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Research Article

Wei Zhuang* and Qiuju Zheng

Double domination in maximal outerplanar graphs

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Abstract: In graph G, a vertex dominates itself and its neighbors. A subset $S \subseteq V(G)$ is said to be a double-dominating set of G if S dominates every vertex of G at least twice. The double domination number $\gamma_{\times 2}(G)$ is the minimum cardinality of a double dominating set of G. We show that if G is a maximal outerplanar graph on $n \geq 3$ vertices, then $\gamma_{\times 2}(G) \leq \left\lfloor \frac{2n}{3} \right\rfloor$. Further, if $n \geq 4$, then $\gamma_{\times 2}(G) \leq \min\left\{\left\lfloor \frac{n+t}{2} \right\rfloor, n - t\right\}$, where t is the number of vertices of degree 2 in G. These bounds are shown to be tight. In addition, we also study the case that G is a striped maximal outerplanar graph.

Keywords: maximal outerplanar graph, striped maximal outerplanar graph, double domination

MSC 2020: 05C69

1 Introduction

For a simple graph $G = (V, E)$, V and E are the sets of vertices and edges of G, respectively. We denote by $|V(G)|$ the *order* of a graph G. If the graph G is clear from the context, we simply write $|G|$ or n rather than $|V(G)|$. For a vertex $v \in V(G)$, let $N_G(v)$ and $N_G[v]$ denote the *open neighborhood* and the *closed neighborhood* of v, respectively; thus, $N_G(v) = \{u | uv \in E(G)\}$ and $N_G[v] = \{v\} \cup N_G(v)$. A graph G is *outerplanar* if it can be embedded in the plane such that all vertices belong to the boundary of its outer face. An outerplanar graph G is *maximal* if $G + uv$ is not outerplanar for any two nonadjacent vertices u and v.

A *dominating set* of a graph G is a set $S \subseteq V(G)$ such that every vertex in G is either in S or is adjacent to a vertex in S. A set $D \subseteq V(G)$ is a double dominating set of G if every vertex in $V(G) - D$ has at least two neighbors in D and every vertex of D has a neighbor in D. The *domination number* (*double domination number*, respectively) of G, denoted by $\gamma(G)$ ($\gamma_{\times 2}(G)$, respectively), is the minimum cardinality of a dominating set (double dominating set, respectively). A double dominating set of G of cardinality $\gamma_{\times 2}(G)$ is called a $\gamma_{\times 2}(G)$-set. If the graph G is clear from the context, we simply write $\gamma_{\times 2}$-set rather than $\gamma_{\times 2}(G)$-set. We say a vertex v in G is double dominated, by a set S, if $|N_G[v] \cap S| \geq 2$.

The concept of double domination was introduced by Harary and Haynes [1] and further studied in, for example, [2–8].

In 1996, Matheson and Tarjan [9] proved that any triangulated disc G with n vertices satisfies $\gamma(G) \leq \left\lfloor \frac{n}{3} \right\rfloor$, and conjectured that $\gamma(G) \leq \left\lfloor \frac{n}{4} \right\rfloor$ for every n-vertex triangulation G with sufficiently large n. In 2013, Campos and Wakabayashi investigated this problem for maximal outerplanar graphs and showed in [10] that if G is a maximal outerplanar graph of order n, then $\gamma(G) \leq \frac{n+k}{4}$, where k is the number of vertices of degree 2 in G.

* **Corresponding author: Wei Zhuang,** School of Applied Mathematics, Xiamen University of Technology, Xiamen, 361024, P. R. China, e-mail: zhuangweixmu@163.com
Qiuju Zheng: School of Applied Mathematics, Xiamen University of Technology, Xiamen, 361024, P. R. China

Tokunaga proved the same result independently in [11]. Li et al. [12] improved the result by showing that $\gamma(G) \le \frac{n+t}{4}$, where t is the number of pairs of consecutive 2-degree vertices with distance at least 3 on the outer cycle. For results on other types of domination in maximal outerplanar graphs, we refer the reader to [13–20].

In this article, we apply the idea of coloring to investigate the same question for double domination. We show that if G is a maximal outerplanar graph on $n \ge 3$ vertices, then $\gamma_{\times 2}(G) \le \left\lfloor \frac{2n}{3} \right\rfloor$. Further, if $n \ge 4$, then $\gamma_{\times 2}(G) \le \min\left\{\left\lfloor \frac{n+t}{2} \right\rfloor, n - t\right\}$, where t is the number of vertices of degree 2 in G. These bounds are shown to be tight. In addition, we also study the case that G is a striped maximal outerplanar graph.

2 Preliminaries

A maximal outerplanar graph G can be embedded in the plane such that the boundary of the outer face is a Hamiltonian cycle and each inner face is a triangle. A maximal outerplanar graph embedded in the plane is called a *maximal outerplane graph*. For such an embedding of G, we denote by H_G the Hamiltonian cycle which is the boundary of the outer face. An inner face of a maximal outerplane graph G is an *internal triangle* if it is not adjacent to outer face. A maximal outerplane graph without internal triangles is called *striped*. The following results are useful for our study.

Proposition 2.1. [10] *Let G be a maximal outerplanar graph of order $n \ge 4$. If G has k internal triangles, then G has $k + 2$ vertices of degree 2.*

From Proposition 2.1, we know that every maximal outerplanar graph has at least two vertices of degree two.

Proposition 2.2. [12] *Let G be a striped maximal outerplanar graph of order $n > 4$; then for any vertex $v \in V(G)$ of degree 2, the degrees of two neighbors of v are 3 and p ($p \ge 4$).*

Proposition 2.3. *If G is a maximal outerplanar graph of order $n \ge 4$, then G is striped if and only if G has exactly two vertices of degree two.*

Proof. Suppose that G is a maximal outerplanar graph with minimum order that satisfy the two properties:
(1) G has exactly two vertices of degree two.
(2) G has at least one internal triangle.

Clearly, $|G| \ge 6$. Let u be a vertex of degree two in G, and v, y be its neighbors. By Proposition 2.2, one of the two neighbors, say v, has degree three. Now, we remove u and denote the resulting graph by G_1. It is easy to see that G_1 is still a maximal outerplanar graph. Moreover, G_1 has exactly two vertices of degree two, and one of them is v (see Figure 1). Assume that C is an internal triangle of G. From the assumption of the graph G, C is not an internal triangle of G_1. It implies that vy is an edge of C. Then, vx is also an edge of C, where x is the remaining neighbor of v in G other than u and y. It contradicts the assumption that C is an internal triangle of G.

On the other hand, from Proposition 2.1, we know that a striped maximal outerplanar graph has exactly two vertices of degree two.

The result holds. □

Proposition 2.4. [15] *If G is a maximal outerplanar graph of order $n \ge 4$, then the set of vertices of G of degree 2 is an independent set of G of size at most $\frac{n}{2}$.*

Proposition 2.5. [1] *Let C be a cycle of order $n \ge 3$, then $\gamma_{\times 2}(C) = \left\lceil \frac{2n}{3} \right\rceil$.*

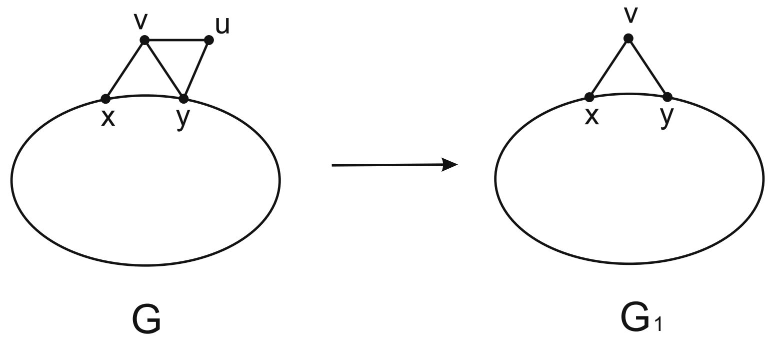

Figure 1: G and G_1.

3 Main result

Let G be any maximal outerplanar graph. We know that there exists a Hamiltonian cycle in G, say H_G. It is easy to see that a $\gamma_{\times 2}$-set of H_G is a double dominating set of G. Hence, the following result is immediate from Proposition 2.5.

Observation 3.1. Let G be a maximal outerplanar graph of order $n \geq 3$, then $\gamma_{\times 2}(G) \leq \left\lceil \frac{2n}{3} \right\rceil$.

This conclusion is obvious, so we are ready to improve it. Next, we will give some definitions that are helpful for our investigations.

A *minor* of a graph G is a graph that can be obtained from G by deleting vertices and deleting or contracting edges. Given a graph H, a graph G is *H-minor free* if no minor of G is isomorphic to H. A K_4-minor free graph G is *maximal* if $G + uv$ is not a K_4-minor free graph for any two nonadjacent vertices u and v of G. A *2-tree* is defined recursively as follows. A single edge is a 2-tree. Any graph obtained from a 2-tree by adding a new vertex and making it adjacent to the end vertices of an existing edge is also a 2-tree. It is well known that the maximal K_4-minor-free graphs are exactly the 2-trees. We know that a maximal outerplanar graph must be a maximal K_4-minor free graph. Hence, before giving the upper bound for the double domination number of a maximal outerplanar graph, we intend to prove the stronger result as follows.

Theorem 3.2. *Let G be a maximal K_4-minor free graph of order $n \geq 3$, then $\gamma_{\times 2}(G) \leq \left\lfloor \frac{2n}{3} \right\rfloor$.*

Proof. Let $G_i = G_{i-1} - v_i$ for $i = 1, 2, \ldots, n-3$, where v_i is a vertex of degree 2 in G_{i-1}, $G_0 = G$. Since a maximal K_4-minor free graph is exactly a 2-tree and according to the definition of 2-tree, we have that G_{n-3} is a K_3. We can give a proper 3-coloring to G_0 by assigning colors 1, 2, and 3 to the three vertices of G_{n-3} and color v_i with the remaining color, which does not appear in $N_{G_{i-1}}(v_i)$ at each stage. Let V_t be the set of vertices assigned color t, where $t = 1, 2, 3$. Choosing two suitable color classes, say V_1 and V_2, such that $|V_1| + |V_2| \leq \left\lfloor \frac{2}{3}|G_0| \right\rfloor$. Let $S = V_1 \cup V_2$. Next, we will show that S is a double dominating set of G.

Clearly, each of the three vertices is double dominated by $S \cap V(G_{n-3})$ in G_{n-3}. In $V(G_{i-1})$ ($i = n-3, n-4, \ldots, 2, 1$), $|N_{G_{i-1}}[v_i] \cap (S \cap V(G_{i-1}))| = 2$. It means that each vertex of G is double dominated by S. □

On the basis of the aforementioned theorem, we have the following conclusion.

Corollary 3.3. *Let G be a maximal outerplanar graph of order $n \geq 3$, then $\gamma_{\times 2}(G) \leq \left\lfloor \frac{2n}{3} \right\rfloor$.*

To show that this bound is tight. We are ready to construct an infinite family of graphs as follows. Let H be any maximal outerplanar graph with $2k$ vertices, and $C = a_1 a_2 \cdots a_{2k-1} a_{2k} a_1$ is the unique Hamiltonian cycle of H. We note that C is the boundary of the outer face of H. Let G_H be the graph obtained from H by adding k new vertices $u_1, u_2, \dots, u_k$ and $2k$ new edges $u_1a_1, u_1a_2, u_2a_3, u_2a_4, \dots, u_ka_{2k-1}, u_ka_{2k}$ (see Figure 2, the nontriangular face can be triangulated in any way so as to obtain a maximal outerplanar graph). Let $\mathcal{U}$ be a family consisting of all such graph G_H. Take a $\gamma_{\times 2}$-set of G_H, say S. Let v be a vertex of degree two in G_H, it is easy to see that $|N_{G_H}[v] \cap S| \geq 2$. It means that $|S| \geq 2k = \frac{2|G_H|}{3}$. On the other hand, it follows from Corollary 3.3 that $|S| \leq \frac{2|G_H|}{3}$. Hence, $\gamma_{\times 2}(G_H) = \frac{2|G_H|}{3}$.

But the result of Corollary 3.3 is still not good enough, it only slightly improves Observation 3.1, so we hope to further improve the result of Corollary 3.3. Next, we will give two upper bounds in terms of the order and the number of 2-degree vertices, and the first one is inspired by [11].

Lemma 3.4. [11] *A maximal outerplanar graph G can be four colored such that every cycle of length 4 in G has all four colors.*

Theorem 3.5. *Let G be a maximal outerplanar graph of order $n \geq 3$. If t is the number of vertices of degree 2 in G, then $\gamma_{\times 2}(G) \leq \left\lfloor \frac{n+t}{2} \right\rfloor$.*

Proof. Let $R = \{a_1, a_2, \dots, a_t\}$ be the set of vertices of G having degree 2. For each a_i, let b_i be one of its neighbors (b_s and b_t are not necessarily distinct when $s \neq t$). We construct a graph G' from G by adding t new vertices $u_1, u_2, \dots, u_t$, and $2t$ new edges $a_1u_1, b_1u_1, a_2u_2, b_2u_2, \dots, a_tu_t, b_tu_t$. Note that G' is also a maximal outerplanar graph. By Lemma 3.4, G' can be four colored such that every cycle of length 4 in G' has all four colors. Let V_p be the set of vertices assigned color p, where $p = 1, 2, 3, 4$. Choosing two suitable color classes, say V_1 and V_2, such that $|V_1| + |V_2| \leq \left\lfloor \frac{n+t}{2} \right\rfloor$. Let $D = V_1 \cup V_2$.

Let $D \cap \{u_1, u_2, \dots, u_t\} = \{u_1', u_2', \dots, u_k'\}$, where $k \leq t$. For any u_i', at least one of its neighbors, say $v_i \in \{a_i, b_i\}$, does not belong to D. We construct a new set $D' = (D \backslash \{u_1', u_2', \dots, u_k'\}) \cup \{v_1, v_2, \dots, v_k\}$.

Now, let x be a vertex of G' of degree at least 3, and x_1, x_2, and x_3 be consecutive vertices of $N_{G'}(x)$ in this order. Clearly, $xx_1x_2x_3$ forms a cycle of length 4, and two of $\{x, x_1, x_2, x_3\}$ belong to D. Since every vertex in $\{x, x_1, x_2, x_3\}$, except possibly one, has degree at least three, we have that at least two of those vertices of degree at least three belong to D'. It means that x is double dominated by D'. Then, D' is a double dominating set of G. □

On the other hand, for a maximal outerplanar graph G of order at least four, of all vertices of G of degree, at least three form a double dominating set of G. Hence, we have the following conclusion.

Observation 3.6. Let G be a maximal outerplanar graph of order $n \geq 4$. If t is the number of vertices of degree 2 in G, then $\gamma_{\times 2}(G) \leq n - t$.

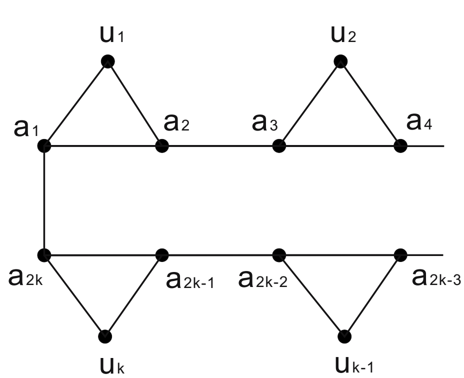

Figure 2: The graph H.

Clearly, each graph of $\mathscr{U}$ achieves equalities in Theorem 3.5 and Observation 3.6. That is, the bounds of Theorem 3.5 and Observation 3.6 are also tight. The following conclusion is immediate from Theorem 3.5 and Observation 3.6.

Corollary 3.7. *Let G be a maximal outerplanar graph of order n ≥ 4. If t is the number of vertices of degree* 2 *in G, then*

$$\gamma_{\times 2}(G) \leq \begin{cases} \left\lfloor \frac{n+t}{2} \right\rfloor, & \text{if } t < \frac{n}{3}; \\ n - t, & \text{otherwise}. \end{cases}$$

Next, we compare the bound in Corollary 3.3 with that in Corollary 3.7. From Proposition 2.4, we know that there are at most $\frac{n}{2}$ vertices of degree 2 in a maximal outerplanar graph. It is easy to see that $\left\lfloor \frac{n+t}{2} \right\rfloor < \left\lfloor \frac{2n}{3} \right\rfloor$ when $t < \frac{n}{3}$, and $n - t < \left\lfloor \frac{2n}{3} \right\rfloor$ when $\frac{n}{3} < t \leq \frac{n}{2}$. Thus, the bound in Corollary 3.7 is better than that in Corollary 3.3.

Based on the aforementioned analysis, it is natural to consider the following question: Can the upper bound for Corollary 3.7 be improved for the striped maximal outerplanar graph? From the aforementioned result, we know that $\left\lfloor \frac{n+t}{2} \right\rfloor < n - t$ when t is far less than n. Then combining Proposition 2.1, we have the following result.

Corollary 3.8. *Let G be a striped maximal outerplanar graph with n ≥ 3 vertices. Then,* $\gamma_{\times 2}(G) \leq \left\lfloor \frac{n}{2} \right\rfloor + 1$.

Next, we construct a striped maximal outerplanar graph G of order $n \geq 6$. Let $C = a_1a_2 \cdots a_{q-1}a_qb_q b_{q-1} \cdots b_2b_1a_1$ is the unique Hamiltonian cycle of G, where $q \equiv 1 \pmod 4$ or $q \equiv 3 \pmod 4$. We know that C is the boundary of the outer face of G. Let $E_1 = \{a_2b_2, a_3b_3, \ldots, a_{q-1}b_{q-1}\}$, $E(G) = E(C) \cup E_1 \cup \{a_1b_2, a_2b_3, a_4b_3, a_5b_4, a_5b_6, a_6b_7, a_8b_7, a_9b_8, \ldots, a_{q-4}b_{q-3}, a_{q-3}b_{q-2}, a_{q-1}b_{q-2}, a_qb_{q-1}\}$, when $q \equiv 1 \pmod 4$ and $E(G) = E(C) \cup E_1 \cup \{a_1b_2, a_2b_3, a_4b_3, a_5b_4, a_5b_6, a_6b_7, a_8b_7, a_9b_8, \ldots, a_{q-3}b_{q-4}, a_{q-2}b_{q-3}, a_{q-2}b_{q-1}, a_{q-1}b_q\}$, when $q \equiv 3 \pmod 4$ (see Figure 3). Note that $|G| \equiv 2 \pmod 4$. Let $\mathscr{A}$ be a family consisting of all such graph G. We will show that each graph of $\mathscr{A}$ attains the bound of Corollary 3.8.

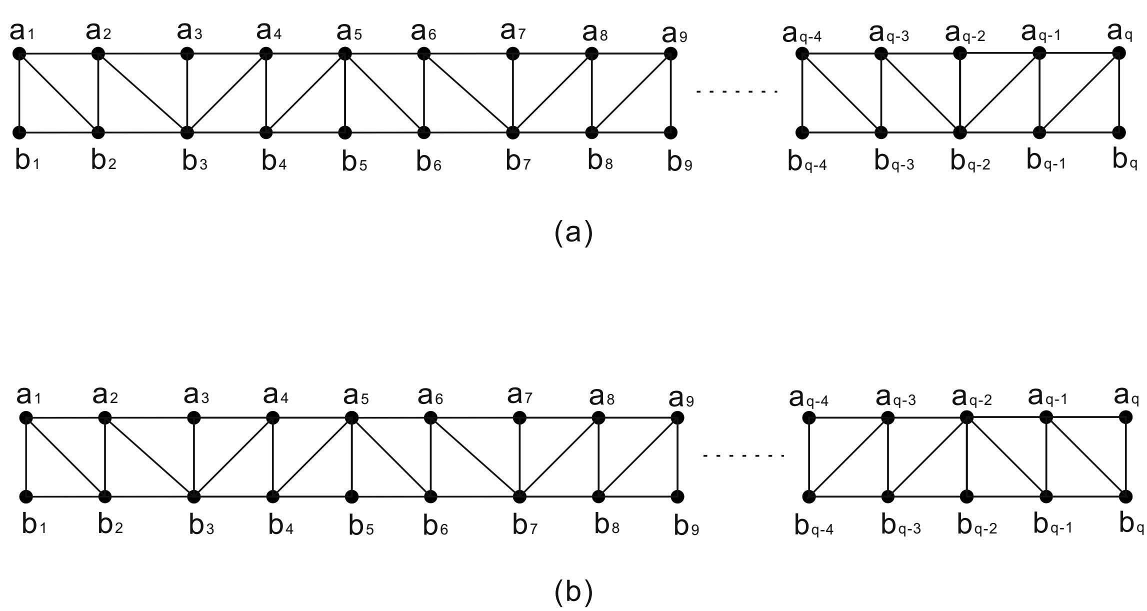

Figure 3: The graph G. (a) $q \equiv 1 \pmod 4$. (b) $q \equiv 3 \pmod 4$.

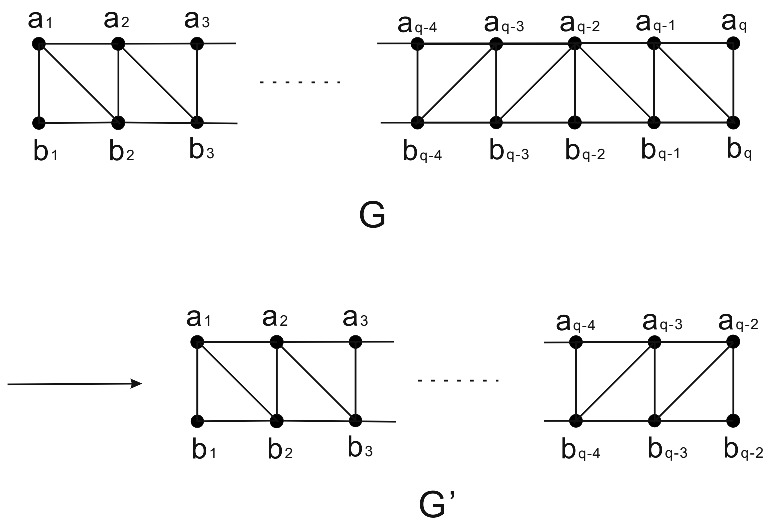

Figure 4: The graphs G and G'.

Theorem 3.9. *If a striped maximal outerplanar graph G belongs to* $\mathscr{A}$, *then* $\gamma_{\times 2}(G) = \frac{|G|}{2} + 1$.

Proof. If $|G| = 6$, the result holds. This establishes the base case. So we let $|G| \geq 10$. (Note that $|G| \equiv 2 \pmod 4$.) Assume that G is a striped maximal outerplanar graph of $\mathscr{A}$ with minimum order, such that $\gamma_{\times 2}(G) < \frac{|G|}{2} + 1$.

Let $C = a_1 a_2 \cdots a_{q-1} a_q b_q b_{q-1} \cdots b_2 b_1 a_1$ be the unique Hamiltonian cycle of G, and S be a $\gamma_{\times 2}$-set of G. It follows from Proposition 2.1 that G has exactly two vertices of degree two, without loss of generality, suppose that a_q is one of them. Clearly, $|N[a_q] \cap S| \geq 2$, and $S_1 = (S \backslash N[a_q]) \cup \{a_{q-1}, b_q\}$ is still a $\gamma_{\times 2}$-set of G. Moreover, $|N[b_{q-2}] \cap S_1| \geq 2$, and $S_2 = (S_1 \backslash N[b_{q-2}]) \cup \{a_{q-2}, b_{q-3}\}$ is also a $\gamma_{\times 2}$-set of G.

Let $G' = G - \{a_{q-1}, b_{q-1}, a_q, b_q\}$ (see Figure 4). Note that G' is also a striped maximal outerplanar graph, which belongs to $\mathscr{A}$, and $S_2 \backslash \{a_{q-1}, b_q\}$ is a double dominating set of G'. Thus, $\gamma_{\times 2}(G') \leq \gamma_{\times 2}(G) - 2 < \frac{|G|}{2} - 1 = \frac{|G'|+4}{2} - 1 = \frac{|G'|}{2} + 1$. It contradicts the assumption that $\gamma_{\times 2}(G') = \frac{|G'|}{2} + 1$. □

From the aforementioned result, we know that Corollary 3.8 is tight for striped maximal outerplanar graphs.

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