Research Article

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Multiplicity solutions for a class of p-Laplacian fractional differential equations via variational methods

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Abstract: While it is known that one can consider the existence of solutions to boundary-value problems for fractional differential equations with derivative terms, the situations for the multiplicity of weak solutions for the p-Laplacian fractional differential equations with derivative terms are less considered. In this article, we propose a new class of p-Laplacian fractional differential equations with the Caputo derivatives. The multiplicity of weak solutions is proved by the variational method and critical point theorem. At the end of the article, two examples are given to illustrate the validity and practicality of our main results.

Keywords: fractional differential equation, boundary-value problem, p-Laplacian operator, variational method

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1 Introduction

Fractional impulsive differential equations have always been a hot research topic in recent years, because the impulse effect is widely present in many real systems, such as signal processing systems, automatic control systems, telecommunications, and multi-agent systems. In other words, it is quite practical to study fractional impulse differential equations (see [1–4] and references therein).

Tian and Zhang [5] used the variational method to study the existence of solutions of differential equations in the case of second-order nonlinearities with instantaneous and non-instantaneous impulses. Zhou et al. [6] not only expanded the linear term into a nonlinear term, but also discussed the integer-order problem into the fractional order problem and given the equivalent criterion of the weak solution and classical solution, and finally gave the existence of solutions for fractional differential equations of p-Laplacian with instantaneous and non-instantaneous impulses. In particular, Ricceri proposed and proved three critical point theorems [7]. Li et al. [8], by using the method of variation and Ricceri's critical point theorem, addressed the existence of at least three solutions for a class of p-Laplacian-type fractional Dirichlet's boundary-value problem (BVP) involving instantaneous and non-instantaneous impulse impacts. It can be seen that the critical point theorem plays a vital role in solving the BVP of fractional differential equations with variational structure.

Since the mountain pass theorem was proposed by Ambrosetti and Rabinowitz in 1973 [9], critical point theory has become one of the most effective and practical tools to tackle the existence of solutions for fractional equations [8,10–13]. In addition, using the variational method under Ambrosetti-Rabinowitz (A-R)

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conditions to study the existence of BVP solutions of p-Laplacian fractional differential equations has become a generally applicable method in recent years. Because when using the variational method, the nonlinear term is usually required to satisfy the A-R condition. And about nonlinear related content, readers can refer to [14].

Under the A-R condition, Chen and Liu [15] obtained the following result that there is at least one weak solution to the boundary-value problem of the p-Laplacian fractional differential equation by using the variational method:

$$\begin{cases} {}_tD_T^\alpha(\Phi_p({}_0D_t^\alpha u(t))) = f(t, u(t)), \quad t \in [0, T], \\ u(0) = u(T) = 0, \end{cases} \tag{1.1}$$

where $0 < \alpha \le 1$, ${}_0D_t^\alpha$ and ${}_tD_T^\alpha$ are the left and right Riemann-Liouville derivatives, respectively. $\varphi_p(s) = s|s|^{p-2}$, $p > 1$, $f : [0, T] \times \mathbb{R} \to \mathbb{R}$.

Zhao et al. [16] investigated the existence of at least one solution to the following problems:

$$\begin{cases} {}_tD_T^\alpha(a(t){}_0^cD_t^\alpha u(t)) + b(t)u(t) = f(t, u(t), {}_0^cD_t^\alpha u(t)), \quad t \ne t_j, \quad a.e.\ t \in [0, T], \\ \Delta(a(t){}_tD_T^{\alpha-1}({}_0^cD_t^\alpha u)(t_j)) = I_j(u(t_j)), \quad j = 1, 2, \dots, l, \\ u(0) = u(T) = 0, \end{cases} \tag{1.2}$$

where $\alpha \in \left(\frac{1}{2}, 1\right]$, $a \in C^1([0, T], \mathbb{R})$, $a_0 = \operatorname{essinf}_{[0,T]}a(t) > 0$, ${}_tD_T^\alpha$ and ${}_0^cD_t^\alpha$ are the right Riemann-Liouville derivatives and left Caputo derivatives of order α, respectively. The researcher obtained the main result to the above problem (1.2) through the variational method and iterative method.

Qiao et al. [17] also used the variational method and the critical point theorem to explore the existence of at least three solutions for a class of p-Laplacian fractional differential equations:

$$\begin{cases} {}_tD_T^\alpha\phi_p({}_0D_t^\alpha x(t)) = \lambda f(t, x(t)), \quad \text{a.e. } t \in [0, T], \\ x(0) = x(T) = 0. \end{cases} \tag{1.3}$$

The difference from previous scholars' research methods is that the A-R condition is not used when using the mountain pass lemma.

Motivated by the above-mentioned work, this article focuses on a class of p-Laplacian-type fractional differential equations with derivative term:

$$\begin{cases} {}_tD_T^\alpha\Phi_p(a(t){}_0^cD_t^\alpha u(t)) + \gamma u(t) = \lambda f(t, u(t), {}_0^cD_t^\alpha u(t)) + h(u(t)), \quad t \ne t_j, t \in [0, T], \\ \Delta({}_tD_T^{\alpha-1}\Phi_p(a(t_j){}_0^cD_t^\alpha u))(t_j) = \mu I_j(u(t_j)), \quad j = 1, 2, \dots, m, \\ u(0) = u(T) = 0, \end{cases} \tag{1.4}$$

where $\alpha \in (0, 1]$, $1 < p < \infty$, λ is a positive constant, $a(t)$ belongs to $L^\infty([0, T], \mathbb{R}^+)$, $\bar{a} = \operatorname{ess\,sup}_{[0,T]}a(t)$, $\tilde{a} = \operatorname{ess\,inf}_{[0,T]}a(t) > 0$, λ and μ are two non-negative real parameters, ${}_tD_T^\alpha$ and ${}_0^cD_t^\alpha$ denote the right Riemann-Liouville fractional derivatives and left Caputo fractional derivatives of order α. The functions $\Phi_p(s) = |s|^{p-2}s$, $(s \ne 0)$, $f : [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous, $h : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function with Lipschitz constant $L > 0$, that is $|h(x) - h(y)| \le L|x - y|$ for every $x, y \in \mathbb{R}$, satisfying $h(0) = 0$, $I_j \in C(\mathbb{R}, \mathbb{R})$, for $j = 0, 1, \dots, m$, $0 = t_1 < t_2 < \dots < t_m < t_{m+1} = T$, the operator Δ is defined as

$$\begin{aligned} \Delta({}_tD_T^{\alpha-1}\Phi_p(a(t){}_0^cD_t^\alpha u))(t_j) &= {}_tD_T^{\alpha-1}\Phi_p(a(t){}_0^cD_t^\alpha u)(t_j^+) - {}_tD_T^{\alpha-1}\Phi_p(a(t){}_0^cD_t^\alpha u)(t_i^-), \\ {}_tD_T^{\alpha-1}\Phi_p(a(t){}_0^cD_t^\alpha u)(t_j^+) &= \lim_{t\to t_j^+} {}_tD_T^{\alpha-1}\Phi_p(a(t){}_0^cD_t^\alpha u(t)), \\ {}_tD_T^{\alpha-1}\Phi_p(a(t){}_0^cD_t^\alpha u)(t_j^-) &= \lim_{t\to t_j^-} {}_tD_T^{\alpha-1}\Phi_p(a(t){}_0^cD_t^\alpha u(t)). \end{aligned}$$

The existence and multiplicity of the solution of BVP (1.4) will be obtained by the critical point theorem.

The rest of the article is organized as follows. In Section 2, some basic knowledge and lemmas used in the latter are presented. In Section 3, on the one hand, we give the variational structure of BVP (1.4) by using the method of variation and the critical point theorem to prove that it satisfies the mountain pass theorem, and obtain the result of the existence of the BVP (1.4) solution. On the other hand, under the guarantee of Theorem 2.12, we obtain that there are at least three solutions to BVP (1.4). In Section 4, we give two examples to describe the validity of the main results. At last, a conclusion is presented in Section 5.

2 Preliminaries

In this section, we review some important properties and lemmas, including the appropriate space that we need to establish using the variational method to obtain the existence of the solution, and the relationship between different norms, and so on. These are all needed in the proofs below, so that we can obtain the existence result of the solution of BVP (1.4).

Definition 2.1. [8,12] Let $n-1<\alpha\le n$, $n\in\mathbb{N}$, $t\in[0,T]$, the left and right Riemann-Liouville fractional derivatives of $u(t)$ denoted by ${}_0D_t^{\alpha}u(t)$ and ${}_tD_T^{\alpha}u(t)$ with definitions can be written as follows:

$$ {}_0D_t^{\alpha}u(t)=\frac{\mathrm{d}^n}{\mathrm{d}t^n}{}_0D_t^{\alpha-n}u(t)=\frac{1}{\Gamma(n-\alpha)}\frac{\mathrm{d}^n}{\mathrm{d}t^n}\int\limits_0^t(t-s)^{n-\alpha-1}u(s)\mathrm{d}s, $$

$$ {}_0D_t^{\alpha}u(t)=(-1)^n\frac{\mathrm{d}^n}{\mathrm{d}t^n}{}_tD_T^{\alpha-n}u(t)=\frac{(-1)^n}{\Gamma(n-\alpha)}\frac{\mathrm{d}^n}{\mathrm{d}t^n}\int\limits_t^T(s-t)^{n-\alpha-1}u(s)\mathrm{d}s. $$

Definition 2.2. [8,12] Let $n-1<\alpha\le n$, $n\in\mathbb{N}$, $t\in[0,T]$, the left and right Caputo fractional derivatives denoted by ${}_0^cD_t^{\alpha}$ and ${}_t^cD_T^{\alpha}$ are given by

$$ {}_0^cD_t^{\alpha}u(t)=\frac{1}{\Gamma(n-\alpha)}\int\limits_0^t(t-s)^{n-\alpha-1}u^n(s)\mathrm{d}s $$

and

$$ {}_t^cD_T^{\alpha}u(t)=\frac{(-1)^n}{\Gamma(n-\alpha)}\int\limits_t^T(s-t)^{n-\alpha-1}u^n(s)\mathrm{d}s, $$

and the following relationships hold:

$$ {}_0^cD_t^{\alpha}u(t)={}_0D_t^{\alpha}u(t)-\sum_{i=0}^{n-1}\frac{u^i(0)}{\Gamma(i-\alpha+1)}(t-0)^{i-\alpha},\quad {}_t^cD_T^{\alpha}u(t)={}_tD_T^{\alpha}u(t)-\sum_{i=0}^{n-1}\frac{u(T)}{\Gamma(i-\alpha+1)}(T-t)^{-\alpha}. \tag{2.1} $$

Proposition 2.3. [6] *Let $\alpha\in(0,1]$, and $u,v\in L^p(a,b)$*

$$ \int\limits_a^b({}_tD_b^{\alpha}u(t))v(t)\mathrm{d}t=\int\limits_a^b({}_aD_t^{\alpha}v(t))u(t)\mathrm{d}t. $$

In order to establish a suitable function space and use the critical point theory to study the existence of BVP (1.4) solutions, we need the following basic annotations.

Proposition 2.4. [6,8,12] *Let $1<p<\infty$, $0<\alpha\le1$, the fractional space*

$$ E_0^{\alpha,p}=\{u(t)\in L^p([0,T],\mathbb{R})|{}_0^cD_t^{\alpha}u\in L^p([0,T],\mathbb{R}),u(0)=u(T)\} \tag{2.2} $$

is defined by the closure of $C_0^{\infty}([0,T],\mathbb{R})$ with the norm

$$\|u\|_{\alpha,p}=\left(\int_0^T|u(t)|^p\mathrm{d}t+\int_0^T|a(t)_0^cD_t^\alpha u(t)|^p\mathrm{d}t\right)^{\frac{1}{p}},\tag{2.3}$$

for any $u(t)\in E_0^{\alpha,p}$.

Lemma 2.5. [12] *Let* $0<\alpha\le 1$, *the fractional derivative space* $E_0^{\alpha,p}$ *is a reflexive as well as separable Banach space.*

Remark 2.6. [8,12] *For any* $u(t)\in E_0^{\alpha,p}$ *with* $u(0)=u(T)=0$, *in view of* (2.1), *we have*

$$_0D_t^\alpha u(t)={}_0^cD_t^\alpha u(t),\quad {}_tD_T^\alpha u(t)={}_t^cD_T^\alpha u(t),\quad \forall t\in[0,T].$$

Lemma 2.7. [12] *Let* $\alpha\in(0,1]$ *and* $p\in(1,\infty)$, *for any* $u\in E_0^{\alpha,p}$, *we have*

$$\|u\|_{L^p}\le\Omega_*\left(\int_0^T a(t)|_0^cD_t^\alpha u(t)|^p\mathrm{d}t\right)^{\frac{1}{p}}.$$

Moreover, if $\alpha>\frac{1}{p}$ *and* $\frac{1}{p}+\frac{1}{q}=1$, *then*

$$\|u\|_\infty\le\Omega\left(\int_0^T a(t)|_0^cD_t^\alpha u(t)|^p\mathrm{d}t\right)^{\frac{1}{p}},\tag{2.4}$$

where $\|u\|_{L^p}=\left(\int_0^T|u(t)|^p\mathrm{d}t\right)^{\frac{1}{p}}$, $\|u\|_\infty=\max_{t\in[0,T]}|u(t)|$, $\Omega_*=\frac{T^\alpha}{\Gamma(\alpha+1)\tilde{a}^{\frac{1}{p}}}$ *and* $\Omega=\frac{T^{\alpha-\frac{1}{p}}}{\Gamma(\alpha)\tilde{a}^{\frac{1}{p}}[(\alpha-1)q+1]^{\frac{1}{q}}}$.

Lemma 2.8. [12] *Let* $0<\alpha\le 1$ *and* $1<p<\infty$. *Assume that the sequence* u_n *converges weakly to* u *in* $E_0^{\alpha,p}$, *i.e.,* $u_n\rightharpoonup u$ *as* $n\to\infty$. *Then,* $u_n\to u$ *in* $C([0,T],\mathbb{R})$, *that is,* $\|u_n-u\|_\infty\to 0$ *as* $n\to\infty$.

Definition 2.9. [16] Let E be a real Banach space and functional $\varphi\in C^1(E,\mathbb{R})$, if any sequence $\{u_k\}_{k\in N}\subset E$ for which $\{\varphi(u_k)\}$ is bounded and $\varphi'(u_k)\to 0$ as $k\to\infty$ possesses a convergent subsequence, then we say that φ satisfies the Palais-Smale (PS) condition.

Lemma 2.10. ([16] Mountain pass theorem) *Let* E *be a real Banach space and* $J:E\to\mathbb{R}$ *satisfy the PS condition. Suppose that*

(1) $J(0)=0$.
(2) *There exist* $\rho>0$ *and* $\sigma>0$ *such that* $J(z)\ge\sigma$ *for every* $z\in E$ *with* $\|z\|=\rho$.
(3) *There exists* $z_*\in E$ *with* $\|z_*\|\ge\rho$ *such that* $J(z_*)<\sigma$.

Then, J *possesses a critical value* $z_0\ge\sigma$. *Moreover,* z_0 *can be characterized as follows*:

$$z_0=\inf_{g\in\Lambda}\max_{z\in g([0,1])}J(z),$$

where $\Lambda=\{g\in C([0,1],E)|g(0)=0,g(1)=z_*\}$.

Lemma 2.11. [12] *For* $u\in E_0^{\alpha,p}$, *we define the norm*

$$\|u\|=\left(\int_0^T(a(t)|_0^cD_t^\alpha u(t)|)^p\mathrm{d}t\right)^{\frac{1}{p}}\tag{2.5}$$

is equivalent to the norm $\|u\|_{\alpha,p}$.

Theorem 2.12 proposed by Ricceri ([7], Theorem 2) is the main tool for us to obtain the main results as follows: If X is a real Banach space, denoted by Γ_X the class of all functionals $\varphi : X \to \mathbb{R}$ possessing the following property: If $\{u_n\}$ is a sequence in X converging weakly to $u \in X$ and $\liminf_{n\to\infty}\varphi(u_n) \le \varphi(u)$, then $\{u_n\}$ has a subsequence converging strongly to u. For example, if X is uniformly convex and $g : [0, +\infty) \to \mathbb{R}$ is a continuous and strictly increasing function, then, by a classical result, the functional $u \to g(\|u\|)$ belongs to the class Γ_X [11].

Theorem 2.12. [11] *Let X be a separable and reflexive real Banach space; let $\varphi : X \to \mathbb{R}$ be a coercive, sequentially weakly lower semicontinuous C^1 functional, belonging to Γ_X, bounded on each bounded subset of X and whose derivative admits a continuous inverse on X^*; $\omega : X \to \mathbb{R}$ is a C^1 functional with compact derivative. Assume that φ has a strict local minimum u_0 with $\varphi(u_0) = \omega(u_0) = 0$. Finally, setting*

$$\rho_1 = \max\left\{0, \limsup_{\|u\|\to+\infty}\frac{\omega(u)}{\varphi(u)}, \limsup_{\|u\|\to u_0}\frac{\omega(u)}{\varphi(u)}\right\}, \quad \rho_2 = \sup_{u\in\varphi^{-1}((0,+\infty))}\frac{\omega(u)}{\varphi(u)},$$

assume that $\rho_1 < \rho_2$. Then, for each compact interval $[c, d] \subset \left(\frac{1}{\rho_2}, \frac{1}{\rho_1}\right)$ (with the conventions $\frac{1}{0} = +\infty$, $\frac{1}{\infty} = 0$), there exists $R > 0$ with the following property: for every $\lambda \in [c, d]$ and every C^1 functional $\phi : X \to \mathbb{R}$ with compact derivative, there exists $\zeta > 0$ such that, for each $\mu \in [0, \zeta]$, $\varphi'(u) - \mu\phi'(u) - \lambda\omega'(u) = 0$ has at least three solutions in X whose norms are less than R.

3 Main results

This section first gives the variational framework of BVP (1.4) and defines the functional Φ_ξ. The critical point of Φ_ξ is proved by the mountain pass lemma as the weak solution of problem (1.4), and then Theorem 2.12 is used to prove that there are at least three solutions to problem (1.4).

In this article, we assume that the following conditions are satisfied:

(H1) There exists a positive constant $\theta \in (p, \tau]$ with $\tau > p$ such that

$$0 < uI_j(u) \le \theta \int_0^{u(t_j)} I(s)\mathrm{d}s, \quad \forall u \in \mathbb{R}\backslash\{0\}$$

for any $s \in \mathbb{R}$, $j = 1, 2, \ldots, m$.

(H2) There exist nonnegative constants K and $\hat{\theta} > p$ such that

$$0 < \hat{\theta}F(t, u, v) \le uf(t, u, v)$$

for any $t \in [0, T]$, $|u| \ge K$, $v \in \mathbb{R}$.

(H3) For any $(u, v) \in \mathbb{R} \times \mathbb{R}$, $\eta = p\Omega^p$, we have

$$\lim_{|u|\to 0}\frac{F(t, u, v)}{|u|^p} < \frac{1}{\eta}.$$

(H4) There exist constants $G > 0$ and $0 < \beta \le p - 1$ such that $I_j(s) \le Gs^\beta$, for $s \in \mathbb{R}$, $j = 1, 2, \ldots, m$.

Definition 3.1. A function $u \in E_0^{\alpha,p}$ is a weak solution of problem (1.4) if

$$\begin{aligned}&\int_0^T \Phi_p(a(t){}_0^cD_t^\alpha u(t)){}_0^cD_t^\alpha v(t) + \gamma u(t)v(t)\mathrm{d}t + \mu\sum_{j=1}^m I_j(u(t))v(t_j) - \int_0^T h(u(t))v(t)\mathrm{d}t \\ &= \lambda\int_0^T f(t, u(t), {}_0^cD_t^\alpha u(t))v(t)\mathrm{d}t\end{aligned} \tag{3.1}$$

holds, for any $v(t) \in E_0^{\alpha,p}$.

Consider the BVP (1.4) for any fixed $\xi(t) \in E_0^{\alpha,p}$ and we have the functional $\Phi_\xi : E_0^{\alpha,p} \to \mathbb{R}$, as follows:

$$\begin{aligned}\Phi_\xi(u(t)) = &\frac{1}{p}\int_0^T a(t)^{p-1}|{}_0^cD_t^\alpha u(t)|^p\mathrm{d}t + \frac{1}{2}\int_0^T \gamma|u(t)|^2\mathrm{d}t + \mu\sum_{j=1}^m\int_0^{u(t_j)} I_j(s)\mathrm{d}s - \int_0^T H(u(t))\mathrm{d}t\\ &- \lambda\int_0^T F(t, u(t), {}_0^cD_t^\alpha\xi(t))\mathrm{d}t,\end{aligned}\tag{3.2}$$

where $u(t) \in E_0^{\alpha,p}$, $F(t, x, y) = \int_0^x f(t, s, y)\mathrm{d}s$ and $F(t, 0, 0) = 0$, $H(x) = \int_0^x h(s)\mathrm{d}s$, for $x, y \in \mathbb{R}$. Since $E_0^{\alpha,p}$ is compactly embedded in $C([0, T], \mathbb{R})$, f and $I_j (j = 1, \ldots, m)$ are continuous, we can infer that Φ_ξ is continuous and *Gâteaux* differentiable function on $E_0^{\alpha,p}$. The *Gâteaux* derivative of Φ_ξ at point $u(t) \in E_0^{\alpha,p}$ is given as

$$\begin{aligned}\langle\Phi_\xi'(u(t)), v(t)\rangle = &\int_0^T \Phi_p(a(t){}_0^cD_t^\alpha u(t)){}_0^cD_t^\alpha v(t) + \gamma u(t)v(t)\mathrm{d}t + \mu\sum_{j=1}^m I_j(u(t_j))v(t_j) - \int_0^T h(u(t))v(t)\mathrm{d}t\\ &- \lambda\int_0^T f(t, u(t), {}_0^cD_t^\alpha\xi(t))v(t)\mathrm{d}t\end{aligned}\tag{3.3}$$

for any $v(t) \in E_0^{\alpha,p}$, $t \in [0, T]$.

Obviously, we establish the weak solution of BVP (1.4) through the critical point, that is, the critical point of Φ_ξ is the weak solution of problem (1.4). In addition, it is similar to the proof method of Remark 3.3 in [12], and the weak solution of problem (1.4) is also the classical solution.

For convenience, we give some special marks to indicate some coefficients that appear in the proof process with them

$$\lambda_1 = \begin{cases}\min\{\bar{a}^{p-2}\}, & 1 < p \le 2,\\ \min\{\tilde{a}^{p-2}\}, & p \ge 2,\end{cases}\quad \lambda_2 = \begin{cases}\max\{\tilde{a}^{p-2}\}, & 1 < p \le 2,\\ \max\{\bar{a}^{p-2}\}, & p \ge 2.\end{cases}\tag{3.4}$$

Furthermore, since $h(0) = 0$, one has $|h(s)| \le L|s|$, and then,

$$H(u(t)) = \int_0^{u(t)} h(x)\mathrm{d}x \le \int_0^{u(t)} L|x|\mathrm{d}x = \frac{1}{2}L|u(t)|^2.\tag{3.5}$$

Lemma 3.2. *Assume that conditions* (*H*1) *and* (*H*2) *hold, then* Φ_ξ *satisfies the PS condition, i.e., for any bounded sequence* $\{u_k\} \subset E_0^{\alpha,p}$ *such that* $\Phi_\xi'(u_k) \to 0$ *as* $k \to \infty$ *admits a convergent subsequence in* $E_0^{\alpha,p}$.

Proof. Assume that $\{u_k\} \subset E_0^{\alpha,p}$ such that $\Phi_\xi(u_k)$ is bounded and $\Phi_\xi'(u_k) \to 0$ as $k \to \infty$. Then, there exists a positive constant M such that $|\Phi_\xi(u_k)| \le M$ and from (3.5) we have

$$\begin{aligned}&\tau\Phi_\xi(u_k(t)) - \langle\Phi_\xi'(u_k(t)), u_k(t)\rangle\\ &= \frac{\tau}{p}\int_0^T a(t)^{p-1}|{}_0^cD_t^\alpha u_k(t)|^p\mathrm{d}t - \int_0^T \Phi_p(a(t){}_0^cD_t^\alpha u_k(t)){}_0^cD_t^\alpha u_k(t)\mathrm{d}t + \left(\frac{\tau}{2} - 1\right)\gamma\int_0^T |u_k(t)|^2\mathrm{d}t\\ &\quad + \tau\mu\sum_{j=1}^m\int_0^{u_k(t_j)} I_j(s)\mathrm{d}s - \mu\sum_{j=1}^m I_j(u_k(t_j))u_k(t_j) + \int_0^T h(u_k(t))u_k(t)\mathrm{d}t - \tau\int_0^T H(u_k(t))\mathrm{d}t\\ &\quad + \lambda\int_0^T f(t, u_k(t), {}_0^cD_t^\alpha\xi(t))u_k(t)\mathrm{d}t - \tau\lambda\int_0^T F(t, u_k(t), {}_0^cD_t^\alpha\xi(t))\mathrm{d}t\end{aligned}$$

$$
\begin{aligned}
&\geq \left(\frac{\tau}{p}-1\right)\lambda_1\|u_k(t)\|^p_{\alpha,p} + (\tau-\theta)\mu\sum_{j=1}^{m}\int_0^{u(t_j)} I_j(s)\mathrm{d}s - LT\|u_k\|^2_\infty - \frac{\tau}{2}LT\|u_k\|^2_\infty \\
&\quad + \left(\frac{\tau}{2}-1\right)\gamma\int_0^T |u_k(t)|^2\mathrm{d}t + \lambda\int_0^T [f(t,u_k(t),{}_0^cD_t^\alpha\xi(t))u_k(t) - \tau F(t,u_k(t),{}_0^cD_t^\alpha\xi(t))]\mathrm{d}t \\
&\geq \left(\frac{\tau}{p}-1\right)\lambda_1\|u_k(t)\|^p_{\alpha,p} - \left(1+\frac{\tau}{2}\right)LT\left(\frac{T^{\alpha-\frac{1}{p}}}{\Gamma(\alpha)(\tilde{a})^{\frac{1}{p}}[(\alpha-1)q+1]^{\frac{1}{q}}}\right)^2\|u_k\|^2_{\alpha,p} \\
&\quad + \left(\frac{\tau}{2}-1\right)\gamma\int_0^T |u_k(t)|^2\mathrm{d}t + \lambda\int_0^T [f(t,u_k(t),{}_0^cD_t^\alpha\xi(t))u_k(t) - \tau F(t,u_k(t),{}_0^cD_t^\alpha\xi(t))]\mathrm{d}t \\
&\geq \left(\frac{\tau}{p}-1\right)\lambda_1\|u_k(t)\|^p_{\alpha,p} + \left[\left(\frac{\tau}{2}-1\right)\gamma - \left(1+\frac{\tau}{2}\right)L\right]T\Omega^2\|u_k\|^2_{\alpha,p}.
\end{aligned}
$$

Noting that $\tau \geq p$, thus $\frac{\tau}{p}-1\geq 0$, we also know that $|\Phi_\xi(u_k)|\leq M$ and $\Phi'_\xi(u_k)\to 0$ as $k\to\infty$, hence $\{u_k\}_{k\in\mathbb{N}}$ is bounded in $E_0^{\alpha,p}$. Considering that $E_0^{\alpha,p}$ is a reflexive space, we assume that $u_k\to u_0$ weakly in $E_0^{\alpha,p}$, from Lemma 2.8, $u_k\to u_0$ on $[0,T]$. Hence, we can obtain

$$
\begin{aligned}
\langle\Phi'_\xi(u_k)-\Phi'_\xi(u_0),(u_k-u_0)\rangle &= \langle\Phi'_\xi(u_k),(u_k-u_0)\rangle - \langle\Phi'_\xi(u_0),(u_k-u_0)\rangle \\
&\leq \|\Phi'_\xi u_k\|\cdot\|u_k-u_0\| - \langle\Phi'_\xi(u_0),(u_k-u_0)\rangle \to 0
\end{aligned}
\tag{3.6}
$$

as $k\to\infty$. Moreover, it follows from Lemma 2.8 and function f continuously differentiable in u and I_j are continuous, we have

$$
\begin{cases}
\lim\limits_{k\to\infty}\int_0^T [\lambda f(t,u(t),{}_0^cD_t^\alpha\xi(t))u_k(t) - \lambda f(t,u_0(t),{}_0^cD_t^\alpha\xi(t))](u_k(t)-u_0(t))\mathrm{d}t = 0,\\
\lim\limits_{k\to\infty}[I_j(u_k(t_j)) - I_j(u_0(t_j))](u_k(t_j)-u_0(t_j)) = 0,\\
\lim\limits_{k\to\infty}[h(u_k(t)) - h(u_0(t))](u_k(t)-u_0(t)) = 0.
\end{cases}
\tag{3.7}
$$

Observe that

$$
\begin{aligned}
\langle\Phi'_\xi(u_k)-\Phi'_\xi(u_0),u_k-u_0\rangle &= \int_0^T(\Phi_p(a(t){}_0^cD_t^\alpha u_k(t)) - \Phi_p(a(t){}_0^cD_t^\alpha u_0(t))){}_0^cD_t^\alpha(u_k(t)-u_0(t))\mathrm{d}t \\
&\quad + \int_0^T \gamma(u_k(t)-u_0(t)) + \mu\sum_{j=i}^{m}(I_j(u_k(t_j)) - I_j(u_0(t_j))(u_k(t_j)-u_0(t_j))) \\
&\quad - \int_0^T (h(u_k(t)) - h(u_0(t)))(u_k(t)-u_0(t))\mathrm{d}t \\
&\quad - \lambda\int_0^T [f(t,u_k(t),{}_0^cD_t^\alpha\xi(t)) - f(t,u_0(t),{}_0^cD_t^\alpha\xi(t))](u_k(t)-u_0(t))\mathrm{d}t \to 0,
\end{aligned}
$$

as $k\to\infty$. Combining (3.6) and (3.7), h is Lipschitz continuous, we obtain

$$
\int_0^T(\Phi_p(a(t){}_0^cD_t^\alpha u_k(t)) - \Phi_p(a(t){}_0^cD_t^\alpha u_0(t))){}_0^cD_t^\alpha(u_k(t)-u_0(t))\mathrm{d}t \to 0, \quad \text{as } k\to\infty. \tag{3.8}
$$

Next, we will use the following inequality [17], for every $s_1, s_2 \in \mathbb{R}^{\mathbb{N}}$,

$$\langle |s_1|^{p-2}s_1 - |s_2|^{p-2}s_2, s_1 - s_2\rangle \geq \begin{cases} j_1|s_1 - s_2|^{p-2}, & p \geq 2, \\ j_1\dfrac{|s_1 - s_2|^2}{(|s_1| + |s_2|)^{2-p}}, & 1 < p \leq 2, \end{cases}$$

where $j_1 \in \mathbb{R}$ and $j_1 > 0$. Using the above formula, we can obtain that there exist positive constants κ_1, κ_2 such that

$$\begin{aligned} &\int_0^T (\Phi_p(a(t){}_0^cD_t^\alpha u_k(t)) - \Phi_p(a(t){}_0^cD_t^\alpha u_0(t))){}_0^cD_t^\alpha(u_k(t) - u_0(t))\mathrm{d}t \\ &\geq \begin{cases} \kappa_1 \displaystyle\int_0^T \frac{1}{a(t)}|a(t)({}_0^cD_t^\alpha u_k(t) - {}_0^cD_t^\alpha u_0(t))|^p\mathrm{d}t, & p \geq 2, \\ \kappa_2 \displaystyle\int_0^T \frac{1}{a(t)}\frac{|a(t)({}_0^cD_t^\alpha u_k(t) - {}_0^cD_t^\alpha u_0(t))|^2}{(|a(t){}_0^cD_t^\alpha u_k(t)| + |a(t){}_0^cD_t^\alpha u_0(t)|)^{2-p}}\mathrm{d}t, & 1 < p \leq 2. \end{cases} \end{aligned} \tag{3.9}$$

When $1 < p < 2$, we obtain the following result by applying the inequality $(b_1 + b_2)^p \leq 2^p(b_1^p + b_2^p)$, where $b_1, b_2 \in [0, \infty)$,

$$\begin{aligned} &\frac{1}{\bar{a}^{\frac{p}{2}}}\int_0^T |a(t)({}_0^cD_t^\alpha u_k(t) - a(t){}_0^cD_t^\alpha u_0(t))|^p\mathrm{d}t \\ &\leq \frac{1}{\bar{a}^{\frac{p}{2}}}\left(\int_0^T\left(\frac{|a(t)({}_0^cD_t^\alpha u_k(t) - a(t){}_0^cD_t^\alpha u_0(t))|^p}{(a(t))^{\frac{p}{2}}(|a(t){}_0^cD_t^\alpha u_k(t)| + |a(t){}_0^cD_t^\alpha u_0(t)|)^{\frac{(2-p)p}{2}}}\right)^{\frac{2}{p}}\mathrm{d}t\right)^{\frac{p}{2}} \\ &\quad\times\left(\int_0^T\left(a(t)^{\frac{p}{2}}(|a(t){}_0^cD_t^\alpha u_k(t)| + |a(t){}_0^cD_t^\alpha u_0(t)|)^{\frac{(2-p)p}{2}}\right)^{\frac{2}{2-p}}\mathrm{d}t\right)^{\frac{2-p}{2}} \\ &\leq 2^{\frac{p(2-p)}{2}}\left(\int_0^T |a(t){}_0^cD_t^\alpha u_k(t)|^p + |a(t){}_0^cD_t^\alpha u_0(t)|^p\mathrm{d}t\right)^{\frac{2-p}{2}} \times \left(\int_0^T \frac{1}{a(t)}\frac{|a(t)({}_0^cD_t^\alpha u_k(t) - {}_0^cD_t^\alpha u_0(t))|^2}{(|a(t){}_0^cD_t^\alpha u_k(t)| + |a(t){}_0^cD_t^\alpha u_0(t)|)^{2-p}}\mathrm{d}t\right)^{\frac{p}{2}}. \end{aligned}$$

So, it is easy to obtain

$$\begin{aligned} &\int_0^T (\Phi_p(a(t){}_0^cD_t^\alpha u_k(t)) - \Phi_p(a(t){}_0^cD_t^\alpha u_0(t))){}_0^cD_t^\alpha(u_k(t) - u_0(t))\mathrm{d}t \\ &\geq \kappa_2\int_0^T \frac{1}{a(t)}\frac{|a(t)({}_0^cD_t^\alpha u_k(t) - {}_0^cD_t^\alpha u_0(t))|^2}{(|a(t){}_0^cD_t^\alpha u_k(t)| + |a(t){}_0^cD_t^\alpha u_0(t)|)^{2-p}}\mathrm{d}t \\ &\geq \kappa_2\frac{2^{(p-2)}}{\bar{a}}\left(\int_0^T |a(t){}_0^cD_t^\alpha u_k(t)|^p + |a(t){}_0^cD_t^\alpha u_0(t)|^p\mathrm{d}t\right)^{\frac{p-2}{p}} \times \left(\int_0^T |a(t)({}_0^cD_t^\alpha u_k(t) - a(t){}_0^cD_t^\alpha u_0(t))|^p\mathrm{d}t\right)^{\frac{2}{p}} \\ &\geq \kappa_3\|u_k - u_0\|_{\alpha,p}^2(\|u_k\|_{\alpha,p}^p + \|u_0\|_{\alpha,p}^p)^{\frac{p-2}{p}}, \end{aligned} \tag{3.10}$$

where $\kappa_3 > 0$. When $p \geq 2$, by (3.9), we have

$$\int_0^T (\Phi_p(a(t){}_0^cD_t^\alpha u_k(t)) - \Phi_p(a(t){}_0^cD_t^\alpha u_0(t))){}_0^cD_t^\alpha(u_k(t) - u_0(t))\mathrm{d}t \geq \kappa_1\lambda_1\|u_k - u_0\|_{\alpha,p}^p. \tag{3.11}$$

According to (3.8), (3.10), and (3.11), we obtain

$$\|u_k - u_0\|_{\alpha,p}^p \to 0, \quad \text{as } k \to \infty,$$

that is, $\{u_k\}$ converges strongly to $\{u_0\}$ in $E_0^{\alpha,p}$. Therefore, we proved that the functional Φ_ξ satisfies the PS condition. The proof is completed. □

Theorem 3.3. *Suppose that* (*H*1)–(*H*4) *hold, then BVP* (1.4) *has at least one weak solution on* $E_0^{\alpha,p}$.

Proof. It is not difficult to see from the definition of Φ_ξ that we have $\Phi_\xi(0) = 0$, according to (H3) there exist $\sigma > 0$ and $0 < \varepsilon < \min\left\{1, \frac{\lambda_1}{\lambda T}\right\}$ such that

$$F(t, u, v) \le \frac{|u|^p \varepsilon}{\eta}, \quad |u| < \sigma. \tag{3.12}$$

Picking out $\rho = \frac{\sigma}{\Omega} > 0$ and $\delta = \left(\frac{\lambda_1 - \lambda T\varepsilon}{p}\right)\rho^p$, and from Proposition 2.5, we have

$$\|u\|_\infty \le \Omega\|u\|_{\alpha,p} = \sigma, \quad \forall u \in E_0^{\alpha,p}, \quad \|u\|_{\alpha,p} = \rho.$$

Hence, for all $u \in E_0^{\alpha,p}$ with $\|u\|_{\alpha,p} = \rho$ and noting $\gamma - L > 0$, from (3.12), we obtain

$$\begin{aligned}
\Phi_\xi(u(t)) &= \frac{1}{p}\int_0^T a(t)^{p-1}|{}_0^cD_t^\alpha u(t)|^p \mathrm{d}t + \mu\sum_{j=1}^m \int_0^{u(t_j)} I_j(s)\mathrm{d}s + \frac{1}{2}\int_0^T \gamma|u(t)|^2\mathrm{d}t \\
&\quad - \int_0^T H(u(t))\mathrm{d}t - \lambda\int_0^T F(t, u(t), {}_0^cD_t^\alpha\xi(t))\mathrm{d}t \\
&\ge \frac{1}{p}\lambda_1\|u\|_{\alpha,p}^p + \mu\sum_{j=1}^m \int_0^{u(t_j)} I_j(s)\mathrm{d}s + \left(\frac{\gamma - L}{2}\right)\int_0^T |u(t)|^2\mathrm{d}t - \lambda\int_0^T F(t, u(t), {}_0^cD_t^\alpha\xi(t))\mathrm{d}t \\
&\ge \frac{\lambda_1}{p}\|u\|_{\alpha,p}^p + \left(\frac{\gamma - L}{2}\right)LT\Omega^2\|u\|_{\alpha,p}^2 - \frac{1}{\eta}\lambda T\varepsilon\Omega^P\|u\|_{\alpha,p}^p \\
&\ge \left(\frac{\lambda_1}{p} - \frac{\lambda T\varepsilon\Omega^P}{\eta}\right)\|u\|_{\alpha,p}^p \\
&= \left(\frac{\lambda_1 - \lambda T\varepsilon}{p}\right)\rho^p \\
&= \delta,
\end{aligned}$$

which means that condition (2) of Lemma 2.10 holds.

From (H2) and A-R condition, we obtain the following inequality

$$F(t, u, v) \ge C_1(|u|^\varsigma + |v|^\varsigma) - C_2 \tag{3.13}$$

holds, where C_1, C_2 are nonnegative constants and $\varsigma > p$. Then, choose $\tilde{\tau} > 1$, by (3.13) and (H4), we infer

$$\begin{aligned}
\Phi_\xi(\tilde{\tau}z_0) &\le \frac{\lambda_2}{p}\tilde{\tau}^p\|z_0\|_{\alpha,p}^p + \sum_{j=1}^m \int_0^{\tilde{\tau}z_0(t_j)} I_j(s)\mathrm{d}s + \frac{\gamma - L}{2}\int_0^T \|\tilde{\tau}z_0\|^2\mathrm{d}t - \lambda C_1\left(\int_0^T (|\tilde{\tau}z_0|^\varsigma + |\tilde{\tau}{}_0^cD_t^\alpha z_0|^\varsigma)\mathrm{d}t\right) + \lambda C_2T \\
&\le \frac{\lambda_2}{p}\tilde{\tau}^p\|z_0\|_{\alpha,p}^p + mG\tilde{\tau}^{\beta+1}\frac{\Omega^{\beta+1}}{\beta+1}\|z_0\|_{\alpha,p}^{\beta+1} + \frac{\gamma - L}{2}LT\tilde{\tau}^2\Omega^2\|z_0\|_{\alpha,p}^2 - \lambda C_1\tilde{\tau}^\varsigma(\|z_0\|_{L^\varsigma}^\varsigma + \|{}_0^cD_t^\alpha z_0\|_{L^\varsigma}^\varsigma) + \lambda C_2T.
\end{aligned}$$

Owing to $\varsigma > p$, $0 < \beta + 1 < p$, we obtain

$$\Phi_\xi(\tilde{\tau}z_0) \to -\infty \quad \text{as } \tilde{\tau} \to \infty,$$

which means that there exists a large enough number τ^* such that $\Phi_\xi(\tau^* z_0) \le 0$ and $\|\tau^* z_0\| > \rho$. That is to say, the condition of Lemma 2.10 is satisfied. Above all, the functional Φ_ξ possesses a critical value $z(t)$ such that $\Phi'_\xi(z(t))(v(t)) = 0$ for any $v(t) \in E_0^{\alpha,p}$. The proof is completed. □

Next, we will apply Theorem 2.12 to prove the existence of multiple solutions to BVP (1.4). Before giving the main results, we define some functions $\varphi, \phi, \omega : X \to \mathbb{R}$ as follows:

$$\varphi(u) = \frac{1}{p}\int_0^T a(t)^{p-1}|{}_0^cD_t^\alpha u(t)|^p \mathrm{d}t + \frac{1}{2}\int_0^T \gamma|u(t)|^2 \mathrm{d}t - \int_0^T H(u(t))\mathrm{d}t, \tag{3.14}$$

$$\phi(u) = -\sum_{j=1}^m \int_0^{u(t_j)} I_j(s)\mathrm{d}s, \tag{3.15}$$

$$\omega(u) = \int_0^T F(t, u(t), {}_0^cD_t^\alpha \xi(t))\mathrm{d}t. \tag{3.16}$$

Standard arguments show that $\varphi - \mu\varphi - \lambda\omega$ is a *Gâteaux* differentiable function with *Gâteaux* derivative at the point $v(t) \in X$ given by

$$\begin{aligned}(\varphi' - \mu\varphi' - \lambda\omega')(u)(v) = &\int_0^T [\Phi_p(a(t){}_0^cD_t^\alpha u(t)){}_0^cD_t^\alpha v(t) + \gamma u(t)v(t)]\mathrm{d}t + \mu\sum_{j=1}^m I_j(u(t_j))v(t_j)\\ &- \int_0^T h(u(t))v(t)\mathrm{d}t - \lambda\int_0^T f(t, u(t), {}_0^cD_t^\alpha \xi(t))v(t)\mathrm{d}t.\end{aligned} \tag{3.17}$$

Hence, the critical point of the functional $\varphi - \mu\varphi - \lambda\omega$ is the weak solution of BVP (1.4), which is also the classical solution and the weak solution of BVP (1.4). The proof is completed.

Theorem 3.4. *Suppose that there is a non-negative constant ε and a function $u(t) \in X$ such that*

$$\max\left\{\lim_{\|u\|\to 0}\frac{F(t, u(t), v(t))}{|u|^p}, \lim_{\|u\|\to\infty}\frac{F(t, u(t), v(t))}{|u|^p}\right\} < \varepsilon \tag{3.18}$$

and

$$\varepsilon T\Omega^p < \frac{\int_0^T F(t, u(t), v(t))\mathrm{d}t}{\lambda_2\|u\|^p - p\int_0^T H(u(t))\mathrm{d}t}. \tag{3.19}$$

Then for each compact interval $[\theta_1, \theta_2] \subset \left(\frac{1}{\rho_2}, \frac{1}{\rho_1}\right)$, there exists $R > 0$ satisfying the property: for every $\lambda \in [\theta_1, \theta_2]$, there exists $\varrho > 0$, such that for each $\mu \in [0, \varrho]$, the BVP (1.4) has at least three solutions whose norms in X are less than R.

Proof. Take $X = E_0^{\alpha,p}$, let M be the boundary of the subset of X, that is, $\|u\| < M$ in the subset of X, from (3.16) one has

$$|\varphi(u)| \le \frac{\lambda_2}{p}\|u\|_{\alpha,p}^p + \frac{\gamma}{2}T\Omega^2\|u\|_{\alpha,p}^2 \le \frac{\lambda_2}{p}M^p + \gamma T\Omega^2 M^2.$$

This means that φ is bounded on every bounded subset. And

$$|\varphi(u)| \geq \frac{\lambda_1}{p}\|u\|_{\alpha,p}^p + \left(\frac{\gamma - L}{2}\right)T\Omega^2\|u\|_{\alpha,p}^2 \geq \frac{\lambda_1}{p}\|u\|_{\alpha,p}^p,$$

which shows that $\varphi(u) \to \infty$ as $\|u\| \to \infty$. Thus, φ is coercive.

Moreover recalling (3.17), we have

$$\begin{aligned}(\varphi'(u) - \varphi'(v))(u - v) = &\int_0^T (\Phi_p(a(t)_0^cD_t^\alpha u(t)) - \Phi_p(a(t)_0^cD_t^\alpha v(t)))_0^cD_t^\alpha(u(t) - v(t))\mathrm{d}t \\ &+ \gamma\int_0^T (u(t) - v(t))^2\mathrm{d}t - \int_0^T h(u(t) - v(t))(u(t) - v(t))\mathrm{d}t\end{aligned}$$

for any $u, v \in X$ with $u - v \neq 0$. Combining (3.10) and (3.11) when $1 < p < 2$, we have

$$(\varphi'(u) - \varphi'(v))(u - v) \geq \kappa_3\|u - v\|_{\alpha,p}^2(\|u\|_p^{\alpha,p} + \|v\|_p^{\alpha,p})^{\frac{p-2}{p}} + \left(\gamma - \frac{L}{2}\right)T\Omega^2\|u - v\|^2 > 0.$$

When $p \geq 2$,

$$(\varphi'(u) - \varphi'(v))(u - v) \geq \kappa_1\lambda_1\|u_k - u_0\|_{\alpha,p}^p + \left(\gamma - \frac{L}{2}\right)T\Omega^2\|u - v\|^2 > 0.$$

Hence, for any $p > 1$, we obtain that $(\varphi'(u) - \varphi'(v))(u - v) > 0$, that is, φ' is a strictly monotone operator. Since X is reflexive, for $u_n \to u$ strongly in X as $n \to +\infty$, we have $\varphi'(u_n) \to \varphi'(u)$ weakly in X^* as $n \to +\infty$. Therefore, φ' is semicontinuous, so by [14] (Theorem 26.A(d)) the inverse operator φ^{-1} of φ exists and it is continuous. In addition, let r_n be a sequence of X^* such that $r_n \to r$ strongly in X^* as $n \to +\infty$. Let u_n and u in X such that $\varphi^{-1}(r_n) = u_n$ and $\varphi^{-1}(r) = u$. Owing to the fact that φ is coercive, the sequence u_n is bounded in the reflexive space X. For an appropriate subsequence, we have $u_n \to \hat{u}$ weakly in X as $n \to +\infty$, it derives

$$\langle\varphi'(u_n) - \varphi'(u), u_n - \hat{u}\rangle = \langle r_n - r, u_n - \hat{u}\rangle = 0.$$

When $u_n \to \hat{u}$ weakly in X as $n \to +\infty$, $\varphi'(u_n) - \varphi'(\hat{u})$ strongly in X^* as $n \to +\infty$, one has $u_n \to \hat{u}$ strongly in X as $n \to +\infty$, and since φ' is continuous, we have $u_n \to \hat{u}$ weakly in X as $n \to +\infty$ and $\varphi'(u_n) \to \varphi'(\hat{u}) = \varphi'(u)$ strongly in X^* as $n \to +\infty$. Therefore, φ' is an injection and we have $u = \hat{u}$.

Moreover, let sequence $\{u_n\} \subset X$, $u_n \to u \in X$ and $\liminf_{n\to\infty}\varphi_{u_n} < \varphi(u)$. Since h is continuous, $\{u_n\}$ has a subsequence converging strongly to u. Hence, $\varphi \in \Gamma_X$. The functionals ϕ and ω are two C^1 functionals with compact derivatives. Moreover, φ has a strict local minimum 0 with $\varphi(0) = \omega(0) = 0$.

Due to (3.18), there exist ι_1, ι_2 with $0 < \iota_1 < \iota_2$ such that

$$F(t, u, v) \leq \varepsilon|u|^p,\ t \in [0, T],\ u \in (0, \iota_1) \cup (\iota_2, \infty). \tag{3.20}$$

In view of the continuity of F, there exist $k > 0$ and $\vartheta > p$ such that

$$F(t, u, v) \leq \varepsilon|u|^p + k|u|^\vartheta, \quad t \in [0, T],\ u \in X. \tag{3.21}$$

Based on (3.21), we obtain

$$\omega(u) = \int_0^T F(t, u(t), {}_0^cD_t^\alpha\xi(t))\mathrm{d}t \leq \int_0^T (\varepsilon|u|^p + k|u|^\vartheta)\mathrm{d}t \leq T\varepsilon\Omega^P\|u\|^p + kT\Omega^\vartheta\|u\|^\vartheta,$$

that is,

$$\limsup_{u\to 0}\frac{\omega(u)}{\varphi(u)} \leq \limsup_{u\to 0}\frac{T\varepsilon\Omega^P\|u\|^p}{\frac{\lambda_2}{p}\|u\|^p} + \frac{kT\Omega^\vartheta\|u\|^\vartheta}{\frac{\lambda_2}{p}\|u\|^p} \leq p\varepsilon T\Omega^P. \tag{3.22}$$

Moreover, from (3.20), for each $u \in X\backslash\{0\}$, we obtain

$$\begin{aligned}\limsup_{u\to\infty}\frac{\omega(u)}{\varphi(u)} &\le \limsup_{u\to\infty}\left(\frac{\int_{|u|\le t_2}F(t,u(t),{}_0^cD_t^\alpha\xi(t))\mathrm{d}t}{\frac{\lambda_2}{p}\|u\|^p}+\frac{\int_{|u|>t_2}F(t,u(t),{}_0^cD_t^\alpha\xi(t))\mathrm{d}t}{\frac{\lambda_2}{p}\|u\|^p}\right)\\ &\le \limsup_{u\to\infty}\frac{\varepsilon T\Omega^p\|u\|^p}{\frac{\lambda_2}{p}\|u\|^p}\\ &\le p\varepsilon T\Omega^p.\end{aligned}\tag{3.23}$$

Hence, by (3.22) and (3.23), we have

$$\rho_1=\max\left\{0,\ \lim_{\|u\|\to 0}\frac{\omega(u)}{|u|^p},\ \lim_{\|u\|\to\infty}\frac{\omega(u)}{|u|^p}\right\}<p\varepsilon T\Omega^p.\tag{3.24}$$

In addition, by applying (3.19), we obtain

$$\begin{aligned}\rho_2 &= \sup_{u\in\varphi^{-1}((0,+\infty))}\frac{\omega(u)}{\varphi(u)}\\ &= \sup_{u\in X\setminus\{0\}}\frac{\omega(u)}{\varphi(u)}\ge\frac{\int_0^T F(t,x(t),{}_0^cD_t^\alpha\xi(t))\mathrm{d}t}{\varphi(x)}\\ &\ge\frac{\int_0^T F(t,x(t),{}_0^cD_t^\alpha\xi(t))\mathrm{d}t}{\frac{\lambda_2}{p}\|x(t)\|^p\mathrm{d}t-\int_0^T H(x(t))\mathrm{d}t}\\ &> p\varepsilon T\Omega^p\ge\rho_1.\end{aligned}\tag{3.25}$$

It follows from Theorem 2.12 that Theorem 3.4 holds. The proof is completed. □

4 Example

Example 4.1. Consider that the following impulsive fractional differential equation with the p-Laplacian operator has at least one weak solution:

$$\begin{cases}{}_tD_1^{\frac{3}{4}}\Phi_3(a(t){}_0^cD_t^{\frac{3}{4}}u(t))+u(t)=4u^3(t)\sin^2({}_0^cD_t^{\frac{3}{4}}u(t))+2u(t)\sin^4({}_0^cD_t^{\frac{3}{4}}u(t))+\dfrac{1}{2}\sin(u(t)),\\ t\in[0,1],\ t\ne t_1,\\ \Delta\left({}_tD_1^{\frac{1}{4}}\Phi_3(a(t){}_0^cD_t^{\frac{3}{4}}u)\right)(t_1)=\dfrac{1}{10}u^{\frac{3}{5}}(t_1),\\ u(0)=u(1)=0,\end{cases}\tag{4.1}$$

where $\alpha=\frac{3}{4}$, $T=1$, $p=3$, $\gamma=\mu=1$, $a(t)=\frac{1}{1+t^2}$ and $h(u(t))=\frac{1}{2}\sin(u(t))\le\frac{1}{2}|u(t)|$, i.e., $L=\frac{1}{2}$, which implies that $\gamma-L>0$. We define the function $F(t,u(t),v(t))=u^4(t)\sin^2(v(t))+u^2(t)\sin^4(v(t))$, and $f(t,u(t),v(t))=4u^3(t)\sin^2(v(t))+2u(t)\sin^4(v(t))$, $I_j(u(t))=\frac{1}{10}u^{\frac{3}{5}}$. We take $\theta=8$, it has

$$0<u(t_1)I_1(u(t_1))=\frac{1}{10}u^{\frac{8}{5}}\le\frac{1}{2}u^{\frac{8}{5}}=\frac{4}{5}\int_0^{u(t_1)}u^{\frac{3}{5}}(s)\mathrm{d}s,$$

which means that condition $(H1)$ is met. If we take $\hat{\theta}=\frac{7}{2}>3$, we have

$$0<\hat{\theta}F(t,u,v)=\frac{7}{2}u^4\sin^2 v+\frac{7}{2}u^2\sin^4 v\le 4u^4\sin^2 v+2u^2 sin^4 v=uf(t,u,v),$$

which means that condition $(H2)$ is met, and it is obvious that the function F satisfied condition $(H3)$. Next, we prove that condition $(H4)$ is also satisfied. We choose $\beta = \frac{3}{2}$, $G = 5$, $I_1(s) = \frac{1}{10}s^{\frac{3}{5}} \leq \frac{1}{2}s^{\frac{3}{2}} = Gs^{\beta}$ holds. It is easy to see that the above conditions satisfy Theorem 3.3, that is, problem (4.1) has at least one weak solution.

Example 4.2. Consider the following fractional differential equation:

$$\begin{cases} {}_tD_1^{\frac{4}{5}}\Phi_3(a(t){}_0^cD_t^{\frac{4}{5}}u(t)) + u(t) = (1+t^2)|u(t)|^2 + |{}_0^cD_t^{\frac{4}{5}}u(t)|^2 + \dfrac{1}{2}\sin(u(t)), \\ t \in [0,1],\, t \neq t_1, \\ \Delta\left({}_tD_1^{\frac{1}{4}}\Phi_3(a(t){}_0^cD_t^{\frac{3}{4}}u)\right)(t_1) = \dfrac{1}{10}u^{\frac{3}{5}}(t_1), \\ u(0) = u(1) = 0, \end{cases} \tag{4.2}$$

where $\alpha = \frac{4}{5}$, $T = 1$, $p = 3$, $a(t) = \left(\frac{5}{8}\right)^3$, $h(x) = \frac{1}{2}\sin(x)$, $x \in \mathbb{R}$, we choose functions $F(t,u,v) = (1+t^2)|u|^2 + |v|^2$ and

$$\tilde{\xi}(t) = \begin{cases} 3t, & t \in \left[0, \dfrac{1}{3}\right), \\ 1, & t \in \left[\dfrac{1}{3}, \dfrac{2}{3}\right], \\ 3(1-t), & t \in \left(\dfrac{2}{3}, 1\right]. \end{cases}$$

Without loss of generality, F is a C^1 functional in u with $F(t,0,0) = 0$, satisfying $\lim_{|u|\to 0}\sup\frac{F(t,u,v)}{|u|^3} = \lim_{|u|\to\infty}\sup\frac{F(t,u,v)}{|u|^3} = 0$. In addition, through direct calculation, we obtain

$${}_0^cD_t^{\frac{4}{5}}\tilde{\xi}(t) = \frac{1}{\Gamma\left(\frac{1}{5}\right)}\begin{cases} 15t^{\frac{1}{5}}, & t \in \left[0, \dfrac{1}{3}\right), \\ 5\sqrt[5]{81}, & t \in \left[\dfrac{1}{3}, \dfrac{2}{3}\right], \\ 5\sqrt[5]{81} - 15\left(t - \dfrac{2}{3}\right)^{\frac{1}{5}}, & t \in \left(\dfrac{2}{3}, 1\right]. \end{cases}$$

And $\int_0^1 H(\tilde{\xi}(t))\mathrm{d}t = -0.05$, $\|\tilde{\xi}\| = \left(\int_0^1 (a(t)|{}_0^cD_t^{\alpha}\tilde{\xi}(t)|)^3\mathrm{d}t\right)^{\frac{1}{3}} \approx 6.15808$, $\|\tilde{\xi}\|^3 \approx 233.5260$, $\Omega^3 = 5.3048$. Choosing $\varepsilon = 10^{-4}$, then

$$\frac{1}{\rho_1} \geq \frac{1}{3 \times 10^{-4} \times 5.3048} \approx 6.289 \times 10^2, \quad \frac{1}{\rho_2} \leq \frac{\left(\frac{5}{8}\right)^3\|\tilde{\xi}\|^3 - 3\int_0^1 H(\tilde{\xi}(t))\mathrm{d}t}{3\int_0^1 F(t, \tilde{\xi}, {}_0^cD_t^{\frac{4}{5}}\tilde{\xi}(t))} \approx 13.2277.$$

Hence, by applying Theorem 3.4, for each compact interval $[\theta_1, \theta_2] \subset (0.132277 \times 10^2, 6.289 \times 10^2)$, there exists $R > 0$ with the following property: for any $\lambda \in [\theta_1, \theta_2]$, there exists $\varrho > 0$ such that, for each $\mu \in [0, \varrho]$, problem (4.2) has at least three solutions whose norms in X are less than R.

5 Conclusion

In the article, we use the variational method to discuss the existence of multiple solutions to the p-Laplacian fractional impulsive differential equations of Dirichlet boundary-value problems, and give the results of the existence of weak solutions to the equations. In addition to extending linear differential operators to nonlinear differential operators, the usage of p-Laplacian operators is more important.

In particular, if $p = 2, \lambda = \mu = 1, h(u(t)) \equiv 0$, then problem (1.4) will return to problem (1.2). As far as the author knows, there is still very little work in studying the multiple solutions of p-Laplacian impulsive fractional differential equations using variational methods. At the same time, we have also noted that there are many kinds of methods for discussing the existence of multiple solutions for such problems, and the following work can continue to expand the research methods. In general, our work summarizes and supplemented some of the results in the previous literature.

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