.6. Since $\mathfrak{P}(S/\rho_t) = [\varepsilon(S/\rho_t), (\varepsilon(S/\rho_t))^T]$ and $\rho_t/\rho_t = \varepsilon(S/\rho_t)$, we only need to verify $\rho^T/\rho_t = (\varepsilon(S/\rho_t))^T$. Note that ρ^T is the greatest (2, 1)-congruence with the same projection trace as ρ_t, and thus ρ^T/ρ_t is the greatest projection-separation (2, 1)-congruence on S/ρ_t. Hence, by using the maximality we obtain $\rho^T/\rho_t = (\varepsilon(S/\rho_t))^T$. Therefore, we complete the proof. □

4 *k*- and *K*-operators

Now suppose $K = \mathrm{Pker}(\rho)$ for $\rho \in C^{2,1}(S)$. Let π_K be an equivalence on S whose equivalent classes are $\{K, S\backslash K\}$. Then the congruence $\pi_K^\flat$, that is the greatest congruence on S contained in π_K, is given by

$$\pi_K^\flat = \{(a, b) \in S \times S : (\forall\, x, y \in S^1) xay \in K \ \Leftrightarrow\ xby \in K\}.$$

As usual, we call $\pi_K^\flat$ the *syntactic congruence* of K; for syntactic congruence, see [22, p. 28]. We define

$$\rho_k = \{(a, a^+) : a \in K\}^*.$$

Theorem 4.1. *With the same notations as above,*
(1) ρ_k *is the least* (2, 1)*-congruence with projection kernel* K *on* S.
(2) $\mathrm{Pker}(\pi_K^\flat) = K$.

Proof. (1). First, we claim $\mathrm{Pker}(\rho_k) = K$. If $x \in K$, then

$$(x, x^+) \in \{(a, a^+) : a \in K\} \subseteq \{(a, a^+) : a \in K\}^* = \rho_k,$$

and whence $K \subseteq \mathrm{Pker}(\rho_k)$. On the other hand, by $x \in K$, we know that $x\rho \in P(S/\rho)$, so that $x\rho = (x\rho)^+ = x^+\rho$ since ρ is a (2, 1)-congruence on S. It follows that $(x, x^+) \in \rho$. Therefore, we have $\{(a, a^+) : a \in K\} \subseteq \rho$, and hence $\rho_k = \{(a, a^+) : a \in K\}^* \subseteq \rho$, thereby $\mathrm{Pker}(\rho_k) \subseteq \mathrm{Pker}(\rho) = K$. Now, $\mathrm{Pker}(\rho_k) = K$, as required.

Now let ξ be a (2, 1)-congruence on S such that $\mathrm{Pker}(\xi) = K$. By the foregoing proof, we have verified that $\{(a, a^+) : a \in K\}^* \subseteq \xi$. Thus, $\rho_k \subseteq \xi$. Consequently, ρ_k is the least (2, 1)-congruence with its projection kernel K on S.

(2). Assume now $a \in S$. If $a \in \mathrm{Pker}(\pi_K^\flat)$, then $(a, e) \in \pi_K^\flat$ for some $e \in P(S)$. Hence $a = a^+a \cdot 1 \in K$ since $\pi_K^\flat$ is the syntactic congruence over K and $a^+e1 = a^+e \in P(S) \subseteq K$, where 1 is the identity of S^1. It follows that $\mathrm{Pker}(\pi_K^\flat) \subseteq K$. Conversely, if $m \in K$, then by the proof of (1), $(m, m^+) \in \rho$; so that $(xmy, xm^+y) \in \rho$ for $x, y \in S^1$; in other words, $(xmy)\rho = (xm^+y)\rho$. This means that

$$xmy \in K \iff xm^+y \in K.$$

So, we have $(m, m^+) \in \pi_K^\flat$, so that $m \in \mathrm{Pker}(\pi_K^\flat)$, leading to $K \subseteq \mathrm{Pker}(\pi_K^\flat)$. Therefore, $\mathrm{Pker}(\pi_K^\flat) = K$. □

Corollary 4.2. *The relation* $\pi_K^\flat$ *is the greatest congruences on S whose projection kernel is* K.

Definition 4.3. On $C^{2,1}(S)$, we define the k-operator by

$$\bullet^k : C^{2,1}(S) \to C^{2,1}(S);\ \rho \mapsto \rho_k$$

and the K-operator by

$$\bullet^K : C^{2,1}(S) \to C^{2,1}(S);\ \rho \mapsto \rho^K = \bigvee\{\xi \in C^{2,1}(S) : \rho \subseteq \xi \subseteq \pi_K^\flat\}.$$

Following [16], if $\alpha, \beta \in C^{2,1}(S)$, then both $\alpha \cap \beta$ and $\alpha \vee \beta$ are in $C^{2,1}(S)$. Moreover, $C^{2,1}(S)$ is a complete subsemilattice of $C(S)$. In this case, the following lemma is evident.

Lemma 4.4. *For any* (2, 1)*-congruence* ρ *on* S *whose projection kernel is* K, *there exists*

$$\rho^K = \bigvee\{\xi \in C^{2,1}(S) : \rho \subseteq \xi \subseteq \pi_K^\flat\},$$

which is the greatest (2, 1)*-congruence on* S *with* K *as its projection kernel. So,* $\bullet^K$ *is well-defined.*

Proposition 4.5. *Let* S *be a left restriction semigroup and* $\rho, \theta \in C^{2,1}(S)$. *Then the following statements are true*:
(1) $[\rho_k, \rho^K]$ *is a complete sublattice of* $C^{2,1}(S)$ *with the same projection kernel as* ρ.
(2) $k^2 = k = Kk$ *and* $K^2 = K = kK$.
(3) *If* $\rho \subseteq \theta$, *then* $\rho_k \subseteq \theta_k$.

Proof. (1) It directly follows from Theorem 4.1 and Definition 4.3 and the fact that any interval of a complete lattice is still complete.

(2) and (3) are immediate from the definitions of k- and K-operators. □

It is worth noting that $\mathrm{Pker}(\varepsilon) = P(S)$ and any (2, 1)-congruence ρ is projection-pure if and only if $\mathrm{Pker}(\rho) = P(S)$. Then we can easily deduce that $\mathfrak{R}(S) = [\varepsilon_k, \varepsilon^K] = [\varepsilon, \varepsilon^K]$ so that by Proposition 4.5, the following corollary is necessary.

Proposition 4.6. *Let S be a left restriction semigroup and $\rho \in C^{2,1}(S)$. Then the mapping*

$$\psi : [\rho_k, \rho^K] \to \mathfrak{R}(S/\rho_k); \xi \mapsto \xi/\rho_k$$

is a complete isotone isomorphism.

Proof. Assume that $\xi \in [\rho_k, \rho^K]$. Then in view of Proposition 2.5, we can see that ξ/ρ_k is necessarily a (2, 1)-congruence on S/ρ_k. Note that $\xi/\rho_k \in \mathfrak{R}(S/\rho_k)$ if and only if $\mathrm{Pker}(\xi) = \mathrm{Pker}(\rho_k)$. Then by Proposition 4.5, it is not hard to check that ψ is well-defined. Furthermore, by using Remark 2.6 we can see that ψ is a complete isotone isomorphism from $[\rho_k, \rho^K]$ onto $[\rho_k/\rho_k, \rho^K/\rho_k]$. On the other hand, because $\mathfrak{R}(S/\rho_k) = [\varepsilon(S/\rho_k), (\varepsilon(S/\rho_k))^K]$ and $\rho_k/\rho_k = \varepsilon(S/\rho_k)$, it suffices to show $\rho^K/\rho_k = (\varepsilon(S/\rho_k))^K$. Observe that ρ^K is the greatest (2, 1)-congruence with the same projection kernel as ρ_k, and whence ρ^K/ρ_k is the greatest projection-pure (2, 1)-congruence on S/ρ_k. Therefore, by the maximality we can obtain $\rho^K/\rho_k = (\varepsilon(S/\rho_k))^K$. This completes the proof. □

5 A min network of (2, 1)-congruences

As applications of k-, K-, t-, and T-operators, in this section we shall give the characterizations of some remarkable (2, 1)-congruences on a left restriction semigroup S.

Let ρ be a (2, 1)-congruence on a left restriction semigroup S. Then ρ is *reduced* if S/ρ is a reduced left restriction semigroup, i.e., $P(S/\rho)$ is a singleton. It is worthy to point out that reduced congruences are generalizations of group and (right)cancellative congruences on inverse and (left) ample semigroups, respectively.

Proposition 5.1. *Let S be a left restriction semigroup. If ρ is a* (2, 1)*-congruence on S, then the following statements equal*:
(1) *ρ is a reduced congruence on S*;
(2) $\mathrm{Ptr}(\rho) = P(S) \times P(S)$;
(3) $\rho \in [\omega_t, \omega^T]$.

Proof. (1) ⇒ (2). Assume that ρ is a reduced congruence on S. Then $|P(S/\rho)| = 1$. Recall from Proposition 2.4 that $P(S/\rho) = \{x^+\rho : x \in S\}$. Then $(e, f) \in \rho$ for any $e, f \in P(S)$. Hence, we have $P(S) \times P(S) \subseteq \mathrm{Ptr}(\rho)$. This, together with the reverse inclusion shows $\mathrm{Ptr}(\rho) = P(S) \times P(S)$.

(2) ⇒ (1). It is immediate from the definition and $P(S/\rho) = \{x^+\rho : x \in S\}$.

(1) ⇒ (3). Let ρ be a reduced congruence on S. Then $\mathrm{Ptr}(\rho) = P(S) \times P(S)$. Note that $\mathrm{Ptr}(\omega) = P(S) \times P(S)$, and thus by Proposition 3.6, $[\omega_t, \omega^T]$ contains all the (2, 1)-congruences whose projection trace are the same as ρ. Thus, $\rho \in [\omega_t, \omega^T]$ holds.

(3) ⇒ (1). It is trivial. □

By Proposition 5.1, we have the following corollary.

Corollary 5.2. *Let S be a left restriction semigroup. Then the smallest reduced* (2, 1)*-congruence σ is equal to ω_t. In this case, each* (2, 1)*-congruence ρ is reduced if and only if $\rho \in [\sigma, \omega]$. Moreover, $\sigma = \{(a, b) : (\exists x \in S)x^+a = x^+b\}$ and $\sigma = \sim^*$.*

Proof. It is evident that $\sigma = \omega_t$ holds. Furthermore, when putting $\rho = \omega$ in Proposition 3.4, the proof is completed. □

A (2, 1)-congruence ρ on a left restriction semigroup S is called a
(i) *semilattice congruence* if S/ρ itself is a semilattice.
(ii) *Clifford congruence* if S/ρ is a Clifford left restriction semigroup.

Proposition 5.3. *Let ρ be a* (2, 1)*-congruence on S. Then the following conditions equal*:
(1) *ρ is a semilattice congruence on S.*
(2) $\mathrm{Pker}(\rho) = S$.
(3) $S/\rho = \{x^+\rho : x \in S\}$.
(4) $\widetilde{\mathcal{R}} \subseteq \rho$.
(5) $\rho \in [\omega_k, \omega^K]$.

Proof. It is obvious that (1), (2), and (3) are equivalent. Now we show the others.

(1) ⇒ (4). Because ρ is a semilattice congruence on S, we can see $(a, a^+) \in \rho$ and $(b^+, b) \in \rho$ for $a, b \in S$. Moreover,

$$a\widetilde{\mathcal{R}}b \;\Rightarrow\; a^+ = b^+ \;\Rightarrow\; a\rho a^+ = b^+\rho b \;\Rightarrow\; a\rho b.$$

Hence, it proves $\widetilde{\mathcal{R}} \subseteq \rho$.

(4) ⇒ (1). It is a direct consequence of $a\widetilde{\mathcal{R}}a^+$ for any $a \in S$.

(1) ⇒ (5). Let ρ be a semilattice congruence on S. Then $\mathrm{Pker}(\rho) = S$. Observe that $\mathrm{Pker}(\omega) = S$ and that $[\omega_k, \omega^K]$ includes all the (2, 1)-congruences whose projection kernels are S, so we have $\rho \in [\omega_k, \omega^K]$.

(5) ⇒ (1). It is immediate from the fact $\mathrm{Pker}(\rho) = \mathrm{Pker}(\omega) = S$. □

Corollary 5.4. *Let S be a left restriction semigroup. Then ω_k equals $\widetilde{\mathcal{R}}^*$ and is the least semilattice congruence η on S. Moreover, any* (2, 1)*-congruence ρ is a semilattice congruence if and only if $\rho \in [\eta, \omega]$.*

Proof. Since ω itself is a semilattice congruence on S, the least semilattice congruence η does exist. Then from Proposition 5.3 (5) it follows that $\eta = \omega_k$. On the other hand, by Proposition 5.3 (4) we can deduce that $\widetilde{\mathcal{R}}^*$ is a semilattice congruence, and thereby $\eta \subseteq \widetilde{\mathcal{R}}^*$. In addition, note $\widetilde{\mathcal{R}} \subseteq \eta$ so that $\widetilde{\mathcal{R}}^* \subseteq \eta$. Therefore, it has proved $\eta = \omega_k = \widetilde{\mathcal{R}}^*$. The rest statement is trivial. □

Theorem 5.5. *Let ρ be a* (2, 1)*-congruence on S and $\eta(S/\rho)$ the least semilattice congruence on S/ρ. Then the following statements are equivalent*:
(1) *ρ is a Clifford congruence on S.*
(2) *ρ^T is a semilattice congruence on S.*
(3) $\rho^T = \rho \vee \eta$.
(4) $\rho^T/\rho = \eta(S/\rho)$.

Proof. (1) ⇒ (2). By hypothesis, we can see $ax^+\rho \quad x^+a$, for $a \in S$ and $x \in S^1$. Because ρ is a (2, 1)-congruence on S and $x^+a^+\rho \quad a^+x^+$, we may obtain

$$(ax)^+ = (ax^+)^+\rho \quad (x^+a)^+ = (x^+a^+)\rho \quad (a^+x^+) = (a^+x)^+,$$

and thus $(ax)^+\rho \quad (a^+x)^+$ so that $a\rho^T a^+$. Therefore, ρ^T is a semilattice congruence on S.

(2) ⇒ (1). Since ρ^T is a semilattice congruence on S, we obtain $a\rho^T a^+$ for $a \in S$, and whence $(ax)^+\rho(a^+x)^+ = a^+x^+\rho x^+a^+$, for $x \in S^1$. Thus, we have $(ax)^+\rho x^+a^+$. Multiplying $(ax)^+\rho x^+a^+$ on the right hand by a, we obtain $(ax)^+a\rho x^+a^+a$, so that $ax^+\rho x^+a$ by $(ax)^+a = ax^+$. Thus, ρ is a Clifford congruence on S.

(2) ⇒ (3). Note that η is the least semilattice congruence on S. By assumption, it follows that $\eta \subseteq \rho^T$, and thus $(\rho \vee \eta) \subseteq \rho^T$ since $\rho \subseteq \rho^T$. According to Theorem 3.2 and Proposition 5.3 (4), we obtain $\rho^T = (\rho \vee \widetilde{\mathcal{R}})^\flat$ and $\widetilde{\mathcal{R}} \subseteq \eta$. Hence, $(\rho \vee \widetilde{\mathcal{R}})^\flat \subseteq (\rho \vee \widetilde{\mathcal{R}}) \subseteq (\rho \vee \eta)$, which is equal to $\rho^T \subseteq (\rho \vee \eta)$. Therefore, we have $\rho^T = \rho \vee \eta$.

(3) ⇒ (2). Clearly, $\eta \subseteq (\rho \vee \eta) = \rho^T$, which together with Corollary 5.4 shows that ρ^T is a semilattice congruence on S.

(3) ⇒ (4). Suppose now that ρ is a (2, 1)-congruence on S. By using Proposition 2.5, $S/(\eta \vee \rho)$ is (2, 1)-isomorphic to $(S/\rho)/((\eta \vee \rho)/\rho)$. Additionally, $\eta \subseteq \eta \vee \rho$, and thus by Corollary 4.6, $\eta \vee \rho$ is a semilattice congruence on S. Hence, $(\eta \vee \rho)/\rho$ is necessarily a semilattice congruence on S/ρ. Note that $\eta(S/\rho)$ is the

least semilattice congruence on S/ρ, so we have $\eta(S/\rho) \subseteq (\eta \vee \rho)/\rho$. On the other hand, define a relation ξ on S as follows:

$$x\xi y \Leftrightarrow (x\rho)\eta(S/\rho)(y\rho).$$

It is not difficult to check that ξ is a (2, 1)-congruence as well as $\rho \subseteq \xi$. Moreover, we can obtain $\xi/\rho = \eta(S/\rho)$. Again by Proposition 2.5, S/ξ is (2, 1)-isomorphic to $(S/\rho)/(\xi/\rho) = (S/\rho)/\eta(S/\rho)$. In this case, ξ is also a semilattice congruence on S, which implies $\eta \subseteq \xi$. Hence, $(\eta \vee \rho) \subseteq \xi$ so that $(\eta \vee \rho)/\rho \subseteq \xi/\rho = \eta(S/\rho)$. From the foregoing argument, we obtain $\eta(S/\rho) = (\eta \vee \rho)/\rho$. Therefore, we conclude $\rho^T/\rho = \eta(S/\rho)$.

(4) $\Rightarrow$ (3). Observing the proof of (3) $\Rightarrow$ (4), it is easy to see $(\eta \vee \rho)/\rho = \eta(S/\rho)$. By hypothesis $\rho^T/\rho = \eta(S/\rho)$, it immediately follows $\rho^T/\rho = (\eta \vee \rho)/\rho$ so that $\rho^T = \rho \vee \eta$. This completes the proof. □

Corollary 5.6. *Let S be a left restriction semigroup. Then η_t is the least Clifford congruence ν on S and $\nu = \eta_t = (\omega_k)_t$, where η is the least semilattice congruence on S. In this case, every* (2, 1)*-congruence ρ is Clifford congruence if and only if $\rho \in [\nu, \omega]$.*

Proof. Obviously, the least Clifford congruence ν exists on S. Note that $\eta \subseteq \eta^T = (\eta_t)^T$, and thus by Corollary 5.4, $(\eta_t)^T$ is a semilattice congruence on S. According to Theorem 5.5 (1), η_t is a Clifford congruence on S so that $\nu \subseteq \eta_t$.

On the other hand, let ρ be any Clifford (2, 1)-congruence on S. By Theorem 5.5 (2), ρ^T is a semilattice congruence on S so that $\eta \subseteq \rho^T$. Using Proposition 3.6, we have

$$\eta_t \subseteq (\rho^T)_t = \rho_t \subseteq \rho,$$

whence $\eta_t \subseteq \nu$ certainly holds. Therefore, we obtain $\nu = \eta_t = (\omega_k)_t$ since $\omega_k = \eta$. □

Proposition 5.7. *Let S be a left restriction semigroup. Then $\omega = \sigma \vee \eta$.*

Proof. In view of Corollaries 5.2 and 5.4, it is not hard to obtain $\sigma = \sim^*$ and $\eta = \widetilde{\mathcal{R}}^*$. Note that $a\omega b$ always holds for $a, b \in S$, and thus

$$a^+ \cdot b^+a = b^+a^+a = (b^+a)^+ \cdot a, \quad (a^+b)^+ \cdot b = a^+b^+b = b^+ \cdot a^+b.$$

This, together with $(b^+a)^+ = (a^+b)^+$, shows $a \sim b^+a\widetilde{\mathcal{R}}a^+b{\sim}b$. so that $\omega \subseteq (\sim\widetilde{\mathcal{R}}\sim)$. By noting that $\sim \subseteq \sigma$ and $\widetilde{\mathcal{R}} \subseteq \eta$, it yields $\omega \subseteq (\sigma\eta\sigma)$, whence $\omega \subseteq (\sigma \vee \eta)$. Consequently, we have proved $\omega = \sigma \vee \eta$. □

A (2, 1)-congruence ρ on S is called *P-unitary congruence*, if S/ρ is P-unitary. In this case, $P(S/\rho) = \{e\rho : e \in P(S)\}$.

Lemma 5.8. *Let S be a left restriction semigroup. If ρ is a* (2, 1)*-congruence on S, then the following conditions are equivalent*:

(1) *S/ρ is P-unitary*;

(2) *$b^+a \in \mathrm{Pker}(\rho)$ implies that $a \in \mathrm{Pker}(\rho)$ for $a, b \in S$*;

(3) *$(x^+y, x^+) \in \rho$ implies $(y, y^+) \in \rho$, for $x, y \in S$.*

Proof. (1) $\Rightarrow$ (2). Let $b^+a \in \mathrm{Pker}(\rho)$. Then $(b^+a)\rho = e\rho \in P(S/\rho)$ for some $e \in P(S)$. Since ρ is a P-unitary congruence, we have $a\rho \in P(S/\rho)$, thereby $a \in \mathrm{Pker}(\rho)$.

(2) $\Rightarrow$ (3). Let $(x^+y, x^+) \in \rho$. Then $x^+y \in \mathrm{Pker}(\rho)$. By means of (2), we have $y \in \mathrm{Pker}(\rho)$ so that $(y, y^+) \in \rho$.

(3) $\Rightarrow$ (1). In view of Proposition 2.4, we only need to show that S/ρ is left P-unitary. Suppose now $(ea)\rho \in P(S/\rho)$ for $e \in P(S)$. Then $(ea, ea^+) \in \rho$, which is equal to $(ea^+a, ea^+) \in \rho$. By hypothesis, we obtain $(a, a^+) \in \rho$ so that $a\rho \in P(S/\rho)$. Therefore, the proof is completed. □

Proposition 5.9. *Let S be a left restriction semigroup. Then $\sigma_k = (\omega_t)_k$ is the least P-unitary (2, 1)-congruence π on S. In addition, if ρ is a P-unitary (2, 1)-congruence, then $\rho \in [\pi, \omega]$.*

Proof. First of all, we claim that σ_k itself is a P-unitary (2, 1)-congruence on S. In fact, let $x, y \in S$. If $x^+y \in \mathrm{Pker}(\sigma_k)$, then from $\mathrm{Pker}(\sigma_k) = \mathrm{Pker}(\sigma)$ it follows $x^+y \in \mathrm{Pker}(\sigma)$. Hence, $(x^+y, x^+y^+) \in \sigma$. According to Corollary 5.2, we obtain $(y, y^+) \in \sigma$ so that $y \in \mathrm{Pker}(\sigma)$. Furthermore, $y \in \mathrm{Pker}(\sigma_k)$ since $\mathrm{Pker}(\sigma) = \mathrm{Pker}(\sigma_k)$. By Lemma 5.8, we can see that σ_k is a P-unitary congruence on S. Hence, the least P-unitary (2, 1)-congruence π exists on S so that $\pi \subseteq \sigma_k$.

Second, we declare that σ_k is contained in any P-unitary (2, 1)-congruence ρ on S. To see this, let $a \in \mathrm{Pker}(\sigma_k)$. Since $\mathrm{Pker}(\sigma_k) = \mathrm{Pker}(\sigma)$, then $a \in \mathrm{Pker}(\sigma)$. Hence, we have $(a, a^+) \in \sigma$. In this case, there is $b \in S$ such that $b^+a = b^+a^+$, which is equal to $(b^+a)^+a = (b^+a)^+$. Clearly, $((b^+a)^+a, (b^+a)^+) \in \rho$. Observe that ρ is a P-unitary congruence on S, using Lemma 5.8 again, it yields $(a, a^+) \in \rho$ so that $a \in \mathrm{Pker}(\rho)$. Thus, we can know $\mathrm{Pker}(\sigma_k) \subseteq \mathrm{Pker}(\rho)$. On the other hand, by using Proposition 4.5, it follows that $(\sigma_k)_k \subseteq \rho_k$. Noting $(\sigma_k)_k = \sigma_k$, it derives $\sigma_k \subseteq \rho_k$. Associating with $\rho_k \subseteq \rho$, thereby $\sigma_k \subseteq \rho$. Because π is also a P-unitary congruence on S, we immediately obtain $\sigma_k \subseteq \pi$. Therefore, $\pi = \sigma_k$ holds.

Finally, the equality $\sigma_k = (\omega_t)_k$ is a direct consequence of $\sigma = \omega_t$. □

Remark 5.10. As already mentioned in the previous paragraphs, for a left restriction semigroup S, the interval $[\sigma, \omega]$ consists of all of its reduced congruences, the interval $[\eta, \omega]$ contains all its semilattice congruences, the interval $[\nu, \omega]$ is the set of all the Clifford congruences and the interval $[\pi, \omega]$ contains all the P-unitary congruences. These (2, 1)-congruences constitute the complete sublattices of $C^{2,1}(S)$, respectively. On the other hand, from the universal algebraic viewpoint, the class of P-unitary left restriction semigroups is a quasivariety. Then by [10, Lemma I.11.14, p. 60], the partially ordered set of all P-unitary (2, 1)-congruences on S under inclusion also constitute a complete sublattice of $C^{2,1}(S)$ with the greatest element ω. Meanwhile, each type of (2, 1)-congruence determines the corresponding left restriction semigroups. In this sense, congruences are usually applied to formulate some remarkable semigroups from a given semigroup and have a great influence on the structure of such a semigroup.

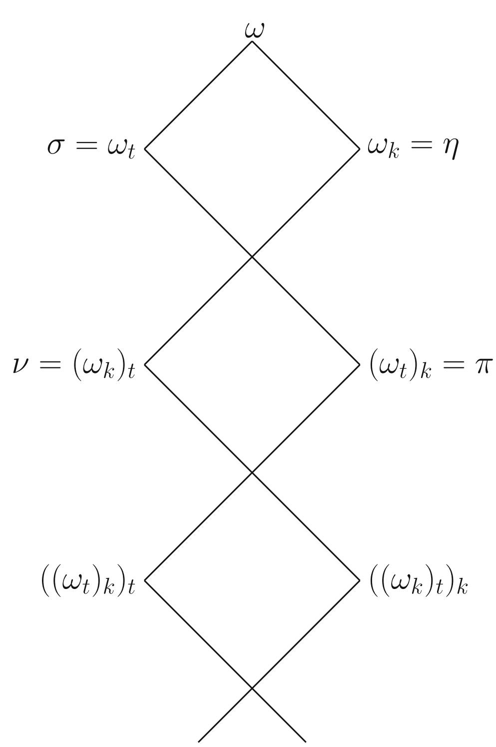

Figure 1: A min network of (2, 1)-congruences.

In closing this article, we shall present a diagram named the min network of (2, 1)-congruences on left restriction semigroups (Figure 1).

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