Research Article

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Generalized Munn rings

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Abstract: Generalized Munn rings exist extensively in the theory of rings. The aim of this note is to answer when a generalized Munn ring is primitive (semiprimitive, semiprime and prime, respectively). Sufficient and necessary conditions are obtained for a generalized Munn ring with a regular sandwich matrix to be primitive (semiprimitive, semiprime and prime, respectively). Also, we obtain sufficient and necessary conditions for a Munn ring over principal ideal domains to be prime (semiprime, respectively). Our results can be regarded as the generalizations of the famous result in the theory of rings that for a ring R, R is primitive (semiprimitive and semiprime, respectively) if and only if so is $M_n(R)$. As applications of our results, we consider the primeness and the primitivity of generalized matrix rings and generalized path algebras. In particular, it is proved that a path algebra is a semiprime if and only if it is semiprimitive.

Keywords: generalized Munn ring, generalized path algebra, semiprimitive ring, semiprime ring, generalized matrix ring

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1 Introduction

Throughout this note, we shall use the standard notions and notations, and each of the considered rings is associative but has possibly no identity.

The class of generalized matrix rings has been extensively studied. Examples of generalized matrix rings include piecewise domains (see [1]), incidence algebras of directed graphs (see, [2,3]), structural matrix rings (see [4] and subsequent papers), endomorphism rings, and Morita context rings. Sands [5] observed that if $[S, V, W, T]$ is a Morita context, then

$$\begin{pmatrix} S & V \\ W & T \end{pmatrix}$$

is a ring. These Morita context rings are precisely generalized matrix rings with idempotent sets E such that $|E| = 2$, and they have been widely studied. In particular, we note Amitsur's paper [6], the survey paper [7], McConnell and Robson's treatment [8] and Müller's computation of the maximal quotient ring [9]. Indeed, cellular algebras, affine cellular algebras and standardly based algebras there exist some "local" structures of generalized Munn algebras (for example, see [10–12]).

Brown [13] considered generalized matrix algebras of finite dimension over a field of characteristic 0. He proved that such a generalized matrix algebra is either simple or nonsemisimple and simple modulo its radical, and it is simple if and only if it possesses an identity. Sands [5] gave the prime radical of generalized matrix rings with a finite idempotent set. Zhang [14] considered the prime radical of the general case.

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In 1989, Wauters and Jespers [15] determined when a generalized matrix ring with a finite idempotent set is semiprime. Classical quotient rings of generalized matrix rings with finite idempotent sets had been attracting due attention. There are a series of papers on this field (see [16–18]).

Indeed, any generalized matrix ring can be viewed as a subring of some generalized Munn ring. It is natural to research generalized Munn rings. Li [19] considered the regularity of Munn rings. The main aim of this study is to answer when a generalized Munn ring is semiprimitive (semiprime and prime, respectively). A sufficient and necessary condition is established for a generalized Munn ring with a regular sandwich matrix to be primitive (semiprimitive, semiprime and prime, respectively) (Theorems 2.5 and 2.7). Moreover, we answer when a Munn ring over a principal ideal domain is semiprime (prime, respectively) (Proposition 2.11). In Section 3, we determine when a generalized matrix ring is primitive (semiprimitive, semiprime and prime, respectively) (Theorems 3.2 and 3.3). Finally, we consider the primeness and the primitivity of generalized path algebras. It is proved that for a quiver Q and a field K, the path algebra $K(Q)$ is semiprime if and only if the path-connected quiver of $\overline{Q}^{PC}$ is the disjoint union of complete quivers; if and only if $K(Q)$ is semiprimitive (Theorem 4.9). And, $K(Q)$ is prime if and only if $\overline{Q}^{PC}$ is a complete quiver (Theorem 4.10).

2 Generalized Munn rings

The aim of this section is to consider the primeness and the primitivity of generalized Munn rings.

2.1 Regular sandwich matrices

To begin with, we give the notion of regular matrices.

Definition 2.1. Let $\mathcal{A}$ be a ring and $X = (x_{im})$ be a generalized $I \times M$ matrix over $\mathcal{A}$. A nonzero entry x_{im} of X is called a *unit entry* of X if there exists a nonzero idempotent $e \in \mathcal{A}$ such that x_{im} is a unit in $e\mathcal{A}e$.

Notice that a group has exactly one idempotent, which is just the identity of the group. This means that in Definition 2.1, the idempotent e is indeed unique. So, the unique idempotent e in Definition 2.1 is denoted by $x_{im}^{\diamond}$.

Definition 2.2. Let $\mathcal{A}$ be a ring. An $I \times M$ matrix $X = (x_{im})$ over $\mathcal{A}$ is said *regular* in $\mathcal{A}$ if the following conditions hold:

(RM1) For any $i \in I$, there exists $m \in M$ such that x_{im} is a unit entry of X.
(RM2) For any $n \in M$, there exists $j \in I$ such that x_{jn} is a unit entry of X.
(RM3) If $x_{i_0m_0}$ is a unit entry of X, then
  (i) $x_{i_0m_0}^{\diamond}x_{i_0m} = x_{i_0m}$ for any $m \in M$;
  (ii) $x_{im_0}x_{i_0m_0}^{\diamond} = x_{im_0}$ for any $i \in I$.

By definition, any $m \times m$ matrix without zero rows and zero columns are regular in the field $\mathbb{C}$ of complex numbers. Also, for a ring $\mathcal{A}$ with unity, any $I \times M$ matrix over $\mathcal{A}$, in which each row and each column contains at least one unit of $\mathcal{A}$, must be regular in $\mathcal{A}$.

Let M, I be nonempty sets, and $\mathcal{A}$ an associative ring and $Q = (q_{mi})$ a generalized $M \times I$ matrix over $\mathcal{A}$. Consider the set $\mathfrak{M}(\mathcal{A}, I, M)$ consisting of all generalized $I \times M$ matrices over $\mathcal{A}$ with only finite nonzero entries, such an $I \times M$ matrix is usually said to be *bounded*. For $C = (a_{im}), D = (b_{im}) \in \mathfrak{M}(\mathcal{A}, I, M)$, define

$$\begin{aligned} C + D &= (e_{im}), \quad \text{where } e_{im} = c_{im} + d_{im} \quad \text{for } i \in I, m \in M; \\ C \circ D &= CQD, \quad \text{where the product on the right side is the product of matrices;} \\ \lambda C &= (\lambda c_{im}) \quad \text{for } \lambda \in R. \end{aligned}$$

By definition, a routine calculation shows that with these operations, $\mathfrak{M}(\mathcal{A}, I, M)$ is an associative ring.

Definition 2.3. The above ring $\mathfrak{M}(\mathcal{A}, I, M)$ is called a *generalized Munn ring* $\mathcal{A}$ with the sandwich matrix Q, in notation, $\mathfrak{M}(\mathcal{A}, I, M; Q)$.

If I is finite, then we identity it with the set $\{1, 2, \ldots, i\}$, where i is the cardinality of I, and we write $\mathfrak{M}(\mathcal{A}, I, M; Q)$ as $\mathfrak{M}(\mathcal{A}, i, M; Q)$. Similarly, the notation $\mathfrak{M}(\mathcal{A}, I, m; Q)$ is used if $|M| = m < \infty$. Denote by $M_{m,n}(\mathcal{A})$ the set of all $m \times n$ matrices over $\mathcal{A}$.

Recall from [20] that the generalized Munn ring $\mathfrak{M}(\mathcal{A}, m, n; Q)$ is called the *Munn $m \times n$ matrix ring* over $\mathcal{A}$ with sandwich matrix Q. It is obvious that $M_n(\mathcal{A})$ is the Munn $n \times n$ matrix ring over $\mathcal{A}$ with sandwich matrix Δ, where Δ is the unit matrix; that is, the diagonal matrix each of whose entries in the diagonal positions is the unity of $\mathcal{A}$.

Definition 2.4. Let $\mathcal{T}$ be a nonempty subset of $\mathfrak{M}(\mathcal{A}, I, M)$. An $M \times I$ matrix $X = (x_{mi})$ over $\mathcal{A}$ is said to be *cancellable* in $\mathcal{T}$ if for any nonzero element $Y \in \mathcal{T}$, YX and XY are neither zero, where YX and XY are usual matrix products.

Evidently, for a ring $\mathcal{A}$ with identity 1, the $I \times I$ unit matrix Δ is cancellable in any subset of $\mathfrak{M}(\mathcal{A}, I, I)$. And, any invertible $n \times n$ matrix must be cancellable in $M_n(\mathcal{A})$, but not all of cancellable matrices in $M_n(\mathcal{A})$ are invertible in the matrix algebra.

Example. Let $\mathbb{Z}$ be the ring of integers. It is easy to check that the matrix

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}$$

is cancellable in the matrix ring $M_2(\mathbb{Z})$. But A is not invertible in $M_2(\mathbb{Z})$.

For convenience, we denote

$(a)_{im}$: the generalized $I \times M$ matrix with a in the (i, m) position and 0 elsewhere;
$(\mathcal{B})_{im}$: the set $\{(b)_{im} : b \in \mathcal{B}\}$ for $\mathcal{B} \subseteq \mathcal{A}$;
$(a_{im})_{i\in J, m\in N}$: the generalized $I \times M$ matrix with a_{im} in the (i, m) position for $i \in J$, $m \in N$ and 0 elsewhere. Especially, if $N = \{n\}$, we simply write $m \in N$ as $m = n$, and the similar sign for the case: $|I| = 1$.
$\mathrm{rad}(\mathcal{A})$: the Jacobson radical of the ring $\mathcal{A}$.

We now arrive at our main results of this note.

Theorem 2.5. *Let $\mathfrak{M} = \mathfrak{M}(\mathcal{A}, I, M; Q)$ be a generalized Munn ring. If Q is regular in $\mathcal{A}$, then $\mathfrak{M}(\mathcal{A}, I, M; Q)$ is semiprime (semiprimitive, respectively) if and only if the following conditions are satisfied:*
(*i*) *Q is cancellable in $\mathfrak{M}(\mathcal{A}, I, M)$;*
(*ii*) *for any unit entries q_{mi}, q_{nj} of Q, if x is a nonzero element of $q_{mi}^{\diamond}\mathcal{A}q_{nj}^{\diamond}$, then $q_{nj}^{\diamond}\mathcal{A}q_{mi}^{\diamond}x \neq 0$ and $xq_{nj}^{\diamond}\mathcal{A}q_{mi}^{\diamond} \neq 0$;*
(*iii*) *for any unit entry q_{mi} of Q, $q_{mi}^{\diamond}\mathcal{A}q_{mi}^{\diamond}$ is semiprime (semiprimitive, respectively).*

Proof. Let q_{mi} be a unit entry of Q and denote by q_{mi}^{-1} the inverse of q_{mi} in $q_{mi}^{\diamond}\mathcal{A}q_{mi}^{\diamond}$, obviously $q_{mi}^{\diamond} = q_{mi}q_{mi}^{-1} = q_{mi}^{-1}q_{mi}$. Then $(q_{mi}^{-1})_{im}$ is an idempotent of $\mathfrak{M}$, and

$$(q_{mi}^{-1})_{im} \circ \mathfrak{M} \circ (q_{mi}^{-1})_{im} = (q_{mi}^{\diamond}\mathcal{A}q_{mi}^{\diamond})_{im} \tag{2.1}$$

since q_{mi} is a unit in $q_{mi}^{\diamond}\mathcal{A}q_{mi}^{\diamond}$. A routine calculation shows that the mapping

$$\phi : (x)_{im} \mapsto xq_{mi} \text{ is an isomorphism from } (q_{mi}^{\diamond}\mathcal{A}q_{mi}^{\diamond})_{im} \text{ onto } q_{mi}^{\diamond}\mathcal{A}q_{mi}^{\diamond}. \tag{2.2}$$

(2.5.1) *The proof for the semiprime case.* If $\mathfrak{M}$ is semiprime, then by (2.1), $(q_{mi}^{\diamond}\mathcal{A}q_{mi}^{\diamond})_{im}$ is semiprime, so that by ϕ is an isomorphism, $q_{mi}^{\diamond}\mathcal{A}q_{mi}^{\diamond}$ is semiprime. It results (iii). To see (i), assume on the contrary that Q is not cancellable in $\mathfrak{M}(\mathcal{A}, I, M)$, then there is a nonzero element $X \in \mathfrak{M}$ such that $XQ = 0$ or $QX = 0$. Without loss of generality, let $XQ = 0$, so that $X \circ U \circ X = XQUQX = 0$ for any $U \in \mathfrak{M}$, whence $X \circ \mathfrak{M} \circ X = 0$, contrary to that $\mathfrak{M}$ is semiprime. Thus, Q is cancellable in $\mathfrak{M}(\mathcal{A}, I, M)$.

We next verify (ii). To the end, we assume contrariwise that there exists a nonzero element $x \in q_{mi}^{\diamond}\mathcal{A}q_{nj}^{\diamond}$ such that $xq_{nj}^{\diamond}\mathcal{A}q_{mi}^{\diamond} = 0$. Obviously, $xq_{nj}^{\diamond} = x = q_{mi}^{\diamond}x$, so that

$$(x)_{im} \circ \mathfrak{M} \circ (x)_{im} \subseteq (x\mathcal{A}x)_{im} = ((xq_{nj}^{\diamond})\mathcal{A}(q_{mi}^{\diamond}x))_{im} = (x(q_{nj}^{\diamond}\mathcal{A}q_{mi}^{\diamond})x)_{im} = 0,$$

contrary to the hypothesis that $\mathfrak{M}$ is semiprime. Therefore, $xq_{nj}^{\diamond}\mathcal{A}q_{mi}^{\diamond} \neq 0$. Similarly, $q_{nj}^{\diamond}\mathcal{A}q_{mi}^{\diamond}x \neq 0$. We have now proved that (ii) is valid.

For the converse, we contrariwise let $w = (w_{im})$ be a nonzero generalized $I \times M$ matrix in $\mathfrak{M}$ such that $w \circ \mathfrak{M} \circ w = 0$. Because Q is cancellable, $QwQ = (u_{mi})_{m\in M, i\in I} \neq 0$, and we assume that $u_{m_0i_0} \neq 0$. If $wQ = (v_{ij})_{i,j\in I}$, then

$$u_{m_0i_0} = \sum_{j\in I} q_{m_0 j}v_{ji_0}. \tag{2.3}$$

When Q is regular, there is $j_0 \in I$ such that $q_{m_0 j_0}$ is a unit entry of Q. It follows that $q_{m_0 j_0}^{\diamond}q_{m_0 j} = q_{m_0 j}$. Now by (2.3),

$$u_{m_0i_0} = \sum_{j\in I} q_{m_0 j_0}^{\diamond}q_{m_0 j}v_{ji_0} = q_{m_0 j_0}^{\diamond}\left(\sum_{j\in I} q_{m_0 j}v_{ji_0}\right) = q_{m_0 j_0}^{\diamond}u_{m_0i_0},$$

and similarly, there exists a unit entry $q_{n_0i_0}$ of Q such that $u_{m_0i_0}q_{n_0i_0}^{\diamond} = u_{m_0i_0}$. Therefore, $u_{m_0i_0} \in q_{m_0 j_0}^{\diamond}\mathcal{A}q_{n_0i_0}^{\diamond}$. Furthermore, by (ii), there is $x \in q_{n_0i_0}^{\diamond}\mathcal{A}q_{m_0 j_0}^{\diamond}$ such that $0 \neq u_{m_0i_0}x \in q_{m_0 j_0}^{\diamond}\mathcal{A}q_{m_0 j_0}^{\diamond}$. Clearly,

$$q_{m_0 j_0}^{\diamond}u_{m_0i_0}x = u_{m_0i_0}x = \left(u_{m_0i_0}x\right)q_{m_0 j_0}^{\diamond}. \tag{2.4}$$

It follows that

$$\begin{aligned} u &:= \left(q_{m_0 j_0}^{\diamond}\right)_{j_0m_0} QwQ(x)_{i_0m_0} \\ &= \left(q_{m_0 j_0}^{\diamond}\right)_{j_0m_0}\left(u_{m_0i_0}\right)_{m_0i_0}(x)_{i_0m_0} \\ &= \left(q_{m_0 j_0}^{\diamond}u_{m_0i_0}x\right)_{j_0m_0} \\ &= \left(u_{m_0i_0}x\right)_{j_0m_0} \\ &= \left(q_{m_0 j_0}^{\diamond}\right)_{j_0m_0} \circ w \circ (x)_{i_0m_0} \\ &\neq 0. \end{aligned} \tag{2.5}$$

Now

$$\begin{aligned} \left(u_{m_0i_0}x \cdot q_{m_0 j_0}^{\diamond}\mathcal{A}q_{m_0 j_0}^{\diamond} \cdot u_{m_0i_0}x\right)_{j_0m_0} &= \left(u_{m_0i_0}xq_{m_0 j_0}^{\diamond} \cdot \mathcal{A} \cdot q_{m_0 j_0}^{\diamond}u_{m_0i_0}x\right)_{j_0m_0} \\ &= \left(u_{m_0i_0}x\right)_{j_0m_0} Q\left(q_{m_0 j_0}^{-1}\mathcal{A}q_{m_0 j_0}^{-1}\right)_{j_0m_0} Q\left(u_{m_0i_0}x\right)_{j_0m_0} \\ &\subseteq \left(u_{m_0i_0}x\right)_{j_0m_0} Q\mathfrak{M}Q\left(u_{m_0i_0}x\right)_{j_0m_0} \\ &= \left(u_{m_0i_0}x\right)_{j_0m_0} \circ \mathfrak{M} \circ \left(u_{m_0i_0}x\right)_{j_0m_0} \\ &= (\left(q_{m_0 j_0}^{\diamond}\right)_{j_0m_0} \circ w \circ (x)_{j_0m_0}) \circ \mathfrak{M} \circ (\left(q_{m_0 j_0}^{\diamond}\right)_{j_0m_0} \circ w \circ (x)_{j_0m_0}) \\ &\subseteq \left(q_{m_0 j_0}^{\diamond}\right)_{j_0m_0} \circ (w \circ \mathfrak{M} \circ w) \circ (x)_{j_0m_0} \\ &= 0, \end{aligned}$$

so that $u_{m_0i_0}x \cdot q_{m_0 j_0}^{\diamond}\mathcal{A}q_{m_0 j_0}^{\diamond} \cdot u_{m_0i_0}x = 0$. This is contrary to that $q_{m_0 j_0}^{\diamond}\mathcal{A}q_{m_0 j_0}^{\diamond}$ is semiprime. Consequently, $\mathfrak{M}$ is semiprime.

(2.5.2) *The proof for the semiprimive case*. It is well known that for a semiprimitive algebra $\mathfrak{A}$ and an idempotent $e \in \mathfrak{A}$, $e\mathfrak{A}e$ is still semiprimitive. So, the same reason as in (2.5.1) shows that the "if" part is valid. With notations in (2.5.1), if $w \in \mathrm{rad}(\mathfrak{M})\backslash\{0\}$, then by (2.5), $0 \neq (u_{m_0i_0}x)_{j_0m_0} \in \mathrm{rad}(\mathfrak{M})$. Notice that $(q_{m_0 j_0}^{-1})_{j_0m_0}$ is an idempotent of $\mathfrak{M}$, we can obtain that

$$\begin{aligned}\big(u_{m_0i_0}x\big)_{j_0m_0} &= \big(q^{\diamond}_{m_0j_0}u_{m_0i_0}xq^{\diamond}_{m_0j_0}\big)_{j_0m_0} \qquad (\text{by } (2.3))\\ &= \big(q^{-1}_{m_0j_0}\big)_{j_0m_0}\circ\big(u_{m_0i_0}x\big)_{j_0m_0}\circ\big(q^{-1}_{m_0j_0}\big)_{j_0m_0}\\ &\in \big(q^{-1}_{m_0j_0}\big)_{j_0m_0}\circ \operatorname{rad}(\mathfrak{M})\circ\big(q^{-1}_{m_0j_0}\big)_{j_0m_0}\\ &= \operatorname{rad}(\big(q^{-1}_{m_0j_0}\big)_{j_0m_0}\circ\mathfrak{M}\circ\big(q^{-1}_{m_0j_0}\big)_{j_0m_0})\\ &= \operatorname{rad}(\big(q^{-1}_{m_0j_0}\mathcal{A}q^{-1}_{m_0j_0}\big)_{j_0m_0})\\ &= \operatorname{rad}(\big(q^{\diamond}_{m_0j_0}\mathcal{A}q^{\diamond}_{m_0j_0}\big)_{j_0m_0}).\end{aligned}$$

This means that $(q^{\diamond}_{m_0j_0}\mathcal{A}q^{\diamond}_{m_0j_0})_{j_0m_0}$ is not semiprimitive. Now by (2.2), we observe that $q^{\diamond}_{m_0j_0}\mathcal{A}q^{\diamond}_{m_0j_0}$ is not semiprimitive. This is contrary to the hypothesis. It results the "only if" part. ☐

Remark 2.6. Let us turn back to the proof of Theorem 2.5. In (2.5.1), the proof of Condition (i) in the direct part has indeed proved that if $\mathfrak{M}(\mathcal{A}, I, M;\ Q)$ is semiprime, then Q is cancellable in $\mathfrak{M}(\mathcal{A}, I, M)$.

Theorem 2.7. *Let $\mathfrak{M} = \mathfrak{M}(\mathcal{A}, I, M;\ Q)$ be a generalized Munn ring. If Q is regular in $\mathcal{A}$, then $\mathfrak{M}(\mathcal{A}, I, M;\ Q)$ is prime (primitive, respectively) if and only if the following conditions are satisfied:*

(i) *Q is cancellable in $\mathfrak{M}(\mathcal{A}, I, M)$;*

(ii) *for any unit entries $q_{mi}, q_{nj}, q_{rk}, q_{sl}$ of Q, if x and y are nonzero elements of $q^{\diamond}_{mi}\mathcal{A}q^{\diamond}_{nj}$ and $q^{\diamond}_{rk}\mathcal{A}q^{\diamond}_{sl}$, respectively, then $xq^{\diamond}_{nj}\mathcal{A}q^{\diamond}_{rk}y \neq 0$;*

(iii) *for any unit entry q_{mi} of Q, $q^{\diamond}_{mi}\mathcal{A}q^{\diamond}_{mi}$ is prime (primitive, respectively).*

Proof. (2.7.1) *The proof for the prime case.* If $\mathfrak{M}$ is prime, then by (2.1), $(q^{\diamond}_{mi}\mathcal{A}q^{\diamond}_{mi})_{im}$ is prime, so that by ϕ is an isomorphism, $q^{\diamond}_{mi}\mathcal{A}q^{\diamond}_{mi}$ is prime. By Remark 2.6, Q is cancellable in $\mathfrak{M}(\mathcal{A}, I, M)$.

We next verify (ii). We contrariwise let $x \in q^{\diamond}_{mi}\mathcal{A}q^{\diamond}_{nj}\backslash\{0\}$, $y \in q^{\diamond}_{rk}\mathcal{A}q^{\diamond}_{sl}\backslash\{0\}$ such that $xq^{\diamond}_{nj}\mathcal{A}q^{\diamond}_{rk}y = 0$. Then, $xq^{\diamond}_{nj} = x$ and $q^{\diamond}_{rk}y = y$. Moreover,

$$(x)_{im}\circ\mathfrak{M}\circ(y)_{im} = (x\mathcal{A}y)_{im} = (x\cdot q^{\diamond}_{nj}\mathcal{A}q^{\diamond}_{rk}\cdot y)_{im} = (0)_{im} = 0.$$

It is contrary to the hypothesis that $\mathfrak{M}$ is prime. We have now proved the necessity.

To see the converse part, we assume conversely that there exist nonzero elements $A, B \in \mathfrak{M}$ such that $A\circ\mathfrak{M}\circ B = 0$. It is not difficult to see that Condition (ii) in Theorem 2.7 implies Condition (ii) in Theorem 2.5. Indeed, by (ii), for any $x \in q^{\diamond}_{mi}\mathcal{A}q^{\diamond}_{nj}$, $xq^{\diamond}_{nj}\mathcal{A}q^{\diamond}_{mi}x \neq 0$, so that $xq^{\diamond}_{nj}\mathcal{A}q^{\diamond}_{ni} \neq 0$; similarly, $q^{\diamond}_{nj}\mathcal{A}q^{\diamond}_{mi}x \neq 0$, and it results immediately in Condition (ii) in Theorem 2.5. Now by (2.5.1) (precisely, see (2.4) and (2.5)), there are $m, n \in M$, $j, k \in I$ and $C_1, C_2, D_1, D_2 \in \mathfrak{M}$ such that

(a) q_{mj}, q_{nk} are unit entries of Q; and

(b) $C_1\circ A\circ D_1 = (a)_{jm}$ and $C_2\circ B\circ D_2 = (b)_{kn}$, where a and b are nonzero elements in $q^{\diamond}_{mj}\mathcal{A}q^{\diamond}_{mj}$ and $q^{\diamond}_{nk}\mathcal{A}q^{\diamond}_{nk}$, respectively.

Furthermore by (ii), we have $u \in q^{\diamond}_{mj}\mathcal{A}q^{\diamond}_{nk}$ such that $aub \neq 0$. Obviously, $q^{\diamond}_{mj}u = u = uq^{\diamond}_{nk}$.

Compute

$$\begin{aligned}(aub)_{jn}\circ\mathfrak{M}\circ(aub)_{jn} &= (aq^{\diamond}_{mj}ub)_{jn}\circ\mathfrak{M}\circ(auq^{\diamond}_{nk}b)_{jn}\\ &= (aq_{mj}q^{-1}_{mj}ub)_{jn}\circ\mathfrak{M}\circ(auq^{-1}_{nk}q_{nk}b)_{jn}\\ &= (a)_{jm}\circ(q^{-1}_{mj}ub)_{jn}\circ\mathfrak{M}\circ(auq^{-1}_{nk})_{jn}\circ(b)_{kn}\\ &\subseteq (a)_{jm}\circ\mathfrak{M}\circ(b)_{kn}\\ &= C_1\circ A\circ D_1\circ\mathfrak{M}\circ C_2\circ B\circ D_2\\ &\subseteq C_1\circ A\circ\mathfrak{M}\circ B\circ D_2\\ &= 0.\end{aligned}\tag{2.6}$$

This shows that $\mathfrak{M}$ is not a semiprime ring. But by Theorem 2.5, $\mathfrak{M}$ is semiprime. It is a contradiction. Therefore, $\mathfrak{M}$ is prime.

(2.7.2) *The proof for the primitive case.* By the well-known result (for example, see [21, Ex. 10, p. 339]): for any primitive algebra $\mathfrak{A}$ and any idempotent e in $\mathfrak{A}$, $e\mathfrak{A}e$ is still primitive, and since any primitive algebra is prime, a similar argument as in (2.5.1) can verify the "if" part. For the converse, if given conditions hold, then by (2.7.1), $\mathfrak{M}$ is prime. The rest follows from a famous result of Lanahi et al. [22] showed that for a prime ring R, if e is a nonzero idempotent in R, then R is primitive if and only if eRe is primitive. □

Based on Theorems 2.5 and 2.7, we may prove the following proposition.

Proposition 2.8. *Let $\mathfrak{M}(\mathcal{A}, I, M; Q)$ be a generalized Munn ring. Assume that*
(1) *$\mathcal{A}$ has a unity*;
(2) *each row and each column of Q contains at least one unit of $\mathcal{A}$.*

Then $\mathfrak{M}(\mathcal{A}, I, M; Q)$ is prime (semiprime, primtive and semiprimitive, respectively) if and only if the following conditions are satisfied:
(*i*) *Q is cancellable in $\mathfrak{M}(\mathcal{A}, I, M)$*;
(*ii*) *$\mathcal{A}$ is prime (semiprime, primtive and semiprimitive, respectively).*

Proof. By definition, $q_{mi}^{\diamond}$ is the unity of $\mathcal{A}$ for any unit entry q_{mi} of Q satisfying Condition (2); in this case, $q_{mi}^{\diamond}\mathcal{A}q_{mi}^{\diamond} = \mathcal{A}$. Obviously, Q is regular in $\mathcal{A}$.

Let q_{mi} be an arbitrary unit entry of Q. By Condition (2), there is an entry q_{mi_0} of Q such that q_{mi_0} is a unit in $\mathcal{A}$. But Q is regular in $\mathcal{A}$, so $q_{mi}^{\diamond}q_{mi_0} = q_{mi_0}$, and it follows that $q_{mi}^{\diamond}$ must be the unity of $\mathcal{A}$. We have now proved that any unit entry of Q is a unit in $\mathcal{A}$. This shows that Condition (ii) in Theorem 2.5 are satisfied and that Condition (ii) in Theorem 2.7 is satisfied whenever $\mathcal{A}$ is prime.

The rest follows immediately from Theorems 2.5 and 2.7. □

For a ring $\mathcal{A}$ with unity, denote by Δ the generalized $I \times I$ matrix over $\mathcal{A}$ each of whose entries in the diagonal positions is the unity of $\mathcal{A}$ and 0 elsewhere. Obviously, Δ is cancellable in $\mathfrak{M}(\mathcal{A}, I, I)$. It is easy to see that the following corollary is an easy consequence of Proposition 2.8.

Corollary 2.9. *Let $\mathcal{A}$ be a ring with unity. Then $\mathcal{A}$ is prime (semiprime, primitive and semiprimitive, respectively) if and only if for any [for some] nonempty set I, $\mathfrak{M}(\mathcal{A}, I, I; \Delta)$ is prime (semiprime, primtive and semiprimitive, respectively).*

Let us turn back to the proof of Theorem 2.7. Assume now that $\mathfrak{M}$ is semiprime and the condition:

(PM) If q_{mj}, q_{nk} are unit entries of Q, then $aq_{mj}^{\diamond}\mathcal{A}q_{nk}^{\diamond}b \neq 0$ for any nonzero elements $a \in q_{mj}^{\diamond}\mathcal{A}q_{mj}^{\diamond}$, $b \in q_{nk}^{\diamond}\mathcal{A}q_{nk}^{\diamond}$.

In this case, u in (2.5) exists in $\mathfrak{M}$. Moreover, we can derive Conditions (a) and (b) in the proof of Theorem 2.7, and whence (2.6). So, we have indeed proved the following theorem.

Theorem 2.10. *Let $\mathfrak{M}(\mathcal{A}, I, M; Q)$ be a generalized Munn ring. If Q is regular in $\mathcal{A}$, then $\mathfrak{M}$ is prime if and only if $\mathfrak{M}$ is semiprime and (PM) is satisfied.*

Comparing with Theorem 2.10, it raises a natural conjecture as follows:

Conjecture 2.11. *Let $\mathfrak{M}(\mathcal{A}, I, M; Q)$ be a generalized Munn ring, and assume that Q is regular in $\mathcal{A}$. Then the following conditions are equivalent*:
(*i*) *$\mathfrak{M}$ is primitive*;
(*ii*) *$\mathfrak{M}$ is semiprimitive and* (PM) *is satisfied*;
(*iii*) *$\mathfrak{M}$ is both semiprimitive and prime.*

2.2 Principal ideal domains

In this subsection, we study the primeness of Munn rings over a principal ideal domain. We first provide one property of cancellable matrices over a principal ideal domain.

Lemma 2.12. *Let $\mathcal{A}$ be a principal ideal domain, and Q a $m \times n$ matrix over $\mathcal{A}$. Then Q is cancellable in $\mathfrak{M}(\mathcal{A}, n, m)$ if and only if $m = n = r_Q$, where r_Q is the rank of Q.*

Proof. (Necessity). Assume that Q is cancellable in $\mathfrak{M}(\mathcal{A}, I, M)$. We suppose contrariwise that $m = n = r_Q$ is not valid. By [23, Proposition III.2.11], there exist invertible matrices U, V such that

$$UQV = \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix},$$

where D is a diagonal $r_Q \times r_Q$ matrix with nonzero diagonal entries. So, there exists a nonzero matrix A_{22} over $\mathcal{A}$ such that

$$\begin{pmatrix} 0 & 0 \\ 0 & A_{22} \end{pmatrix} UQV = 0.$$

Moreover,

$$\begin{pmatrix} 0 & 0 \\ 0 & A_{22} \end{pmatrix} UQ = 0,$$

so that

$$\begin{pmatrix} 0 & 0 \\ 0 & A_{22} \end{pmatrix} = 0$$

since, by hypothesis, Q is cancellable in $\mathfrak{M}(\mathcal{A}, I, M)$. It follows that $A_{22} = 0$. It is a contradiction. Thus, $m = r_Q = n$.

(Sufficiency). If $m = r_Q = n$, then by [23, Proposition III.2.11], there exist invertible matrices U, V such that $UQV = \operatorname{diag}(d_1, d_2, \ldots, d_m)$, where $d_i \neq 0$ for $i = 1, 2, \ldots, m$. For any $X = (x_{ij}) \in M_n(\mathcal{A})$, we have

$$\begin{aligned} XQ = 0 &\Leftrightarrow (XU^{-1})UQV = (y_{ij}d_j) = 0, \quad \text{where } XU^{-1} = (y_{ij}); \\ &\Leftrightarrow y_{ij}d_j = 0 \quad \text{for } i, j = 1, 2, \ldots, n; \\ &\Leftrightarrow y_{ij} = 0 \quad \text{for } i, j = 1, 2, \ldots, n; \\ &\Leftrightarrow XU^{-1} = 0; \\ &\Leftrightarrow X = 0, \end{aligned}$$

and similarly, $QX = 0$ if and only if $X = 0$. Therefore, Q is cancellable in $\mathfrak{M}(\mathcal{A}, m, n)$. □

Proposition 2.13. *Let $\mathfrak{M}(\mathcal{A}, m, n; Q)$ be a Munn ring. If $\mathcal{A}$ is a principal ideal domain with unity* 1, *then the following conditions are equivalent*:

(*i*) $\mathfrak{M}$ *is semiprime*;
(*ii*) $m = r_Q = n$;
(*iii*) $\mathfrak{M}$ *is prime*.

Proof. (i) ⇒ (ii). By Remark 2.6, Q is cancellable in $\mathfrak{M}(\mathcal{A}, m, n)$. Now Lemma 2.12 results (ii).

(ii) ⇒ (iii). Let A_1 and A_2 be an arbitrary nonzero $n \times n$ matrices over $\mathcal{A}$. By [23, Proposition III.2.11], there exist invertible matrices U_i, V_i, $i = 1, 2$ such that

$$U_iA_iV_i = \operatorname{diag}\left(d_1^{(i)}, d_2^{(i)}, \ldots, d_{m_i}^{(i)}, 0, \ldots, 0\right),$$

where $d_k^{(i)} \neq 0$ for any $1 \leq k \leq m_i$. Let U and V be invertible matrices such that $Q = U\mathrm{diag}(c_1, c_2, \ldots, c_n)V$, where $c_j \neq 0$ for $j = 1, 2, \ldots, n$. Because

$$V_1^{-1}U = (x_{ij}),\ VU_2^{-1} = (y_{ij})$$

are both invertible, there exist $1 \leq j_0, i_0 \leq n$ such that $x_{1j_0} \neq 0$, $y_{i_01} \neq 0$. Compute

$$\begin{aligned}(1)_{11}U_1A_1QV^{-1}(1)_{j_0i_0}U^{-1}QA_2V_2(1)_{11} &= (1)_{11}(U_1A_1V_1)(V_1^{-1}U)\mathrm{diag}(c_1, \ldots, c_n)(1)_{j_0i_0}(U^{-1}QV^{-1})(VU_2^{-1})(U_2A_2V_2)(1)_{11}\\ &= (1)_{11}\mathrm{diag}\left(d_1^{(1)}, \ldots, d_{m_1}^{(1)}, 0 \cdots, 0\right)(x_{ij})\mathrm{diag}(c_1, c_2, \ldots, c_n)\\ &\quad \cdot(1)_{j_0i_0}\mathrm{diag}(c_1, c_2, \ldots, c_n)(y_{ij})\mathrm{diag}\left(d_1^{(2)}, \ldots, d_{m_2}^{(2)}, 0 \cdots, 0\right)(1)_{11}\\ &= \left(d_1^{(1)}x_{1j_0}c_{j_0}c_{i_0}y_{i_01}d_1^{(2)}\right)_{11}\\ &\neq 0,\end{aligned}$$

so that the entry in the $(1, 1)$ position of

$$U_1A_1QV^{-1}(1)_{j_0i_0}U^{-1}QA_2V_2 = U_1A_1 \circ V^{-1}(1)_{j_0i_0}U^{-1} \circ A_2V_2$$

is equal to the nonzero element $d_1^{(1)}x_{1j_0}c_{j_0}c_{i_0}y_{i_01}d_1^{(2)}$. It follows that $A_1 \circ V^{-1}(1)_{j_0i_0}U^{-1} \circ A_2 \neq 0$ since $\mathcal{A}$ is a principal ideal domain, giving $d_1^{(1)}x_{1j_0}c_{j_0}c_{i_0}y_{i_01}d_1^{(2)} \neq 0$. Thus, $A_1 \circ \mathfrak{M} \circ A_2 \neq 0$ and whence $\mathfrak{M}$ is prime.

(iii) ⇒ (i). It is obvious. □

Notice that for a principal ideal domain, the unity is the only nonzero idempotent. We observe that a generalized matrix over a principal ideal domain is regular if and only if each of its rows and each of its columns contain at least one unit. By Propositions 2.8 and 2.13, the following corollary is immediate from that any domain is prime.

Corollary 2.14. *Let $\mathfrak{M}(\mathcal{A}, m, n; Q)$ be a Munn ring. If*
(1) *$\mathcal{A}$ is a principal ideal domain*;
(2) *Q is regular in $\mathcal{A}$*,

then $\mathfrak{M}$ is prime if and only if $m = r_Q = n$.

3 Generalized matrix rings

In this section, we shall consider the primeness and the primitivity of generalized matrix rings. We first recall the definition of generalized matrix rings.

Let I be a nonempty set. For any $i, j, l \in I$, let A_{ii} be a ring with identity 1_i, and A_{ij} a unitary (A_{ii}, A_{jj})-bimodule. Assume that there is a module homomorphism $\mu_{ijl} : A_{ij} \times A_{jl}$ into A_{il}, written $\mu_{ijl}(x, y) = xy$, for any $i, j, l \in I$. If the following conditions hold:
(G1) $(x + y)z = xz + yz, \quad w(x + y) = wx + wy$;
(G2) $w(xz) = (wx)z$,

for any $x, y \in A_{ij}, z \in A_{jl}, w \in A_{li}$, then the triple (A_{ij}, I, μ_{ijl}) is called a Γ-*system* with index I.

Given a Γ-system (A_{ij}, I, μ_{ijl}), let $\mathcal{GM} = \mathcal{GM}(A_{ij}, I, \mu_{ijl})$ be the external direct sum of $\{A_{ij} : i, j \in I\}$. We shall use $\{x_{ij}\}$ to denote the external direct sum of x_{ij} with $i, j \in I$. Now we define the multiplication in $\mathcal{GM}$ as

$$xy = \left\{\sum_k x_{ik}y_{kj}\right\},$$

where $x = \{x_{ij}\}$ and $y = \{y_{ij}\}$. It is easy to check that $\mathcal{GM}$ is a ring (possibly without unity). We call $\mathcal{GM}$ a *generalized matrix ring*, or a *gm ring* for short, written $\mathcal{GM}(A_{ij}, I, \mu_{ijk})$ or $\mathcal{GM}$ for short. If e_{ii} is a nonzero element of A_{ii} satisfying that $e_{ii}x = x = xe_{jj}$ for all $x \in A_{ij}$, then the set $\{e_{ii} : i \in I\}$ is called a *generalized matrix unit* of the Γ-system (A_{ij}, I, μ_{iju}) (for example, see [14]).

In what follows, we still write the element $x = \{x_{ij}\}$ satisfying that $x_{ij} = 0$ if $i \neq i_0, j \neq j_0$ and $x_{i_0 j_0} = u$ as $\{u\}_{i_0 j_0}$, especially, write $\{1_{i_0}\}_{i_0 i_0} = 1_{i_0}$. Also, we use $\{A\}_{ij}$ to stand for the set $\{\{a\}_{ij} : a \in A\}$. And, we write $x = \{x_{ij}\}_{i \in A, j \in B}$ if $x_{ij} = 0$ whenever $i \in I \backslash A$ or $j \in I \backslash B$. It is easy to check that the set $\{1_i : i \in I\}$ is a generalized matrix unit of the Γ-system (A_{ij}, I, μ_{iju}).

Proposition 3.1. *The generalized matrix ring $\mathcal{GM}(A_{ij}, I, \mu_{ijk})$ is a subring of the generalized Munn ring $\mathfrak{M}(\mathcal{GM}, I, I; \Xi)$, where Ξ is the generalized $I \times I$ matrix in which any entry in the (i, i) position is 1_i, for any $i \in I$, and 0 elsewhere.*

Proof. Consider the mapping

$$\phi : \mathcal{GM} \to \mathfrak{M}(\mathcal{GM}, I, I; \Xi); \{x_{ij}\} \mapsto \sum (x_{ij})_{ij}.$$

A routine calculation shows that ϕ is an injective homomorphism, and here, we omit the detail. □

We can now describe the main results of this section.

Theorem 3.2. *Let $\mathcal{GM} = \mathcal{GM}(A_{ij}, I, \mu_{ijl})$ be a generalized matrix ring. Then $\mathcal{GM}$ is semiprime (semiprimitive, respectively) if and only if the following conditions are satisfied:*
(i) for any $i, j \in I$, if x is a nonzero element in A_{ij}, then $xA_{ji} \neq 0$ and $A_{ji}x \neq 0$;
(ii) for any $i \in I$, A_{ii} is semiprime (semiprimtive, respectively).

Proof. A routine calculation shows that $\{1_i\}_{ii}$ is an idempotent, for all $i \in I$, and the mapping $\varphi : \{x\}_{ii} \mapsto x$ is an isomorphism from $\{A_{ii}\}_{ii}$ onto A_{ii}. Compute

$$\{1_i\}_{ii}\mathcal{GM}\{1_i\}_{ii} = \{A_{ii}\}_{ii}.$$

So,

$$\{1_i\}_{ii}\mathcal{GM}\{1_i\}_{ii} \cong A_{ii}. \tag{3.1}$$

(3.2.1) *The proof for the semiprime case.* If $\mathcal{GM}$ is semiprime, then as for all $i \in I$, $\{1_i\}_{ii}$ is an idempotent in $\mathcal{GM}$, we obtain that $\{1_i\}_{ii}\mathcal{GM}\{1_i\}_{ii}$ is semiprime, so that by (3.1), A_{ii} is semiprime.

We contrariwise let $i_0, j_0 \in I$ such that $A_{i_0 j_0} \neq 0$ but $A_{j_0 i_0} = 0$. Pick a nonzero element a in $A_{i_0 j_0}$. Of course, $\{a\}_{i_0 j_0}$ is a nonzero element in $\mathcal{GM}$. Compute

$$\{a\}_{i_0 j_0}\mathcal{GM}\{a\}_{i_0 j_0} = \{a\}_{i_0 j_0}\left\{A_{j_0 i_0}\right\}_{j_0 i_0}\{a\}_{i_0 j_0} = \left\{aA_{j_0 i_0}a\right\}_{i_0 j_0} \subseteq \left\{A_{j_0 i_0}\right\}_{i_0 j_0} = 0, \tag{3.2}$$

contrary to the hypothesis that $\mathcal{GM}$ is semiprime. Therefore we have now proved that for any $i, j \in I$,

$$A_{ij} \neq 0 \Leftrightarrow A_{ji} \neq 0. \tag{3.3}$$

To see (i), we assume contrarily that x is a nonzero element in $A_{i_1 j_1}$ such that $xA_{j_1 i_1} = 0$. By (3.2), we have $\{x\}_{i_1 j_1}\mathcal{GM}\{x\}_{i_1 j_1} = \{xA_{j_1 i_1}x\}_{i_1 j_1} = 0$, contrary to the hypothesis that $\mathcal{GM}$ is semiprime. So, $xA_{ji} \neq 0$. Dually, we may prove that $A_{ji}x \neq 0$.

For the converse, assume that given Conditions (i) and (ii) hold. We oppositely let $u = \{u_{ij}\}$ be a nonzero element in $\mathcal{GM}$ such that $u\mathcal{GM}u = 0$. Notice that

- Ξ is regular in $\mathcal{GM}$ and cancellable in $\mathfrak{M}(\mathcal{GM}, I, I)$;
- $\{1_i\}_{ii}$, $i \in I$ are all unit entries of Ξ. Obviously, $\{1_i\}_{ii}^{\diamond} = \{1_i\}_{ii}$ and furthermore, $\{1_i\}_{ii}\mathcal{GM}\{1_j\}_{jj} = \{A_{ij}\}_{ij}$. Together with Condition (i), it is easy to see that Condition (ii) in Theorem 2.5.

By Theorem 2.5, the generalized Munn ring $\mathfrak{M}(\mathcal{GM}, I, I; \Xi)$ is semiprime. Denote

$$J = \{k \in I : u_{kj} \neq 0 \quad \text{for some } j \in I, \quad \text{or} \quad u_{ik} \neq 0 \quad \text{for some } i \in I\}.$$

It is not difficult to check that

(a) $\varepsilon = \sum_{i \in J}(1_i)_{ii}$ is an idempotent. Moreover, $\varepsilon\mathfrak{M}(\mathcal{GM}, I, I; \Xi)\varepsilon$ is semiprime.

(b) $\varepsilon\phi(u) = \phi(u) = \phi(u)\varepsilon$, where ϕ has the same meanings as in the proof of Proposition 3.1.

Moreover,

$$\phi(u)(\varepsilon\mathfrak{M}(\mathcal{GM}, I, I; \Xi)\varepsilon)\phi(u) \subseteq \phi(u)\phi(\mathcal{GM})\phi(u) = \phi(u\mathcal{GM}u) = 0,$$

contrary to the foregoing proof that $\varepsilon\mathfrak{M}(\mathcal{GM}, I, I; \Xi)\varepsilon$ is semiprime. Therefore $u = 0$ and whence $\mathcal{GM}$ is semiprime.

(3.2.2) *The proof for the semiprimitive case*. Similar as (3.2.1), we may prove the necessity.

For the converse, we contrariwise assume that u is a nonzero element in $\text{rad}(\mathcal{GM})$. With notations in (3.2.1), we denote $X = \{\{x_{ij}\}_{i \in J, j \in J}\} \subseteq \mathcal{GM}$. It is easy to see that

(a) X is a subalgebra of $\mathcal{GM}$;

(b) $\tau = \sum_{i \in J}\{1_i\}_{ii}$ is an idempotent in $\mathcal{GM}$. Moreover, $\phi(\tau) = \varepsilon$, $\tau u = u = u\tau$ and $\tau\mathcal{GM}\tau = X$;

(c) $\phi(X) = \varepsilon\mathfrak{M}(\mathcal{GM}, I, I; \Xi)\varepsilon$.

Therefore $u \in \text{rad}(\tau\mathcal{GM}\tau) = \text{rad}(X)$. Notice that ϕ is an injective homomorphism. We observe that $\varepsilon\mathfrak{M}(\mathcal{GM}, I, I; \Xi)\varepsilon$ is isomorphic to X. It follows that

$$\phi(u) \in \text{rad}(\varepsilon\mathfrak{M}(\mathcal{GM}, I, I; \Xi)\varepsilon),$$

so that $\varepsilon\mathfrak{M}(\mathcal{GM}, I, I; \Xi)\varepsilon$ is not semiprimitive. Indeed, by the proof of the converse part in (3.2.1), we can obtain that $\mathfrak{M}(\mathcal{GM}, I, I; \Xi)$ is semiprimitive if for any $i \in I$, A_{ii} is semiprimtive. In this case, $\varepsilon\mathfrak{M}(\mathcal{GM}, I, I; \Xi)\varepsilon$ is semiprimitive, contrary to the foregoing proof that $\varepsilon\mathfrak{M}(\mathcal{GM}, I, I; \Xi)\varepsilon$ is not semiprimitive. Consequently, $\mathcal{GM}$ is semiprimitive. □

Theorem 3.3. *Let $\mathcal{GM} = \mathcal{GM}(A_{ij}, I, \mu_{ijl})$ be a generalized matrix ring. Then $\mathcal{GM}$ is prime (primitive, respectively) if and only if the following conditions are satisfied:*

(i) for any $i, j, k, l \in I$, if x and y are respectively nonzero elements in A_{ij} and in A_{kl}, then $xA_{jk}y \neq 0$;

(ii) for any $i \in I$, A_{ii} is prime (primtive, respectively).

Proof. (3.3.1) *The proof for the prime case*. Similar as in (3.2.1), we may prove the necessity.

For the sufficiency, we contrarily assume that $u = \{u_{ij}\}$ and $v = \{v_{kl}\}$ are nonzero elements in $\mathcal{GM}$ such that $u\mathcal{GM}v = 0$. Obviously, there are i_0, j_0, k_0, and $l_0 \in I$ such that $u_{i_0 j_0}$ and $v_{k_0 l_0}$ are neither equal to 0. Further, by Condition (i), there is $x \in A_{j_0 k_0}$ such that $u_{i_0 j_0}xv_{k_0 l_0} \neq 0$. By the same reason, we have $y \in A_{l_0 i_0}$ such that $u_{i_0 j_0}xv_{k_0 l_0}yu_{i_0 j_0}xv_{k_0 l_0} \neq 0$. So that $0 \neq u_{i_0 j_0}xv_{k_0 l_0}y \in A_{i_0 i_0}$. Compute

$$\begin{aligned}
\left\{u_{i_0 j_0}xv_{k_0 l_0}y \cdot A_{i_0 i_0} \cdot u_{i_0 j_0}xv_{k_0 l_0}y\right\}_{i_0 i_0} &= \left\{u_{i_0 j_0}\right\}_{i_0 j_0}\left\{xv_{k_0 l_0}y\right\}_{j_0 i_0}\mathcal{GM}\left\{u_{i_0 j_0}x\right\}_{i_0 k_0}\left\{v_{k_0 l_0}\right\}_{k_0 l_0}\{y\}_{l_0 i_0} \\
&\subseteq \left\{u_{i_0 j_0}\right\}_{i_0 j_0}\mathcal{GM}\left\{v_{k_0 l_0}\right\}_{k_0 l_0}\{y\}_{l_0 i_0} \\
&= \left(\left\{1_{i_0}\right\}_{i_0 i_0}u\left\{1_{j_0}\right\}_{j_0 j_0}\right)\mathcal{GM}\left(\left\{1_{k_0}\right\}_{k_0 k_0}v\left\{1_{l_0}\right\}_{l_0 l_0}\right)\{y\}_{l_0 i_0} \\
&\subseteq \left\{1_{i_0}\right\}_{i_0 i_0} \cdot u\mathcal{GM}v \cdot \left\{1_{l_0}\right\}_{l_0 l_0}\{y\}_{l_0 i_0} \\
&= \left\{1_{i_0}\right\}_{i_0 i_0} \cdot 0 \cdot \left\{1_{l_0}\right\}_{l_0 l_0}\{y\}_{l_0 i_0} = 0.
\end{aligned}$$

It follows that $u_{i_0 j_0}xv_{k_0 l_0}y \cdot A_{i_0 i_0} \cdot u_{i_0 j_0}xv_{k_0 l_0}y = 0$. This means that $A_{i_0 i_0}$ is not semiprime, contrary to Condition (ii). Therefore, $\mathcal{GM}$ is prime.

(3.3.2) *The proof for the primitive case*. It follows from the proof in (2.5.2). □

4 Generalized path algebras

In this section, we consider the primeness and the primitivity of generalized path algebras. We first provide some results on quivers.

4.1 Quivers

We start with the basic definitions. A *quiver* $Q = (V, E)$ is an oriented graph, where V is the vertex set and E is the arrow set. We denote by $\mathfrak{S} : E \to V$ and $\mathfrak{T} : E \to V$ the mappings, where $\mathfrak{S}(\alpha) = i$ and $\mathfrak{T}(\alpha) = j$ when $\alpha : i \to j$ is an arrow from i to j. A *path* in the quiver Q is an ordered sequence of arrows $p = \alpha_n \cdots \alpha_1$ with $\mathfrak{T}(\alpha_l) = \mathfrak{S}(\alpha_{l+1})$ for $1 < l < n$, or the symbol e_i for $i \in V$. We call the path e_i *trivial path* and define $\mathfrak{S}(e_i) = i = \mathfrak{T}(e_i)$. For a nontrivial path $p = \alpha_n \cdots \alpha_1$, we define $\mathfrak{S}(p) = \mathfrak{S}(\alpha_n)$ and $\mathfrak{T}(p) = \mathfrak{T}(\alpha_1)$. A nontrivial path $p = \alpha_n \cdots \alpha_1$ is said to be
(i) an *oriented cycle* if $\mathfrak{S}(p) = \mathfrak{T}(p)$;
(ii) a *loop* from i to i if $n = 1$ and $\mathfrak{S}(p) = i = \mathfrak{T}(p)$.

Definition 4.1.
(i) A quiver G with vertex set V is said to be a *complete quiver* if for any $a, b \in V$ with $a \neq b$, there are one arrow from a to b and one arrow from b to a.
(ii) Let G_1 and G_2 be quivers with vertex set V_1 and arrow set E_1, and with vertex set V_2 and arrow set E_2, respectively. A quiver G is said to be a *union* of G_1 and G_2 if the vertex set of G is $V_1 \sqcup V_2$ and the arrow set of G is $E_1 \sqcup E_2$. If, in addition, both $V_1 \sqcup V_2$ and $E_1 \sqcup E_2$ are disjoint unions, then we shall call G to be a *disjoint union* of G_1 and G_2.

By an *empty graph*, we mean a graph without arrows. Obviously, we have the following observations:
(OB1) The empty graph is a complete quiver if and only if it it has exactly one vertex.

Also,
(OB2) Let $Q = (V, E)$ be a quiver without loops. Q is a disjoint union of complete quivers if and only if for any $a, b \in V$ with $a \neq b$, if there is a path from a to b, then there is one arrow from b to a.

Indeed, by definition, the necessity is evident. Conversely, we define a relation on the vertex set V as follows:

$$a\mathcal{D}b \quad \text{if } a = b; \text{ or there is a path from } a \text{ to } b.$$

It is not difficult to see that $\mathcal{D}$ is an equivalence on V. Consider the quotient $V/\mathcal{D} = \{V_\alpha : \alpha \in A\}$ and for V_α, construct a subquiver $Q_\alpha = (V_\alpha, E_\alpha)$ of Q as follows: for $a, b \in V_\alpha$,

there is an arrow from a to b in Q_α if and only if there is an arrow from a to b in Q.

It follows that Q is a disjoint union of the quivers Q_α with $\alpha \in A$. We next prove that each Q_α is a complete quiver. We consider the following two cases:
- If Q_α has exactly one vertex, then Q_α is an empty graph because it has no loops; thus, Q_α is a complete quiver.
- If Q_α has more than two vertices, then for any two vertices u, v of Q_α, there is a path from u to v, and furthermore by hypothesis, there is an arrow from v to u in Q. Therefore, there is an arrow from v to u in Q_α, and by definition, Q_α is a complete quiver.

However, Q_α is a complete quiver. Consequently, Q is a disjoint union of complete quivers.

Definition 4.2. Let $Q = (V, E)$ be a quiver with vertex set V and arrow set E. Construct a quiver $\overline{Q}^{PC}$ with vertex set V and in which for $u, v \in V$, there is an arrow from u to v in $\overline{Q}^{PC}$ if $u \neq v$ and there is a path from u to v in Q. The quiver $\overline{Q}^{PC}$ is called the *path-connected quiver* of Q, written $\overline{Q}^{PC}$.

By definition, it is easy to know that the path-connected quiver of a quiver always has no loops.

4.2 Generalized path algebras

We recall the definition of generalized path algebras.

Let I be a nonempty set and K a field. For any $i, j, u, v \in I$, A_{ij} is a vector space over the field K, and there, exists K-linear mapping μ_{iju} from $A_{ij} \otimes_K A_{ju}$ into A_{iu}, written $\mu_{iju}(x \otimes y) = xy$, such that $x(yz) = (xy)z$ for any $x \in A_{ij}, y \in A_{ju}, z \in A_{uv}$, then the set $\{A_{ij}, I, \mu_{iju}\}$ is a Γ-system with index I over the field K. Similar to the generalized matrix ring, we obtain a K-algebra, called a *generalized matrix algebra*, or a *gm algebra* in short, and written as $\mathcal{GMA}(A_{ij}, I, \mu_{iju})$, or $\mathcal{GMA}$ in short.

Assume that $D = (V, E)$ is a quiver (possibly an infinite quiver and also not a simple graph) with vertex set V and arrow set E. Let $\Omega = \mathcal{GMA}(\Omega_{ij}, V, \mu_{iju})$ be a generalized matrix algebra over the field K satisfying the following conditions:

(O1) Ω has a generalized matrix unit $\{e_{ii} : i \in V\}$.

(O2) $\Omega_{ij} = 0$ for any $i, j \in V$ with $i \neq j$.

The sequence $x = a_{i_0}x_{i_0i_1}a_{i_1}x_{i_1i_2} \cdots x_{i_{n-1}i_n}a_{i_n}$ is called a *generalized path*, or an Ω-*path*, from i_0 to i_n via arrows $x_{i_0i_1}, x_{i_1i_2}, \ldots, x_{i_{n-1}i_n}$, where $0 \neq a_{i_p} \in \Omega_{i_pi_p}$ for $p = 0, 1, 2, \ldots, n$. In this case, n is called the *length* of x, written $l(x)$.

For two Ω-paths $x = a_{i_0}x_{i_0i_1}a_{i_1}x_{i_1i_2} \cdots x_{i_{m-1}i_m}a_{i_m}$ and $y = b_{j_0}y_{j_0j_1}b_{j_1}y_{j_1j_2} \cdots y_{j_{n-1}j_n}b_{j_n}$ with $i_m = j_0$, we define the multiplication of x and y as follows:

$$xy = a_{i_0}x_{i_0i_1}a_{i_1}x_{i_1i_2} \cdots x_{i_{m-1}i_m}\left(a_{i_m}b_{j_0}\right)y_{j_0j_1}b_{j_1}y_{j_1j_2} \cdots y_{j_{n-1}j_n}b_{j_n}. \tag{4.1}$$

Denote by A'_{ij} the vector space over the field K with basis consisting of all Ω-paths from i to j with length ≥ 1. Let B_{ij} be the subspace spanned by all elements:

$$a_{i_0}x_{i_0i_1}a_{i_1}x_{i_1i_2} \cdots x_{i_{k-1}i_k}\left(\sum_{r=1}^{n} a_{i_k}^{(r)}\right)x_{i_{k+1}i_{k+2}} \cdots x_{i_{m-1}i_m}a_{i_m} - \sum_{r=1}^{n} a_{i_0}x_{i_0i_1}a_{i_1}x_{i_1i_2} \cdots x_{i_{k-1}i_k}a_{i_k}^{(r)}x_{i_{k+1}i_{k+2}} \cdots x_{i_{m-1}i_m}a_{i_m}, \tag{4.2}$$

where $i_0 = i$, $i_m = j$, $a_{i_k}^{(l)} \in \Omega_{i_ki_k}$, and $x_{i_pi_{p+1}}$ is an arrow, $p = 0, 1, \ldots, m-1$. Let $A_{ij} = A'_{ij}/B_{ij}$ when $i \neq j$ and $A_{ii} = (A'_{ii} + \Omega_{ii})/B_{ii}$, written $[\alpha] = \alpha + B_{ij}$ for any generalized path α from i to j. We can obtain a K-linear mapping κ_{iju} from $A_{ij} \otimes_K A_{ju}$ to A_{iu} induced by (4.1). We write a instead of $[a]$ when $a \in \Omega$. So, $(A_{ij}, V, \kappa_{iju}\}$ is a Γ-system. It is not difficult to know that $e_{ii}x_{ij} = x_{ij} = x_{ij}e_{jj}$ for any x_{ij} from i to j. Moreover, $\{e_{ii} : i \in V\}$ is a generalized matrix unit of the Γ-system $(A_{ij}, V, \kappa_{iju})$.

The notion of generalized path algebras is originally defined in [24]. For generalized path algebras, also see [25].

Definition 4.3. The aforementioned generalized matrix algebra $\mathcal{GMA}(A_{ij}, V, \kappa_{iju})$ is called the *generalized path algebra* of the quiver D over the generalized matrix algebra Ω, or the Ω-*path algebra*, written $K(D, \Omega)$. If, in addition, $\Omega_{ii} = Ke_{ii}$ for any $i \in V$, then $K(D, \Omega)$ is called a *path algebra* of the quiver D over the field K, written $K(D)$.

It is worthy to record here that for a generalized path algebra $K(D, \Omega)$, by (4.2), it follows that for any nonzero elements,

$$x = a_{i_0}x_{i_0i_1}a_{i_1}x_{i_1i_2} \cdots x_{i_{m-1}i_m}a_{i_m},\ y \in K(D, \Omega),$$

we have

(E1) $x = 0$ if and only if $a_{i_r} = 0$ for some $0 \leq r \leq m$;

(E2) $[x] = [y]$ in $K(D, \Omega)$ if and only if $x = y$ regarded as sequences.

Let $\mathcal{GM} = \mathcal{GM}(A_{ij}, I, \mu_{ijl})$ be a generalized matrix ring, and construct a quiver $\mathfrak{Q}(\mathcal{GM})$ with vertex set I and in which there is an arrow from i to j if and only if $i \neq j$ and $A_{ij} \neq 0$. We call the quiver $\mathfrak{Q}(\mathcal{GM})$ the Γ-*quiver* of the generalized matrix algebra $\mathcal{GM}(A_{ij}, I, \mu_{ijl})$. Obviously, $\mathfrak{Q}(\mathcal{GM})$ is a quiver without loops.

We next establish the relationship between a quiver and the Γ-quiver of its generalized path algebra.

Lemma 4.4. *Let $D = (V, E)$ be a quiver. If $K(D, \Omega)$ is a generalized path algebra of D over the generalized matrix ring Ω, then*

(*i*) *$\mathfrak{Q}(K(D, \Omega))$ is just the path connected quiver $\overline{D}^{PC}$ of D.*

(*ii*) *For any $i, j, u \in V$, if x is a nonzero element of A_{ij}, then $xA_{ju} \neq 0$ whenever $A_{ju} \neq 0$.*

Proof. (i). Notice that $\mathfrak{Q}(K(D, \Omega))$ and $\overline{D}^{PC}$ have the same vertex set. So, we need only to see whether $\mathfrak{Q}(K(D, \Omega))$ and $\overline{D}^{PC}$ have the same arrow set. It follows from the following implications: There is an arrow from u to v in $\mathfrak{Q}(K(D, \Omega))$ if and only if $u \neq v$ and $A_{uv} \neq \varnothing$; if and only if $u \neq v$ and $A'_{uv} \neq \varnothing$; if and only if there is a Ω-path $a_{i_0}x_{i_0i_1}a_{i_1}x_{i_1i_2} \cdots x_{i_{n-1}i_n}a_{i_n}$, where $i_0 = u$, $i_n = v$; if and only if there is a Ω-path $e_{i_0i_0}x_{i_0i_1}e_{i_1i_1}x_{i_1i_2} \cdots x_{i_{n-1}i_n}e_{i_ni_n}$, where $i_0 = u$, $i_n = v$; if and only if there is a path $x_{i_0i_1}x_{i_1i_2} \cdots x_{i_{n-1}i_n}$, where $i_0 = u$, $i_n = v$; if and only if there is an edge from u to v in $\overline{D}^{PC}$.

(ii). Let $x = a_{i_0}x_{i_0i_1}a_{i_1}x_{i_1i_2} \cdots x_{i_{n-1}i_n}a_{i_n}$, where $x_{i_0i_1}, x_{i_1i_2}, \ldots, x_{i_{n-1}i_n} \in E$, $i_0 = i$, $i_n = j$ and $0 \neq a_{i_p}$ for $p = 0, 1, 2, \ldots, n-1$. Obviously, $y = e_{j_0j_0}y_{j_0j_1}e_{j_1j_1}y_{j_1j_2} \cdots y_{j_{n-1}j_n}e_{j_nj_n}$, where $j_0 = j$, $j_n = u$ and $y_{j_pj_{p+1}} \in E$ for $p = 0, 1, \ldots, n-1$, is a nonzero element of A_{ju}. Then,

$$\begin{aligned} xy &= a_{i_0}x_{i_0i_1}a_{i_1}x_{i_1i_2} \cdots x_{i_{n-1}i_n}\big(a_{i_n}e_{j_0j_0}\big)y_{j_0j_1}e_{j_1j_1}y_{j_1j_2} \cdots y_{j_{n-1}j_n}e_{j_nj_n} \\ &= a_{i_0}x_{i_0i_1}a_{i_1}x_{i_1i_2} \cdots x_{i_{n-1}i_n}a_{i_n}y_{j_0j_1}e_{j_1j_1}y_{j_1j_2} \cdots y_{j_{n-1}j_n}e_{j_nj_n} \neq 0, \end{aligned}$$

which results (ii). □

By Theorem 3.2 and Lemma 4.4, we may prove the following theorem.

Theorem 4.5. *Let $K(D, \Omega)$ be a generalized path algebra. Then $K(D, \Omega)$ is semiprime (semiprimitive, respectively) if and only if*

(*i*) *$\overline{D}^{PC}$ of D is a disjoint union of complete quivers;*

(*ii*) *for any $i \in V$, A_{ii} is semiprime (semiprimitive, respectively).*

Proof. Suppose that $K(D, \Omega)$ is semiprime (semiprimitive, respectively). Theorem 3.2 immediately results (ii). Notice that $A_{ij} \neq 0$ if and only if there is a Ω-path from i to j; if and only if there is a path from i to j in the quiver D; if and only if there is an arrow from i to j in $\overline{D}^{PC}$. We can observe that if in $\overline{D}^{PC}$, there is a path $i_1 \to i_2 \to \cdots \to i_n$, then $A_{i_ki_{k+1}} \neq 0$ for $i = 1, 2, \ldots, n-1$, so that by Lemma 4.4, $A_{i_1i_2}A_{i_2i_3} \cdots A_{i_{n-1}i_n} \neq 0$. This means that there is a Ω-path from i_1 to i_n. It follows that $A_{i_1i_n} \neq 0$. By (3.3) in the proof of Theorem 3.2, this implies that $A_{i_ni_1} \neq 0$, thereby there is an arrow from i_n to i_1 in $\mathfrak{Q}(K(D, \Omega))$. It follows from Lemma 4.4 (i) that there is an arrow in $\overline{D}^{PC}$. Now by (OB2), $\overline{D}^{PC}$ is a disjoint union of complete quivers.

Conversely, assume that given conditions hold. For a nonzero element $x \in A_{ij}$, we have $A_{ij} \neq 0$, so that there is a path from i to j in $\overline{D}^{PC}$, it follows from (OB2) that there is an arrow from j to i in $\overline{D}^{PC}$, thus $A_{ji} \neq 0$. Again by Lemma 4.4, $xA_{ji} \neq 0$ and similarly, $A_{ji}x \neq 0$. Now by Theorem 3.2, $K(D, \Omega)$ is semiprime (semiprimitive, respectively). □

Also, by Theorem 3.3 and Lemma 4.4, we have

Theorem 4.6. *Let $K(D, \Omega)$ be a generalized path algebra. Then $K(D, \Omega)$ is prime (primitive, respectively) if and only if*

(*i*) *$\overline{D}^{PC}$ is a complete quiver;*

(*ii*) *for any $i \in V$, A_{ii} is prime (primitive, respectively).*

Proof. For the necessity, it suffices to verify that $\overline{D}^{PC}$ is a complete quiver.

- If $|I| = 1$, then $\mho(K(D, \Omega))$ is an empty graph since it has no loops, and by Lemma 4.4, $\overline{D}^{PC}$ is a complete quiver.
- Assume that $|I| \geq 2$. For any $i, j \in I$ with $i \neq j$, by definition, $A_{ii} \neq 0$ and $A_{jj} \neq 0$, and by Theorem 3.3 (i), $A_{ii}A_{ij}A_{jj} \neq 0$. It follows that $A_{ij} \neq 0$, so that there is an arrow form i to j in $\mho(K(D, \Omega))$. Similarly, we may prove that there is an arrow form j to i in $\mho(K(D, \Omega))$. Therefore, $\mho(K(D, \Omega))$ is a complete quiver, and by Lemma 4.4 (i), $\overline{D}^{PC}$ is a complete quiver.

To verify the sufficiency, we consider the following two cases:

- If $|I| = 1$, then $\mho(K(D, \Omega))$ is an empty graph, and $K(D, \Omega) \cong A_{ii}$, and this means that Condition (i) in Theorem 3.3 holds since each A_{ii} has a unity. It follows that $K(D, \Omega)$ is prime.
- Assume that $|I| \geq 2$. In this case, by $\mho(K(D, \Omega))$ is a complete quiver, there is an arrow from j to k in $\mho(K(D, \Omega))$ for any $j, k \in I$. This shows that $A_{jk} \neq 0$. By Lemma 4.4, $xA_{jk}y \neq 0$ for any $i, j, k, l \in I$ and nonzero elements $x \in A_{ij}$, $y \in A_{kl}$. Now by Theorem 3.3, $K(D, \Omega)$ is prime.

However, $K(D, \Omega)$ is prime. Similarly, we may verify the primitive case. We complete the proof. □

We may now prove the following proposition.

Proposition 4.7. *Let $K(D, \Omega)$ be a generalized path algebra and $i \in V$.*

(*i*) *If D has no paths from i to i, then A_{ii} is prime* (*primitive, semiprime and semiprimitive, respectively*) *if and only if so is Ω_{ii}.*

(*ii*) *If D has paths from i to i, then the following conditions are equivalent*:

(1) *A_{ii} is semiprime*;

(2) $\mathrm{ann}_\ell(\Omega_{ii}) = 0$ *and* $\mathrm{ann}_r(\Omega_{ii}) = 0$, *where* $\mathrm{ann}_\ell(X)$ ($\mathrm{ann}_r(X)$) *is the left* (*right*) *annihilator of X*;

(3) *A_{ii} is prime.*

Proof. (i). If D has no paths from i to i, then $A_{ii}' = 0$ and so $A_{ii} = [\Omega_{ii}]$. It follows that A_{ii} is isomorphic to Ω_{ii}, which results (i).

(ii). Assume that D has paths from i to i. We need only to verify that (1) ⇒ (2) and (2) ⇒ (3) since a prime ring is semiprime.

(1) ⇒ (2). Suppose that A_{ii} is semiprime. We assume contrariwise at least one of $\mathrm{ann}_\ell(\Omega_{ii}) \neq 0$ and $\mathrm{ann}_r(\Omega_{ii}) \neq 0$ holds. Without loss of generality, we let $\mathrm{ann}_\ell(\Omega_{ii}) \neq 0$ and $u \in \mathrm{ann}_\ell(\Omega_{ii})$, so that $u\Omega_{ii} = 0$. Consider the generalized path

$$x = a_{i_0}x_{i_0i_1}a_{i_1}x_{i_1i_2}\cdots x_{i_{m-1}i_m}u$$

via arrows $x_{i_0i_1}, x_{i_1i_2}, \ldots, x_{i_{n-1}i_n}$ with $i_0 = i = i_m$, and for any generalized path,

$$y = b_{j_0}y_{j_0j_1}b_{j_1}y_{j_1j_2}\cdots y_{j_{n-1}j_n}b_{j_n}$$

via arrows $y_{j_0j_1}, y_{j_1j_2}, \ldots, y_{j_{n-1}j_n}$ with $j_0 = i = j_n$. Obviously, $b_{j_0} \in \Omega_{ii}$. Therefore,

$$\begin{aligned} xy &= a_{i_0}x_{i_0i_1}a_{i_1}x_{i_1i_2}\cdots x_{i_{m-1}i_m}(ub_{j_0})y_{j_0j_1}b_{j_1}y_{j_1j_2}\cdots y_{j_{n-1}j_n}b_{j_n} \\ &= a_{i_0}x_{i_0i_1}a_{i_1}x_{i_1i_2}\cdots x_{i_{m-1}i_m}0y_{j_0j_1}b_{j_1}y_{j_1j_2}\cdots y_{j_{n-1}j_n}b_{j_n} = 0, \end{aligned}$$

and hence, $xb_{j_0} = 0$, thereby by the arbitrariness of b_{j_0}, $xA_{ii}' = 0$. It follows that $xA_{ii}x = 0$, contrary to the hypothesis that A_{ii} is semiprime. It results (2).

(2) ⇒ (3). Assume that (2) is satisfied. We let contrarily $w, z \in K(D, \Omega)\backslash\{0\}$ such that

$$wA_{ii}z = 0. \tag{4.3}$$

Let $w = \sum_{k=1}^{r} w_k$ and $z = \sum_{k=1}^{s} z_k$, where w_k, z_k are generalized paths, and

- $l(w_1) \geq l(w_2) \geq \cdots \geq l(w_r)$;
- $l(z_1) \geq l(z_2) \geq \cdots \geq l(z_s)$;

– $w_1 \notin \Omega_{ii} w_k \Omega_{ii}$ for $2 \le k \le r$;
– $z_1 \notin \Omega_{ii} z_l \Omega_{ii}$ for $2 \le l \le s$.

The equality (4.3) can derive that $w\Omega_{ii}z = 0$, so that for any $d \in \Omega_{ii}$,

$$w_1dz_1 + w_1dz_2 + \cdots + w_rdz_s = 0,$$

and hence, by (E2) and comparing the lengths of generalized paths w_idz_j, $w_1dz_1 = 0$. It follows that

$$0 = w_1\Omega_{ii}z_1 = a_{i_0}x_{i_0i_1}a_{i_1}x_{i_1i_2}\cdots x_{i_{m-1}i_m}(u\Omega_{ii}v)y_{j_0j_1}b_{j_1}y_{j_1j_2}\cdots y_{j_{n-1}j_n}b_{j_n}, \tag{4.4}$$

where $w_1 = a_{i_0}x_{i_0i_1}a_{i_1}x_{i_1i_2}\cdots x_{i_{m-1}i_m}u$ and $z_1 = vy_{j_0j_1}b_{j_1}y_{j_1j_2}\cdots y_{j_{n-1}j_n}b_{j_n}$. Again by (E1), the equality (4.4) can imply that $u\Omega_{ii}v = 0$. We have $u\Omega_{ii} = 0$ and $\Omega_{ii}v = 0$ by picking $u = 1_i$ or $v = 1_i$. This is contrary to (2). Therefore, A_{ii} is prime. □

Lemma 4.8. *Let Q be a quiver and K a field. Then the following statements are true for the path algebra $K(Q)$:*
(*i*) *If Q has no paths from i to i, then $A_{ii} \cong K$.*
(*ii*) *If Q has paths from i to i, then A_{ii} is semiprimitive.*

Proof. (i). If Q has no paths from i to i, then $A_{ii} = [\Omega_{ii}] = [Ke_{ii}] \cong Ke_{ii}$. But $Ke_{ii} \cong K$, so $A_{ii} \cong K$.

(ii). We assume on the contrary that A_{ii} is not semiprimitive. With notations in the proof of (2) ⇒ (3) in Proposition 4.7, assume that w is a nonzero element in $radA_{ii}$, and further, let z be a nonzero element of A_{ii} such that $wz + w + z = 0$. Without the loss of generality, we let $z = \sum_{k=1}^{s} z_k$, and each z_k has the same properties as (2) ⇒ (3) in Proposition 4.7. So,

$$w_1z_1 + \cdots + w_rz_s + z_1 + \cdots + z_s + w_1 + \cdots + w_r = wz + w + z = 0. \tag{4.5}$$

Consider that the length of w_iz_j is bigger than those of w_i and z_j, equation (4.5) derives that $w_1z_1 + \cdots + w_rz_s = 0$. Notice that the length of w_1z_1 is maximum among all w_iz_j, and this equation implies that $w_1z_1 = 0$. It follows that $w_1 = 0$ or $z_1 = 0$ since w_1 and z_1 are both Ω-paths from i to i. Now by the maximality of $l(w_1)$ and $l(z_1)$, all w_i are zero or all z_j are zero. Therefore, $w = 0$ or $z = 0$, contrary to that w, z are neither zero elements. Consequently, each A_{ii} is semiprimitive. □

For a path algebra $K(Q)$, by Lemma 4.8, each Ω_{ii} is semiprimitive and of course, semiprime. Now, the following theorem is an immediate consequence of Theorem 4.5.

Theorem 4.9. *Let Q be a quiver and K a field. Then, the following conditions are equivalent:*
(*i*) *$K(Q)$ is semiprime;*
(*ii*) *$\overline{Q}^{PC}$ is the disjoint union of complete quivers;*
(*iii*) *$K(Q)$ is semiprimitive.*

By Proposition 4.7 and since a field is prime, each A_{ii} of the path algebra $K(Q)$ is prime. Theorem 4.6 results in the following theorem.

Theorem 4.10. *Let Q be a quiver and K a field. Then $K(Q)$ is prime if and only if $\overline{Q}^{PC}$ is a complete quiver.*

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