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Research Article

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Spectra and reticulation of semihoops

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Abstract: In this article, we further study the filter theory of semihoops. Moreover, we use the prime (maximal) filters to construct the prime (maximal) spectrum on semihoops, and prove that the prime spectrum is a compact T_0 topological space and that the maximal spectrum is a compact T_2 topological space. As an application, in order to study the relationship between the spectrum of semihoops and the spectrum of lattices, we introduce the reticulations of semihoops and obtain some related results.

Keywords: semihoop, prime filter, maximal filter, spectra, reticulation, bounded distributive lattice

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1 Introduction

It is well known that logic is not only an important tool in mathematics and information science but also a basic technology. A host of logical algebras have been proposed as the semantical systems of non-classical logic systems, for example, MV-algebras, BL-algebras, MTL-algebras, and residuated lattices. Semihoops are the fundamental residuated structures and contain all logical algebras based on residuated lattices. Semihoops are generalizations of hoops that were originally introduced by Bosbach under the name of complementary semigroups. In recent years, many scholars have conducted research on semihoops. For example, in 2015, Borzooei and Kologani [1] studied the relationships among various filters on semihoops. In 2017, He et al. studied the states and internal states on semihoops [2]. In 2019, Niu and Xin further studied the tense operators on bounded semihoops [3], and Zhang and Xin further studied the derivations and differential filters on semihoops [4].

Algebra and topology are two basic fields in mathematics, which play complementary roles. Algebra and topology are naturally related in some applications or higher mathematics, such as analytic functions, dynamical systems, etc. The establishment of topological structures in logic algebra is always a hot topic in mathematical research. Currently, there are three main construction methods of topologies on logic algebra: First, the structure of the distance functions of the logic algebra is induced by filter theory to define an open set and then the topological structure is established [5–7]; second, filter system induced by filter theory is used to define open sets and then establish a topological structure [8,9]; finally, open sets are defined by the family of prime (maximal) filters (or ideals) in logical algebra, and then topological structures (such topological structures are called spectra) are established. In 1980, Simmons defined the reticulation of a ring, and then it was extended by Belluce to non-commutative rings in 1991. As for the fuzzy algebras, Belluce et al. constructed the reticulation of MV-algebras in 1994 [10]. In the twenty-first century, Leustean defined the reticulations of BL-algebras [11]. In June 2021, Zhang and Yang discussed the reticulations of EQ-algebras [12] and then Georgescu defined the reticulations of quantales [13]. However, we find

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that there is no research about the spectra and reticulations on semihoops, although semihoops are the fundamental residuated structures.

Based on the above analysis and summary, on one hand, semihoops are the most basic residuated structures, and the research results on topological structures can be extended to any residuated structure. On the other hand, by studying the reticulations of semihoops, some questions of algebraic structures can be transformed into questions of lattices to study, which can reduce the difficulty of the study. Therefore, we will study the topological structures (spectra) of semihoops by giving the related concepts and proposing propositions of semihoops.

This article is organized as follows: In Section 2, we give some basic facts on semihoops, which will be used in the sequel sections. In Section 3, we study how to construct topological spaces (prime and maximal spectrum) on semihoops A by using prime filters and maximal filters. And we prove that prime spectrum $\mathrm{Spec}(A)$ is a compact T_0 topological space and maximal spectrum $\mathrm{Max}(A)$ is a compact T_2 topological space. In Section 4, we introduce reticulations of semihoops, obtain some relationships between semihoops and bounded distributive lattices, and prove that the prime spectrum of semihoops is a homeomorphism of topological.

2 Preliminaries

In this section, we recollect a few definitions and propositions, which will be used in the following sections.

Definition 2.1. [14] An algebra $(A, \odot, \to, \wedge, 1)$ of type $(2, 2, 2, 0)$ is called a semihoop if it satisfies the following conditions:
(D1) $(A, \wedge, 1)$ is a $\wedge$-semilattice with upper bound 1;
(D2) $(A, \odot, 1)$ is a commutative monoid;
(D3) $(x \odot y) \to z = x \to (y \to z)$, for all $x, y, z \in A$ (Galois connect).

Let A be a semihoop, we say that it is prelinear if for all $a, b \in A$, 1 is the unique upper bound in A of the set $\{a \to b, b \to a\}$.

Example 2.2. [1] Let $A = \{0, a, b, 1\}$ be a chain. We define $\odot$ and $\to$ on A as follows:

$\odot$	0	a	b	1
0	0	0	0	0
a	0	0	a	a
b	0	a	b	b
1	0	a	b	1

$\to$	0	a	b	1
0	1	1	1	1
a	a	1	1	1
b	0	a	1	1
1	0	a	b	1.

Then $(A, \odot, \to, \wedge, 1)$ is a semihoop, where $x \wedge y = \min\{x, y\}$, for all $x, y \in A$.

Example 2.3. [1] Let $A = \{0, a, b, c, 1\}$. Define $\odot$ and $\to$ as follows:

$\odot$	0	a	b	c	1
0	0	0	0	0	0
a	0	a	a	a	a
b	0	a	b	a	b
c	0	a	a	c	c
1	0	a	b	c	1

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	0	1	1	1	1
b	0	c	1	c	1
c	0	b	b	1	1
1	0	a	b	c	1.

Then $(A, \odot, \to, \wedge, 1)$ is a semihoop, where $x \wedge y = x \odot y$, for all $x, y \in A$.

Proposition 2.4. [14] *Let A be a semihoop, then the following hold, for any $x, y, z \in A$:*
(1) $x \odot y \leq z$ *if and only if* $x \leq y \to z$;
(2) $x \odot y \leq x, y$;
(3) $1 \to x = x, x \to 1 = 1$;
(4) $x^n \leq x$, *for all* $n \in \mathbb{N}^+$;
(5) $x \odot (x \to y) \leq y$;
(6) *if* $x \leq y$, *then* $x \odot z \leq y \odot z$, $y \to z \leq x \to z$ *and* $z \to x \leq z \to y$.

On a semihoop A, we define $x \leq y$ if and only if $x \to y = 1$. It is easy to check that $\leq$ is a partial order relation on A.

Definition 2.5. [1] Let A be a semihoop and F be a non-empty subset of A. Then F is said be a filter of A, if it satisfies:
(F1) $x \odot y \in F$, for any $x, y \in F$;
(F2) $x \leq y$ and $x \in F$ imply $y \in F$, for any $x, y \in A$.

A filter F of A is called proper filter if $F \neq A$. We can see that $\{1\}$ and A are filters of A. We call they are ordinary filters. And we denote the set of all filters of A by $\mathcal{F}(A)$.

Lemma 2.6. (MP rule) *Let A be a semihoop and F be a non-empty subset of A. Then $F \in \mathcal{F}(A)$ if and only if $1 \in F$ and if $x, x \to y \in F$, for some $x, y \in A$, then $y \in F$.*

Let A be a semihoop and $X \subseteq A$. We denote by $\langle X \rangle$ the filter generated by X that is the intersection of all filters of A that contain X. If $X = \{x\}$, then the filter generated by X will be denoted by $\langle x \rangle$.

Lemma 2.7. [1] *Let A be a semihoop and $x \in A$. Then $\langle x \rangle = \{a \in A | x^n \leq a, \exists n \in \mathbb{N}^+\}$.*

Corollary 2.8. [1] *Let A be a semihoop, $F \in \mathcal{F}(A)$, and $x \in A$. Then $\langle F \cup \{x\} \rangle = \{a \in A | y \odot x^n \leq a, \exists n \in \mathbb{N}^+, y \in F\}$.*

Definition 2.9. [1] A proper filter F of semihoop A is called a maximal filter of A, if it is not properly contained in any other proper filters of A.

We consider $\mathrm{Max}(A)$ the set of all maximal filters of semihoop A.

Lemma 2.10. [1] *Let A be a semihoop and F be a proper filter of A . Then the following conditions are equivalent:*
(1) $F \in \mathrm{Max}(A)$.
(2) *if* $x \notin F$, *then* $\langle F \cup \{x\} \rangle = A$.

Definition 2.11. [1] A proper filter F of semihoop A is called prime filter of A, if any $H, G \in \mathcal{F}(A)$ such that $H \cap G \subseteq F$, then $H \subseteq F$ or $H \subseteq F$.

Lemma 2.12. [1] *Let A be a semihoop and for any $x, y \in A$, we define: $x \vee y = [(x \to y) \to y] \wedge [(y \to x) \to x]$. Then the following conditions are equivalent:*
(1) $\vee$ *is an associative operation on* A;
(2) $x \leq y$ *imply* $x \vee z \leq y \vee z$, *for all* $x, y, z \in A$;
(3) $x \vee (y \wedge z) \leq (x \vee y) \wedge (x \vee z)$, *for all* $x, y, z \in A$;
(4) $\vee$ *is the join operation on* A.

Lemma 2.13. [1] *A semihoop is a $\vee$-semihoop if it satisfies one of the equivalent conditions of* Lemma 2.12.

Definition 2.14. [1] Let A be a $\vee$-semihoop and F be a proper filter of A. Then the following conditions are equivalent:
(1) F is a prime filter.
(2) if $x \vee y$, for some $x, y \in A$, then $x \in F$ or $y \in F$.

We shall denote the set of all prime filters of A by Spec(A).

A semihoop A is bounded if there exists an element $0 \in A$ such that $0 \leq x$ for all $x \in A$. In a bounded semihoop A, we define the negation $'$ on A by $x' = x \to 0$, for all $x \in A$. If $x'' = x$, for any $x \in A$, then the bounded semihoop A is said to have the double negation property (DNP).

Definition 2.15. [15] In a bounded semihoop A, the binary operation $\oplus$ is defined by $x \oplus y = x' \to y$, for any $x, y \in A$.

Lemma 2.16. [15] *Let A be a bounded semihoop, then the following hold, for any $x, y, z \in A$:*
(1) *if $x \leq y$, then $x \oplus z \leq y \oplus z$;*
(2) $x, y \leq x \oplus y$;
(3) $x \oplus x' = 1$;
(4) $x \oplus y = 1$ *if and only if* $x' \leq y$.

In a bounded semihoop A with DNP, the operation $\oplus$ is commutative and associative.

3 Prime and maximal spectra of semihoops

Proposition 3.1. *Let A be a $\vee$-semihoop. Then A satisfies $x \odot (y \vee z) = (x \odot y) \vee (x \odot z)$, for any $x, y, z \in A$.*

Proof. For any $x, y, z \in A$. Clearly, $y, z \leq y \vee z$, that is, $x \odot y, x \odot z \leq x \odot (y \vee z)$. Then $(x \odot y) \vee (x \odot z) \leq x \odot (y \vee z)$. Conversely, since $x \odot y$, $x \odot z \leq (x \odot y) \vee (x \odot z)$, by Gaolis Connect, we obtain $y, z \leq x \to [(x \odot y) \vee (x \odot z)]$, that is, $y \vee z \leq x \to [(x \odot y) \vee (x \odot z)]$. Then, we obtain $x \odot (y \vee z) \leq (x \odot y) \vee (x \odot z)$. □

Proposition 3.2. *Let A be a $\vee$-semihoop and F be a proper filter of A. Suppose that $x, y \in A$, and $x, y \notin F$, then $\langle F \cup \{x\}\rangle \cap \langle F \cup \{y\}\rangle = \langle F \cup \{x \vee y\}\rangle$.*

Proof. Let $a \in \langle F \cup \{x\}\rangle \cap \langle F \cup \{y\}\rangle$. Hence, by Corollary 2.8, there exist $f_1, f_2 \in F$, $n_1, n_2 \in \mathbb{Z}^+$ such that $a \geq f_1 \odot x^{n_1}$, $a \geq f_2 \odot y^{n_2}$. Take $f = f_1 \oplus f_2$, $n = max\{n_1, n_2\}$, then we obtain $a \geq f \odot x^n$, $a \geq f \odot y^n$. With Proposition 3.1, $a = a \vee a \geq f \odot (x^n \vee y^n) = (f \odot x^n) \vee (f \odot y^n)$. Since $(x \vee y)^{n \times n} \leq x^n \vee y^n$, then we obtain $a \geq f \odot (x \vee y)^{n \times n}$. It follows that $a \in \langle F \cup \{x \vee y\}\rangle$. Conversely, suppose that $b \in \langle F \cup \{x \vee y\}\rangle$, then there exist $t \in F$, $s \in \mathbb{Z}^+$ such that $b \geq t \odot (x \vee y)^s$. Obviously, $b \geq t \odot x^s$ and $b \geq t \odot y^s$. It follows that $b \in \langle F \cup \{x\}\rangle$ and $b \in \langle F \cup \{y\}\rangle$, that is, $b \in \langle F \cup \{x\}\rangle \cap \langle F \cup \{y\}\rangle$. □

Proposition 3.3. *Let A be a $\vee$-semihoop and $M \in$ Max(A), then M is a prime filter of A.*

Proof. Let A be a $\vee$-semihoop and $M \in$ Max(A). Suppose that $H, G \in \mathcal{F}(A)$ such that $H \cap G \subseteq M$. If $H, G \nsubseteq M$, then there exist $x \in H$, $y \in G$ such that $x, y \notin M$. By Lemma 2.10, we have $\langle M \cup \{x\}\rangle = \langle M \cup \{y\}\rangle = A$. Moreover, since $x, y \leq x \vee y$, so we have $x \vee y \in H \cap G \subseteq M$. By Proposition 3.2, $A = \langle M \cup \{x\}\rangle \cap \langle M \cup \{y\}\rangle = \langle M \cup \{x \vee y\}\rangle = M$, which is a contradiction. Therefore, M is a prime filter. □

Lemma 3.4. *Let A be a semihoop, then the union of any chain of proper filters of A is a proper filter of A.*

Proof. Let $F = \cup\{F_i | F_1 \subseteq F_2 \subseteq \cdots \subseteq F_s\}$, $1 \le i \le s$. It is the union of any proper filters of A. Since any F_i is a proper filter of A, then $\{1\} \in F_i \Rightarrow \{1\} \subseteq F \neq \emptyset$. Suppose that $x \in F$, $y \in A$, and $x \le y$. Since $x \in F$, there exists t such that $x \in F_t$. Hence, $y \in F_t$, that is, $y \in F$. Finally, suppose that $x, y \in F$, then there exist $1 \le m$ and $n \le s$ such that $x \in F_m$, $y \in F_n$. Without losing generality, assume $F_m \subseteq F_n$. If F is a chain, then $x, y \in F_n$. Since F_n is proper, then $x \odot y \in F_n$, and we obtain $x \odot y \in F$. In summary, F is a proper filter of A. □

Proposition 3.5. *Let A be a $\vee$-semihoop and $a \in A$, $a \neq 1$. Then there exists a prime filter P of A such that $a \notin P$.*

Proof. Suppose that $a \in A$ and $a \neq 1$, let $\mathcal{F}(a) = \{F | F \in \mathcal{F}(A)\}$, F be a proper filter of A, and $a \notin F$. Since $a \neq 1$, then $\{1\} \in \mathcal{F}(a) \neq \emptyset$. Applying Lemma 3.4 and Zorn's lemma on $(\mathcal{F}(a), \subseteq)$, we obtain a maximal element P on $(\mathcal{F}(a), \subseteq)$. That means P is a maxmial filter of A with $a \notin P$. By Proposition 3.3, then P is a prime filter of A. □

Proposition 3.6. *Let A be a $\vee$-semihoop and F be a proper filter of A, then $F = \cap\{P \in \mathrm{Spec}(A) | F \subseteq P\}$.*

Proof. Obviously, $F \subseteq \cap \{P \in \mathrm{Spec}(A) | F \subseteq P\}$. Suppose that $a \notin F$, $a \neq 1$. By Proposition 3.5, there exists a prime filter P such that $a \notin P$ and $F \subseteq P$. It follows that $\cap\{P \in \mathrm{Spec}(A) | F \subseteq P\} \subseteq F$. □

Lemma 3.7. *If A is a $\vee$-semihoop, then every proper filter F of A is contained in a maximal filter.*

Corollary 3.8. *If A is a $\vee$-semihoop, then every proper filter F of A is contained in a prime filter.*

Proof. It directly follows Propositions 3.5 and 3.6. □

Corollary 3.9. *Any semihoop has a maximal, prime filter.*

In the sequel, we study the prime spectrum $\mathrm{Spec}(A)$ and maximal spectrum $\mathrm{Max}(A)$ of a semihoop.

Given a semihoop A and $X \subseteq A$, we define $E(X) = \{P \in \mathrm{Spec}(A) | X \subseteq P\}$.

Proposition 3.10. *Let A be a $\vee$-semihoop. The following propositions hold*:
(1) *if $X \subseteq Y \subseteq A$, then $E(Y) \subseteq E(X) \subseteq \mathrm{Spec}(A)$;*
(2) $E(\{0\}) = \emptyset$, $E(\emptyset) = E(\{1\}) = \mathrm{Spec}(A)$;
(3) $E(X) = E(\langle X \rangle)$, *for any $X \subseteq A$;*
(4) $E(X) = \emptyset \Leftrightarrow \langle X \rangle = A$;
(5) $E(X) = \mathrm{Spec}(A) \Leftrightarrow X = \emptyset$ or $X = \{1\}$;
(6) *if $\{X_i\}_{i \in I}$ is a family of subsets of A, then $E(\cup_{i \in I} X_i) = \cap_{i \in I} E(X_i)$;*
(7) *if $X, Y \subseteq A$, then $E(X) \cup E(Y) = E(\langle X \rangle \cap \langle Y \rangle)$;*
(8) *if $X, Y \subseteq A$, then $\langle X \rangle = \langle Y \rangle$ if and only if $E(X) = E(Y)$;*
(9) *if F, G are filters of A, then $F = G$ if and only if $E(F) = E(G)$.*

Proof. (1), (2), and (3) are obviously established.

(4) ($\Rightarrow$) Let $E(X) = \emptyset$. Suppose that $\langle X \rangle \neq A$, namely $\langle X \rangle$ is a proper filter of A. Applying Corollary 3.8, there exists a prime filter P such that $X \subseteq \langle X \rangle \subseteq P$, that is, $P \in E(X) \neq \emptyset$, which is a contradiction. Therefore, $\langle X \rangle = A$.

($\Leftarrow$) By (2) and (3), we have $E(X) = E(\langle X \rangle) = E(A) = \emptyset$.

(5) Let $E(X) = \mathrm{Spec}(A)$. Suppose that $X \neq \emptyset$ and $X \neq \{1\}$. It means that there exists $a \neq 1 \in X$, by Proposition 3.5, there exists a prime filter P such that $a \notin P$, which is a contradiction. Conversely, it directly follows from (2).

(6) Clearly, $X_i \subseteq \cup_{i \in I} X_i$, by (1), it follows that $E(\cup_{i \in I} X_i) \subseteq E(X_i)$. Thus, $E(\cup_{i \in I} X_i) \subseteq \cap_{i \in I} E(X_i)$. Conversely, suppose that $P \in \cap_{i \in I} E(X_i)$, then $X_i \subseteq P$ for any $i \in I$. Therefore, we obtain $\cup_{i \in I} X_i \subseteq P$, that is, $P \in E(\cup_{i \in I} X_i)$.

(7) First, $\langle X\rangle \cap \langle Y\rangle \subseteq \langle X\rangle, \langle Y\rangle$, by (1) and (3), then we obtain $E(X) \cup E(Y) \subseteq E(\langle X\rangle \cap \langle Y\rangle)$. Conversely, suppose that $P \in E(\langle X\rangle \cap \langle Y\rangle)$, that is, $\langle X\rangle \cap \langle Y\rangle \subseteq P$. If $P \notin E(X) \cup E(Y)$, then $P \notin E(X) = E(\langle X\rangle)$ and $P \notin E(Y) = E(\langle Y\rangle)$. It follows that $\langle X\rangle \nsubseteq P$ and $\langle Y\rangle \nsubseteq P$. Thus, there exist $x \in \langle X\rangle$ and $y \in \langle Y\rangle$ such that $x \notin P$ and $y \notin P$. Moreover, $x, y \leq x \vee y$, since $\langle X\rangle$ and $\langle Y\rangle$ are filters, so $x \vee y \in \langle X\rangle \cap \langle Y\rangle \subseteq P$. Since $P \in \mathrm{Spec}(A)$, we have $x \in P$ or $y \in P$, which is a contradiction. Hence, $E(\langle X\rangle \cap \langle Y\rangle) \subseteq E(X) \cup E(Y)$.

(8) ($\Rightarrow$) Let $\langle X\rangle = \langle Y\rangle$. Applying (3), it directly follows that $E(X) = E(Y)$.($\Leftarrow$) Suppose that $E(X) = E(Y)$. First, if $0 \in \langle X\rangle$, then $E(Y) = E(X) = E(A) = \varnothing$. By (4), we obtain $\langle X\rangle = \langle Y\rangle = A$. Moreover, if $\langle X\rangle$ is a proper filter of A, by Proposition 3.6, then $\langle X\rangle = \cap\{P \in \mathrm{Spec}(A) | \langle X\rangle \subseteq P\} = \cap\{P \in \mathrm{Spec}(A) | P \in E(\langle X\rangle) = E(X)\} = \cap\{P \in \mathrm{Spec}(A) | P \in E(Y) = E(\langle Y\rangle)\} = \cap\{P \in \mathrm{Spec}(A) | \langle Y\rangle \subseteq P\} = \langle Y\rangle$.

(9) It follows by (8). □

By Proposition 3.10 (2), (6), and (7), it follows that the family $\{E(X)\}_{X\subseteq A}$ of subsets of $\mathrm{Spec}(A)$ satisfies the axioms for close sets in a topological space, which is called Zarkiski topology and topological space $\mathrm{Spec}(A)$ is called the prime spectrum of A.

In the following, we consider the form of open sets of this topology. For any $X \subseteq A$, let us denote the complement of $E(X)$ by $D(X)$. Thus, $D(X) = \{P \in \mathrm{Spec}(A) | X \nsubseteq P\}$. By duality, from Proposition 3.10 we obtain the following.

Proposition 3.11. *Let A be a $\vee$-semihoop. The following propositions hold*:
(1) *if $X \subseteq Y \subseteq A$, then $D(X) \subseteq D(Y) \subseteq \mathrm{Spec}(A)$*;
(2) $D(\{0\}) = \mathrm{Spec}(A)$, $D(\varnothing) = D(\{1\}) = \varnothing$;
(3) *$D(X) = D(\langle X\rangle)$, for any $X \subseteq A$*;
(4) *$D(X) = \varnothing \Leftrightarrow X = \varnothing$ or $X = \{1\}$*;
(5) $D(X) = \mathrm{Spec}(A) \Leftrightarrow \langle X\rangle = A$;
(6) *if $\{X_i\}_{i\in I}$ is a family of subsets of A, then $D(\cup_{i\in I}X_i) = \cup_{i\in I}D(X_i)$*;
(7) *if $X, Y \subseteq A$, then $D(X) \cup D(Y) = D(\langle X\rangle \cup \langle Y\rangle)$*;
(8) *if $X, Y \subseteq A$, then $\langle X\rangle = \langle Y\rangle$ if and only if $D(X) = D(Y)$*;
(9) *if F, G are filters of A, then $F = G$ if and only if $D(F) = D(G)$.*

For any $a \in A$, let us denote $E(\{a\})$ by $E(a)$ and $D(\{a\})$ by $D(a)$. It follows that $E(a) = \{P \in \mathrm{Spec}(A) | a \in P\}$ and $D(a) = \{P \in \mathrm{Spec}(A) | a \notin P\}$.

Proposition 3.12. *Let A be a $\vee$-semihoop and $a, b \in A$. The following statements hold*:
(1) *$D(a) = \mathrm{Spec}(A)$ if and only if $\langle a\rangle = A$*;
(2) *$D(a) = \varnothing$ if and only if $a = 1$*;
(3) *$D(a) = D(b)$ if and only if $\langle a\rangle = \langle b\rangle$*;
(4) if *$a \leq b$, then $D(b) \subseteq D(a)$*;
(5) $D(a) \cap D(b) = D(a \vee b)$;
(6) $D(a) \cup D(b) = D(a \wedge b) = D(a \odot b)$.

Proof. By Proposition 3.11 (4), (5), and (8), we know (1), (2), and (3) hold obviously.

(4) Let $a \leq b$. Suppose that $P \in D(b)$, then $b \notin P$. If $P \notin D(a)$, then $a \in P$ and from $a \leq b$, we have $b \in P$, which is a contradiction.

(5) For any prime filter P of A, we have $a \vee b \notin P$ iff $a \notin P$ and $b \notin P$. Thus, $P \in D(a \vee b)$ iff $a \vee b \notin P$ iff $a \notin P$ and $b \notin P$ iff $P \in D(a)$ and $P \in D(b)$ iff $P \in D(a) \cap D(b)$.

(6) Since $a \odot b \leq a \wedge b \leq a, b$, by (4), it follows that $D(a \odot b) \supseteq D(a \wedge b) \supseteq D(a) \cup D(b)$. Conversely, suppose that $P \in D(a \odot b)$, that is, $a \odot b \notin P$. Since P is a filter, we can obtain $a \notin P$ or $b \notin P$. Thus, $D(a) \cup D(b) \supseteq D(a \odot b)$. □

Proposition 3.13. *Let A be a $\vee$-semihoop. Then family $\{D(a)\}_{a\in A}$ is a basis for the topology of* $\mathrm{Spec}(A)$.

Proof. Let $X \subseteq A$. Then $D(X) = D(\cup_{a\in X}\{a\}) = \cup_{a\in X} D(a)$ by Proposition 3.11(6). Hence, any open subset of Spec(A) is the union of subsets from the family $\{D(a)|a \in A\}$. □

The sets $\{D(a)|a \in A\}$ will be called basic open sets of Spec(A).

Proposition 3.14. *Let A be a $\vee$-semihoop. Then for any $a \in A$, $D(a)$ is compact in* Spec(A).

Proof. Obviously, we can obtain any cover of $D(a)$ with basic open sets containing a finite cover of $D(a)$. Suppose that $D(a) = \cup_{i\in I} D(a_i)$, by Proposition 3.11 (6), we have $D(a) = \cup_{i\in I} D(a_i) = D(\cup_{i\in I}\{a_i\})$. According to Proposition 3.11 (8), then $\langle a\rangle = \langle \cup_{i\in I}\{a_i\}\rangle$, so $a \in \langle \cup_{i\in I}\{a_i\}\rangle$. By the definition of generated filter, there are $n \geq 1$ and $i_1, \ldots, i_n \in I$ such that $a_{i_1} \odot \cdots \odot a_{i_n} \leq a$. We should prove that $D(a) = D(a_{i_1} \odot \cdots \odot a_{i_n})$. Applying Proposition 3.12 (4) and (6), we have $D(a) \subseteq D(a_{i_1} \odot \cdots \odot a_{i_n}) = D(a_{i_1}) \cup \cdots \cup D(a_{i_n})$. The other inclusion is clear, since $D(a_{i_1}) \cup \cdots \cup D(a_{i_n}) \subseteq \cup_{i\in I} D(a_i) = D(a)$. □

Proposition 3.15. Spec(A) *is a T_0 topological space.*

Proof. It remains to prove that for any different prime filters $P \neq Q \in$ Spec(A), there exists an open set U such that $Q \in U$, $P \notin U$, or $Q \notin U$, $P \in U$. Since $P \neq Q$, we obtain $P \nsubseteq Q$ or $Q \nsubseteq P$. There is no loss of generality in assuming $P \nsubseteq Q$, then there exists $a \in A$ such that $a \in P$ and $a \notin Q$. Take $U = D(a)$. Then $Q \in U$ and $P \notin U$. □

In the sequel, let A be a $\vee$-semihoop. By Proposition 3.3, we obtain Max(A) $\subseteq$ Spec(A). Now we consider a new topology space induced by Zariski topology on Max(A), which is called the maximal spectrum of A.

For any $X \subseteq A$ and $a \in A$, let us define: $E_{\text{Max}}(X) = E(X) \cap \text{Max}(A) = \{M \in \text{Max}(A)|X \subseteq M\}$, $D_{\text{Max}}(X) = D(X) \cap \text{Max}(A) = \{M \in \text{Max}(A)|X \nsubseteq M\}$, and $E_{\text{Max}}(a) = E(a) \cap \text{Max}(A) = \{M \in \text{Max}(A)|a \in M\}$, $D_{\text{Max}}(a) = D(a) \cap \text{Max}(A) = \{M \in \text{Max}(A)|a \notin M\}$.

It follows that the family $\{E_{\text{Max}}(X)\}_{X\subseteq A}$ is the family of closed sets of the maximal spectrum, the family $\{D_{\text{Max}}(X)\}_{X\subseteq A}$ is the family of open sets of the maximal spectrum, and the family $\{D_{\text{Max}}(a)\}_{a\in A}$ is a base for topology of Max(A).

Proposition 3.16. *Let A be a $\vee$-semihoop, $X, Y \subseteq A$, $\{X_i\}_{i\in I}$ be a family of subsets of A and $a, b \in A$. Then*:

(1) *if $X \subseteq Y \subseteq A$, then $D_{\text{Max}}(X) \subseteq D_{\text{Max}}(Y) \subseteq \text{Max}(A)$*;
(2) $D_{\text{Max}}(0) = \text{Max}(A)$, $D_{\text{Max}}(\varnothing) = D_{\text{Max}}(1) = \varnothing$;
(3) $D_{\text{Max}}(X) = \text{Max}(A)$ *if and only if* $\langle X\rangle = A$;
(4) $D_{\text{Max}}(\cup_{i\in I} X_i) = \cup_{i\in I} D_{\text{Max}}(X_i)$;
(5) $D_{\text{Max}}(X) = D_{\text{Max}}(\langle X\rangle)$;
(6) $D_{\text{Max}}(X) \cup D_{\text{Max}}(Y) = D_{\text{Max}}(\langle X\rangle \cup \langle Y\rangle)$;
(7) $D_{\text{Max}}(a) = \text{Max}(A)$ *if and only if* $\langle a\rangle = A$;
(8) if $a \leq b$, *then* $D_{\text{Max}}(b) \subseteq D_{\text{Max}}(a)$;
(9) $D_{\text{Max}}(a) \cap D_{\text{Max}}(b) = D_{\text{Max}}(a \vee b)$;
(10) $D_{\text{Max}}(a) \cup D_{\text{Max}}(b) = D_{\text{Max}}(a \wedge b) = D_{\text{Max}}(a \odot b)$.

Proof. We only prove (3), and other statements are immediate consequences of Proposition 3.11.

(3) Let $D_{\text{Max}}(X) = \text{Max}(A)$. If $\langle X\rangle \neq A$, then $\langle X\rangle$ is a proper filter of A, by Lemma 3.7, there exists a maximal filter M of A such that $\langle X\rangle \subseteq M$. Thus, $M \notin D_{\text{Max}}(X)$. This is a contradiction. Conversely, let $\langle X\rangle = A$. By Proposition 3.11 (5), we have $D(X) = \text{Spec}(A)$. Therefore, $D_{\text{Max}}(X) = D(X) \cap \text{Max}(A) = \text{Max}(A)$. □

Proposition 3.17. *If A is a $\vee$-semihoop with negation, then* Max(A) *is a compact space.*

Proof. Since there exists $\{a_i\}_{i\in I} \subseteq A$ such that $\text{Max}(A) = \cup_{i\in I} D_{\text{Max}}(a_i)$, then by Proposition 3.16 (4), we have $\text{Max}(A) = D_{\text{Max}}(\cup_{i\in I}\{a_i\})$. And applying Proposition 3.16 (3), we obtain $A = \langle \cup_{i\in I}\{a_i\}\rangle$ with negation, so

$0 \in \cup_{i \in I}\{a_i\}$. It follows that there are $n \geq 1$ and $i_1, \ldots, i_n \in I$ such that $a_{i_1} \odot \cdots \odot a_{i_n} = 0$. According to Proposition 3.16 (2), we obtain $\mathrm{Max}(A) = D_{\mathrm{Max}}(0) = D_{\mathrm{Max}}(a_{i_1} \odot \cdots \odot a_{i_n}) = \cup_{j=1}^{n} D_{\mathrm{Max}}(a_{i_j})$. Therefore, $\mathrm{Max}(A)$ is compact. □

Proposition 3.18. *If A is a prelinear $\vee$-semihoop, then* $\mathrm{Max}(A)$ *is a Hausdorff space.*

Proof. Suppose that M, N are different maximal filters of A. Since $M \neq N$, then there exist $x \in M \backslash N$ and $y \in N \backslash M$. Take $a = x \to y$, $b = y \to x$. If $a \in M$, then $x \odot (x \to y) = x \odot a \in M$, so we have $y \in M$, which is a contradiction. Thus, $a \notin M$ and $b \notin N$, that is, $M \in D_{\mathrm{Max}}(a)$ and $N \in D_{\mathrm{Max}}(b)$. Since A is prelinear, then $D_{\mathrm{Max}}(a) \cap D_{\mathrm{Max}}(b) = D_{\mathrm{Max}}(a \vee b) = D_{\mathrm{Max}}(1) = \varnothing$. Therefore, $\mathrm{Max}(A)$ is a Hausdorff space. □

4 Reticulation of semihoops

In this section, we give a definition of the reticulation of semihoops.

Lemma 4.1. *The map $f : X \to Y$ is surjective if and only if $f^{-1} : \mathcal{P}(Y) \to \mathcal{P}(X)$ is injective.*

Definition 4.2. Let A be a bounded semihoop, L be a bounded distributive lattice, and $\lambda : A \to L$ be a map. A reticulation of A is a pair (L, λ), which satisfies the following conditions:
(R1) λ is isotone and surjective;
(R2) $\lambda(a \odot b) = \lambda(a) \wedge \lambda(b)$, for any $a, b \in A$;
(R3) $\lambda(a \oplus b) = \lambda(a) \vee \lambda(b)$, for any $a, b \in A$;
(R4) $\lambda(0) = 0$, $\lambda(1) = 1$;
(R5) $\lambda(a) \leq \lambda(b)$ if and only if $\exists\, n \in \mathbb{N}^+$ such that $a^n \leq b$.

Example 4.3. Let $A = \{0, a, b, 1\}$ be a chain. We define $\odot$, $\to$, and $\oplus$ on A as follows:

$\odot$	0	a	b	1
0	0	0	0	0
a	0	a	a	a
b	0	a	b	b
1	0	a	b	1

$\to$	0	a	b	1
0	1	1	1	1
a	b	1	1	1
b	a	a	1	1
1	0	a	b	1.

Then $(A, \odot, \to, \wedge, 0, 1)$ is a bounded semihoop with (DNP), where $x \wedge y = \min\{x, y\}$ for all $x, y \in A$. Moreover, we define $L = \{0, 1\}$ is a bounded distributive lattice and the map $\lambda : A \to L$ as follows:

$$\lambda(x) = \begin{cases} 0 & x = 0 \\ 1 & x \neq 0. \end{cases}$$

One can easily check that (L, λ) is a reticulation of bounded semihoop A with DNP.

Example 4.4. Let $A = \{0, a, b, c, d, 1\}$ with $0 \leq b, d \leq a \leq 1$, $0 \leq d \leq c \leq 1$, where a and c are incomparable, b and d are incomparable. Define operations $\odot$ and $\to$ on A as follows:

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	b	b	d	0	a
b	0	b	b	0	0	b
c	0	d	0	c	d	c
d	0	0	0	d	0	d
1	0	a	b	c	d	1

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	a	c	c	1
b	c	1	1	c	c	1
c	b	a	b	1	a	1
d	a	1	a	1	1	1
1	0	a	b	c	d	1.

Define $x \wedge y = x \odot (x \to y)$, for any $x, y \in A$. We can see $(A, \odot, \to, \wedge, 0, 1)$ is a bounded semihoop with DNP. Moreover, we define $L = \{0, 1\}$ is a bounded distributive lattice and the map $\lambda : A \to L$ as follows:

$$\lambda(x) = \begin{cases} 0 & x = 0, c, d \\ 1 & x = a, b, 1. \end{cases}$$

Then we can easily see that (L, λ) is a reticulation of bounded semihoop A with DNP.

Second, we study a reticulation of semihoop A without DNP.

Example 4.5. Let $A = \{0, a, b, c, d, 1\}$. Define operations $\odot$ and $\to$ on A as follows:

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	c	c	d	a
b	0	c	b	c	d	b
c	0	c	c	c	d	c
d	0	d	d	d	0	d
1	0	a	b	c	d	1

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	0	1	b	b	d	1
b	0	a	1	a	d	1
c	0	1	1	1	d	1
d	d	1	1	1	1	1
1	0	a	b	c	d	1.

If $x \wedge y = x \odot (x \to y)$, then A with these operations is a bounded semihoop without (DNP). Moreover, we define $L = \{0, 1\}$ is a bounded distributive lattice and the map $\lambda : A \to L$ as follows:

$$\lambda(x) = \begin{cases} 0 & x = 0, d \\ 1 & \text{others}. \end{cases}$$

Then we can easily check that (L, λ) is a reticulation of bounded semihoop A without (DNP).

Example 4.6. (1) Let Godel t-norm:$A = [0, 1]$, then we define: $x \odot y = \min\{x, y\}$,

$$x \to y = \begin{cases} 1 & x \leq y \\ y & \text{others} \end{cases}$$

and

$$x \oplus y = \begin{cases} y & x = 0 \\ 1 & x \neq 1 \end{cases}$$

for any $x, y \in A$. Then $(A, \odot, \to, \wedge, \oplus, 0, 1)$ is a bound semihoop, where $x \wedge y = \min\{x, y\}$. Moreover, we define $L = \{0, 1\}$ is a bounded distributive lattice and the map $\lambda_1 : A \to L$ as follows:

$$\lambda_1(x) = \begin{cases} 0 & x = 0 \\ 1 & x \neq 0. \end{cases}$$

Then we can easily see that (L, λ_1) is a reticulation of bounded semihoop A.

(2) Let us define product t-norm: $A = [0, 1]$, then we define: $x \odot y = xy$,

$$x \to y = \begin{cases} 1 & x \leq y \\ y/x & \text{others} \end{cases}$$

and

$$x \oplus y = \begin{cases} y & x = 0 \\ 1 & x \neq 1 \end{cases}$$

for any $x, y \in A$. Then $(A, \odot, \to, \wedge, \oplus, 0, 1)$ is a bound semihoop, where $x \wedge y = \min\{x, y\}$. Moreover, we define $L = \{0, 1\}$ is a bounded distributive lattice and the map $\lambda_2 : A \to L$ as follows:

$$\lambda_2(x) = \begin{cases} 0 & x = 0 \\ 1 & x \neq 0. \end{cases}$$

Therefore, we can easily see that (L, λ_2) is a reticulation of bounded semihoop A.

Remark 4.7. Through Example 4.6 (1) and (2), we can know different semihoops may have the same reticulation.

Proposition 4.8. *If $\lambda : A \to L$ satisfies conditions $(R1)$, $(R2)$, and $(R3)$, then*
(1) *$\lambda(a \wedge b) = \lambda(a) \wedge \lambda(b)$ for any $a, b \in A$;*
(2) *if $x \vee y$ exists, then $\lambda(a \vee b) = \lambda(a) \vee \lambda(b)$ for any $a, b \in A$;*
(3) *$\lambda(a^n) = \lambda(a)$ for any $a \in A$, $n \in \mathbb{N}^+$.*

Proof.
(1) $\forall a, b \in A$, $a \wedge b \leq a, b$. Since λ is isotone, then $\lambda(a \wedge b) \leq \lambda(a), \lambda(b)$, that is, $\lambda(a \wedge b) \leq \lambda(a) \wedge \lambda(b)$. And $a \odot b \leq a, b$, it follows that $a \odot b \leq a \wedge b \Rightarrow \lambda(a \odot b) \leq \lambda(a \wedge b)$. Applying condition $(R2)$, then we obtain $\lambda(a) \wedge \lambda(b) \leq \lambda(a \wedge b)$.
(2) Suppose that $x \vee y$ exists, since $a, b \leq a \vee b$ and λ is isotone. Then $\lambda(a), \lambda(b) \leq \lambda(a \vee b)$, that is, $\lambda(a) \vee \lambda(b) \leq \lambda(a \vee b)$. And $a, b \leq a \oplus b$, we obtain $a \vee b \leq a \oplus b \Rightarrow \lambda(a \vee b) \leq \lambda(a \oplus b)$. Applying condition $(R3)$, we obtain $\lambda(a \vee b) \leq \lambda(a) \vee \lambda(b)$.
(3) Suppose that $a \in A$, $n \in \mathbb{N}^+$, by (1) and condition $(R2)$, we have $\lambda(a^n) = \lambda(a \odot a \cdots \odot a) = \lambda(a \wedge a \cdots \wedge a) = \lambda(a)$. □

Proposition 4.9. *If F is a filter of L, then $\lambda^{-1}(F)$ is a filter of A.*

Proof. Since F is a filter of L, then $1 \in F \neq \emptyset$, we have $\lambda^{-1}(F) \neq \emptyset$. Let $a, b \in \lambda^{-1}(F)$, then $\lambda(a), \lambda(b) \in F \Rightarrow \lambda(a) \wedge \lambda(b) \in F$. By condition $(R2)$, we obtain $\lambda(a \odot b) \in F$, then $a \odot b \in \lambda^{-1}(F)$. Suppose that $a, b \in A$, $a \in \lambda^{-1}(F)$, and $a \leq b$. Since λ is isotone, then $\lambda(a) \leq \lambda(b)$. By $\lambda(a) \in F$ and F is a filter of L, we have $\lambda(b) \in F \Rightarrow b \in \lambda^{-1}(F)$. Therefore, $\lambda^{-1}(F)$ is a filter of A. □

Proposition 4.10. *If A is a bounded $\vee$-semihoop and P is a prime filter of L, then $\lambda^{-1}(P)$ is a prime filter of A.*

Proof. By Proposition 4.9, we obtain $\lambda^{-1}(P)$ is a filter of A. Since $P \neq L$ and the fact that λ^{-1} is injective, we have $\lambda^{-1}(P) \neq A$. Suppose that $a, b \in A$ such that $a \oplus b \in \lambda^{-1}(P)$. Applying Proposition 4.8(2), we obtain $\lambda(a) \vee \lambda(b) = \lambda(a \oplus b) \in P$. Since P is prime, it follows that $\lambda(a) \in P$ or $\lambda(b) \in P$, so $a \in \lambda^{-1}(P)$ or $b \in \lambda^{-1}(P)$. Therefore, $\lambda^{-1}(P)$ is a prime filter of A. □

Lemma 4.11. *Let F be a filter of A and $a, b \in A$ such that $\lambda(a) = \lambda(b)$, then $a \in F$ if and only if $b \in F$.*

Proof. Let $a, b \in A$ such that $\lambda(a) = \lambda(b)$. Suppose that $a \in F$, by condition $(R5)$, since $\lambda(a) \leq \lambda(b)$, there exists $n \in \mathbb{N}^+$ such that $a^n \leq b$. And $a \in F$, so $a^n \in F$, that is, $b \in F$. Using $\lambda(b) \leq \lambda(a)$, one obtains the proof of the converse implication. □

Lemma 4.12. *For any $a, b \in A$, $\lambda(a) = \lambda(b)$ if and only if $\langle a \rangle = \langle b \rangle$.*

Proof. Let $\forall a, b \in A$ such that $\lambda(a) = \lambda(b)$. Since $\lambda(a) \leq \lambda(b)$, by condition $(R5)$, there exists $n \in \mathbb{N}^+$ such that $a^n \leq b$, so $\langle b \rangle \subseteq \langle a \rangle$. Similarly, since $\lambda(b) \leq \lambda(a)$, we obtain $\langle a \rangle \subseteq \langle b \rangle$. Thus, $\langle a \rangle = \langle b \rangle$. Conversely, let $\langle a \rangle = \langle b \rangle$, then $b \in \langle a \rangle$, by the definition of generated filter, $a^n \leq b$, we obtain $\lambda(a^n) \leq \lambda(b)$, that is, $\lambda(a) \leq \lambda(b)$. Similarly, by $a \in \langle b \rangle$, we obtain $\lambda(b) \leq \lambda(a)$. Therefore, $\lambda(a) = \lambda(b)$. □

Lemma 4.13. *Let F be a filter of A and $a \in A$, then $\lambda(a) \in \lambda(F)$ if and only if $a \in F$.*

Proof. Suppose that F is a filter of A and $a \in A$. Let $\lambda(a) \in \lambda(F)$, then there exists $b \in F$ such that $\lambda(a) = \lambda(b)$. Applying Lemma 4.11, we obtain $a \in F$. The converse implication is obvious. □

Lemma 4.14. *Let F be a filter of A, then $\lambda^{-1}\lambda(F) = F$.*

Proof. By Lemma 4.13 and condition ($R1$), it follows that $a \in F \Leftrightarrow \lambda(a) \in \lambda(F) \Leftrightarrow a \in \lambda^{-1}\lambda(F) = F$. □

Proposition 4.15. *If F is a filter of A, then $\lambda(F)$ is a filter of L.*

Proof. Since F is a filter of A, then $1 \in F$. Applying condition ($R1$), it follows that $1 = \lambda(1) \in \lambda(F) \neq \varnothing$. Let $x, y \in \lambda(F)$, then there exist $a, b \in F$ such that $x = \lambda(a)$, $y = \lambda(b)$. Since F is a filter of A, we have $a \odot b \in F$. By condition ($R2$), we obtain $x \wedge y = \lambda(a) \wedge \lambda(b) = \lambda(a \odot b) \in \lambda(F)$. Let $z \in L$ such that $x \leq z$. Since λ is surjective, there exists $c \in F$ such that $\lambda(c) = z$. Thus, $\lambda(a) \leq \lambda(c)$, by condition ($R5$), there exists $n \in N^+$ such that $a^n \leq c$. By $a \in F$, we have $a^n \in F$, then $c \in F$. Therefore, $z = \lambda(c) \in \lambda(F)$. □

Proposition 4.16. *If A is a bounded $\vee$-semihoop and Q is a prime filter of A, then $\lambda(Q)$ is a prime filter of L.*

Proof. According to Proposition 4.15, we obtain $\lambda(Q)$ is a filter of L. Since $Q \neq A$, there exists $a \in A$ such that $a \in Q$. By Lemma 4.13, we obtain $\lambda(a) \notin \lambda(Q)$, so $\lambda(Q)$ is a proper filter of L. Suppose that $u, v \in L$ such that $u \vee v \in \lambda(Q)$. Using condition ($R3$), there exist $a, b \in A$ such that $\lambda(a) = u$, $\lambda(b) = v$. Thus, $\lambda(a) \vee \lambda(b) \in \lambda(Q)$, by condition ($R3$), it follows that $\lambda(a \oplus b) = \lambda(a) \vee \lambda(b) \in \lambda(Q)$. Since λ is surjective, there exists $c \in Q$ such that $\lambda(a \oplus b) = \lambda(c)$. By Lemma 4.11, we have $a \vee b \in Q$. Since Q is prime, then $a \in Q$ or $b \in Q$. Thus, $\lambda(a) \in \lambda(Q)$ or $\lambda(b) \in \lambda(Q)$, that is, $u \in \lambda(Q)$ or $v \in \lambda(Q)$. Therefore, $\lambda(Q)$ is a prime filter of L. □

Proposition 4.17. *The map $\lambda^{-1} : \mathrm{Spec}(L) \to \mathrm{Spec}(A)$ is bijective.*

Proof. Let $Q \in \mathrm{Spec}(A)$. By Proposition 4.9, we have $\lambda(Q) \in \mathrm{Spec}(L)$ and according to Lemma 4.14, we obtain $\lambda^{-1}\lambda(Q) = Q$. Therefore, λ^{-1} is surjective. By Lemma 4.1, λ^{-1} is injective, so λ^{-1} is a bijective map. □

Proposition 4.18. *The map $\lambda^{-1} : \mathrm{Spec}(L) \to \mathrm{Spec}(A)$ is continuous and open.*

Proof. Let $a \in A$, we obtain $(\lambda^{-1})^{-1}(D(a)) = \{P \in \mathrm{Spec}(L) | \lambda^{-1}(P) \in D(a)\} = \{P \in \mathrm{Spec}(L) | a \notin \lambda^{-1}(P)\} = \{P \in \mathrm{Spec}(L) | \lambda(a) \notin P\} = D(\lambda(a))$. Thus, λ^{-1} is continuous. Let $u \in L$. Since λ is surjective, there exists $a \in A$ such that $\lambda(a) = u$. So $\lambda^{-1}(D(u)) = \lambda^{-1}(D(\lambda(a))) = \{\lambda^{-1} | P \in D(\lambda(a))\} = \{\lambda^{-1} | P \in \mathrm{Spec}(L), \lambda(a) \notin P\} = \{\lambda^{-1} | P \in \mathrm{Spec}(L), a \notin \lambda^{-1}(P)\} = \{Q \in \mathrm{Spec}(A) | a \notin Q\} = D(a)$. Therefore, λ^{-1} is open. □

Theorem 4.19. *If $\lambda : A \to L$ satisfying conditions, then the map $\lambda^{-1} : \mathrm{Spec}(L) \to \mathrm{Spec}(A)$ is a homeomorphism of topological spaces.*

Proof. It directly follows from Propositions 4.17 and 4.18. □

5 Conclusion

In the first part of this article, we first presented the proof process that maximal filters in $\vee$-semihoops are prime filters, and got some conclusions about prime filters and maximum filters on semihoops. Second, we constructed the prime spectra (maximal spectra) on semihoops by defining the open sets by prime filters (maximal filters). Finally, it is proved that prime spectra is a compact T_0 topological space and maximal spectra is a compact T_2 topological space. In the second part, we first defined the concept of bounded semihoops and give some examples. The relationship between the filters of bounded semihoops and bounded distributive lattices is presented. It turns out that the map λ^{-1} : Spec(L) → Spec(A) is a homeomorphism of topological spaces.

As far as we know, L-algebras is a kind of logical algebra with quantum properties. Therefore, the study of L-algebras can better show the relationship between topology and quantum algebras and we also can enrich the research of topological structure in different logic algebras. Hence, we will devote ourselves to the study of the spectra and reticulation on L-algebras in future work.

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