Research Article

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Regularity of models associated with Markov jump processes

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Abstract: We consider a jump Markov process $X = (X_t)_{t\geq 0}$, with values in a state space $(E, \mathcal{E})$. We suppose that the corresponding infinitesimal generator $\pi_\theta(x, \mathrm{d}y)$, $x \in E$, hence the law $\mathbb{P}_{x,\theta}$ of X, depends on a parameter $\theta \in \Theta$. We prove that several models (filtered or not) associated with X are linked, by their regularity according to a certain scheme. In particular, we show that the regularity of the model $(\pi_\theta(x, \mathrm{d}y))_{\theta\in\Theta}$ is equivalent to the local regularity of $(\mathbb{P}_{x,\theta})_{\theta\in\Theta}$.

Keywords: Fisher information matrix, Hellinger integrals, infinitesimal generator, isomorphism, jump Markov process, likelihood processes, local regularity, randomization, regularity of models

MSC 2020: 65C20, 62M20

1 Introduction and main results

Jump Markov processes, have found application in Bayesian statistics, chemistry, economics, information theory, finance, physics, population dynamics, speech processing, signal processing, statistical mechanics, traffic modeling, thermodynamics, and many others [1]. Regularity plays a significant role in the classical asymptotic statistics for parametric statistical models for jump Markov processes; see [2–4] for recent developments. Asymptotic normality or Bernstein-von Mises-type theorems impose several regularity conditions so that their results hold rigorously. In this article, we focus on the regularity conditions of several statistical models associated with a jump Markov process X with values E being an arbitrary space state, endowed with a σ-field $\mathcal{E}$. Let Ω be the canonical space of piecewise constant functions $\omega : \mathbb{R}_+ \longrightarrow E$, right continuous for the discrete topology. Let $X = (X_t)_{t\geqslant 0}$ be the canonical process, $(\mathcal{F}_t)_{t\geqslant 0}$ the canonical filtration, and $\mathcal{F} = \vee_{t\geq 0} \mathcal{F}_t$. Let $T_0 = 0$ and $(T_n)_{n\geq 0}$ be the sequence of the jump times of X, which are almost surely increasing to ∞. To each $\theta \in \Theta \subset \mathbb{R}^d$ and $x \in E$, we associate

$$\begin{cases} \mu_\theta : E \longrightarrow (0, \infty), & \text{an } \mathcal{E}\text{-measurable function}, \\ Q_\theta(x, \mathrm{d}y), & \text{a Markov kernel (also called a transition probability) on } E. \end{cases}$$

We assume that, under $\mathbb{P}_{x,\theta}$, the process $(X_t)_{t\geqslant 0}$ is Markovian, starts from $x \in E$, is non-exploding, and admits the infinitesimal generator

$$\pi_\theta(x, \mathrm{d}y) = \mu_\theta(x) Q_\theta(x, \mathrm{d}y).$$

The existence of the probabilities $\mathbb{P}_{x,\theta}$ is guaranteed by the boundedness of the function μ_θ for instance. The following facts clarify our focus on the different statistical models that will be presented later on:

- under $\mathbb{P}_{x,\theta}$, and conditionally to $\mathcal{F}_{T_{n-1}}$, the distribution of $T_n - T_{n-1}$ is exponential with parameter $\mu_\theta(X_{T_{n-1}})$;
- $Q_\theta(x, \mathrm{d}y) = \mathbb{P}_{x,\theta}(X_{T_1} \in \mathrm{d}y)$ is the transition probability of the embedded Markov chain $(X_{T_n})_{n\geq 0}$;

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- $\overline{Q}^k_\theta(x, \mathrm{d}y, \mathrm{d}t)$, $k \in \mathbb{N}$, is the distribution of (X_{T_k}, T_k) under $\mathbb{P}_{x,\theta}$;
- The associated sub-Markovian transition kernels $(P^\theta_t)_{t\geq 0}$ satisfy the *backward Kolmogorov* equations:

$$\frac{\partial}{\partial t}P^\theta_t(x, A) = \int_E (P^\theta_t(y, A) - P^\theta_t(x, A))\pi_\theta(x, \mathrm{d}y), \quad s, t \geq 0,\ x \in E,\ A \in \mathcal{E}. \tag{1}$$

The Markov process X is simple if $\pi_\theta(x, \mathrm{d}y)$ is a Markov kernel, i.e., for every $x \in E$, $\pi_\theta(x, \mathrm{d}y)$ is a probability measure on $(E, \mathcal{E})$. In this case, the transition functions are also Markov and satisfy the *Chapman-Kolmogorov* equation

$$P^\theta_{s+t}(x, A) = \int_E P^\theta_t(y, A)P^\theta_s(x, \mathrm{d}x), \quad x \in E,\ A \in \mathcal{E},$$

and

$$\mathbb{P}_{x,\theta}(X_{s+t} \in A|\mathcal{F}_t) = P^\theta_s(X_t, A), \quad \mathbb{P}_{x,\theta}\text{-almost surely},$$

cf. [5] for more account.
- The multivariate point process associated with the process $(X_t)_{t\geqslant 0}$ is

$$\lambda(\cdot, \mathrm{d}t, \mathrm{d}y) = \sum_{k\geq 1} \varepsilon_{(T_k, X_{T_k})}(\mathrm{d}t, \mathrm{d}y),$$

and its compensator, under $\mathbb{P}_{x,\theta}$, is

$$\nu^\theta(\cdot, \mathrm{d}t, \mathrm{d}y) = \pi_\theta(X_t, \mathrm{d}y)\mathbb{1}_{\mathbb{R}_+}(t)\mathrm{d}t.$$

Cf. Höpfner et al. [6] for instance. Note that in our study, we will not use the transition functions nor the multivariate point process and its compensator. In fact, we aim to show that the regularity of each model for the following statistical models is linked to the others, according to a certain scheme:

$$\begin{aligned}\mathcal{E}_x &= (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geqslant 0}, (\mathbb{P}_{x,\theta})_{\theta\in\Theta}) \\ &= \text{the filtered model associated with } (X_t)_{t\geqslant 0},\end{aligned} \tag{2}$$

$$\begin{aligned}\mathrm{E}_x &= (\mathrm{E}, \mathcal{E}, (\pi_\theta(x, \mathrm{d}y))_{\theta\in\Theta}) \\ &= \text{the model associated with the generator of } (X_t)_{t\geqslant 0},\end{aligned} \tag{3}$$

$$\begin{aligned}\mathrm{E}'_x &= (\mathrm{E}, \mathcal{E}, (Q_\theta(x, \mathrm{d}y))_{\theta\in\Theta}) \\ &= \text{the model associated with the observation of } X_{T_1},\end{aligned} \tag{4}$$

$$\begin{aligned}\mathrm{E}^k_x &= \left(\mathrm{E} \times \mathbb{R}_+, \mathcal{E} \otimes \mathcal{B}_{\mathbb{R}_+}, (\overline{Q}^k_\theta(x, \mathrm{d}y, \mathrm{d}t))_{\theta\in\Theta}\right) \\ &= \text{the model associated with the observation of } \left(X_{T_k}, T_k\right).\end{aligned} \tag{5}$$

The model E_x is not a proper statistical model since $(\pi_\theta(x, \mathrm{d}y))_{\theta\in\Theta}$ is not a probability measure. Nevertheless, the extension of the notion of regularity to models associated with families of finite positive measures is also feasible and is described as follows. Let $(R_\theta)_{\theta\in\Theta}$ be a family of finite positive measures in $(E, \mathcal{E})$. For $\theta, \xi \in \Theta$, we denote by $\Pi^{\theta,\xi}$ a measure that dominates R_0, R_θ, and R_ξ, and by $z^{\theta,\xi}$, $z^{\xi,\theta}$, and $Z^{\theta,\xi}$, be Radon-Nikodym derivatives, respectively, of R_θ and R_ξ according to $\Pi^{\theta,\xi}$ and of R_θ according to R_ξ. The Lebesgue decomposition of R_θ, with respect to R_ξ, is given by the pair $(N^{\theta,\xi}, Z^{\theta,\xi})$,

$$N^{\theta,\xi} = \{u / z^{\xi,\theta}(u) = 0\}, \quad Z^{\theta,\xi} = \begin{cases} \dfrac{z^{\theta,\xi}}{z^{\xi,\theta}}, & \text{outside } N^{\theta,\xi}, \\ 0, & \text{on } N^{\theta,\xi}. \end{cases}$$

We start by recalling the notion of "*error functions*" which was introduced in [6] as follows.

Definition 1. A function $f : [0, \infty) \to [0, \infty)$ is called an error function, if $\lim_{u\searrow 0} f(u) = 0$. More generally, an error function is any positive function $f : E \times (0, \infty) \to [0, \infty)$, such that

$$f(\cdot, u) \text{ is } \mathcal{E}\text{-measurable}, \quad \forall u > 0, \quad \text{and} \quad \lim_{u \searrow 0} f(x, u) = 0, \ \forall x \in E.$$

Definition 2. (Regularity of non-filtered models). The model $(E, \mathcal{E}, (R_\theta)_{\theta\in\Theta})$ is regular at $\theta = 0$, if the random function

$$\begin{aligned} \Theta &\longrightarrow \mathbb{L}^2(R_0) \\ \theta &\longmapsto \sqrt{Z^{\theta,0}}, \end{aligned}$$

is differentiable at $\theta = 0$, i.e., there exists a random vector $V = (V_i)_{1\le i\le d}$ and an error function $f : [0, \infty) \to [0, \infty)$, such that

$$R_\theta(N^{\theta,0}) + \left\| \sqrt{Z^{\theta,0}} - 1 - \frac{1}{2} V \cdot \theta \right\|^2_{\mathbb{L}^2(R_0)} \le |\theta|^2 f(|\theta|). \tag{6}$$

Note that if the regularity of the model $(E, \mathcal{E}, (R_\theta)_{\theta\in\Theta})$ holds, then V is necessarily (R_0)-square integrable. Furthermore, if $(R_\theta)_{\theta\in\Theta}$ is a family of probability measures, then $\mathbb{E}_{R_0}(V) = 0$. The *Hellinger integral* of order $\frac{1}{2}$ between the measures R_θ and R_ξ,, is defined by

$$H^{\theta,\xi} := \Pi^{\theta,\xi}(\sqrt{z^{\theta,\xi} z^{\xi,\theta}})$$

and is independent of the dominating measure $\Pi^{\theta,\xi}$. The regularity of the model $(E, \mathcal{E}, (R_\theta)_{\theta\in\Theta})$ is equivalent to one of these two assertions:

(i) There exist an error function f_1 and a random vector V (the same as before), such that

$$\left\| \sqrt{z^{\theta,0}} - \sqrt{z^{0,\theta}} - \frac{1}{2}\sqrt{z^{0,\theta}} V \cdot \theta \right\|_{\mathbb{L}^2(\Pi^{\theta,0})} \le |\theta| f_1(|\theta|). \tag{7}$$

(ii) There exists an error function f_2 and a matrix $I = [I^{ij}]_{1\le i,j\le d}$, such that

$$\left| H^{0,0} + H^{\theta,\xi} - H^{0,\theta} - H^{0,\xi} - \frac{1}{4}\theta \cdot I \cdot \xi \right| \le |\theta||\xi| f_2(|\theta| \vee |\xi|).$$

The matrix I is positive definite and is called the *Fisher information matrix* of the model at $\theta = 0$. It is linked to the vector V by

$$I^{ij} = R_0(V^i V^j),$$

cf. [6,7].

Let $(\Omega, \mathcal{F})$ be a sample space endowed with a filtration $(\mathcal{F}_t)_{t\geqslant 0}$, and a family of probability measures $(\mathbb{P}_\theta)_{\theta\in\Theta}$ coinciding on $\mathcal{F}_0$. The regularity of the statistical filtered model

$$(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geqslant 0}, (\mathbb{P}_\theta)_{\theta\in\Theta}) \tag{8}$$

mimics the one in Definition 2 and is expressed in terms of likelihood processes [8,7]. For a clear presentation, we need to introduce the *likelihood process* of $\mathbb{P}_\theta$ with respect to $\mathbb{P}_\xi$, $\theta, \xi \in \Theta$, defined in Jacod and Shiryaev's book [9], by

$$Z_t^{\theta,\xi} = \frac{d\mathbb{P}_\theta|\mathcal{F}_t}{d\mathbb{P}_\xi|\mathcal{F}_t}, \quad t \ge 0.$$

The process $Z_t^{\theta,\xi}$ is a positive $(\mathbb{P}_\xi, \mathcal{F}_t)$-supermartingale and is a martingale if

$$\mathbb{P}_\theta \overset{loc}{\ll} \mathbb{P}_\xi, \ \text{(i.e. if } \mathbb{P}_\theta|\mathcal{F}_t \ll \mathbb{P}_\xi|\mathcal{F}_t, \quad \forall t \ge 0).$$

For any probability measure $\mathbb{K}^{\theta,\xi}$, locally dominating $\mathbb{P}_\theta$ and $\mathbb{P}_\xi$, the $(\mathbb{K}^{\theta,\xi}, \mathcal{F}_t)$-martingales

$$z_t^{\theta,\xi} = \frac{d\mathbb{P}_\theta|\mathcal{F}_t}{d\mathbb{K}^{\theta,\xi}|\mathcal{F}_t}, \quad z_t^{\xi,\theta} = \frac{d\mathbb{P}_\xi|\mathcal{F}_t}{d\mathbb{K}^{\theta,\xi}|\mathcal{F}_t}$$

and the stopping times

$$\tau^{\theta,\xi} = \inf\{t \geq 0 \text{ s.t. } z_t^{\theta,\xi} = 0\}, \quad \tau^{\xi,\theta} = \inf\{t \geq 0 \text{ s.t. } z_t^{\xi,\theta} = 0\}$$

provide this version of $Z^{\theta,\xi}$:

$$Z_t^{\theta,\xi} = \begin{cases} \frac{z_t^{\theta,\xi}}{z_t^{\xi,\theta}}, & \text{if } t < \tau^{\theta,\xi} \wedge \tau^{\xi,\theta} \\ 0, & \text{if } t \geq \tau^{\theta} \wedge \tau^{\xi}. \end{cases}$$

As for non-filtered models, we have the following definition.

Definition 3. (Regularity of filtered models). Let T be a stopping time relative to $(\mathcal{F}_t)_{t\geqslant 0}$. The model $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geqslant 0}, (\mathbb{P}_\theta)_{\theta\in\Theta})$ is said to be regular (or differentiable) at time T and at $\theta = 0$, if the model $(\Omega, \mathcal{F}_T, (\mathbb{P}_\theta)_{\theta\in\Theta})$ is regular in the sense of Definition 2. That means that there exists an $\mathcal{F}_T$-measurable, $\mathbb{P}_0$-square-integrable, and centered random vector $V_T = [V_T^i]_{1\leq i\leq d}$ and two error functions $f_{1,T}, f_{2,T}$, such that

$$\mathbb{E}_{\mathbb{P}_0}[1 - Z_T^{\theta,0}] \leq |\theta|^2 f_{1,T}(|\theta|) \tag{9}$$

and

$$\mathbb{E}_{\mathbb{P}_0}\left[\left(\sqrt{Z_T^{\theta,0}} - 1 - \frac{1}{2}\theta \cdot V_T\right)^2\right] \leq |\theta|^2 f_{2,T}(|\theta|). \tag{10}$$

As in Definition 2 and according to [7, point 3.12], the regularity of the model is equivalent to the existence of a positive definite $d \times d$ matrix J_T and of an error function f_T, such that

$$\left| H_T^{0,0} + H_T^{\theta,\xi} - H_T^{0,\theta} - H_T^{0,\xi} - \frac{1}{4}\theta \cdot J_T \cdot \xi \right| \leq |\theta||\xi| f_T(|\theta| \vee |\xi|),$$

where

$$H_T^{\theta,\xi} = \mathbb{E}_{\mathbb{K}^{\theta,\xi}}[\sqrt{z_T^{\theta,\xi} z_T^{\xi,\theta}}] = \mathbb{E}_{\mathbb{P}_0}[\sqrt{Z_T^{\theta,\xi} Z_T^{\xi,\theta}}\, \mathbb{1}_{(T<\tau^\theta\wedge\tau^\xi)}]$$

is the Hellinger integral of order $\frac{1}{2}$, at time T, and which is independent of the choice of the dominating probability measure. The Fisher information matrix of the model is then

$$J_T = [\mathbb{E}_{\mathbb{P}_0}[V_T^i V_T^j]]_{1\leq i,j\leq d}.$$

It is worth noting that if the regularity at a time T implies the regularity at any stopping time $S \leq T$. In particular, if the regularity holds along a sequence S_p, $p \in \mathbb{N}$, increasing to infinity, then there exists a local martingale $(V_t)_{t\geq 0}$, locally square-integrable, null at zero, such that if $T \leq S_p$ for some p, then V_T is a version of the random variable in (10). In particular, if (9) and (10) are satisfied for all $t \geq 0$, then $(V_t)_{t\geq 0}$ is a square-integrable martingale, null at 0, cf. [7, Corollary 3.16].

We are now able to introduce the notion of local regularity, which is less restrictive than the preceding one.

Definition 4. (Local regularity of filtered models). A sequence $(S_p)_{p\in\mathbb{N}}$ of stopping times is called a *localizing sequence* if it is $\mathbb{P}_0$-almost surely increasing to ∞. A localizing family is a sequence formed by the pair $(S_p, S_{n,p})_{p\in\mathbb{N},n\geq 1}$, where $(S_p)_{p\in\mathbb{N}}$ is a localizing sequence and $(S_{n,p})_{n\geq 1}$ is a sequence of stopping times, satisfying

$$S_{n,p} \leq S_p \quad \text{and} \quad \lim_{n\to\infty} \mathbb{P}_0(S_{n,p} < S_p) = 0. \tag{11}$$

The model (8) is said to be locally regular (or locally differentiable at $\theta = 0$), if there exists a right continuous, left limited process $(V_t)_{t\geq 0}$ on $\mathbb{R}^d$, such that, for all (θ_n, θ) satisfying

$$\lim_{n\to\infty} \theta_n = 0 \quad \text{and} \quad \lim_{n\to\infty} \frac{\theta_n}{|\theta_n|} = \theta, \tag{12}$$

there exists a localizing family $(S_p, S_{n,p})_{p\in\mathbb{N},n\geq 1}$, satisfying

$$\lim_{n\to\infty} \frac{\mathbb{E}_{\mathbb{P}_0}[1 - Z^{\theta_n,0}_{S_{n,p}}]}{|\theta_n|^2} = 0, \quad \forall p \in \mathbb{N}, \tag{13}$$

and

$$\frac{\sqrt{Z^{\theta_n,0}_{t\wedge S_{n,p}}} - 1}{|\theta_n|} \xrightarrow{\mathbb{L}^2(\mathbb{P}_0)} \frac{1}{2}\theta \cdot V_{t\wedge S_p}, \quad \text{as } n \to +\infty, \quad \forall p \in \mathbb{N}, \ \forall t \geq 0. \tag{14}$$

Note that if the model is regular along a localizing sequence, then it is necessarily locally regular. By Theorem [7, Theorem 4.6], the process $(V_t)_{t\geq 0}$ is a locally square-integrable $(\mathbb{P}_0, \mathcal{F}_t)$-local martingale and the Fisher information process $(I_t)_{t\geq 0}$, at $\theta = 0$, is defined as the predicable quadratic covariation of $(V_t)_{t\geq 0}$:

$$I_t := [I^{ij}_t]_{1\leq i,j\leq d} = [\langle V^i, V^j\rangle_t]_{1\leq i,j\leq d}.$$

The local regularity does guarantee the integrability of I; however, it is the minimal condition we require to obtain the property of *local asymptotic normality* (LAN) for statistical models. In this case, the Fisher information quantities provide the lower bound of the variance of any estimator of the unknown parameters intervening in the models, see [10–12] for instance.

According to [7, Theorem 6.2], the local regularity is equivalent to the two following conditions:

$$\frac{1}{|\theta||\xi|}\operatorname{Var}\left\{h^{0,0} + h^{\theta,\xi} - h^{0,\theta} - h^{0,\xi} - \frac{1}{4}\theta \cdot I \cdot \xi\right\}_t \xrightarrow{\mathbb{P}_0} 0, \quad \text{as } \theta, \xi \to 0, \quad \forall t \geq 0, \tag{15}$$

and for all $t \geq 0$,

$$\frac{A^\theta_t}{|\theta|^2} \xrightarrow{\mathbb{P}_0} 0, \quad \text{as } \theta \to 0, \tag{16}$$

where

- $(\operatorname{Var}\{.\}_t)_{t\geq 0}$ is the variation process of $\{.\}$.
- $(h^{\theta,\xi}_t)_{t\geq 0}$ is a version of the Hellinger process of order $\frac{1}{2}$ between $\mathbb{P}_\theta$ and $\mathbb{P}_\xi$, i.e., is a predictable nondecreasing process, null at zero, such that

$$\sqrt{z^\theta z^\xi} + [\sqrt{z^\theta_- z^\xi_-}\,] \circ h^{\theta,\xi} \ \text{ is a } (\mathbb{K}^{\theta,\xi}, \mathcal{F}_t)\text{-martingale}. \tag{17}$$

- $(A^\theta_t)_{t\geq 0}$ is the predictable nondecreasing process intervening in the Doob-Meyer decomposition of the supermartingale $(Z^\theta_t)_{t\geq 0}$. Since $\mathbb{P}_\theta$ and $\mathbb{P}_0$ coincide on $\mathcal{F}_0$, then necessarily $Z^\theta_0 = 1$ and there exists a $(\mathbb{P}_0, \mathcal{F}_t)$-local martingale $(M^\theta_t)_{t\geq 0}$ such that

$$Z^\theta = 1 + M^\theta - A^\theta.$$

The results that we obtain complete those of Höpfner et al. [6], who proved that if $(\pi_\theta(x, dy))_{\theta\in\Theta}$ is regular and if the process X satisfies a condition of positive recurrence (resp. null recurrence), then the model $(\mathbb{P}_{x,\theta})_{\theta\in\Theta}$ localized around the parameter $\theta = 0$ is or locally asymptotically normal or is locally asymptotically mixed normal. The main result is as follows.

Theorem 5. *The model $\mathcal{E}_x$ is locally regular for all $x \in E$, if, and only if, E_y is regular for all $y \in E$.*

Models (2)–(5) are described in depth in Section 2. We also provide a full scheme linking them by their regularity, see Theorems 6–8 and 10. The proofs are given in Section 3.

2 Additional regularity properties

Our notations and the calculus of the Hellinger integrals and of the likelihood processes are borrowed from Höpfner [13] and Höpfner et al. [6]. For $x \in E$ and $\theta, \xi \in \theta$, the following measures will be used in the sequel.

(a) $\Pi_x^{\theta,\xi}(\mathrm{d}y)$ is a measure dominating $\pi_\theta(x,\mathrm{d}y)$, $\pi_\xi(x,\mathrm{d}y)$, and $\pi_0(x,\mathrm{d}y)$. Thus, $\Pi_x^{\theta,\xi}(\mathrm{d}y)$ also dominates $Q_\theta(x,\mathrm{d}y)$, $Q_\xi(x,\mathrm{d}y)$, and $Q_0(x,\mathrm{d}y)$;
(b) $Q_x^{\theta,\xi}(\mathrm{d}y,\mathrm{d}t)$ is a transition probability on $E\times\mathbb{R}_+$ dominating $\bar{Q}_\theta(x,\mathrm{d}y,\mathrm{d}t)$, $\bar{Q}_\xi(x,\mathrm{d}y,\mathrm{d}t)$, and $\bar{Q}_0(x,\mathrm{d}y,\mathrm{d}t)$;
(c) $\mathbb{K}_x^{\theta,\xi}$ is a probability measure, locally dominating $\mathbb{P}_{x,\theta}$, $\mathbb{P}_{x\xi}$, and $\mathbb{P}_{x,0}$;
(d) $\Pi_x^\theta(\mathrm{d}y)=\Pi_x^{\theta,0}(\mathrm{d}y)$, $\quad Q_x^\theta(\mathrm{d}y,\mathrm{d}t)=\mathcal{P}_x^{\theta,0}(\mathrm{d}y,\mathrm{d}t)$, and $\mathbb{K}_x^\theta=\mathbb{K}_x^{\theta,0}$.

The Radon-Nikodym derivatives are denoted by

$$\begin{aligned}
&\chi_{\theta,\xi}(x,\cdot)=\frac{\mathrm{d}\pi_\theta(x,\cdot)}{\mathrm{d}\Pi_x^{\theta,\xi}(\cdot)}, && \rho_{\theta,\xi}(x,\cdot)=\frac{\mathrm{d}Q_\theta(x,\cdot)}{\mathrm{d}\Pi_x^{\theta,\xi}(\cdot)}, && \rho^1_{\theta,\xi}(x,\cdot,\cdot)=\frac{\mathrm{d}\bar{Q}_\theta(x,\cdot,\cdot)}{\mathrm{d}Q_x^{\theta,\xi}(\cdot,\cdot)}\\
&\chi_\theta(x,\cdot)=\chi_{\theta,0}(x,\cdot), && \rho_\theta(x,\cdot)=\rho_{\theta,0}(x,\cdot), && \rho^1_\theta(x,\cdot,\cdot)=\rho^1_{\theta,0}(x,\cdot,\cdot)\\
&\chi_0(x,\cdot)=\chi_{0,\theta}(x,\cdot), && rho_0(x,\cdot)=\rho_{0,\theta}(x,\cdot), && \rho^1_0(x,\cdot,\cdot)=\rho^1_{0,\theta}(x,\cdot,\cdot).
\end{aligned}$$

If we choose

$$\Pi_x^{\theta,\xi}(\mathrm{d}y)=\pi_\theta(x,\mathrm{d}y)+\pi_\xi(x,\mathrm{d}y)+\pi_0(x,\mathrm{d}y),$$

and if $\mathbb{K}_x^{\theta,\xi}$ is the probability under which the canonical process $(X_t)_{t\geqslant 0}$, starts from x, and has the infinitesimal generator $\Pi_x^{\theta,\xi}(\mathrm{d}y)$, then we have

$$\mathbb{P}_{x,0}\overset{loc}{\ll}\mathbb{K}_x^{\theta,\xi},\quad \mathbb{P}_{x,\theta}\overset{loc}{\ll}\mathbb{K}_x^{\theta,\xi}\quad\text{and}\quad \mathbb{P}_{x,0}\overset{loc}{\ll}\mathbb{K}_x^{\theta,\xi}.$$

With the convention $\prod_{j=1}^0=1$, a version of the likelihood processes of $\mathbb{P}_{x,\theta}$, with respect to $\mathbb{K}_x^{\theta,\xi}$ and to $(\mathcal{F}_t)_{t\geqslant 0}$, is given in [6] by

$$z_t^{\theta,\xi}=\frac{\mathrm{d}\mathbb{P}_{x,\theta}|\mathcal{F}_t}{\mathrm{d}\mathbb{K}_x^{\theta,\xi}|\mathcal{F}_t}=\left\{\prod_{j\geq 1:\,T_j\leq t}\chi_{\theta,\xi}\big(X_{T_{j-1}},X_{T_j}\big)\right\}\exp\int_0^t\int_E(1-\chi_{\theta,\xi})(X_s,y)\Pi_{X_s}^{\xi,\theta}(\mathrm{d}y)\mathrm{d}s.$$

With the notations

$$z_t^\theta\coloneqq z_t^{\theta,0},\quad z_t^0\coloneqq z_t^{0,\theta},\quad\text{and}\quad \tau^\theta\coloneqq\inf\{t\geq 0/z_t^\theta=0\},$$

a version of the likelihood processes of $\mathbb{P}_{x,\theta}$, relative to $\mathbb{P}_{x,0}$ and $(\mathcal{F}_t)_{t\geqslant 0}$, is explicitly given by

$$Z_t^\theta=\frac{z_t^\theta}{z_t^0}=\begin{cases}\exp\left(\int_0^t(\mu_0-\mu_\theta)(X_s)\mathrm{d}s\right)\prod_{j\geq 1:\,T_j\leq t}\frac{\chi_\theta}{\chi_0}\big(X_{T_{j-1}},X_{T_j}\big), & \text{if } t<\tau^0\wedge\tau^\theta\\ 0, & \text{if } t\geq\tau^0\wedge\tau^\theta.\end{cases}$$

The Hellinger integral of order $\frac{1}{2}$ between $\pi_\theta(x,\mathrm{d}y)$ and $\pi_\xi(x,\mathrm{d}y)$ is then

$$H^{\theta,\xi}(x)=\int_E\sqrt{\chi_{\theta,\xi}(x,y)\chi_{\xi,\theta}(x,y)}\,\Pi_x^{\theta,\xi}(\mathrm{d}y),$$

and the Hellinger integral of order $\frac{1}{2}$ at time t between $\mathbb{P}_{x,\theta}$ and $\mathbb{P}_{x,\xi}$ relative to the filtration $(\mathcal{F}_t)_{t\geqslant 0}$, is expressed by

$$H_t^{\theta,\xi}(x)=\mathbb{E}_{\mathbb{K}_x^{\theta,\xi}}\big[\sqrt{z_t^{\theta,\xi}z_t^{\xi,\theta}}\,\big].\tag{18}$$

We also consider the quantities

$$\bar{H}^{\theta,\xi}(x)=\frac{\mu_\theta(x)+\mu_\xi(x)}{2}-H^{\theta,\xi}(x),$$

which are used to define the Hellinger process $(h_t^{\theta,\xi})_{t\geq 0}$, of order $\frac{1}{2}$, between $\mathbb{P}_{x,\theta}$ and $\mathbb{P}_{x,\xi}$, and relative to $(\mathcal{F}_t)_{t\geqslant 0}$. It is expressed by

$$h_t^{\theta,\xi} = \int_0^t \bar{H}^{\theta,\xi}(X_s)\mathrm{d}s. \tag{19}$$

Finally, we define the function

$$g(x,\theta,\xi) \coloneqq \frac{1}{|\theta||\xi|}\left(\bar{H}^{0,\theta}(x) + \bar{H}^{0,\xi}(x) - \bar{H}^{\theta,\xi}(x) - \frac{1}{4}\theta\cdot I(x)\cdot\xi\right), \tag{20}$$

where $I(x)$ is the Fisher information matrix of the model E_x at $\theta = 0$, whenever it is regular. Consequently, $\bar{H}^{0,\theta}(x)$ is expressed by

$$\bar{H}^{0,\theta}(x) = \frac{1}{8}\theta\cdot I(x)\cdot\theta + \frac{1}{2}|\theta|^2 g(x,\theta,\theta). \tag{21}$$

Observe that the function g in (20) is such that the function

$$f(x,u) \coloneqq \sup_{|\theta|,|\xi|\le u} |g(x,\theta,\xi)|, \quad x \in E,$$

is nondecreasing in u, satisfies $|g(x,\theta,\xi)| \le f(x,|\theta|\vee|\xi|)$. Thus, f has the vocation to be an error function.

We can now state a first technical but intuitive result.

Theorem 6. *Let $x \in E$. Then the following assertions are equivalent.*
(1) *E_x is regular;*
(2) *E_x' is regular and $\underset{\cdot}{\mu}(x)$ is differentiable at $\theta = 0$;*
(3) *E_x^1 is regular;*
(4) *$\mathcal{E}_x$ is regular at time T_1.*

In the three following theorems, we complete our results by studying the regularity of the filtered model $\mathcal{E}_x$, at fixed times $t > 0$, or at the jump times T_k, $k \in \mathbb{N}$. In this direction, we obtain only partial results appealing to some additional conditions of integrability.

Theorem 7. *Let $t > 0$. For all $x \in E$, assume the following.*

Condition $\mathbf{A}_t(x)$. There exists $u_t > 0$, an error function f_1 and a measurable function $f_2 : E \to [0,\infty)$, satisfying the following:

$$\bar{H}^{\theta,\xi}(x) \le |\theta - \xi|^2 f_2(x), \quad \text{if } |\theta|, |\xi| \le u_t,$$

and

$$\int_0^t \mathbb{E}_{\mathbb{K}_x^\theta}[f_1(X_s,u_t)^2]\mathrm{d}s < +\infty, \quad \int_0^t \mathbb{E}_{\mathbb{K}_x^{\theta,\xi}}[f_2(X_s)^2]\mathrm{d}s < +\infty.$$

Then, $\mathcal{E}_y$ is regular at the time t, for all $y \in E$.

Theorem 8. *For all $x \in E$, assume the following. The model E_x is regular, and*

Condition $\mathbf{B}(x)$. The error function f in (7), associated with the model E_x^1, satisfies the following. There exists $r > 0$ such that

$$Q_x^\theta[f(\cdot,r)](x) = \int_{E\times\mathbb{R}_+} f(y,r)Q_x^\theta(x,\mathrm{d}y,\mathrm{d}t) < +\infty, \quad \text{if } |\theta| \le r.$$

Then, the model E_y^k is regular for all $y \in$ E, and all $k \in \mathbb{N}$.

Remark 9.
(i) The control in the first integral in condition $\mathbf{A}_t(x)$ is exactly the required condition for E_x to be regular. The finiteness of the second integral will ensure integrability conditions in the proof of Theorem 7.

(ii) Equivalently, we could replace the error function in condition $\mathbf{B}(x)$ by the one in (6). The integrability condition becomes

$$\bar{Q}_0[f(\cdot,r)](x) = \int_{E\times\mathbb{R}_+} f(y,r)\bar{Q}_0(x,\mathrm{d}y,\mathrm{d}t) < +\infty,$$

and the only difference is that we would have to check two inequalities instead of one.

We conclude with our last result.

Theorem 10. 1. *Let $x \in E$. If $\mathcal{E}_x$ is regular at a time $t > 0$, then E_x is regular.*
2. *Furthermore, if $\mathcal{E}_x$ is regular at a time $t > 0$, for all $x \in E$, then, $\mathcal{E}_y$ is locally regular, for all $y \in E$.*

3 Proofs of the theorems

We will sometimes use the notion of *isomorphism* between two statistical models. Referring to Strasser's book [14], we say that two models $\mathcal{G} = (A, \mathcal{A}, (\mathbb{P}_\theta)_{\theta\in\Theta})$ and $\mathcal{H} = (B, \mathcal{B}, (Q_\theta)_{\theta\in\Theta})$ are *isomorphic* if they are *randomized* of each other. To illustrate this notion, assume for instance that $\mathcal{G}$ and $\mathcal{H}$ are, respectively, dominated by P and Q. Then, the model $\mathcal{H}$ is randomized from $\mathcal{G}$, if there exists a Markovian operator $M : L^\infty(A, \mathcal{A}, P_\theta) \to L^\infty(B, \mathcal{B}, Q_\theta)$, such that

$$\frac{\mathrm{d}Q_\theta}{\mathrm{d}Q} = M\frac{\mathrm{d}P_\theta}{\mathrm{d}P}, \quad \forall\theta \in \Theta.$$

The models $\mathcal{G}$ and $\mathcal{H}$ are randomized of each other if they are mutually exhaustive, which is always the case in our study, each time an isomorphism holds, cf. [14, Lemma 23.5 and Theorem 24.11]. When computing expectations, these isomorphisms allow us to handle at our convenience, one of the two likelihoods of the models $\mathcal{G}$ and $\mathcal{H}$. The latter is justified by the fact that they have the same law under the respective probability quotient.

Proof of Theorem 6. (1) ⇒ (2): (a) By (7), the regularity of E_x at $\theta = 0$ is equivalent to the existence of a random vector $V(x,\cdot) \in \mathrm{L}^2(\pi_0(x,\mathrm{d}y))$, and of an error function f_χ, such that

$$h(x,\theta) := \int_E \Big(\sqrt{\chi_\theta(x,y)} - \sqrt{\chi_0(x,y)} - \frac{1}{2}\sqrt{\chi_0(x,y)}\,\theta\cdot V(x,y)\Big)^2 \Pi_x^\theta(\mathrm{d}y) \le |\theta|^2 f_\chi(x,|\theta|). \tag{22}$$

The latter implies

$$\begin{aligned}\int_E (\sqrt{\chi_\theta(x,y)} - \sqrt{\chi_0(x,y)})^2 \Pi_x^\theta(\mathrm{d}y) &\le |\theta|^2\left\{2f_\chi(x,|\theta|) + \frac{1}{2}\int_E |V(x,z)|^2\pi_0(x,\mathrm{d}z)\right\}\\ &= |\theta| f'_\chi(x,|\theta|) \quad\text{and}\quad f'_\chi \text{ is an error function}.\end{aligned} \tag{23}$$

(b) The implication "E_x *is regular* at $\theta = 0 \Rightarrow \underset{\cdot}{\mu}(x)$ *is differentiable at* $\theta = 0$" is shown in [6], using the fact that the differentiability of $\sqrt{\chi_\theta(x,\cdot)}$ in L^2 implies the differentiability of $\chi_\theta(x,\cdot)$ in L^1. Furthermore, the derivative at $\theta = 0$ of $\underset{\cdot}{\mu}(x)$ is

$$\int_E V(x,z)\pi_0(x,\mathrm{d}z) = \mu_0(x)\int_E V(x,z)Q_0(x,\mathrm{d}z),$$

which gives,

$$\sqrt{\frac{\mu_0(x)}{\mu_\theta(x)}} = 1 - \frac{1}{2}\theta\cdot\int_E V(x,z)Q_0(x,\mathrm{d}z) + \theta\cdot F_\mu(x,\theta), \tag{24}$$

where

$$f_\mu(x,u) := \sup_{|\theta|\le u} |F_\mu(x,\theta)| \text{ is an error function}$$

(c) Let us define

$$h'(x,\theta) := \int_E \Big(\sqrt{\rho_\theta(x,y)} - \sqrt{\rho_0(x,y)} - \frac{1}{2}\sqrt{\rho_0(x,y)}\,\theta\cdot V'(x,y)\Big)^2 \Pi_x^\theta(\mathrm{d}y),$$

where the function

$$V'(x,y) := V(x,y) - \int_E V(x,z)Q_0(x,\mathrm{d}z) \in \mathrm{L}^2(Q_0(x,\mathrm{d}y)) \tag{25}$$

satisfies

$$\int_E V'(x,y)Q_0(x,\mathrm{d}y) = 0.$$

Then, we can write

$$h'(x,\theta) = \int_E \Bigg(\sqrt{\frac{\chi_\theta(x,y)}{\mu_\theta(x)}} - \sqrt{\frac{\chi_0(x,y)}{\mu_0(x)}} - \frac{1}{2}\sqrt{\frac{\chi_0(x,y)}{\mu_0(x)}}\,\theta\cdot V'(x,y)\Bigg)^2 \Pi_x^\theta(\mathrm{d}y),$$

and use (24) and (25) to obtain

$$\begin{aligned}
h'(x,\theta) = \frac{1}{\mu_0(x)} \int_{\mathrm{E}} \Bigg[& \sqrt{\chi_\theta(x,y)} - \sqrt{\chi_0(x,y)} - \frac{1}{2}\sqrt{\chi_0(x,y)}\,\theta\cdot V(x,y) \\
& - \frac{1}{2}\Big\{\theta\cdot \int_E V(x,z)Q_0(x,\mathrm{d}z)\Big\}\{\sqrt{\chi_\theta(x,y)} - \sqrt{\chi_0(x,y)}\} + \theta\cdot F_\mu(x,\theta)\sqrt{\chi_\theta(x,y)}\Bigg]^2 \Pi_x^\theta(\mathrm{d}y) \\
\le \frac{3}{\mu_0(x)} \int_E \Bigg[& \Big|\sqrt{\chi_\theta(x,y)} - \sqrt{\chi_0(x,y)} - \frac{1}{2}\sqrt{\chi_0(x,y)}\,\theta\cdot V(x,y)\Big|^2 + \frac{1}{4}\Big|\theta\cdot\int_E V(x,z)Q_0(x,\mathrm{d}z)\Big|^2 \\
& \times |\sqrt{\chi_\theta(x,y)} - \sqrt{\chi_0(x,y)}|^2 + |\theta|^2(f_\mu(x,|\theta|))^2\chi_\theta(x,y)\Bigg]\Pi_x^\theta(\mathrm{d}y).
\end{aligned}$$

Finally, according to (22) and (23), we have

$$\begin{aligned}
h'(x,\theta) &\le \frac{3}{\mu_0(x)}\Bigg[|\theta|^2 f_\chi(x,|\theta|) + \frac{1}{4}\Big|\theta\cdot\int_E V(x,z)Q_0(x,\mathrm{d}z)\Big|^2 f'_\chi(x,|\theta|) + \mu_\theta(x)|\theta|^2(f_\mu(x,|\theta|))^2\Bigg] \\
&:= |\theta|^2 f_\rho(x,|\theta|), \quad \text{where } f_\rho \text{ is an error function.}
\end{aligned}$$

(2) ⇒ (1): (a) Under the condition of differentiability of $\mu(x)$ at 0, we obtain

$$\sqrt{\mu_\theta(x)} = \sqrt{\mu_0(x)}\Bigg[1 + \frac{1}{2}\theta\cdot\frac{\mu_0'(x)}{\mu_0(x)}\Bigg] + \theta\cdot F'_\mu(x,\theta), \tag{26}$$

where

$$f'_\mu(x,u) = \sup\nolimits_{|\theta|\le u}|F'_\mu(x,\theta)| \text{ is an error function.}$$

(b) According to (7), the regularity of E'_x, at $\theta = 0$, is equivalent to the existence of a centered vector $V'(x,\cdot) \in L^2(Q_0(x, dy))$, and of an error function f_ρ such that

$$h'(x,\theta) = \int_E \left(\sqrt{\rho_\theta(x,y)} - \sqrt{\rho_0(x,y)} - \frac{1}{2}\sqrt{\rho_0(x,y)}\,\theta\cdot V'(x,y)\right)^2 \Pi_x^\theta(dy) \le |\theta|^2 f_\rho(x,|\theta|).$$

The vector $V(x,\cdot)$ is defined by

$$V(x,y) := V'(x,y) + \frac{\mu_0'(x)}{\mu_0(x)},$$

which belongs to $L^2(\pi_0(x, dy))$ and satisfies

$$\mu_0'(x) := \int_E V(x,z)\pi_0(x, dz).$$

(c) Let us define

$$h(x,\theta) = \int_E \left(\sqrt{\chi_\theta(x,y)} - \sqrt{\chi_0(x,y)} - \frac{1}{2}\sqrt{\chi_0(x,y)}\,\theta\cdot V(x,y)\right)^2 \Pi_x^\theta(dy). \tag{27}$$

By (26), we have

$$\begin{aligned} h(x,\theta) = \int_E \Bigg[&\left\{1 + \frac{1}{2}\theta\cdot\int_E V(x,z)Q_0(x,dz) + \theta\cdot F'_\mu(x,\theta)\right\}\sqrt{\mu_0(x)\rho_\theta(x,y)} - \sqrt{\mu_0(x)\rho_0(x,y)} \\ &- \frac{1}{2}\sqrt{\mu_0(x)\rho_0(x,y)}\,\theta\cdot V(x,y)\Bigg]^2 \Pi_x^\theta(dy) \\ = \mu_0(x)\int_E \Bigg[&\sqrt{\rho_\theta(x,y)} - \sqrt{\rho_0(x,y)} - \frac{1}{2}\sqrt{\rho_0(x,y)}\,\theta\cdot V'(x,y) \\ &+ \frac{1}{2}\theta\cdot\int_E V(x,z)Q_0(x,dz)\left\{\sqrt{\rho_\theta(x,y)} - \sqrt{\rho_0(x,y)}\right\} + \theta\cdot F'_\mu(x,\theta)\sqrt{\rho_\theta(x,y)}\Bigg]^2 \Pi_x^\theta(dy). \end{aligned}$$

With the same arguments as in (1) ⇒ (2) (c), we retrieve

$$\begin{aligned} h(x,\theta) \le 3\mu_0(x)\Bigg[&|\theta|^2 f_\rho(x,|\theta|) + \frac{1}{4}\left\{\theta\cdot\int_E V(x,z)Q_0(x,dz)\right\}^2 \times \left\{2|\theta|^2 f_\rho(x,|\theta|) + \frac{1}{2}\int_E |\theta\cdot V(x,z)|^2 Q_0(x,dz)\right\} \\ &+ \frac{|\theta|^2}{\mu_\theta(x)} f'_\mu(x,|\theta|)\Bigg]. \end{aligned}$$

Consequently, there exists an error function f_χ such that

$$h(x,\theta) \le |\theta|^2 f_\chi(x,|\theta|).$$

(2) ⇒ (3): Let $x \in E$. Since

$$\bar{Q}_\theta(x, dy, dt) = Q_\theta(x, dy)\mu_\theta(x)e^{-\mu_\theta(x)t}\mathbb{1}_{\mathbb{R}_+}(t)dt$$

is the tensorial product of two probability measures, then E_x^1 is statistically isomorphic to $E'_x \times E''_x$, where

$$E''_x = \left(\mathbb{R}_+, \mathcal{B}_{\mathbb{R}_+}, \left(\mu_\theta(x)e^{-\mu_\theta(x)t}\mathbb{1}_{\mathbb{R}_+}(t)dt\right)_{\theta\in\Theta}\right). \tag{28}$$

The differentiability of $\mu(x)$, at $\theta = 0$, is equivalent to the differentiability of the model E''_x. The assertion is then a consequence of [15, Corollary I.7.1] in Ibragimov and Has'minskii's book.

(3) $\Rightarrow$ (2): As in the proceeding implication, observe that E^1_x is statistically isomorphic to $E'_x \times E''_x$, and the result becomes a simple consequence of [15, Theorem I.7.2].

(3) $\Leftrightarrow$ (4): This equivalence is deduced from the fact that $\bar{Q}_\theta(x, \mathrm{d}y, \mathrm{d}t)$ is the distribution of (X_{T_1}, T_1), then $\bar{Q}_\theta(x, \mathrm{d}y, \mathrm{d}t)$ is identified with $\mathbb{P}_{x,\theta}$ restricted to the σ-field $\mathcal{F}_{T_1}$. Thus, E^1_x and $(\Omega, \mathcal{F}_{T_1}, (\mathbb{P}_\theta)_{\theta\in\Theta})$ are statistically isomorphic. □

Proof of Theorem 5. (1) For the necessity condition, we will check (15) and (16), as it was done for the Markov chains in [7]. For fixed $x \in E$, we choose the dominating probability $\mathbb{K}^{\theta,\xi}_x$, the one for which the process $(X_t)_{t\geqslant 0}$ has the generator

$$\Pi^{\theta,\xi}_x(\mathrm{d}y) = \pi_\theta(x, \mathrm{d}y) + \pi_\xi(x, \mathrm{d}y) + \pi_0(x, \mathrm{d}y).$$

(1)(a) Using the function g in (20), we have

$$h^{0,\theta}_t + h^{0,\xi}_t - h^{\theta,\xi}_t - \frac{1}{4}\int_0^t \theta \cdot I(X_s)\cdot \xi \mathrm{d}s = |\theta||\xi| \int_0^t g(X_s, \theta, \xi)\mathrm{d}s.$$

Then, the convergence (15) holds if $\mathbf{A}_t(y)$ is true for all $y \in E$, which, by Remark 9, is equivalent to the regularity of E_y, which is regular for all $y \in E$.

(1)(b) The Doob-Meyer decomposition of the supermartingale Z^θ asserts that

$$Z^\theta = 1 + M^\theta - A^\theta,$$

where M^θ is a local martingale and A^θ is a predictable nondecreasing process. Since the jump times of the process $(X_t)_{t\geqslant 0}$ are totally inaccessible, then Z^θ is left-quasi continuous and A^θ has necessarily continuous paths, cf. [16, Theorem 14]. From the decomposition of the additive functional $\log Z^\theta$ on the event $(t < \tau^0 \wedge \tau^\theta)$, into a local martingale N^θ, and a process with finite variation B^θ (see [5, p. 40]), we may write Z^θ in the form

$$Z^\theta_t = \left\{ \prod_{i\geq 1, T_i\leq t} \frac{\chi_\theta}{\chi_0}\big(X_{T_{i-1}}, X_{T_i}\big)\right\} \exp\int_0^t (\mu_0 - \mu_\theta)(X_s)\mathrm{d}s = e^{N^\theta_t + B^\theta_t},$$

where,

$$N^\theta_t = \left\{ \sum_{s\leq t, X_s\neq X_{s-}} \log\frac{\chi_\theta}{\chi_0}\big(X_{s-}, X_s\big)\right\} - \int_0^t \int_E \log\frac{\chi_\theta}{\chi_0}(X_s, y)\pi_0(X_s, \mathrm{d}y)\mathrm{d}s,$$

$$B^\theta_t = \int_0^t \int_E \left(1 - \frac{\mu_\theta}{\mu_0}(X_s) + \log\frac{\chi_\theta}{\chi_0}(X_s, y)\right)\pi_0(X_s, \mathrm{d}y)\mathrm{d}s.$$

Applying Ito's formula to the semimartingale Z^θ, we obtain

$$\begin{aligned} Z^\theta_t &= 1 + \int_0^t Z^\theta_{s-} dN^\theta_s + \int_0^t Z^\theta_{s-}\mathrm{d}B^\theta_s + \sum_{s\leq t} Z^\theta_{s-}(e^{\Delta N^\theta_s} - 1 - \Delta N^\theta_s) \\ &= 1 + \int_0^t Z^\theta_{s-} dN^\theta_s + \int_0^t Z^\theta_{s-}\mathrm{d}B^\theta_s + \sum_{i\geq 1, T_i\leq t} Z^\theta_{T_{i-1}}\left(\frac{\chi_\theta}{\chi_0}\big(X_{T_{i-1}}, X_{T_i}\big) - 1 - \log\frac{\chi_\theta}{\chi_0}\big(X_{T_{i-1}}, X_{T_i}\big)\right) \\ &= 1 + M^\theta_t - A^\theta_t, \end{aligned}$$

where

$$M_t^\theta = \int_0^t Z_{s\_}^\theta dN_s^\theta + \int_0^t \int_E Z_{s\_}^\theta \left(\frac{\chi_\theta}{\chi_0}(X_s, y) - 1 - \log \frac{\chi_\theta}{\chi_0}(X_s, y) \right) (\mu - \nu^0)(\cdot, \mathrm{d}s, \mathrm{d}y),$$
$$A_t^\theta = -\int_0^t Z_{s\_}^\theta \mathrm{d}B_s^\theta - \int_0^t \int_E Z_{s\_}^\theta \left(\frac{\chi_\theta}{\chi_0}(X_s, y) - 1 - \log \frac{\chi_\theta}{\chi_0}(X_s, y) \right) \nu^0(\cdot, \mathrm{d}s, \mathrm{d}y).$$

Then, we may write

$$A_t^\theta = \int_0^t Z_{s\_}^\theta \mu_\theta(X_s) \int_E \left(1 - \frac{\rho_\theta}{\rho_0}(X_s, y)\right) Q_0(X_s, \mathrm{d}y) \mathrm{d}s.$$

Using [7, point 7.5], we retrieve that there exists an error function $f(z, \cdot)$, $z \in E$, such that

$$0 \le \int_E \left(1 - \frac{\rho_\theta}{\rho_0}(z, y)\right) Q_0(z, \mathrm{d}y) \le |\theta|^2 f(z, |\theta|)$$

$$\frac{A_t^\theta}{|\theta|^2} \le F_t^\theta := Y_t^\theta \sup_{s \le t} Z_s^\theta, \quad Y_t^\theta := \sum_{k \ge 0} (T_{k+1} \wedge t - T_k \wedge t) \mu_\theta\big(X_{T_k}\big) f\big(X_{T_k}, |\theta|\big). \tag{29}$$

Then, observing that $Y_t^\theta(\omega)$, $\omega \in \Omega$, is a finite sum, that $\theta \mapsto \mu_\theta(\cdot)$ is continuous at $\theta = 0$ and using the fact that f is an error function, we deduce that for all $t \ge 0$,

$$Y_t^\theta \longrightarrow 0, \quad \mathbb{P}_{x,0}\text{-a.s.}, \quad \text{as } \theta \to 0. \tag{30}$$

On the other hand, the Doob inequality for positive supermartingales yields

$$\mathbb{P}_{x,0}((Z^\theta)_t^* \ge A) \le \frac{\mathbb{E}_{x,0}[Z_0^\theta]}{A} = \frac{1}{A}, \quad \text{for all } t \ge 0,\ A > 0, \quad \text{and} \quad \theta \in \Theta.$$

We deduce that if θ_n is a sequence going to 0 and if $\varepsilon > 0$, then

$$\begin{aligned}\mathbb{P}_{x,0}\big(F_t^{\theta_n} > \varepsilon\big) &= \mathbb{P}_{x,0}\big(\big(Z^{\theta_n}\big)_t^* Y_t^{\theta_n} > \varepsilon, \big(Z^{\theta_n}\big)_t^* > A\big) + \mathbb{P}_{x,0}\big(\big(Z^{\theta_n}\big)_t^* Y_t^{\theta_n} > \varepsilon, \big(Z^{\theta_n}\big)_t^* \le A\big) \\ &\le \mathbb{P}_{x,0}\big(\big(Z^{\theta_n}\big)_t^* > A\big) + \mathbb{P}_{x,0}\left(Y_t^{\theta_n} > \frac{\varepsilon}{A}\right) \le \frac{1}{A} + \mathbb{P}_{x,0}\left(Y_t^{\theta_n} > \frac{\varepsilon}{A}\right).\end{aligned}$$

Since A may be chosen arbitrarily big, then the latter and (30) show that for all $t \ge 0$,

$$F_t^{\theta_n} \overset{\mathbb{P}_{x,0}}{\longrightarrow} 0, \quad \text{as } n \to +\infty,$$

which, by (29), gives (16).

(2) For the sufficient condition, we write the conditions of local regularity, then we express them at the time T_1.

(2)(a) The local regularity of $\mathcal{E}_x$ at $\theta = 0$ implies that there exists a $(\mathbb{P}_{x,0}, \mathcal{F}_t)$-local martingale $(V_t)_{t \ge 0}$, locally square-integrable, null at zero, satisfying (14) and represented by

$$V_t = \int_0^t \int_E \upsilon(s, y)(\lambda - \nu^0)(\cdot, \mathrm{d}s, \mathrm{d}y), \tag{31}$$

where the function $\upsilon : \Omega \times \mathbb{R}_+ \times E \longmapsto \mathbb{R}^d$ is predictable and satisfies

$$\int_0^t \int_E |\upsilon(s, y)|(\lambda + \nu^0)(\cdot, \mathrm{d}s, \mathrm{d}y) < +\infty, \quad \mathbb{P}_{x,0} - \text{a.s.}, \quad \text{for all } t \ge 0.$$

Let $(\theta_n, \theta)_n$ satisfy (12), and $(S_p, S_{n,p})_{p \in \mathbb{N}, n \ge 1}$ be the corresponding localizing family. By [7, Theorem 4.6], we obtain that for all $p \in \mathbb{N}$:

$$\lim_{n\to+\infty}\mathbb{E}_{x,0}\left[\sup_{t\geq 0}\left|\frac{\sqrt{Z^{\theta_n,0}_{t\wedge S_{n,p}}}-1}{|\theta_n|}-\frac{1}{2}\theta\cdot V_{t\wedge S_p}\right|^2\right]=0.$$

Furthermore, we can choose $(S_p)_{p\in\mathbb{N}}$ independent of $(\theta_n)_{n\in\mathbb{N}}$ and $S_p\leq p$ (this is what we will do in the sequel). We deduce that

$$\lim_{n\to+\infty}\mathbb{E}_{x,0}\left[\left|\frac{\sqrt{Z^{\theta_n,0}_{T_1\wedge S_{n,p}}}-1}{|\theta_n|}-\frac{1}{2}\theta\cdot V_{T_1\wedge S_p}\right|^2\right]=0,$$

then, using [7, Lemma 3.17], we obtain

$$\lim_{n\to+\infty}\mathbb{E}_{x,0}\left[\left|\frac{\sqrt{\mathbb{E}_{x,0}[Z^{\theta_n,0}_{T_1\wedge S_{n,p}}|\sigma(X_{T_1\wedge S_{n,p}})]}-1}{|\theta_n|}-\frac{1}{2}\theta\cdot\mathbb{E}_{x,0}[V_{T_1\wedge S_p}|\sigma\big(X_{T_1\wedge S_{n,p}}\big)]\right|^2\right]=0. \tag{32}$$

(2)(b) By [5], we may write

$$T_1\wedge S_{n,p}=T_1\wedge R_{n,p},\quad T_1\wedge S_p=T_1\wedge R_p,$$

where $R_{n,p}=r_{n,p}(X_0)$, $R_p=r_p(X_0)$, the sequence $(R_p)_{p\in\mathbb{N}}$ does not depend on $(\theta_n)_{n\in\mathbb{N}}$ and the functions $r_{n,p},r_p:E\longrightarrow(0,p]$ are $\mathcal{E}$-measurable. Moreover, by (11), we deduce the following inequalities and inclusions:

(*i*) $T_1\wedge R_{n,p}\leq T_1\wedge R_p$ and $R_p\leq p$;
(*ii*) $(S_{n,p}\geq T_1)=(R_{n,p}\geq T_1)\subseteq(S_p\geq T_1)=(R_p\geq T_1)$;
(*iii*) $(S_p<T_1,S_p=S_{n,p})=(R_p<T_1,R_p=R_{n,p})$;
(*iv*) $\lim_{n\to+\infty}\mathbb{P}_{x,0}(R_p<T_1,R_p=R_{n,p})=\mathbb{P}_{x,0}(R_p<T_1)>0$;
(*v*) $\lim_{n\to+\infty}\mathbb{P}_{x,0}(R_{n,p}\geq T_1)=\lim_{n\to+\infty}\mathbb{P}_{x,0}(R_p\geq T_1,R_p=R_{n,p})=\mathbb{P}_{x,0}(R_p\geq T_1)>0$.

(2)(c) Let us define the quantities

$$\begin{aligned}k_{x,p}(n)&:=\mathbb{E}_{x,0}\left[\left|\frac{\sqrt{\mathbb{E}_{x,0}[Z^{\theta_n,0}_{T_1\wedge R_{n,p}}|\sigma(X_{T_1\wedge R_{n,p}})]}-1}{|\theta_n|}-\frac{1}{2}\theta\cdot\mathbb{E}_{x,0}[V_{T_1\wedge R_p}|\sigma\big(X_{T_1\wedge R_{n,p}}\big)]\right|^2\mathbb{1}_{(R_p<T_1,R_p=R_{n,p})}\right]\\&=\mathbb{E}_{x,0}\left[\left|\frac{e^{\frac{1}{2}(\mu_0-\mu_{\theta_n})(X_0)r_p(X_0)}-1}{|\theta_n|}-\frac{1}{2}\theta\cdot\mathbb{E}_{x,0}[V_{r_p(X_0)}|\sigma(X_0)]\right|^2\mathbb{1}_{(R_p<T_1,R_p=R_{n,p})}\right],\\l_{x,p}(n)&:=\mathbb{E}_{x,0}\left[\left|\frac{\sqrt{\mathbb{E}_{x,0}[Z^{\theta_n,0}_{T_1\wedge R_{n,p}}|\sigma(X_{T_1\wedge R_{n,p}})]}-1}{|\theta_n|}-\frac{1}{2}\theta\cdot\mathbb{E}_{x,0}[V_{T_1\wedge R_p}|\sigma\big(X_{T_1\wedge R_{n,p}}\big)]\right|^2\mathbb{1}_{(R_p\geq T_1,R_p=R_{n,p})}\right]\\&=\mathbb{E}_{x,0}\left[\left|\frac{\sqrt{\mathbb{E}_{x,0}[Z^{\theta_n,0}_{T_1}|\sigma(X_{T_1})]}-1}{|\theta_n|}-\frac{1}{2}\theta\cdot\mathbb{E}_{x,0}[V_{T_1}|\sigma\big(X_{T_1}\big)]\right|^2\mathbb{1}_{(R_p\geq T_1,R_p=R_{n,p})}\right],\end{aligned}$$

and the $\mathcal{E}$-measurable function $w_{1,p}:E\longrightarrow\mathbb{R}^d$ given by

$$w_{1,p}(x)=\mathbb{E}_{x,0}\big[V_{r_p(x)}\big].$$

There exists an $\mathcal{E}\otimes\mathcal{E}$-measurable function $w_2:E\times E\longrightarrow\mathbb{R}^d$, such that

$$\mathbb{E}_{x,0}\big[V_{T_1}|\sigma\big(X_{T_1}\big)\big]=w_2\big(x,X_{T_1}\big). \tag{33}$$

(2)(d) Using (*iv*) in (2) (b), and by (32), we obtain that for all $p \in \mathbb{N}$,

$$\lim_{n\to+\infty} k_{x,p}(n) = \lim_{n\to+\infty} \left| \frac{e^{\frac{1}{2}(\mu_0-\mu_{\theta_n})(x)r_p(x)} - 1}{|\theta_n|} - \frac{1}{2}\theta \cdot w_{1,p}(x) \right|^2 \mathbb{P}_{x,0}(R_p < T_1) = 0.$$

Since $R_p \le p$, then $\mathbb{P}_{x,0}(R_p < T_1) > 0$. We deduce from the latter that $\theta \mapsto \mu_\theta(x)$ is differentiable at 0, and that its derivative

$$\mu_0'(x) = \frac{w_{1,p}(x)}{r_p(x)},$$

is independent of p and also of the functions r_p.

(2)(e) Similarly, by (*v*) in 2(b), and by (32), for all $p \in \mathbb{N}$, we have

$$\lim_{n\to+\infty} l_{x,p}(n) = \lim_{n\to+\infty} \mathbb{E}_{x,0}\left[\left| \frac{\sqrt{\frac{\rho_{\theta_n}}{\rho_0}(x, X_{T_1})} - 1}{|\theta_n|} - \frac{1}{2}\theta \cdot w_2\big(x, X_{T_1}\big) \right|^2 \right] \mathbb{P}_{x,0}(R_p \ge T_1) = 0,$$

and since

$$\mathbb{P}_{x,0}(R_p \ge T_1) = \mathbb{P}_{x,0}(r_p(x) \ge T_1) > 0,$$

we obtain the differentiability, in $L^2(Q_0(x, dy))$, at $\theta = 0$, of $\theta \mapsto \sqrt{\frac{\rho_\theta}{\rho_0}(x, \cdot)}$.

(2)(f) To prove the regularity of the model E_x', it remains to show that

$$\lim_{n\to+\infty} \frac{1}{|\theta_n|^2} \mathbb{E}_{x,0}\left[1 - \frac{\rho_{\theta_n}}{\rho_0}\big(x, X_{T_1}\big)\right] = 0. \tag{34}$$

Observe that, for all $p \in \mathbb{N}$, we have

$$\begin{aligned}\mathbb{E}_{x,0}[1 - Z^{\theta_n}_{T_1\wedge R_{n,p}}] &= \mathbb{E}_{x,0}[\mathbb{E}_{x,0}\big[1 - Z^{\theta_n}_{T_1}|\sigma\big(X_{T_1}\big)\big]\mathbb{1}_{(R_{n,p}\ge T_1)}] + \mathbb{E}_{x,0}[\mathbb{E}_{x,0}\big[1 - Z^{\theta_n}_{R_{n,p}}|\sigma\big(X_{R_{n,p}}\big)\big]\mathbb{1}_{(R_{n,p}<T_1)}] \\ &= \mathbb{E}_{x,0}\left[\left(1 - \frac{\rho_{\theta_n}}{\rho_0}\right)\big(x, X_{T_1}\big)\mathbb{1}_{(R_{n,p}\ge T_1)}\right] + [1 - e^{(\mu_0-\mu_{\theta_n})(x)r_{n,p}(x)}]\mathbb{P}_{x,0}(R_{n,p} < T_1).\end{aligned}$$

On the other hand, by (13), by (*iv*) in (2)(b), and by the fact that $\theta \mapsto \mu_\theta(x)$ is differentiable at 0 (which is equivalent to the regularity of the model E_x'' in (28)), we obtain

$$0 \le \lim_{n\to+\infty} \frac{\mathbb{E}_{x,0}[1 - Z^{\theta_n}_{T_1\wedge R_{n,p}}]}{|\theta_n|^2} \le \lim_{n\to+\infty} \frac{\mathbb{E}_{x,0}[1 - Z^{\theta_n}_{S_{n,p}}]}{|\theta_n|^2} = 0$$

and

$$\lim_{n\to+\infty} \frac{[1 - e^{(\mu_0-\mu_{\theta_n})(x)r_p(x)}]}{|\theta_n|^2}\mathbb{P}_{x,0}(R_{n,p} < T_1) = 0.$$

The latter gives

$$\lim_{n\to+\infty} \frac{\mathbb{E}_{x,0}\left[1 - \frac{\rho_{\theta_n}}{\rho_0}(x, X_{T_1})\right]}{|\theta_n|^2}\mathbb{P}_{x,0}(R_p \ge T_1) = 0.$$

(2)(g) The regularity of the model E_x is deduced by steps (2)(d), (2)(e), (2)(f) and by Theorem 6. ☐

Proof of Theorem 7. Fix $x \in E$ and $t > 0$. Assume that $A_t(y)$ is satisfied for all $y \in E$ and that $|\theta|, |\xi| < u_t$. The dominating probability measure $\mathbb{K}_x^{\theta,\xi}$ is

$$\mathbb{K}_x^{\theta,\xi} = \frac{1}{3}(\mathbb{P}_{x,0} + \mathbb{P}_{x,\theta} + \mathbb{P}_{x,\xi}).$$

In this proof, we simplify some notations as follows:

$$z_s^\alpha = \frac{\mathrm{d}\mathbb{P}_{x,\alpha}|\mathcal{F}_s}{\mathrm{d}K_x^{\alpha,\xi}|\mathcal{F}_s}, \quad \text{for } \alpha = 0, |\theta|, |\xi| < u_t. \tag{35}$$

Due to the choice of $\mathbb{K}_x^{\theta,\xi}$, we have

$$z_s^0 + z_s^\theta + z_s^\xi = 3.$$

(1) First note that the regularity of E_y is equivalent to

$$|g(y,\theta,\xi)| = \frac{1}{|\theta||\xi|}\left|\bar{H}^{0,\theta}(y) + \bar{H}^{0,\xi}(y) - \bar{H}^{\theta,\xi}(y) - \frac{1}{4}\theta\cdot I(y)\cdot\xi\right| \le f_1(y,|\theta|\vee|\xi|),$$

where f_1 is the error function in condition $\mathbf{A}_t(y)$. Moreover, we have

$$\left\{\int_0^t \mathbb{E}_{\mathbb{K}_x^{\theta,\xi}}[g(X_s,\theta,\xi)^2]\mathrm{d}s\right\}^{\frac{1}{2}} \le F_{1,t}(x,|\theta|\vee|\xi|), \tag{36}$$

where the error function $F_{1,t}$ is

$$F_{1,t}(x,|\theta|\vee|\xi|) := \left\{\int_0^t \mathbb{E}_{\mathbb{K}_x^{\theta,\xi}}[f(X_s,|\theta|\vee|\xi|)^2]\mathrm{d}s\right\}^{\frac{1}{2}}.$$

On the other hand, (21) and the condition $\mathbf{A}_t(y)$ yield

$$\sup_{|\theta|,|\xi|\le u_t}\int_0^t \mathbb{E}_{\mathbb{K}_x^{\theta,\xi}}[|I(X_s)|^2]\mathrm{d}s < \infty. \tag{37}$$

(2) We will show the existence of an error function F_t, for which

$$L_t(x,\theta,\xi) := 1 + H_t^{\theta,\xi} - H_t^{0,\theta} - H_t^{0,\xi} - \frac{1}{4}\mathbb{E}_{x,0}\left[\int_0^t \theta\cdot I(X_s)\cdot\xi\,\mathrm{d}s\right], \quad x\in E$$

satisfies

$$|L_t(x,\theta,\xi)| \le |\theta||\xi|F_t(x,|\theta|\vee|\xi|). \tag{38}$$

Using (17), (19), and the fact that $(z_{t-}^0)_{t\ge 0}$ is a $(\mathbb{K}_x^{\theta,\xi},\mathcal{F}_{t-})$-martingale, we decompose $L_t(x,\theta,\xi)$ into

$$\begin{aligned} L_t(x,\theta,\xi) &= \mathbb{E}_{\mathbb{K}_x^{\theta,\xi}}\left[-\int_0^t\sqrt{z_{s-}^\theta z_{s-}^\xi}\,\mathrm{d}h_s^{\theta,\xi} + \int_0^t\sqrt{z_{s-}^0 z_{s-}^\theta}\,\mathrm{d}h_s^{0,\theta} + \int_0^t\sqrt{z_{s-}^0 z_{s-}^\xi}\,\mathrm{d}h_s^{0,\xi} - \frac{1}{4}z_{t-}^0\int_0^t\theta\cdot I(X_s)\cdot\xi\,\mathrm{d}s\right] \\ &= \int_0^t \mathbb{E}_{\mathbb{K}_x^{\theta,\xi}}[A_s + B_s + C_s + D_s]\mathrm{d}s, \end{aligned} \tag{39}$$

where

$$\begin{aligned} A_s &= \left(\bar{H}^{0,\theta}(X_s) + \bar{H}^{0,\xi}(X_s) - \bar{H}^{\theta,\xi}(X_s) - \frac{1}{4}\theta\cdot I(X_s)\cdot\xi\right)\sqrt{z_{s-}^\theta z_{s-}^\xi}, \\ B_s &= \bar{H}^{0,\theta}(X_s)(\sqrt{z_{s-}^\theta z_{s-}^0} - \sqrt{z_{s-}^\theta z_{s-}^\xi}), \\ C_s &= \bar{H}^{0,\xi}(X_s)(\sqrt{z_{s-}^\xi z_{s-}^0} - \sqrt{z_{s-}^\theta z_{s-}^\xi}), \\ D_s &= \frac{1}{4}\theta\cdot I(X_s)\cdot\xi(\sqrt{z_{s-}^\theta z_{s-}^\xi} - z_{s-}^0). \end{aligned}$$

(2)(a) By (35) and (36), we obtain

$$\left|\int_0^t \mathbb{E}_{\mathbb{K}_x^{\theta,\xi}}[A_s]\mathrm{d}s\right| \leq 3|\theta||\xi|F_{1,t}(x, |\theta| \vee |\xi|). \tag{40}$$

(2)(b) Applying Cauchy-Schwarz's inequality twice, and using (35), we obtain

$$\left|\int_0^t \mathbb{E}_{\mathbb{K}_x^{\theta,\xi}}[B_s]\mathrm{d}s\right| \leq \sqrt{3}\left\{\int_0^t \mathbb{E}_{\mathbb{K}_x^{\theta,\xi}}[\bar{H}^{0,\theta}(X_s)^2]\mathrm{d}s\right\}^{\frac{1}{2}} \times \left\{\int_0^t \mathbb{E}_{\mathbb{K}_x^{\theta,\xi}}[(\sqrt{z_{s-}^0} - \sqrt{z_{s-}^{\xi}})^2]\mathrm{d}s\right\}^{\frac{1}{2}}. \tag{41}$$

Then, the condition $\mathbf{A}_t$ implies

$$\int_0^t \mathbb{E}_{\mathbb{K}_x^{\theta,\xi}}[\bar{H}^{0,\theta}(X_s)^2]\mathrm{d}s \leq |\theta|^4 \int_0^t \mathbb{E}_{\mathbb{K}_x^{\theta,\xi}}[f_2(X_s)^2]\mathrm{d}s.$$

By (17), we have

$$\mathbb{E}_{\mathbb{K}_x^{\theta,\xi}}[(\sqrt{z_{s-}^0} - \sqrt{z_{s-}^{\xi}})^2] = 2\mathbb{E}_{\mathbb{K}_x^{\theta,\xi}}\left[\int_0^{s_-} \sqrt{z_{r-}^0 z_{r-}^{\xi}}\,\mathrm{d}h_r^{0,\xi}\right] \leq 6\mathbb{E}_{\mathbb{K}_x^{\theta,\xi}}[h_{s_-}^{0,\xi}],$$

hence,

$$\int_0^t \mathbb{E}_{\mathbb{K}_x^{\theta,\xi}}[(\sqrt{z_{s-}^0} - \sqrt{z_{s-}^{\xi}})^2]\mathrm{d}s \leq 6\int_0^t\int_0^{s_-} \mathbb{E}_{\mathbb{K}_x^{\theta,\xi}}[\bar{H}^{0,\xi}(X_r)]\mathrm{d}r\mathrm{d}s \leq 6t|\xi|^2 \int_0^t \mathbb{E}_{\mathbb{K}_x^{\theta,\xi}}[f_2(X_s)]\mathrm{d}s.$$

Finally, condition $\mathbf{A}_t$, implies that

$$F_{2,t}(x,u) = 3u\sqrt{2t} \sup_{|\theta|,|\xi|\leq u_t} \left\{\int_0^t \mathbb{E}_{\mathbb{K}_x^{\theta,\xi}}[f_2(X_s)^2]\mathrm{d}s\right\}^{\frac{3}{4}}$$

is an error function. Then, by (41) and (3) we obtain

$$\left|\int_0^t \mathbb{E}_{\mathbb{K}_x^{\theta,\xi}}[B_s]\mathrm{d}s\right| \leq |\theta||\xi|F_{2,t}(x, |\theta| \vee |\xi|).$$

(2)(c) As in (4), there exists an error function $F_{3,t}$ such that

$$\left|\int_0^t \mathbb{E}_{\mathbb{K}_x^{\theta,\xi}}[C_s]\mathrm{d}s\right| \leq |\theta||\xi|F_{3,t}(x, |\theta| \vee |\xi|).$$

(2)(d) For the control of the fourth integral in (39), it suffices to observe that the inequality

$$|\sqrt{z_{s-}^{\theta}z_{s-}^{\xi}} - z_{s-}^0| \leq \sqrt{z_{s-}^0} \times |\sqrt{z_{s-}^{\theta}} - \sqrt{z_{s-}^0}| + \sqrt{z_{s-}^{\theta}} \times |\sqrt{z_{s-}^{\xi}} - \sqrt{z_{s-}^0}|$$

implies

$$\mathbb{E}_{\mathbb{K}_x^{\theta,\xi}}[(\sqrt{z_{s-}^{\theta}z_{s-}^{\xi}} - z_{s-}^0)^2] \leq 6\{\mathbb{E}_{\mathbb{K}_x^{\theta,\xi}}[(\sqrt{z_{s-}^0} - \sqrt{z_{s-}^{\theta}})^2] + \mathbb{E}_{\mathbb{K}_x^{\theta,\xi}}[(\sqrt{z_{s-}^0} - \sqrt{z_{s-}^{\xi}})^2]\}.$$

Then, using (3) one obtains

$$\left\{\int_0^t \mathbb{E}_{\mathbb{K}_x^{\theta,\xi}}[(\sqrt{z_{s-}^{\theta}z_{s-}^{\xi}} - z_{s-}^0)^2]\mathrm{d}s\right\}^{\frac{1}{2}} \leq 6\sqrt{t}(|\theta| \vee |\xi|)\left\{\int_0^t \mathbb{E}_{\mathbb{K}_x^{\theta,\xi}}[f_2(X_s)]\mathrm{d}s\right\}^{\frac{1}{2}}.$$

By (37) and by condition $\mathbf{A}_t$ conclude that

$$F_{4,t}(x,u) = 6u\sqrt{t}\sup_{|\theta|,|\xi|\le u_t}\left\{\int_0^t \mathbb{E}_{\mathbb{K}_x^{\theta,\xi}}[|I(X_s)|^2]\mathrm{d}s \times \Big[\int_0^t \mathbb{E}_{\mathbb{K}_x^{\theta,\xi}}[f_2(X_s)]\mathrm{d}s\Big]^{\frac{1}{2}}\right\}$$

is an error function satisfying

$$\left|\int_0^t \mathbb{E}_{\mathbb{K}_x^{\theta,\xi}}[D_s]\mathrm{d}s\right| \le |\theta||\xi|F_{4,t}(x,|\theta|\vee|\xi|).$$

(2)(e) The control (38) is obtained with $F_t = 3F_{1,t} + F_{2,t} + F_{3,t} + F_{4,t}$. □

For the proof of Theorem 8, we need a lemma which generalizes [15, Corollary I.7.1], hence the situation of Theorem 8. Let $(\mathrm{F},\mathcal{F})$ be an arbitrary state space and $R_\theta(x,\mathrm{d}y)$, $S_\theta(x,dy)$, $\theta\in\Theta$, $x\in\mathrm{F}$, be two Markovian kernels and $\Pi_x^\theta(dy)$ be a kernel dominating

$$R_\theta(x,\mathrm{d}y),\quad R_0(x,\mathrm{d}y),\quad S_\theta(x,\mathrm{d}y),\quad \text{and}\quad S_0(x,\mathrm{d}y).$$

We consider the statistical models:

$$\begin{aligned}F_x &= (\mathrm{F},\mathcal{F},(R_\theta(x,\mathrm{d}y))_{\theta\in\Theta}),\quad G_x = (\mathrm{F},\mathcal{F},(S_\theta(x,\mathrm{d}y))_{\theta\in\Theta}),\\ H_x &= (\mathrm{F},\mathcal{F},(R_\theta S_\theta(x,\mathrm{d}y))_{\theta\in\Theta}),\quad \bar{H}_x(\mathrm{F}^2,\mathcal{F}^{2\otimes},(R_\theta(x,\mathrm{d}y_1)S_\theta(y_1,\mathrm{d}y_2))_{\theta\in\Theta}),\end{aligned}$$

where the product $R_\theta S_\theta$ is the Markovian product of the kernels R_θ and S_θ, i.e.,

$$R_\theta S_\theta(x,A) = \int_E R_\theta(x,\mathrm{d}y)S_\theta(y,A),\quad A\in\mathcal{F}.$$

The Radon-Nikodym densities associated with the models F_x and G_x, relative to $\Pi_x^\theta(\mathrm{d}y)$, are

$$\begin{aligned}\alpha_\theta(x,\cdot) &= \frac{\mathrm{d}R_\theta(x,\cdot)}{\mathrm{d}\Pi_x^\theta(\cdot)},\quad \beta_\theta(x,\cdot) = \frac{\mathrm{d}S_\theta(x,\cdot)}{\mathrm{d}\Pi_x^\theta(\cdot)}\\ \alpha_0(x,\cdot) &= \frac{\mathrm{d}R_0(x,\cdot)}{\mathrm{d}\Pi_x^\theta(\cdot)},\quad \beta_0(x,\cdot) = \frac{\mathrm{d}S_0(x,\cdot)}{\mathrm{d}\Pi_x^\theta(\cdot)}.\end{aligned}$$

Choosing

$$\Pi_x^\theta(\cdot) = \frac{1}{4}\{R_\theta(x,\cdot)+R_0(x,\cdot)+S_\theta(x,\cdot)+S_0(x,\cdot)\},$$

we have $\alpha_., \beta_. \le 4$. We introduce the realizations of the last kernels as follows. Let Y_1 and Y_2 be two random variables on a probability space $(\Omega,\mathcal{A})$ with values on the state space $(\mathrm{F},\mathcal{F})$. For $\theta\in\Theta$ and $x\in\mathrm{F}$, let us define the probability measure $\mathbb{P}_{x,\theta}$ on Ω such that

$$\mathcal{L}_{\mathbb{P}_{x,\theta}}(Y_1) = R_\theta(x,\mathrm{d}y)\quad\text{and}\quad \mathcal{L}_{\mathbb{P}_{x,\theta}}(Y_2|Y_1=y) = S_\theta(y,\mathrm{d}z), \tag{42}$$

hence,

$$\mathcal{L}_{\mathbb{P}_{x,\theta}}(Y_2) = R_\theta S_\theta(x,\mathrm{d}y).$$

We also define the probability measure $\mathbb{K}_x^\theta$, on Ω, enjoying the same properties as in (42), when replacing $\mathbb{P}_{x,\theta}$ by $\mathbb{K}_x^\theta$ (respectively, $R_\theta(x,\mathrm{d}y)$ and $S_\theta(x,\mathrm{d}y)$ by $\Pi_x^\theta(\mathrm{d}y)$). With these choices, and by [9, Theorem IV 4.16], we see that

$$\mathbb{P}_{x,\theta} \ll \mathbb{K}_x^\theta\quad\text{and}\quad \mathbb{P}_{x,0}\ll\mathbb{K}_x^\theta.$$

We can now state that

$$\begin{aligned}&F_x \text{ is statistically isomorphic to } \mathcal{F}_x := (\Omega,\sigma(Y_1),(\mathbb{P}_{x,\theta})_{\theta\in\Theta}),\\ &H_x \text{ is statistically isomorphic to } \mathcal{H}_x := (\Omega,\sigma(Y_2),(\mathbb{P}_{x,\theta})_{\theta\in\Theta}),\\ &\bar{H}_x \text{ is statistically isomorphic to } \bar{\mathcal{H}}_x := (\Omega,\sigma(Y_1,Y_2),(\mathbb{P}_{x,\theta})_{\theta\in\Theta}).\end{aligned}$$

Consequently, the Radon-Nikodym densities of $\mathcal{F}_x$ and $\bar{\mathcal{H}}_x$, with respect to $\mathbb{K}_x^\theta$, are expressed by

$$z^\theta = \frac{d\mathbb{P}_{x,\theta}|\sigma(Y_1)}{d\mathbb{K}_x^\theta|\sigma(Y_1)} = \alpha_\theta(x, Y_1) \quad \text{and} \quad \bar{z}^\theta = \frac{d\mathbb{P}_{x,\theta}|\sigma(Y_1, Y_2)}{d\mathbb{K}_x^\theta|\sigma(Y_1, Y_2)} = \alpha_\theta(x, Y_1)\beta_\theta(Y_1, Y_2).$$

The regularity of the models $\mathcal{F}_x$ and $\mathcal{G}_x$ is equivalent to the following: there exists two error functions f_α and f_β, an $R_0(x, dy)$-centered random vector $V_\alpha(x, \cdot) \in L^2(R_0(x, dy))$, and an $S_0(x, dy)$-centered random vector $V_\beta(x, \cdot) \in L^2(S_0(x, dy))$, such that

$$\begin{aligned} a(x, \theta) &:= \int_F \Big(\sqrt{\alpha_\theta(x, y)} - \sqrt{\alpha_0(x, y)} - \frac{1}{2}\sqrt{\alpha_0(x, y)}\,\theta \cdot V_\alpha(x, y)\Big)^2 \Pi_x^\theta(dy) \\ &= \mathbb{E}_{\mathbb{K}_x^\theta}\Big[\Big|\sqrt{\alpha_\theta(x, Y_1)} - \sqrt{\alpha_0(x, Y_1)} - \frac{1}{2}\sqrt{\alpha_0(x, Y_1)}\,\theta \cdot V_\alpha(x, Y_1)\Big|^2\Big] \\ &\le |\theta|^2 f_\alpha(x, |\theta|) \end{aligned} \tag{43}$$

and

$$b(x, \theta) := \int_F \Big(\sqrt{\beta_\theta(x, y)} - \sqrt{\beta_0(x, y)} - \frac{1}{2}\sqrt{\beta_0(x, y)}\,\theta \cdot V_\beta(x, y)\Big)^2 \Pi_x^\theta(dy) \le |\theta|^2 f_\beta(x, |\theta|). \tag{44}$$

We are now able to state the fundamental lemma.

Lemma 11. *Let $x \in F$. Assume that F_x is regular and that G_y is regular for all $y \in F$. Also assume that there exists $r > 0$, such that the error function f_β in (44) satisfies*

$$\Pi_x^\theta[f_\beta(\cdot, r)](x) < +\infty, \quad \text{if } |\theta| < r.$$

Then, the model $\bar{H}_x$ is regular, and so is H_x (as a sub-model of $\bar{H}_x$).

Proof. We need to show that there exists an error function f_γ such that

$$c(x, \theta) := \mathbb{E}_{\mathbb{K}_x^\theta}\Big[\Big|\sqrt{\bar{z}^\theta} - \sqrt{\bar{z}^0} - \frac{1}{2}\sqrt{\bar{z}^0}\,\theta \cdot (V_\alpha(x, Y_1) + V_\beta(Y_1, Y_2))\Big|^2\Big] \le |\theta|^2 f_\gamma(x, |\theta|), \tag{45}$$

for all θ satisfying $|\theta| < r$. To this end, we split $c(x, \theta)$ as follows:

$$\begin{aligned} c(x, \theta) &= \mathbb{E}_{\mathbb{K}_x^\theta}\Big[\Big|\sqrt{\alpha_\theta(x, Y_1)\beta_\theta(Y_1, Y_2)} - \sqrt{\alpha_0(x, Y_1)\beta_0(Y_1, Y_2)} \\ &\quad - \frac{1}{2}\sqrt{\alpha_0(x, Y_1)\beta_0(Y_1, Y_2)}\,\theta \cdot (V_\alpha(x, Y_1) + V_\beta(Y_1, Y_2))\Big|^2\Big] \\ &= \mathbb{E}_{\mathbb{K}_x^\theta}\Big[\Big|\Big(\sqrt{\alpha_\theta(x, Y_1)} - \sqrt{\alpha_0(x, Y_1)} - \frac{1}{2}\sqrt{\alpha_0(x, Y_1)}\,\theta \cdot V_\alpha(x, Y_1)\Big)\sqrt{\beta_\theta(Y_1, Y_2)} \\ &\quad + \Big(\sqrt{\beta_\theta(Y_1, Y_2)} - \sqrt{\beta_0(Y_1, Y_2)} - \frac{1}{2}\sqrt{\beta_0(Y_1, Y_2)}\,\theta \cdot V_\beta(Y_1, Y_2)\Big)\sqrt{\alpha_0(x, Y_1)} \\ &\quad + \frac{1}{2}\sqrt{\alpha_0(x, Y_1)}\,\theta \cdot V_\alpha(x, Y_1)(\sqrt{\beta_\theta(Y_1, Y_2)} - \sqrt{\beta_0(Y_1, Y_2)})\Big|^2\Big]. \end{aligned}$$

Then, using the fact that $\alpha_\cdot, \beta_\cdot \le 4$, we obtain

$$\begin{aligned} c(x, \theta) \le 12\Big\{&\mathbb{E}_{\mathbb{K}_x^\theta}\Big[\Big|\sqrt{\alpha_\theta(x, Y_1)} - \sqrt{\alpha_0(x, Y_1)} - \frac{1}{2}\sqrt{\alpha_0(x, Y_1)}\,\theta \cdot V_\alpha(x, Y_1)\Big|^2\Big] \\ &+ \mathbb{E}_{\mathbb{K}_x^\theta}\Big[\Big|\sqrt{\beta_\theta(Y_1, Y_2)} - \sqrt{\beta_0(Y_1, Y_2)} - \frac{1}{2}\sqrt{\beta_0(Y_1, Y_2)}\,\theta \cdot V_\beta(Y_1, Y_2)\Big|^2\Big] \\ &+ \frac{|\theta|^2}{16}\mathbb{E}_{x,0}[|V_\alpha(x, Y_1)|^2]^{\frac{1}{2}}\mathbb{E}_{\mathbb{K}_x^\theta}[|\sqrt{\beta_\theta(Y_1, Y_2)} - \sqrt{\beta_0(Y_1, Y_2)}|^2]^{\frac{1}{2}}\Big\}. \end{aligned}$$

Using inequalities (43), (44) and the fact that $\Pi_x^\theta[f_\beta(\cdot,r)](x) < +\infty$, we obtain

$$\mathbb{E}_{x,0}[|V_\alpha(x, Y_1)|^2] = R_0[|V_\alpha|^2](x) < \infty \quad \int_{F\times F} |\sqrt{\beta_0(y,z)}\, V_\beta(y,z)|^2 \Pi_y^\theta(dz)\Pi_x^\theta(dy) < \infty.$$

Thus,

$$g_\gamma(x, |\theta|) := |\theta|^2 \left[2\Pi_x^\theta[f_\beta(\cdot,|\theta|)](x) + \int_{F\times F} |V_\beta(y,z)|^2 \Pi_y^\theta(dz)\Pi_x^\theta(dy) \right] \to 0, \quad \text{as } \theta \to 0.$$

Finally, since

$$\mathbb{E}_{\mathbb{K}_x^\theta}[|\sqrt{\beta_\theta(Y_1,Y_2)} - \sqrt{\beta_0(Y_1,Y_2)}|^2] = \int_{F\times F} |\sqrt{\beta_\theta(y,z)} - \sqrt{\beta_0(y,z)}|^2 \Pi_y^\theta(dz)\Pi_x^\theta(dy),$$

then, (45) holds with the error function

$$f_\gamma(x, |\theta|) := 12 f_\alpha(x, |\theta|) + 12 R_0[f_\beta(.,|\theta|)](x) + \frac{3}{4}\mathbb{E}_{x,0}[|V_\alpha(x, Y_1)|^2] g_\gamma(x, |\theta|).$$ □

Proof of Theorem 8. The proof is a simple application of Lemma 11, by taking

$$F = E \times \mathbb{R}_+, \quad \mathcal{F} = \mathcal{E} \otimes \mathcal{B}_{\mathbb{R}_+}, \quad R_\theta = S_\theta = \bar{Q}_\theta,$$

and by making an induction on the index k, using the same condition of integrability of the error function. □

Proof of Theorem 10.

(1) First, we note that

$$\begin{aligned}(\Omega, \mathcal{F}_t, (\mathbb{P}_{x,\theta})_{\theta\in\Theta}) \text{ regular} &\Rightarrow \big(\Omega, \mathcal{F}_{t\wedge T_1}, (\mathbb{P}_{x,\theta})_{\theta\in\Theta}\big) \text{ regular} \\ &\Rightarrow \big(\Omega, \sigma\big(X_{t\wedge T_1}\big), (\mathbb{P}_{x,\theta})_{\theta\in\Theta}\big) \text{ regular}.\end{aligned}$$

Using the Bayes theorem, we express the likelihood of the model by

$$\mathbb{E}_{x,0}\big[Z_{t\wedge T_1}^\theta | \sigma\big(X_{t\wedge T_1}\big)\big].$$

Since the model $\mathcal{E}_x$ is regular at each time $s \in [0, t]$, then the derivative $(V_s)_{0\le s\le t}$ of the model $(\Omega, \mathcal{F}_s, (\mathbb{P}_{x,\theta})_{\theta\in\Theta})$ is given by (31). Using [7, Lemma 3.13], we obtain the derivative at $\theta = 0$ of the likelihood $\mathbb{E}_{x,0}[Z_{t\wedge T_1}^\theta | \sigma(X_{t\wedge T_1})]$ in the form

$$\mathbb{E}_{x,0}\big[V_{t\wedge T_1} | \sigma\big(X_{t\wedge T_1}\big)\big].$$

(1)(a) There exists then an error function $F_{1,t}$, such that

$$\begin{aligned} k(t,x,\theta) &:= \mathbb{E}_{x,0}\left[\left(\sqrt{\mathbb{E}_{x,0}\big[Z_{t\wedge T_1}^\theta|\sigma\big(X_{t\wedge T_1}\big)\big]} - 1 - \frac{1}{2}\theta\cdot\mathbb{E}_{x,0}\big[V_{t\wedge T_1}|\sigma\big(X_{t\wedge T_1}\big)\big]\right)^2\right] \\ &= \mathbb{E}_{x,0}\left[\left(\sqrt{\mathbb{E}_{x,0}[Z_t^\theta|\sigma(X_t)]} - 1 - \frac{1}{2}\theta\cdot\mathbb{E}_{x,0}[V_t|\sigma(X_t)]\right)^2 \mathbb{1}_{(t<T_1)}\right] \\ &\quad + \mathbb{E}_{x,0}\left[\left(\sqrt{\mathbb{E}_{x,0}\big[Z_{T_1}^\theta|\sigma\big(X_{T_1}\big)\big]} - 1 - \frac{1}{2}\theta\cdot\mathbb{E}_{x,0}\big[V_{T_1}|\sigma\big(X_{T_1}\big)\big]\right)^2 \mathbb{1}_{(t\ge T_1)}\right] \\ &= k_1(t,x,\theta) + k_2(t,x,\theta) \le |\theta|^2 F_{1,t}(x,|\theta|). \end{aligned}$$

(1)(b) As in the last point, we see that there exists an error function $F_{2,t}$ such that

$$l(t,x,\theta) := \mathbb{E}_{x,0}\left[1 - \frac{\rho_\theta}{\rho_0}\big(x, X_{T_1}\big)\right]\mathbb{P}_{x,0}(t \ge T_1) \le \mathbb{E}_{x,0}[1 - Z_{T_1}] \le F_{2,t}(x,|\theta|). \tag{46}$$

(1)(c) We use the same arguments as in the proof of Theorem 5 by taking $S_{n,p} = R_{n,p} = S_p = R_p = t$, and we obtain that

- $k_1(t, x, \theta) \leq |\theta|^2 F_{1,t}(x, |\theta|)$ expresses the differentiability of $\theta \mapsto \mu_\theta(x)$ at $\theta = 0$,
- $k_2(t, x, \theta) \leq |\theta|^2 F_{1,t}(x, |\theta|)$ and (46) express the regularity of the model E'_x.

In virtue of Theorem 6, the latter is equivalent to the regularity of E_x.
(2) The second assertion is an immediate consequence of Theorem 6. □

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