Research Article

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Existence of positive periodic solutions for first-order nonlinear differential equations with multiple time-varying delays

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Abstract: This study elucidates the sufficient conditions for the first-order nonlinear differential equations with periodic coefficients and time-varying delays to have positive periodic solutions. Our results are proved using the Krasnosel'skii fixed point theorem. In this article, we have identified two sets Δ and ∇ and proved that at least one positive periodic solution exists in the interval between the point belonging to Δ and the point belonging to ∇. We propose simple conditions that guarantee the existence of sets Δ and ∇. In addition, we obtain the necessary conditions for the existence of positive periodic solutions of the first-order nonlinear differential equations when the periodic coefficients satisfy certain conditions. Finally, examples and numerical simulations are used to illustrate the validity of our results.

Keywords: positive periodic solutions, time-varying delays, Krasnosel'skii fixed point theorem, differential equations

MSC 2020: 34A34, 34K13, 47H10, 92D25

1 Introduction

Consider the first-order nonlinear equation

$$x'(t) = -a(t)x(t) + \sum_{i=1}^{n} b_i(t)f(x(t-\tau_i(t))) - \mu\sum_{i=1}^{n} H_i(t)x(t-\tau_i(t)), \tag{1.1}$$

where $t \in \mathbb{R}$. Let $\omega > 0$ and $\mu \in [0, 1]$.

We assume that

(H1) $a : \mathbb{R} \to [0, 1]$ and $b_i : \mathbb{R} \to [0, \infty)$ $(1 \le i \le n)$ are ω-periodic continuous functions satisfying $a(t) = a(t+\omega)$ and $b_i(t) = b_i(t+\omega)$ for $t \in \mathbb{R}$;

(H2) $\tau_i : \mathbb{R} \to [0, \infty)(1 \le i \le n)$ and $H_i : \mathbb{R} \to [0, \infty)(1 \le i \le n)$ are ω-periodic continuous functions satisfying $\tau_i(t) = \tau_i(t+\omega)$ and $H_i(t) = H_i(t+\omega)$ for $t \in \mathbb{R}$;

(H3) $f : [0, \infty) \to [0, \infty)$ is a Lipschitz continuous function, i.e., there is a non-negative constant L, for any x, y, the following inequality holds:

$$|f(x) - f(y)| \le L|x-y|.$$

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Equation (1.1) is a delay differential equation. Delay differential equations are mainly used to describe dynamic systems that rely on current and past historical states. The time-delay phenomenon is widely used in the fields of population dynamics [1–4] and infectious diseases [5,6]. Scholars have shown that delay differential equations can more accurately reflect the changing laws of objective things than differential equations without time delay. In the past few decades, delay differential equations have received the attention of many scholars and achieved certain results.

For equation (1.1), when the nonlinear term $f(x)$ has different expressions, model (1.1) also has different application backgrounds. For example, when the nonlinear term $f(N) = Ne^{-\alpha N}$, then equation (1.1) will degenerate into the following Nicholson's blowflies model [7] through certain assumptions:

$$N'(t) = -\delta N(t) + BN(t-\tau)e^{-\gamma N(t-\tau)},$$

where $N(t)$ is the size of the population at time t, B is the maximum per capita daily egg production rate, δ is per capita daily adult death rate, τ denotes the approximate time of the life cycle, and $\frac{1}{\gamma}$ is the size at which the population reproduces at its maximum rate. Nicholson's blowflies model has received wide attention due to its extensive practical significance, and its theoretical achievements have made remarkable progress in the past few decades, see [8–12]. For example, Li and Du [13] studied the existence of positive periodic solutions of the generalized Nicholson's blowflies model,

$$x'(t) = -\delta(t)x(t) + \sum_{i=1}^{m} p_i(t)x(t-\tau_i(t))e^{-q_i(t)x(t-\tau_i(t))},$$

where m is a positive integer, $p_i, \delta, q_i \in C(\mathbb{R}^+, (0,\infty))$, and $\tau_i \in C(\mathbb{R}^+, \mathbb{R}^+)$ are T-periodic functions for $i = 1, 2, 3, \ldots, m$.

When the nonlinear term $f(x) = \frac{x^m}{1+x^l}$, then model (1.1) represents the hematopoiesis model. The term $\sum_{i=1}^{n} b_i(t)\frac{x^m(t-\tau_i(t))}{1+x^l(t-\tau_i(t))}$ represents the current production (density) of blood cells that is affected by the past blood cell count (density). The hematopoiesis model was proposed by Mackey and Glass in 1977 [14]. Since then, hematopoiesis models have been studied by many scholars [15–19]. For example, Liu et al. [20] used the fixed point theorem to study the existence of the positive periodic solution of the following hematopoiesis model and gave the necessary conditions for the existence of a unique positive periodic solution,

$$x'(t) = -a(t)x(t) + \sum_{i=1}^{n} \frac{b_i(t)}{1 + x^m(t-\tau_i(t))},$$

where $m > 0$ is a constant, a, b_i, τ_i are positive continuous ω-periodic functions.

In recent years, many authors have studied the existence of positive ω-periodic solutions for periodic ecological models with multiple delays. See [21–24] and references therein. However, in this article, we will study the existence of positive periodic solutions of the first-order nonlinear differential equation with multiple time-varying delays. When nonlinear term f takes some special forms, problem (1.1) can degrade into some well-known models. Therefore, it is more universal than literature [7,13,20]. First, we give the existence condition of positive periodic solutions of equation (1.1), then give the uniqueness condition of positive periodic solution by using the contraction mapping principle, and finally give the oscillation condition of periodic solutions.

Here, we introduce the notations required to describe our main results. For each $i = 1, 2, \ldots, n$, let

$$\underline{b}_i = \min_{0\le t\le\omega} b_i(t), \quad \overline{b}_i = \max_{0\le t\le\omega} b_i(t),$$

$$\underline{H}_i = \min_{0\le t\le\omega} H_i(t), \quad \overline{H}_i = \max_{0\le t\le\omega} H_i(t).$$

Define

$$\alpha = \frac{1}{e^{\int_0^\omega a(r)\mathrm{d}r} - 1} \quad \text{and} \quad \beta = \frac{e^{\int_0^\omega a(r)\mathrm{d}r}}{e^{\int_0^\omega a(r)\mathrm{d}r} - 1}.$$

Let

$$\Delta = \left\{ x \in \mathbb{R} : \frac{\alpha\omega\sum_{i=1}^{n}\underline{b}_i \min_{\rho x \le u \le x} f(u)}{\alpha\mu\omega\sum_{i=1}^{n}\overline{H}_i + 1} \ge x > 0 \right\} \quad \text{and} \quad \nabla = \left\{ x \in \mathbb{R} : \frac{\beta\omega\sum_{i=1}^{n}\overline{b}_i \max_{\rho x \le u \le x} f(u)}{\beta\rho\mu\omega\sum_{i=1}^{n}\underline{H}_i + 1} \le x \right\},$$

where

$$\rho = \frac{\alpha}{\beta} = \frac{1}{e^{\int_0^{\omega} a(r)\mathrm{d}r}} \in (0, 1).$$

In this article, we also assume that (H4) holds:

(H4) $\beta\omega\sum_{i=1}^{n}\overline{b}_i > \beta\rho\omega\mu\sum_{i=1}^{n}\underline{H}_i + 1$.

For simplicity, we refer to points belonging to Δ (resp., ∇) as Δ-points (resp., ∇-points) of (1.1). Note that there are no positive real numbers belonging to both Δ and ∇.

2 Auxiliary lemmas and preparations

For convenience, in this section, we would like to introduce some notations, definitions, lemmas, and assumptions which are used in what follows.

Definition 1. [25] Let M be a real Banach space. A nonempty, closed, convex set $P \subset M$ is a cone if it satisfies the following two conditions:

(i) $x \in P, \lambda \ge 0$ imply $\lambda x \in P$;

(ii) $x \in P, -x \in P$ imply $x = \theta$, where θ is the zero element of P.

Definition 2. [25] An operator $K : M \to M$ is completely continuous if it is continuous and maps bounded sets into relatively compact set.

The following is the well-known Krasnoselskii's fixed point theorem in a cone.

Lemma 1. [25] *Let M be a Banach space, and let $P \subset M$ be a cone. Assume that Ω_1 and Ω_2 are open subset of M with $\theta \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$, and let*

$$K : P \cap (\overline{\Omega}_2 \backslash \Omega_1) \to P$$

be a completely continuous operator such that

(i) $\|Ku\| \le \|u\|, u \in P \cap \partial\Omega_1$, *and* $\|Ku\| \ge \|u\|, u \in P \cap \partial\Omega_2$; *or*

(ii) $\|Ku\| \ge \|u\|, u \in P \cap \partial\Omega_1$, *and* $\|Ku\| \le \|u\|, u \in P \cap \partial\Omega_2$.

Then K has a fixed point in $P \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

Let

$$M = \{x \in C(\mathbb{R}, \mathbb{R}) : x(t + \omega) = x(t) \text{ for } t \in \mathbb{R}\}$$

be the Banach space of ω-periodic continuous functions equipped with the norm

$$\|x\| = \max_{t\in\mathbb{R}}|x(t)| = \max_{t\in[0,\omega]}|x(t)|.$$

Define a subset in M by

$$P = \{x \in M : x(t) \ge \rho\|x\|, t \in [0, \omega]\}.$$

It is easy to see that P is a cone in M.

Lemma 2. *The positive ω-periodic function x is a solution of* (1.1) *if and only if $x \in M$ satisfies that*

$$x(t) = \int_t^{t+\omega} \left[\sum_{i=1}^n b_i(s) f(x(s - \tau_i(s))) - \mu \sum_{i=1}^n H_i(s) x(s - \tau_i(s)) \right] G(t, s) \mathrm{d}s, \tag{2.1}$$

where

$$G(t, s) = \frac{e^{\int_t^s a(r)\mathrm{d}r}}{e^{\int_0^\omega a(r)\mathrm{d}r} - 1} \quad \textit{for } t \le s \le t + \omega.$$

Proof. "Only if" part: Let $x \in M$ be a solution of equation (1.1). We have

$$(x'(t) + a(t)x(t))e^{\int_0^t a(r)\mathrm{d}r} = \left[\sum_{i=1}^n b_i(t) f(x(t - \tau_i(t))) - \mu \sum_{i=1}^n H_i(t) x(t - \tau_i(t)) \right] e^{\int_0^t a(r)\mathrm{d}r},$$

which is equivalent to

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(x(t) e^{\int_0^t a(r)\mathrm{d}r} \right) = \left[\sum_{i=1}^n b_i(t) f(x(t - \tau_i(t))) - \mu \sum_{i=1}^n H_i(t) x(t - \tau_i(t)) \right] e^{\int_0^t a(r)\mathrm{d}r}.$$

The integration from t to $t + \omega$ gives

$$x(t + \omega) e^{\int_0^{t+\omega} a(r)\mathrm{d}r} - x(t) e^{\int_0^t a(r)\mathrm{d}r} = \int_t^{t+\omega} \left[\sum_{i=1}^n b_i(s) f(x(s - \tau_i(s))) - \mu \sum_{i=1}^n H_i(s) x(s - \tau_i(s)) \right] e^{\int_0^s a(r)\mathrm{d}r} \mathrm{d}s.$$

By the periodic properties, we obtain that

$$x(t) e^{\int_0^t a(r)\mathrm{d}r} \left[e^{\int_t^{t+\omega} a(r)\mathrm{d}r} - 1 \right] = \int_t^{t+\omega} \left[\sum_{i=1}^n b_i(s) f(x(s - \tau_i(s))) - \mu \sum_{i=1}^n H_i(s) x(s - \tau_i(s)) \right] e^{\int_0^s a(r)\mathrm{d}r} \mathrm{d}s.$$

Thus,

$$x(t) = \int_t^{t+\omega} \left[\sum_{i=1}^n b_i(s) f(x(s - \tau_i(s))) - \mu \sum_{i=1}^n H_i(s) x(s - \tau_i(s)) \right] \frac{e^{\int_t^s a(r)\mathrm{d}r}}{e^{\int_0^\omega a(r)\mathrm{d}r} - 1} \mathrm{d}s$$

for $t \in \mathbb{R}$. Hence, we obtain the expression (2.1).

"If" part: Take the derivative of (2.1) with respect to t to obtain (1.1).

Thus, the proof of Lemma 2 is complete. □

Define a functional $G : \mathbb{R} \times \mathbb{R} \to [0, \infty)$ by

$$G(t, s) = \frac{e^{\int_t^s a(r)\mathrm{d}r}}{e^{\int_0^\omega a(r)\mathrm{d}r} - 1} \quad \text{for } t \le s \le t + \omega.$$

It is clear that G is strictly increasing with respect to the second component $s \in [t, t + \omega]$. Also, we see that

$$G(t, t) = \frac{1}{e^{\int_0^\omega a(r)\mathrm{d}r} - 1} = \alpha$$

and

$$G(t, t + \omega) = \frac{e^{\int_0^\omega a(r)\mathrm{d}r}}{e^{\int_0^\omega a(r)\mathrm{d}r} - 1} = \beta.$$

Hence, when $s \in [t, t+\omega]$ and $t \in \mathbb{R}$, we have

$$\alpha = G(t,t) \le G(t,s) \le G(t,t+\omega) = \beta. \tag{2.2}$$

Moreover, G has the relationship

$$G(t+\omega, s+\omega) = \frac{e^{\int_{t+\omega}^{s+\omega} a(r)\mathrm{d}r}}{e^{\int_0^{\omega} a(r)\mathrm{d}r} - 1} = \frac{e^{\int_t^{s} a(r)\mathrm{d}r}}{e^{\int_0^{\omega} a(r)\mathrm{d}r} - 1} = G(t,s).$$

Define an operator $K : M \to M$ by:

$$(Kx)(t) = \int_t^{t+\omega} \left[\sum_{i=1}^n b_i(s) f(x(s-\tau_i(s))) - \mu \sum_{i=1}^n H_i(s) x(s-\tau_i(s))\right] G(t,s)\mathrm{d}s. \tag{2.3}$$

The operator K has the following properties.

Lemma 3. *The operator K defined by* (2.3) *maps P into P.*

Proof. Any element x of P is ω-periodic because $P \subset M$. According to conditions (H1), (H2), and (2.3), we have

$$\begin{aligned}
(Kx)(t+\omega) &= \int_{t+\omega}^{t+2\omega} \left[\sum_{i=1}^n b_i(s) f(x(s-\tau_i(s))) - \mu \sum_{i=1}^n H_i(s) x(s-\tau_i(s))\right] G(t+\omega, s)\mathrm{d}s \\
&= \int_t^{t+\omega} \left[\sum_{i=1}^n b_i(s+\omega) f(x(s+\omega-\tau_i(s+\omega))) - \mu \sum_{i=1}^n H_i(s+\omega) x(s+\omega-\tau_i(s+\omega))\right] G(t+\omega, s+\omega)\mathrm{d}s \\
&= \int_t^{t+\omega} \left[\sum_{i=1}^n b_i(s) f(x(s-\tau_i(s))) - \mu \sum_{i=1}^n H_i(s) x(s-\tau_i(s))\right] G(t,s)\mathrm{d}s \\
&= (Kx)(t).
\end{aligned}$$

This means that $Kx \in M$. By (2.2), we have

$$(Kx)(t) \le \beta \int_t^{t+\omega} \left[\sum_{i=1}^n b_i(s) f(x(s-\tau_i(s))) - \mu \sum_{i=1}^n H_i(s) x(s-\tau_i(s))\right] \mathrm{d}s \tag{2.4}$$

and

$$(Kx)(t) \ge \alpha \int_t^{t+\omega} \left[\sum_{i=1}^n b_i(s) f(x(s-\tau_i(s))) - \mu \sum_{i=1}^n H_i(s) x(s-\tau_i(s))\right] \mathrm{d}s. \tag{2.5}$$

Form (2.4) and (2.5), we see that

$$(Kx)(t) \ge \frac{\alpha}{\beta}\|Kx\| = \rho\|Kx\|.$$

Hence,

$$KP \subset P.$$

The proof of Lemma 3 is complete. □

Lemma 4. *The operator $K : P \to P$ is completely continuous.*

Proof. We need to verify the following two points:
(i) K is continuous;
(ii) K maps any bounded subset of P into a relatively compact subset of P.

Point (i): Let $\{x_n\}$ converge to x in P. From (2.1), we obtain

$$\begin{aligned}|(Kx_n)(t) - (Kx)(t)| &= \Bigg| \int_t^{t+\omega} \left[\sum_{i=1}^n b_i(s) f(x_n(s - \tau_i(s))) - \mu \sum_{i=1}^n H_i(s) x_n(s - \tau_i(s)) \right] G(t,s) \mathrm{d}s \\ &\quad - \int_t^{t+\omega} \left[\sum_{i=1}^n b_i(s) f(x(s - \tau_i(s))) - \mu \sum_{i=1}^n H_i(s) x(s - \tau_i(s)) \right] G(t,s) \mathrm{d}s \Bigg| \\ &\le \beta \int_t^{t+\omega} \sum_{i=1}^n b_i(s) |f(x_n(s - \tau_i(s))) - f(x(s - \tau_i(s)))| \mathrm{d}s \\ &\quad + \mu\beta \int_t^{t+\omega} \sum_{i=1}^n H_i(s) |x_n(s - \tau_i(s)) - x(s - \tau_i(s))| \mathrm{d}s \\ &\le \left[L\beta\omega \sum_{i=1}^n \overline{b}_i + \mu\beta\omega \sum_{i=1}^n \overline{H}_i \right] \|x_n - x\|,\end{aligned}$$

that is,

$$|(Kx_n)(t) - (Kx)(t)| \to 0 \text{ as } n \to \infty.$$

Therefore, operator K is continuous.

Point (ii): For any $x \in P$, $t \in \mathbb{R}$, we have

$$\begin{aligned}|(Kx)(t)| &= \left| \int_t^{t+\omega} \left[\sum_{i=1}^n b_i(s) f(x(s - \tau_i(s))) - \mu \sum_{i=1}^n H_i(s) x(s - \tau_i(s)) \right] G(t,s) \mathrm{d}s \right| \\ &\le \beta \left[\int_t^{t+\omega} \sum_{i=1}^n |b_i(s) f(x(s - \tau_i(s)))| \mathrm{d}s + \int_t^{t+\omega} \mu \sum_{i=1}^n |H_i(s) x(s - \tau_i(s))| \mathrm{d}s \right] \\ &\le \beta\omega \sum_{i=1}^n (\overline{b}_i f^* + \mu \overline{H}_i x_i^*) < +\infty,\end{aligned}$$

where

$$f^* = \max_{x \in P} f(x), \quad x_i^* = \max_{t \in [0,\omega]} x(t - \tau_i(t)).$$

Therefore, KP is uniformly bounded.

On the other hand, let $x \in P$ and $t_1, t_2 \in [0, \omega]$ with $t_1 < t_2$. We have

$$\begin{aligned}|(Kx)(t_2) - (Kx)(t_1)| &= \Bigg| \int_{t_2}^{t_2+\omega} \left[\sum_{i=1}^n b_i(s) f(x(s - \tau_i(s))) - \mu \sum_{i=1}^n H_i(s) x(s - \tau_i(s)) \right] G(t_2,s) \mathrm{d}s \\ &\quad - \int_{t_1}^{t_1+\omega} \left[\sum_{i=1}^n b_i(s) f(x(s - \tau_i(s))) - \mu \sum_{i=1}^n H_i(s) x(s - \tau_i(s)) \right] G(t_1,s) \mathrm{d}s \Bigg| \\ &= \left| \int_{t_2}^{t_1} \left[\sum_{i=1}^n b_i(s) f(x(s - \tau_i(s))) - \mu \sum_{i=1}^n H_i(s) x(s - \tau_i(s)) \right] G(t_2,s) \mathrm{d}s \right|\end{aligned}$$

$$+\left|\int_{t_1}^{t_1+\omega}\left[\sum_{i=1}^{n}b_i(s)f(x(s-\tau_i(s)))-\mu\sum_{i=1}^{n}H_i(s)x(s-\tau_i(s))\right]G(t_2,s)\mathrm{d}s\right|$$
$$+\left|\int_{t_1+\omega}^{t_2+\omega}\left[\sum_{i=1}^{n}b_i(s)f(x(s-\tau_i(s)))-\mu\sum_{i=1}^{n}H_i(s)x(s-\tau_i(s))\right]G(t_2,s)\mathrm{d}s\right|$$
$$-\left|\int_{t_1}^{t_1+\omega}\left[\sum_{i=1}^{n}b_i(s)f(x(s-\tau_i(s)))-\mu\sum_{i=1}^{n}H_i(s)x(s-\tau_i(s))\right]G(t_1,s)\mathrm{d}s\right|$$
$$\leq 2\beta\sum_{i=1}^{n}(\overline{b}_i f^*+\mu\overline{H}_i x_i^*)|t_2-t_1|+\beta\sum_{i=1}^{n}(\overline{b}_i f^*+\mu\overline{H}_i x_i^*)\int_{t_1}^{t_1+\omega}|G(t_2,s)-G(t_1,s)|\mathrm{d}s.$$

We can write

$$|G(t_2,s)-G(t_1,s)|=\frac{1}{e^{\int_0^{\omega}a(r)\mathrm{d}r}-1}\left|e^{\int_{t_2}^{s}a(r)\mathrm{d}r}-e^{\int_{t_1}^{s}a(r)\mathrm{d}r}\right|\leq\frac{e^{\int_{t_2}^{s}a(r)\mathrm{d}r}}{e^{\int_0^{\omega}a(r)\mathrm{d}r}-1}\left|1-e^{\int_{t_1}^{t_2}a(r)\mathrm{d}r}\right|.$$

This immediately implies that

$$\int_{t_1}^{t_1+\omega}|G(t_2,s)-G(t_1,s)|\mathrm{d}s\leq\int_{t_1}^{t_1+\omega}\frac{e^{\int_{t_2}^{s}a(r)\mathrm{d}r}}{e^{\int_0^{\omega}a(r)\mathrm{d}r}-1}\left|1-e^{\int_{t_1}^{t_2}a(r)\mathrm{d}r}\right|\mathrm{d}s\leq\omega\beta|t_2-t_1|.$$

Therefore, for any $x\in P$, we obtain

$$\begin{aligned}|(Kx)(t_2)-(Kx)(t_1)|&\leq 2\beta\sum_{i=1}^{n}(\overline{b}_i f^*+\mu\overline{H}_i x_i^*)|t_2-t_1|+\beta\sum_{i=1}^{n}(\overline{b}_i f^*+\mu\overline{H}_i x_i^*)\omega\beta|t_2-t_1|\\&\leq\left(2\beta\sum_{i=1}^{n}(\overline{b}_i f^*+\mu\overline{H}_i x_i^*)+\beta\sum_{i=1}^{n}(\overline{b}_i f^*+\mu\overline{H}_i x_i^*)\omega\beta\right)|t_2-t_1|\\&\leq\beta\sum_{i=1}^{n}(\overline{b}_i f^*+\mu\overline{H}_i x_i^*)(2+\omega\beta)|t_2-t_1|,\end{aligned}$$

which implies that the operator K is equicontinuous. By using the Ascoli-Arzela theorem, $K:P\to P$ is relatively compact. Hence, $K:P\to P$ is completely continuous. □

3 Simple conditions to ensure the existence of ∇-point and Δ-point

We can prove that (1.1) has sufficient conditions for ∇-point and Δ-point. For ∇-point, we can choose ε such that

$$\frac{\beta\omega\sum_{i=1}^{n}\overline{b}_i}{\beta\rho\omega\mu\sum_{i=1}^{n}\underline{H}_i+1}<\frac{1}{\varepsilon}.$$

If function f satisfies

$$\lim_{x\to 0}\frac{f(x)}{x}=0, \tag{3.1}$$

then we can find a sufficiently small number $x^*>0$ such that $f(x)<\varepsilon x$ for $0\leq x\leq x^*$. Since f is continuous on $[0,\infty)$, we can see that $f(x^*)<\varepsilon x^*$. Hence, from (3.1), we have

$$\frac{\beta\omega\sum_{i=1}^{n}\overline{b}_i\max_{\rho x^*\leq x\leq x^*}f(x)}{\beta\rho\omega\mu\sum_{i=1}^{n}\underline{H}_i+1}<\frac{1}{\varepsilon}\max_{\rho x^*\leq x\leq x^*}f(x)\leq x^*.$$

Hence, we see that $x^*\in\nabla$.

For Δ-point, we can choose G such that

$$\frac{\alpha\omega\sum_{i=1}^{n}\underline{b}_i}{\alpha\omega\mu\sum_{i=1}^{n}\overline{H}_i + 1} > \frac{1}{G}.$$

If for given G, there exists $M > 0$, so that when $x > M = \rho x_*$, there is $f(x) > Gx$. Then, we have

$$\frac{\alpha\omega\sum_{i=1}^{n}\underline{b}_i \min_{\rho x_*\le x\le x_*} f(x)}{\alpha\omega\mu\sum_{i=1}^{n}\overline{H}_i + 1} > \frac{1}{G}\min_{\rho x_*\le x\le x_*} f(x) \ge x_*.$$

Hence, we see that $x_* \in \Delta$.

4 Main results

Theorem 1. *Under conditions* $(H1)$–$(H3)$, *if there exist a* Δ*-point* x_* *and a* ∇*-point* x^* *of* (1.1), *then equation* (1.1) *has at least one positive* ω*-periodic solution* $\hat{x}$ *satisfying* $\min\{x_*, x^*\} \le \|\hat{x}\| \le \max\{x_*, x^*\}$.

Proof. To apply Lemma 1, we have to only find open bounded subsets Ω_1 and Ω_2 of M that satisfy either conditions (i) and (ii). Here, we define the subsets Ω_1 and Ω_2 of M by

$$\Omega_1 = \{x \in M : \|x\| < \min\{x_*, x^*\}\}$$

and

$$\Omega_2 = \{x \in M : \|x\| < \max\{x_*, x^*\}\},$$

respectively. Any ∇-point x^* and any Δ-point x_* never have the same value. Hence, the inclusion relation $\theta \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$ holds. And from Lemma 4, $K : P \cap (\overline{\Omega}_2\backslash\Omega_1) \to P$ is completely continuous. There are two cases to consider depending on which is larger, x_* or x^*. We present the proof only of the case $x_* < x^*$ because the proofs of both cases are essentially the same.

Any ω-periodic function x of $P \cap \partial\Omega_1 \subset M$ satisfies $x(t) \ge \rho\|x\| = \rho x_*$ for $t \in \mathbb{R}$. Hence, we have

$$\rho x_* \le x(s - \tau_i(s)) \le x_* \quad \text{for } s \in \mathbb{R} \quad \text{and} \quad i = 1, 2, \ldots, n. \tag{4.1}$$

By (2.2) and (4.1), we obtain

$$\begin{aligned}(Kx)(t) &\ge \alpha \int_t^{t+\omega} \left[\sum_{i=1}^{n} b_i(s)f(x(s - \tau_i(s))) - \mu\sum_{i=1}^{n} H_i(s)x(s - \tau_i(s))\right]\mathrm{d}s \\ &\ge \alpha\omega\sum_{i=1}^{n}\underline{b}_i \min_{\rho x_*\le x\le x_*} f(x) - \alpha\omega\mu\sum_{i=1}^{n}\overline{H}_i x_* \\ &\ge x_*\end{aligned}$$

for $t \in \mathbb{R}$. From these inequalities, we can see that $\|Kx\| \ge x_* = \|x\|$ for $x \in P \cap \partial\Omega_1$.

Any ω-periodic function x of $P \cap \partial\Omega_2 \subset M$ satisfies $x(t) \ge \rho\|x\| = \rho x^*$ for $t \in \mathbb{R}$. Hence, we have

$$\rho x^* \le x(s - \tau_i(s)) \le x^* \quad \text{for } s \in \mathbb{R} \quad \text{and} \quad i = 1, 2, \ldots, n. \tag{4.2}$$

By (2.2) and (4.2), we obtain

$$\begin{aligned}(Kx)(t) &\le \beta \int_t^{t+\omega} \left[\sum_{i=1}^{n} b_i(s)f(x(s - \tau_i(s))) - \mu\sum_{i=1}^{n} H_i(s)x(s - \tau_i(s))\right]\mathrm{d}s \\ &\le \beta\omega\sum_{i=1}^{n}\overline{b}_i \max_{\rho x^*\le x\le x^*} f(x) - \beta\rho\omega\mu\sum_{i=1}^{n}\underline{H}_i x^* \\ &\le x^*\end{aligned}$$

for $t \in \mathbb{R}$. From these inequalities, we can see that $\|Kx\| \le x^* = \|x\| $ for $x \in P \cap \partial\Omega_2$.

Thus, we have confirmed that assumption (ii) of Lemma 1 is satisfied in the case $x_* \le x^*$ (the assumption (i) of Lemma 1 holds in the case $x_* \ge x^*$). From Lemma 1, we can conclude that the operator K has a fixed point $\hat{x}$ in $P \cap \partial(\overline{\Omega}_2 \backslash \Omega_1)$. Hence, the fixed point $\hat{x}$ is a positive ω-periodic solution that satisfied the properties

$$\hat{x}(t) \ge \rho\|\hat{x}\| \quad \text{for } t \in \mathbb{R} \quad \text{and} \quad x_* \le \|\hat{x}\| \le x^*.$$

Then, $\hat{x}$ is a positive ω-periodic solution to equation (1.1). The proof is complete. □

Theorem 2. *If $L\beta\omega\sum_{i=1}^n \overline{b}_i + \mu\beta\omega\sum_{i=1}^n \overline{H}_i < 1$ holds, then equation* (1.1) *has a unique positive ω-periodic solution x.*

Proof. From Lemma 4, we know that K is a continuous operator from P to P. For any $x, y \in P$, we have

$$\begin{aligned}\|(Kx)(t) - (Ky)(t)\| &= \Bigg\| \int_t^{t+\omega} \Bigg[\sum_{i=1}^n b_i(s) f(x(s-\tau_i(s))) - \mu \sum_{i=1}^n H_i(s) x(s-\tau_i(s))\Bigg] G(t,s)\mathrm{d}s \\ &\quad - \int_t^{t+\omega} \Bigg[\sum_{i=1}^n b_i(s) f(y(s-\tau_i(s))) - \mu \sum_{i=1}^n H_i(s) y(s-\tau_i(s))\Bigg] G(t,s)\mathrm{d}s \Bigg\| \\ &\le \beta \int_t^{t+\omega} \sum_{i=1}^n b_i(s) \|f(x(s-\tau_i(s))) - f(y(s-\tau_i(s)))\| \mathrm{d}s \\ &\quad + \mu\beta \int_t^{t+\omega} \sum_{i=1}^n H_i(s) \|x(s-\tau_i(s)) - y(s-\tau_i(s))\| \mathrm{d}s \\ &\le \Bigg[L\beta\omega \sum_{i=1}^n \overline{b}_i + \mu\beta\omega \sum_{i=1}^n \overline{H}_i\Bigg] \|x-y\|.\end{aligned}$$

According to condition $L\beta\omega\sum_{i=1}^n \overline{b}_i + \mu\beta\omega\sum_{i=1}^n \overline{H}_i < 1$, K is a compressed map. Thus, from the contraction mapping principle, equation (1.1) has a unique positive ω-periodic solution x. The proof of Theorem 2 is complete. □

5 Necessary condition

For convenience, let us make an assumption:

$$H_{\inf}(t) = \liminf_{x\to 0^+ i\in I_n} \frac{\mu\sum_{i=1}^n H_i(t) x(t-\tau_i(t))}{x(t)}, \quad H_{\sup}(t) = \limsup_{x\to 0^+ \, i\in I_n} \frac{\mu\sum_{i=1}^n H_i(t) x(t-\tau_i(t))}{x(t)},$$

where $I_n = \{1, 2, 3, \dots, n\}$.

Theorem 3. *Assume that* $(H1)$–$(H4)$ *hold and that*

$$a(t) + H_{\sup}(t) \ge a(t) + H_{\inf}(t) \ge \sum_{i=1}^n b_i(t) \tag{5.1}$$

for all t. Then every positive solution of equation (1.1) *tends to zero as $t \to \infty$.*

Proof. Let $x(t)$ be any positive solution of equation (1.1). Equation (1.1) can be changed into

$$\frac{\mathrm{d}}{\mathrm{d}t}\Bigg(x(t) e^{\int_0^t \left[a(r) + \frac{\mu\sum_{i=1}^n H_i(r) x(r-\tau_i(r))}{x(r)}\right]\mathrm{d}r}\Bigg) = \sum_{i=1}^n b_i(t) f(x(t-\tau_i(t))) e^{\int_0^t [a(r) + \frac{\mu\sum_{i=1}^n H_i(r) x(r-\tau_i(r))}{x(r)}]\mathrm{d}r}.$$

Integrating the above from $t_0 > 0$ to $t > t_0$, we have

$$x(t) = x(t_0)e^{-\int_{t_0}^{t}\left[a(r)+\frac{\mu\sum_{i=1}^{n}H_i(r)x(r-\tau_i(r))}{x(r)}\right]\mathrm{d}r} + \int_{t_0}^{t}\sum_{i=1}^{n}b_i(s)f(x(s-\tau_i(s)))e^{\int_t^s\left[a(r)+\frac{\mu\sum_{i=1}^{n}H_i(r)x(r-\tau_i(r))}{x(r)}\right]\mathrm{d}r}\mathrm{d}s.$$

From (3.1) and (5.1),

$$\begin{aligned} x(t) &\le x(t_0)e^{-\int_{t_0}^{t}\left[a(r)+\frac{\mu\sum_{i=1}^{n}H_i(r)x(r-\tau_i(r))}{x(r)}\right]\mathrm{d}r} + \varepsilon\int_{t_0}^{t}\sum_{i=1}^{n}b_i(s)x(s-\tau_i(s))e^{\int_t^s[a(r)+H_{\sup}(r)]\mathrm{d}r}\mathrm{d}s \\ &\le x(t_0)e^{-\int_{t_0}^{t}[a(r)+H_{\inf}(r)]\mathrm{d}r} + \varepsilon\int_{t_0}^{t}[a(s)+H_{\sup}(s)]x(s-\tau_i(s))e^{\int_t^s[a(r)+H_{\sup}(r)]\mathrm{d}r}\mathrm{d}s. \end{aligned} \tag{5.2}$$

Let $\zeta = \lim_{t\to\infty}\sup x(t)$, then $0 \le \zeta < \infty$. Below we prove that $\zeta = 0$. We divide it into three cases.

Case 1. When $x'(t) > 0$. Choose $t_0 > 0$ such that $x'(t) > 0$ for $t > t_0$. Then $0 < x(t_0 - \tau_i(t_0)) < x(t - \tau_i(t)) < x(t)$ for $t > t_0$. From (1.1),

$$\begin{aligned} 0 &< -a(t)x(t) + \sum_{i=1}^{n}b_i(t)f(x(t-\tau_i(t))) - \mu\sum_{i=1}^{n}H_i(t)x(t-\tau_i(t)) \\ &\le -x(t)\left[a(t) + \frac{\mu\sum_{i=1}^{n}H_i(t)x(t-\tau_i(t))}{x(t)}\right] + \sum_{i=1}^{n}b_i(t)f(x(t-\tau_i(t))) \\ &\le -x(t)[a(t)+H_{\inf}(t)] + \varepsilon\sum_{i=1}^{n}b_i(t)x(t-\tau_i(t)) \\ &\le -x(t)[a(t)+H_{\inf}(t)] + \varepsilon\sum_{i=1}^{n}b_i(t)x(t) \\ &\le x(t)[\varepsilon\sum_{i=1}^{n}b_i(t) - a(t) - H_{\inf}(t)] \\ &\le 0. \end{aligned}$$

This contradiction shows that Case 1 is impossible.

Case 2. When $x'(t) < 0$. Choose $t_0 > 0$ such that $x'(t) < 0$ for $t > t_0$. Then $\zeta < x(t - \tau_i(t)) < x(t_0 - \tau_i(t_0))$ for $t > t_0$. From (5.1) and (5.2), we have

$$\begin{aligned} x(t) &\le x(t_0)e^{-\int_{t_0}^{t}[a(r)+H_{\inf}(r)]\mathrm{d}r} + \varepsilon\int_{t_0}^{t}[a(s)+H_{\sup}(s)]x(s-\tau_i(s))e^{\int_t^s[a(r)+H_{\sup}(r)]\mathrm{d}r}\mathrm{d}s \\ &\le x(t_0)e^{-\int_{t_0}^{t}[a(r)+H_{\inf}(r)]\mathrm{d}r} + \varepsilon\max_{i\in I_n}x(t_0-\tau_i(t_0))\left[1 - e^{-\int_{t_0}^{t}[a(r)+H_{\sup}(r)]\mathrm{d}r}\right]. \end{aligned} \tag{5.3}$$

Let $t \to \infty$ in (5.3), we obtain

$$\zeta \le \varepsilon\max_{i\in I_n}x(t_0-\tau_i(t_0)).$$

Again let $t_0 \to \infty$ in the above, we have that $\zeta \le \varepsilon\zeta$. Because $\varepsilon \in (0, 1)$, which implies that $\zeta = 0$.

Case 3. When $x'(t)$ is oscillatory. In this case, there is t_n with $t_n \to \infty$ as $n \to \infty$ such that

$$x'(t_n) = 0 \quad \text{for } n = 1, 2, \ldots, \quad \text{and} \quad \lim_{n\to\infty}x(t_n) = \zeta.$$

From (1.1) and (3.1), we have

$$a(t_n)x(t_n) + \mu\sum_{i=1}^{n}H_i(t_n)x(t_n-\tau_i(t_n)) = \sum_{i=1}^{n}b_i(t_n)f(x(t_n-\tau_i(t_n))) \le \varepsilon\sum_{i=1}^{n}b_i(t_n)x(t_n-\tau_i(t_n)).$$

Transforming the above formula, we have

$$x(t_n)[a(t_n) + H_{\inf}(t_n)] \le \varepsilon \sum_{i=1}^{n} b_i(t_n) x(t_n - \tau_i(t_n)). \tag{5.4}$$

Choose $l = l(n) \in I_n$ such that

$$x(t_n) - \tau_l(t_n) = \max_{i \in I_n} x(t_n - \tau_i(t_n)). \tag{5.5}$$

From (5.4) and (5.5), we have

$$x(t_n)[a(t_n) + H_{\inf}(t_n)] \le \varepsilon x(t_n - \tau_l(t_n)) \sum_{i=1}^{n} b_i(t_n). \tag{5.6}$$

Set $\xi = \lim_{n\to\infty} \sup x(t_n - \tau_l(t_n))$, then $\xi \le \zeta$. Thus, (5.6) becomes

$$\zeta[a(t_n) + H_{\inf}(t_n)] \le \varepsilon \xi \sum_{i=1}^{n} b_i(t_n) \le \varepsilon \zeta \sum_{i=1}^{n} b_i(t_n),$$

which is a contradiction. So we have that $\zeta = 0$. The proof is complete. □

From Theorem 3, we have the following results immediately.

Corollary 1. *Let* (*H*1)–(*H*4) *and* (5.1) *hold. Then equation* (1.1) *has no positive* ω*-periodic solution.*

Corollary 2. *Let* (*H*1)–(*H*4) *hold, and let* $a(t) + H_{\sup}(t) < \sum_{i=1}^{n} b_i(t)$. *Then equation* (1.1) *has at least one positive* ω*-periodic solution.*

6 Example and numerical simulation

In this section, we give an example to illustrate the correctness of our main results.

Example 1. Consider the delayed periodic Nicholson's blowflies models with a time-varying delay:

$$x'(t) = -\left(\frac{1}{2} + \frac{1}{3}\sin t\right)x(t) + (5 + 2\sin t)x(t - \tau(t))e^{-\frac{1}{5}(2+\cos t)x(t-\tau(t))} - (1 + \sin t)x(t - \tau(t)). \tag{6.1}$$

Proof. Note that $a(t) = \frac{1}{2} + \frac{1}{3}\sin t$, $b(t) = 5 + 2\sin t$, $\tau(t) = 1 - \cos t$, $H(t) = 2 + 2\sin t$, $\mu = \frac{1}{2}$, and $\omega = 2\pi$. There are

$$\bar{b}(t) = 7, \quad \underline{b}(t) = 3, \quad \overline{H}(t) = 4, \quad \underline{H}(t) = 0.$$

Then

$$\alpha = \frac{1}{e^{\int_0^\omega a(r)\mathrm{d}r} - 1} = \frac{1}{e^\pi - 1}, \quad \beta = \frac{e^{\int_0^\omega a(r)\mathrm{d}r}}{e^{\int_0^\omega a(r)\mathrm{d}r} - 1} = \frac{e^\pi}{e^\pi - 1}.$$

Thus,

$$P = \left\{x \in M : x(t) \ge \frac{1}{e^\pi}\|x\|, \quad t \in [0, 2\pi]\right\}.$$

According to the definition of sets Δ and ∇, we can easily find points belonging to sets Δ and ∇ satisfy all the conditions of Theorem 1. Hence, equation (6.1) has a positive 2π-periodic solution. This fact is verified by the numerical simulation in Figure 1. □

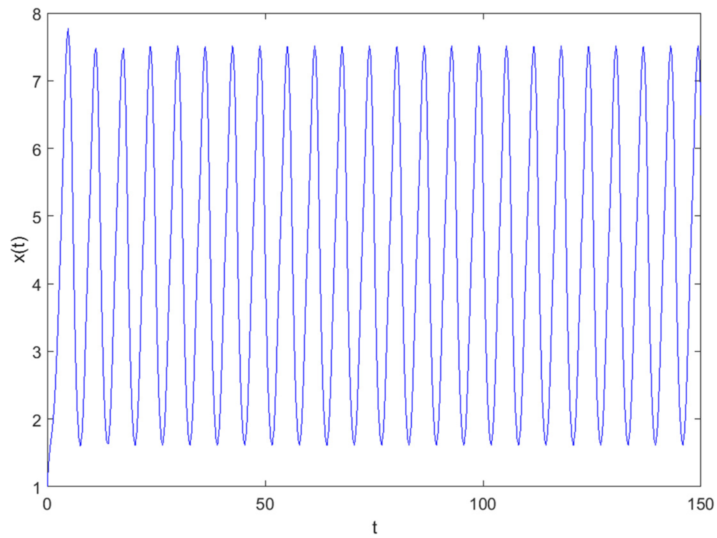

Figure 1: Numerical solution $x(t)$ of equation (6.1) for initial value $\varphi(t) \equiv 1$.

Example 2. Consider the delayed periodic Nicholson's blowflies models with time delay:

$$x'(t) = -(2 + \sin t)x(t) + (1 + \sin t)x(t - 2)e^{-\frac{1}{5}(2+\cos t)x(t-2)} - (2 + \sin t)x(t - 2). \tag{6.2}$$

Proof. Let $\tau = 2$ and $\mu = 1$, then we note

$$a(t) = 2 + \sin t, \quad b(t) = 1 + \sin t, \quad H(t) = 2 + \sin t.$$

Then, we have

$$a(t) + H(t) = 4 + 2\sin t > 1 + \sin t = b(t).$$

Therefore, from Theorem 3, every positive solution of equation (6.2) tends to zero as $t \to \infty$. This fact is verified by the numerical simulation in Figure 2. □

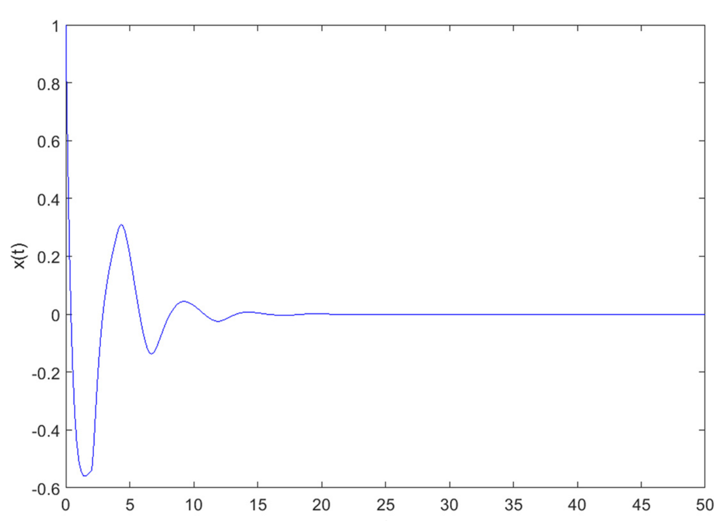

Figure 2: Numerical solution $x(t)$ of equation (6.2) for initial value $\phi(t) \equiv 1$.

7 Conclusion

In this work, we used Ascoli-Arzela and Krasnosel'skii fixed point theorems and some useful properties of Green's function to establish the existence of at least one positive periodic solution for our equation. In biology, equation (1.1) can be used to describe the relevant dynamic behavior of different single species, such as Nicholson's blowflies model and hematopoiesis model. The research work of this article enriches and supplements the findings in the literature and differ from those of [7,13,14,20] in two aspects.

First, when f takes some special forms, equation (1.1) can degrade into some well-known models. And equation (1.1) contains many influencing factors such as periodic coefficient, death term, harvesting term and time-varying delays, so the research results of equation (1.1) are applicable to many problems.

Second, we prove the existence, uniqueness, and oscillations of the period solutions of equation (1.1) and give the relationship between birth rate, death and harvesting rate when the periodic solution tends to zero.

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