Research Article

Wei Cheng*, Yi-Liang Liu, and Qi Zhao

A posteriori regularization method for the two-dimensional inverse heat conduction problem

https://doi.org/10.1515/math-2022-0489
received March 25, 2021; accepted August 9, 2022

Abstract: In this article, we consider a two-dimensional inverse heat conduction problem that determines the surface temperature distribution from measured data at the fixed location. This problem is severely ill-posed, i.e., the solution does not depend continuously on the data. A quasi-boundary value regularization method in conjunction with the *a posteriori* parameter choice strategy is proposed to solve the problem. A Hölder-type error estimate between the approximate solution and its exact solution is also given. The error estimate shows that the regularized solution is dependent continuously on the data.

Keywords: ill-posed problem, inverse heat conduction problem, regularization, *a posteriori* parameter choice strategy, error estimate

MSC 2020: 65M30, 35R30, 35R25

1 Introduction

The inverse heat conduction problem (IHCP) arises from many physical and engineering problems such as nuclear physics, aerospace, food science, metallurgy, and nondestructive testing. It is well known that the IHCP is severely ill-posed in Hadamard's sense [1], i.e., the solution does not depend continuously on the data, any small error in the measurement can induce an enormous error in computing the unknown solution. Therefore, some regularization techniques are needed to restore the stability of the solution to the problem [2–5].

As we know, many authors have studied IHCPs with different regularization methods. These methods include the Fourier method [6–8], the Tikhonov method [9–11], the method of fundamental solutions [12,13], the mollification method [14–16], the wavelet-Galerkin method [17,18], the wavelet method [19–22], the variational method [23], and so on. However, to the authors' knowledge, most of the aforementioned methods focus on the one-dimensional IHCP. A few works based on numerical methods have been presented for the two-dimensional IHCP. This article will investigate the following two-dimensional IHCP:

$$\begin{cases} u_t = u_{xx} + u_{yy}, & 0 < x < 1,\ y > 0,\ t > 0, \\ u(0, y, t) = g(y, t), & y \ge 0,\ t \ge 0, \\ u_x(0, y, t) = 0, & y \ge 0,\ t \ge 0, \\ u(x, 0, t) = 0, & 0 \le x \le 1,\ t \ge 0, \\ u(x, y, 0) = 0, & 0 \le x \le 1,\ y \ge 0, \end{cases} \tag{1.1}$$

*** Corresponding author: Wei Cheng,** College of Science, Henan University of Technology, Zhengzhou 450001, P. R. China, e-mail: chwei2007@163.com
Yi-Liang Liu: School of Air Transport and Engineering, Nanhang Jincheng College, Nanjing, 211156, P. R. China, e-mail: 542967565@qq.com
Qi Zhao: College of Science, Henan University of Technology, Zhengzhou 450001, P. R. China, e-mail: zhaoqi155155@163.com

where g denotes the temperature history at fixed $x = 0$. We want to recover the temperature distribution $u(x, \cdot, \cdot)$ for $0 < x < 1$ from temperature measurement $g^\delta(y, t)$. In this article, we will apply a quasi-boundary value method to solve the problem (1.1) and provide a Hölder-type error estimate between the approximate solution and its exact solution.

The quasi-boundary value method is a regularization technique that replaces the boundary condition or final condition with a new approximate condition. This regularization method has been used for solving the backward heat conduction problem [24–27], the Cauchy problem for elliptic equations [28,29], the IHCP [30], and the inverse source identification problem [31,32]. In this article, we will use a quasi-boundary value method to solve the ill-posed problem (1.1).

Kurpisz and Nowak [33] used the boundary element method for solving the two-dimensional IHCP. Qian and Fu [34] applied a quasi-reversibility method and a Fourier method to solve the two-dimensional IHCP (1.1) and gave some quite sharp error estimates for the regularized solution. A differential-difference regularization method was used to deal with the two-dimensional IHCP (1.1) [35]. Wei and Gao [36] solved a two-dimensional IHCP by a meshless manifold method, which is based on the moving least-square method and the finite cover approximation theory in the mathematical manifold. Bergagio et al. [37] used the iterative finite-element algorithm to solve two-dimensional nonlinear IHCPs. It is worth noting that most of the aforementioned works apply *a priori* regularization parameter choice rule, which usually depends on both the noise level and the *a priori* bound. In practice, the an *a priori* bound cannot be known exactly. In this article, we will apply a quasi-boundary value method combined with *a posteriori* regularization parameter choice rule to solve the ill-posed problem (1.1).

Some researchers are dealing with the error estimate under an *a posteriori* parameter choice strategy. Engl and Gfrerer [38] applied the *a posteriori* parameter choice for general regularization methods to solve linear ill-posed problems. Shi et al. [39] gave *a posteriori* parameter choice strategy for the convolution regularization method. Adler et al. [40] used an *a posteriori* parameter choice strategy for the weak Galerkin least squares method. Trong and Hac [41] applied a modified version of the quasi-boundary value method with *a priori* and *a posteriori* parameter choice strategies to solve time-space fractional diffusion equations. Duc et al. [42] gave *a posteriori* parameter choice strategy for the Tikhonov-type regularization to deal with the backward heat equations with a time-dependent coefficient.

The widely used method for the *a posteriori* parameter choice is Morozov's discrepancy principle, i.e., matching the error of the approximate solution with the accuracy of the initial data of the ill-posed problem. This discrepancy principle is first seen in [43]. Then the discrepancy principle has been used for solving different problems. Scherzer [44] used it for the Tikhonov regularization for the nonlinear ill-posed problems. Bonesky [45] applied it to select the regularization parameter for the Tikhonov regularization method for the linear operator equation. Fu et al. [46] considered it for the Cauchy problem for the Helmholtz equation with application to the Fourier regularization method. Feng et al. [47] investigated a backward problem for a time-space fractional diffusion equation, and obtained the order optimal convergence rates by using Morozov's discrepancy principle and an *a priori* regularization parameter choice rule. In this article, we will use Morozov's discrepancy principle to select the regularization parameter for a quasi-boundary value regularization method.

In order to use the Fourier transform technique, we extend the functions $u(x, \cdot, \cdot,), g(\cdot, \cdot), g^\delta(\cdot, \cdot)$, to be whole real (y, t) plane by defining them to be zero everywhere in (y, t), $y < 0$, $t < 0$. We assume that these functions are in $L^2(\mathbb{R}^2)$ and wish to determine the temperature distribution $u(x, \cdot, \cdot) \in L^2(\mathbb{R}^2)$ for $0 < x < 1$ from the temperature measurement $g^\delta(\cdot, \cdot) \in L^2(\mathbb{R}^2)$. We also use the corresponding L^2 norm as follows:

$$\|f\| = \left(\int_{\mathbb{R}^2} |f(y, t)|^2 \mathrm{d}y\mathrm{d}t \right)^{\frac{1}{2}}. \tag{1.2}$$

Let

$$\hat{h}(\xi, \eta) = \frac{1}{2\pi} \int_{\mathbb{R}^2} h(y, t) e^{-i(\xi y + \eta t)} \mathrm{d}y\mathrm{d}t$$

be the Fourier transform of function $h(y,t) \in L^2(\mathbb{R}^2)$. Using Fourier transform on both sides of (1.1) with respect to the variable y and t, we can obtain the formal solution of problem (1.1) in the frequency domain as:

$$\hat{u}(x,\xi,\eta) = \hat{g}(\xi,\eta)\cosh(x\vartheta(\xi,\eta)), \tag{1.3}$$

then using the inverse Fourier transform on (1.3), we have the formal solution of problem (1.1):

$$u(x,y,t) = \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{i(\xi y+\eta t)}\hat{g}(\xi,\eta)\cosh(x\vartheta(\xi,\eta))\mathrm{d}\xi\mathrm{d}\eta, \tag{1.4}$$

where ξ and η are the variables of Fourier transform on y and t, respectively, and

$$\vartheta(\xi,\eta) = \sqrt{\xi^2 + i\eta}. \tag{1.5}$$

From (1.5), we obtain

$$\vartheta(\xi) = \sqrt{\frac{\sqrt{\xi^4+\eta^2}+\xi^2}{2}} + i\,\mathrm{sign}(\eta)\sqrt{\frac{\sqrt{\xi^4+\eta^2}-\xi^2}{2}}. \tag{1.6}$$

Due to the Parseval formula and (1.3), we have

$$\|u(x,\cdot,\cdot)\| = \|\hat{u}(x,\cdot,\cdot)\| = \left(\int_{\mathbb{R}^2} |\cosh(x\vartheta(\xi,\eta))|^2\,|\hat{g}(\xi,\eta)|^2\mathrm{d}\xi\mathrm{d}\eta\right)^{\frac{1}{2}}. \tag{1.7}$$

Note that, for fixed $0 < x \le 1$, $|\cosh(x\vartheta(\xi,\eta))|$ tends to infinity when $|\xi| \to \infty$ or $|\eta| \to \infty$. Formula (1.7) implies a rapid decay of $\hat{g}(\xi,\eta)$ at high frequencies. But such decay is not likely to occur in the measured noisy data $g^\delta(y,t)$ at $x = 0$. Therefore, small perturbation of $g^\delta(y,t)$ in high-frequency components can blow up and completely destroy the temperature $u(x,y,t)$, i.e., problem (1.1) is severely ill-posed. So an effective regularization method is necessary for solving the problem (1.1).

In fact, in practice the data function $g(y,t)$ is given only by measurement and measurement errors exist in $g(y,t)$. We assume that the exact data $g(y,t)$ and the noisy data $g^\delta(y,t)$ satisfy the following noise level:

$$\|g - g^\delta\| \le \delta. \tag{1.8}$$

The constant $\delta > 0$ denotes a bound on the measurement error.

We also assume that there exists an *a priori* condition for problem (1.1):

$$\left(\int_{\mathbb{R}^2} |e^{|\vartheta(\xi,\eta)|}\hat{g}(\xi,\eta)|^2\mathrm{d}\xi\mathrm{d}\eta\right)^{\frac{1}{2}} \le E, \tag{1.9}$$

where $E > 0$ is constant.

The main aim of this article is to solve the two-dimensional IHCP (1.1) by using the *a posteriori* quasi-boundary value method. This article is organized as follows. In Section 2, we provide a quasi-boundary value regularization method to formulate a regularized solution and give *a posteriori* choice strategy of regularization parameter based on Morozov's discrepancy principle. In Section 3, a Hölder-type error estimate between the approximate solution and its exact solution is presented under the *a posteriori* regularization parameter choice rule. The article ends with a brief conclusion in Section 4.

2 An *a posteriori* parameter choice strategy for a quasi-boundary value method and some auxiliary results

In this section, we solve the ill-posed problems (1.1) by a quasi-boundary value method and give some auxiliary results under an *a posteriori* regularization parameter choice strategy.

The quasi-boundary value method is a regularization technique by replacing the boundary condition or final condition with a new approximate condition. So we add a perturbation term in the boundary condition and consider the following boundary conditions instead:

$$u(0, y, t) + \alpha u(1, y, t) = g^{\delta}(y, t), \quad y \geq 0, \quad t \geq 0, \tag{2.1}$$

where α plays a role of regularization parameter and the noisy data g^{δ} are the measured data of functions g.

Let $u_{\alpha}^{\delta}(x, y, t)$ be the solution of the following regularized problem:

$$\begin{cases} \dfrac{\partial v}{\partial t} = \dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2}, & 0 < x < 1, \ y > 0, \ t > 0, \\ v(0, y, t) + \alpha v(1, y, t) = g^{\delta}(y, t), & y \geq 0, \ t \geq 0, \\ \dfrac{\partial v}{\partial x}(0, y, t) = 0, & y \geq 0, \ t \geq 0, \\ v(x, 0, t) = 0, & 0 \leq x \leq 1, \ t \geq 0, \\ v(x, y, 0) = 0, & 0 \leq x \leq 1, \ y \geq 0. \end{cases} \tag{2.2}$$

By taking Fourier transform on both sides of (2.2) with respect to the variable y and t, we can obtain the following form:

$$\hat{u}_{\alpha}^{\delta}(x, \xi, \eta) = \frac{\cosh(x\vartheta(\xi, \eta))}{1 + \alpha \cosh(\vartheta(\xi, \eta))} \hat{g}^{\delta}(\xi, \eta). \tag{2.3}$$

Comparing formula (1.3) for the exact solution with formula (2.3) for its quasi-boundary value approximation, we can see that the regularization procedure consists in replacing $\hat{g}(\xi, \eta)$ with an appropriately filtered Fourier transform of noisy data $g^{\delta}(y, t)$. The filter in (2.3) attenuates the high frequencies in $\hat{g}^{\delta}(\xi, \eta)$. From this, we can replace the original filter $\frac{1}{1+\alpha \cosh(\vartheta(\xi, \eta))}$ with another filter $\frac{1}{1+\alpha\, |\cosh(\vartheta(\xi, \eta))|}$ and introduce a new approximation $u_{\alpha,*}^{\delta}(x, y, t)$ of problem (1.1)

$$\hat{u}_{\alpha,*}^{\delta}(x, \xi, \eta) = \frac{\cosh(x\vartheta(\xi, \eta))}{1 + \alpha |\cosh(\vartheta(\xi, \eta))|} \hat{g}^{\delta}(\xi, \eta). \tag{2.4}$$

We call $u_{\alpha,*}^{\delta}(x, y, t)$ given by (2.4) a quasi-boundary value approximation of the exact solution $u(x, y, t)$.

We define an operator $K_x : u(x, \cdot, \cdot) \to g(\cdot, \cdot)$, $x \in [0, 1)$. Then problem (1.1) can be rewritten as the following operator equation:

$$K_x u(x, y, t) = g(y, t), \quad x \in [0, 1), \tag{2.5}$$

with a linear operator $K_x \in \mathcal{L}(L^2(\mathbb{R}^2), L^2(\mathbb{R}^2))$. From (1.3) and (2.5), we have

$$\widehat{K_x u}(x, \xi, \eta) = \hat{g}(\xi, \eta) = \hat{u}(x, \xi, \eta)[\cosh(x\vartheta(\xi, \eta))]^{-1}, \quad x \in [0, 1). \tag{2.6}$$

We apply Morozov's discrepancy principle as *a posteriori* regularization parameter choice rule. Recalling the definition of Morozov's discrepancy principle, the classical Morozov's discrepancy principle chooses the regularization parameter $\alpha > 0$ such that [43,48]

$$\|K_x u_{\alpha,*}^{\delta} - g^{\delta}\| = \delta. \tag{2.7}$$

Scherzer [44] extended Morozov's discrepancy principle and chose the regularization parameter $\alpha > 0$ such that

$$\|K_x u_{\alpha,*}^{\delta} - g^{\delta}\| = \tau\delta, \tag{2.8}$$

where $\tau > 1$ is a constant. In this article, we select the regularization parameter $\alpha > 0$ satisfied equation (2.8), because equation (2.7) will not fit in our framework.

In order to establish the existence and uniqueness of the solution of equation (2.8), the following lemma is needed.

Lemma 2.1. *Let* $d(\alpha) = \|K_x u^\delta_{\alpha,*} - g^\delta\|$. *If* $\|g^\delta\| > \delta > 0$, *then the following results exist*:

(1) $d(\alpha)$ *is a continuous function*;

(2) $\lim_{\alpha\to 0} d(\alpha) = 0$;

(3) $\lim_{\alpha\to +\infty} d(\alpha) = \|g^\delta\|$;

(4) $d(\alpha)$ *is a strictly increasing function over* $(0, \infty)$.

Proof. Due to the Parseval formula and (2.4), (2.6), we have

$$d(\alpha) = \left(\int\limits_{\mathbb{R}^2} \left| \frac{\alpha|\cosh(\vartheta(\xi,\eta))|}{1+\alpha|\cosh(\vartheta(\xi,\eta))|} \right|^2 |\hat{g}^\delta(\xi,\eta)|^2 \mathrm{d}\xi \mathrm{d}\eta \right)^{\frac{1}{2}}.$$

From the above expression, the results of Lemma 2.1 are straightforward. □

Lemma 2.2. *For* $0 < \alpha < 1$, $0 < x < 1$, $\vartheta(\xi,\eta) = \sqrt{\xi^2 + i\eta}$, *then there holds*

$$\sup_{(\xi,\eta)\in\mathbb{R}^2} \left| \frac{\cosh(x\vartheta(\xi,\eta))}{1+\alpha|\cosh(\vartheta(\xi,\eta))|} \right| \le (c\alpha)^{-x}. \tag{2.9}$$

Proof. Using the inequality $|\cosh(z)| \le \cosh(|z|)$, $\forall z \in \mathbb{C}$, we have

$$|\cosh(x\vartheta(\xi,\eta))| \le \cosh(x|\vartheta(\xi,\eta)|) = \frac{e^{x|\vartheta(\xi,\eta)|} + e^{-x|\vartheta(\xi,\eta)|}}{2} \le e^{x|\vartheta(\xi,\eta)|}. \tag{2.10}$$

From Lemmas 2.1 and 2.2 in [49], we know there exists a positive constant c such that

$$\sup_{(\xi,\eta)\in\mathbb{R}^2} \left| \frac{\cosh(x\vartheta(\xi,\eta))}{1+\alpha|\cosh(\vartheta(\xi,\eta))|} \right| = \sup_{(\xi,\eta)\in\mathbb{R}^2} \frac{|\cosh(x\vartheta(\xi,\eta))|}{1+c\alpha e^{|\vartheta(\xi,\eta)|}} = \sup_{(\xi,\eta)\in\mathbb{R}^2} \frac{e^{x|\vartheta(\xi,\eta)|}}{1+c\alpha e^{|\vartheta(\xi,\eta)|}} = (c\alpha)^{-x}.$$ □

3 Error estimate for the *a posteriori* quasi-boundary value method

In this section, we will give a Hölder-type error estimate between the exact solution of temperature and its regularized solution by using an *a posteriori* choice rule for the regularization parameter.

Theorem 3.1. *Suppose that the noise assumption* (1.8) *and the a priori condition* (1.9) *hold. If the regularization parameter* $\alpha > 0$ *is chosen by Morozov discrepancy principle* (2.8), *then we have the following error estimate*:

$$\|u^\delta_{\alpha,*}(x,\cdot,\cdot) - u(x,\cdot,\cdot)\| \le ((\tau+1)^{1-x} + (c(\tau-1))^{-x})E^x\delta^{1-x}. \tag{3.1}$$

Proof. Using the Parseval formula and the triangle inequality, we have

$$\begin{aligned}\|u^\delta_{\alpha,*}(x,\cdot,\cdot) - u(x,\cdot,\cdot)\| &= \|\hat{u}^\delta_{\alpha,*}(x,\cdot,\cdot) - \hat{u}(x,\cdot,\cdot)\| \\ &\le \|\hat{u}^\delta_{\alpha,*}(x,\cdot,\cdot) - \hat{u}_{\alpha,*}(x,\cdot,\cdot)\| + \|\hat{u}_{\alpha,*}(x,\cdot,\cdot) - \hat{u}(x,\cdot,\cdot)\|.\end{aligned} \tag{3.2}$$

We first give an estimate for the second term.

From (1.3), (2.4), and (2.10), with the Hölder inequality, we obtain

$$\|\hat{u}_{\alpha,*}(x,\cdot,\cdot) - \hat{u}(x,\cdot,\cdot)\|^2 = \left\| \frac{\cosh(x\vartheta(\cdot,\cdot))}{1+\alpha|\cosh(\vartheta(\cdot,\cdot))|}\hat{g}(\cdot,\cdot) - \cosh(x\vartheta(\cdot,\cdot))\hat{g}(\cdot,\cdot) \right\|^2$$

$$
\begin{aligned}
&= \left\| \frac{-\alpha|\cosh(\vartheta(\cdot,\cdot))|\cosh(x\vartheta(\cdot,\cdot))}{1+\alpha|\cosh(\vartheta(\cdot,\cdot))|}\hat{g}(\cdot,\cdot) \right\|^2 \\
&= \int_{\mathbb{R}^2} \left| \frac{\alpha|\cosh(\vartheta(\xi,\eta))|}{1+\alpha|\cosh(\vartheta(\xi,\eta))|} \right|^2 |\cosh(x\vartheta(\xi,\eta))|^2 |\hat{g}(\xi,\eta)|^2 \mathrm{d}\xi\mathrm{d}\eta \\
&\le \int_{\mathbb{R}^2} \left| \frac{\alpha|\cosh(\vartheta(\xi,\eta))|}{1+\alpha|\cosh(\vartheta(\xi,\eta))|} \right|^2 (e^{x|\vartheta(\xi,\eta)|})^2 |\hat{g}(\xi,\eta)|^2 \mathrm{d}\xi\mathrm{d}\eta \\
&= \int_{\mathbb{R}^2} \left| \frac{\alpha|\cosh(\vartheta(\xi,\eta))|}{1+\alpha|\cosh(\vartheta(\xi,\eta))|} \right|^2 |\hat{g}(\xi,\eta)|^{2(1-x)} |e^{|\vartheta(\xi,\eta)|}\hat{g}(\xi,\eta)|^{2x} \mathrm{d}\xi\mathrm{d}\eta \\
&\le \int_{\mathbb{R}^2} \left[\left| \frac{\alpha|\cosh(\vartheta(\xi,\eta))|}{1+\alpha|\cosh(\vartheta(\xi,\eta))|} \right| |\hat{g}(\xi,\eta)| \right]^{2(1-x)} |e^{|\vartheta(\xi,\eta)|}\hat{g}(\xi,\eta)|^{2x} \mathrm{d}\xi\mathrm{d}\eta \\
&\le \left(\int_{\mathbb{R}^2} \left| \frac{\alpha|\cosh(\vartheta(\xi,\eta))|}{1+\alpha|\cosh(\vartheta(\xi,\eta))|}\hat{g}(\xi,\eta) \right|^2 \mathrm{d}\xi\mathrm{d}\eta \right)^{(1-x)} \left(\int_{\mathbb{R}^2} |e^{|\vartheta(\xi,\eta)|}\hat{g}(\xi,\eta)|^2 \mathrm{d}\xi\mathrm{d}\eta \right)^{x}.
\end{aligned}
$$

From (2.4) and (2.8), we obtain

$$
\tau\delta = \|K_x u^{\delta}_{\alpha,*} - g^{\delta}\| = \|\widehat{K_x u^{\delta}_{\alpha,*}} - \hat{g}^{\delta}\| = \left\| \frac{\alpha|\cosh(\vartheta(\cdot,\cdot))|}{1+\alpha|\cosh(\vartheta(\cdot,\cdot))|}\hat{g}^{\delta}(\cdot,\cdot) \right\|, \tag{3.3}
$$

with the noise assumption (1.8) and the *a priori* condition (1.9), we have

$$
\begin{aligned}
\|\hat{u}_{\alpha,*}(x,\cdot,\cdot) - \hat{u}(x,\cdot,\cdot)\| &\le \left\| \frac{\alpha|\cosh(\vartheta(\cdot,\cdot))|}{1+\alpha|\cosh(\vartheta(\cdot,\cdot))|}\hat{g}(\cdot,\cdot) \right\|^{(1-x)} E^x \\
&\le E^x \left[\left\| \frac{\alpha|\cosh(\vartheta(\cdot,\cdot))|}{1+\alpha|\cosh(\vartheta(\cdot,\cdot))|}(\hat{g}(\cdot,\cdot) - \hat{g}^{\delta}(\cdot,\cdot)) \right\| + \left\| \frac{\alpha|\cosh(\vartheta(\cdot,\cdot))|}{1+\alpha|\cosh(\vartheta(\cdot,\cdot))|}\hat{g}^{\delta}(\cdot,\cdot) \right\| \right]^{(1-x)} \\
&\le E^x [\|\hat{g}(\cdot,\cdot) - \hat{g}^{\delta}(\cdot,\cdot)\| + \tau\delta]^{(1-x)} \le E^x[(\tau+1)\delta]^{(1-x)}.
\end{aligned} \tag{3.4}
$$

Now we give the bound for the first term. Due to (2.4) and (2.9), we have

$$
\begin{aligned}
\|\hat{u}^{\delta}_{\alpha,*}(x,\cdot,\cdot) - \hat{u}_{\alpha,*}(x,\cdot,\cdot)\| &= \left\| \frac{\cosh(x\vartheta(\cdot,\cdot))}{1+\alpha|\cosh(\vartheta(\cdot,\cdot))|}(\hat{g}^{\delta}(\cdot,\cdot) - \hat{g}(\cdot,\cdot)) \right\| \\
&\le \sup_{(\xi,\eta)\in\mathbb{R}^2} \left| \frac{\cosh(x\vartheta(\xi,\eta))}{1+\alpha|\cosh(\vartheta(\xi,\eta))|} \right| \|\hat{g}^{\delta}(\cdot,\cdot) - \hat{g}(\cdot,\cdot)\| \\
&\le (c\alpha)^{-x}\delta.
\end{aligned} \tag{3.5}
$$

From (3.3) and (1.9), we know

$$
\begin{aligned}
\tau\delta &= \left\| \frac{\alpha|\cosh(\vartheta(\cdot,\cdot))|}{1+\alpha|\cosh(\vartheta(\cdot,\cdot))|}\hat{g}^{\delta}(\cdot,\cdot) \right\| \\
&\le \left\| \frac{\alpha|\cosh(\vartheta(\cdot,\cdot))|}{1+\alpha|\cosh(\vartheta(\cdot,\cdot))|}(\hat{g}^{\delta}(\cdot,\cdot) - \hat{g}(\cdot,\cdot)) \right\| + \left\| \frac{\alpha|\cosh(\vartheta(\cdot,\cdot))|}{1+\alpha|\cosh(\vartheta(\cdot,\cdot))|}\hat{g}(\cdot,\cdot) \right\| \\
&\le \|\hat{g}^{\delta}(\cdot,\cdot) - \hat{g}(\cdot,\cdot)\| + \left\| \frac{\alpha}{1+\alpha|\cosh(\vartheta(\cdot,\cdot))|}|\cosh(\vartheta(\cdot,\cdot))|\hat{g}(\cdot,\cdot) \right\| \\
&\le \delta + \alpha\||\cosh(\vartheta(\cdot,\cdot))|\hat{g}(\cdot,\cdot)\| \le \delta + \alpha\|e^{|(\vartheta(\cdot,\cdot))|}\hat{g}(\cdot,\cdot)\| \\
&\le \delta + \alpha E.
\end{aligned}
$$

This yields

$$
\alpha \ge \frac{\delta}{E}(\tau - 1). \tag{3.6}
$$

Substituting (3.6) into (3.5), we have

$$\|\hat{u}_{\alpha,*}^{\delta}(x,\cdot,\cdot)-\hat{u}_{\alpha,*}(x,\cdot,\cdot)\|\leq(c(\tau-1))^{-x}E^{x}\delta^{1-x}. \tag{3.7}$$

Combining (3.2), (3.4), and (3.7), we obtain the error estimate (3.1). □

Estimate (3.1) is a Hölder-type stability estimate, and the error bounds in (3.1) are similar to error bounds in (4.14) of Theorem 4.2 in [46].

4 Conclusion

In this article, we investigate a two-dimensional IHCP, which determines the surface temperature distribution from measured data at the fixed location. We propose a quasi-boundary value method for obtaining a regularized solution. The Hölder-type error estimate between the approximate solution and its exact solution is obtained under Morozov's discrepancy principle. Error analysis shows that our regularization method is effective.

Acknowledgments: The authors would like to thanks the editor and the anonymous referees for their valuable comments and suggestions on this article.

Funding information: This work was supported by the National Natural Science Foundation of China (Grant Nos 11961044 and 11561045).

Author contributions: All authors have accepted responsibility for the entire content of this manuscript and approved its submission.

Conflict of interest: The authors state no conflict of interest.

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