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Research Article

Cui Wang, Ze-Hua Zhou, and Liang Zhang*

Disjoint diskcyclicity of weighted shifts

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Abstract: In this article, we will discuss disjoint diskcyclicity for finitely many operators acting on a separable, infinite dimensional Fréchet space X. More precisely, we provide disjoint disk blow-up/collapse property and disjoint diskcyclicity criterion. In addition, we characterize the disjoint diskcyclicity for weighted shifts both in the bilateral and unilateral cases.

Keywords: disjoint diskcyclic, criterion, weighted shift, operator

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1 Introduction

Let X denote a separable, infinite dimensional Fréchet space over the real or complex scalar field $\mathbb{K}$. $L(X)$ denotes the space of linear continuous operators on X. As usual, $\mathbb{Z}$ is the set of integers and $\mathbb{N}$ is the set of nonnegative integers, and let $\mathbb{C}$ be the complex plane.

An operator $T \in L(X)$ is said to be hypercyclic, if there exists a vector $x \in X$ such that its orbit under the operator

$$orb(T, x) = \{x, Tx, T^2x, \ldots\}$$

is norm dense in X. Such a vector x is said to be a hypercyclic vector for the operator T. An operator $T \in L(X)$ is supercyclic if there is a vector x for which the orbit $\{\lambda T^n x;\ \lambda \in \mathbb{C},\ n \geq 0\}$ is dense in X. Hypercyclicity and supercyclicity have been studied in recent decades, see [1,2].

The diskcyclic phenomenon was introduced by Zeana in [3]. Let $T \in L(X)$, T is called diskcyclic if there is a vector $x \in X$ such that the set $\{\alpha T^n x : \alpha \in \mathbb{C},\ |\alpha| \leq 1,\ n \geq 0\}$ is dense in X, see [4]. The vector x is called a diskcyclic vector for T. The following diagram shows the relations among cyclic operators:

$$\text{Hypercyclicity} \Rightarrow \text{Diskcyclicity} \Rightarrow \text{Supercyclicity}.$$

Since contractive operators cannot be diskcyclic, Supercyclicity $\not\Rightarrow$ Diskcyclicity. Bamerni et al. [5] gave an example of diskcyclic operator, which is not hypercyclic.

The following definitions are from Definition 1.1 in [6] and Section 1.3 in [7].

Definition 1.1. For $N \geq 2$, the operators $T_1, \ldots, T_N$ in $L(X)$ are disjoint hypercyclic or d-hypercyclic (disjoint supercyclic or d-supercyclic, respectively), if there is a vector $z \in X$ such that $(z, z, \ldots, z) \in X^N$ is a hypercyclic (supercyclic, respectively) vector for the direct sums $\oplus_{i=1}^N T_i$.

*** Corresponding author: Liang Zhang,** School of Marine Science and Technology, Tianjin University, Tianjin 300072, P. R. China, e-mail: 168zhangliang2011@163.com
Cui Wang: College of Mathematics and Information Science, Henan Normal University, Xinxiang 453007, P. R. China, e-mail: cuiwang2016@126.com
Ze-Hua Zhou: Department of Mathematics, Tianjin University, Tianjin 300350, P. R. China, e-mail: zehuazhoumath@aliyun.com, zhzhou@tju.edu.cn

The research about the disjoint diskcyclicity is still in the blank state. So it is our goal in this article to give a new subject called disjoint diskcyclicity.

Definition 1.2. For $N \geq 2$, the operators $T_1, \ldots, T_N$ in $L(X)$ are disjoint diskcyclic, if there is a vector $z \in X$ such that $(z, z, \ldots, z) \in X^N$ is a diskcyclic vector for the direct sums $\oplus_{i=1}^N T_i$. The vector z is called a disjoint diskcyclic vector associated with the operators $T_1, \ldots, T_N$.

Similarly, the following holds true:

$$\text{Disjoint hypercyclicity} \Rightarrow \text{Disjoint diskcyclicity} \Rightarrow \text{Disjoint supercyclicity}$$

but Disjoint supercyclicity $\nRightarrow$ Disjoint diskcyclicity. Moreover, we will provide an example of disjoint diskcyclic system but not disjoint hypercyclic in Section 3.

The article is organized as follows: In Section 2, we provide some basic definitions associated with disjoint diskcyclicity. In addition, the related properties are obtained, which play a key role in the theory of disjoint diskcyclicity. In Section 3, we characterize the disjoint diskcyclicity for distinct powers of weighted bilateral(unilateral) shifts. In Section 4, we characterize disjoint diskcyclicity of weighted shift operators.

2 Disjoint diskcyclicity

Definition 2.1. We say that $N \geq 2$ sequences of operators $(T_{1,j})_{j=1}^\infty, \ldots, (T_{N,j})_{j=1}^\infty$ in $L(X)$ are disjoint disk-topologically transitive, if for every nonempty open subsets $V_0, \ldots, V_N$ of X, there exist $m \in \mathbb{N}$, $\alpha_m \in \mathbb{C}$ with $|\alpha_m| \geq 1$ such that $\varnothing \neq V_0 \cap T_{1,m}^{-1}(\alpha_m V_1) \cap \cdots \cap T_{N,m}^{-1}(\alpha_m V_N)$. Also, we say that $N \geq 2$ operators $T_1, \ldots, T_N$ in $L(X)$ are disjoint disk-topologically transitive, provided $(T_1^j)_{j=1}^\infty, \ldots, (T_N^j)_{j=1}^\infty$ are disjoint disk-topologically transitive sequences.

Definition 2.2. We say that $N \geq 2$ sequences of operators $(T_{1,j})_{j=1}^\infty, \ldots, (T_{N,j})_{j=1}^\infty$ in $L(X)$ are disjoint disk-universal, if

$$\{\alpha(T_{1,j}z, T_{2,j}z, \ldots, T_{N,j}z); j \in \mathbb{N}, \alpha \in \mathbb{C}, |\alpha| \leq 1\}$$

is dense in X^N for some vector $z \in X$. Also, we say that $(T_{1,j})_{j=1}^\infty, \ldots, (T_{N,j})_{j=1}^\infty$ are disk-hereditarily universal, provided for each increasing sequence of positive integers (n_k), the sequences $(T_{1,n_k})_{k=1}^\infty, \ldots, (T_{N,n_k})_{k=1}^\infty$ are disjoint disk-universal.

The operator $T \in L(X)$ is topologically transitive if for each pair U, V of nonempty open subsets of X, there exists n such that $U \cap T^{-n}(V) \neq \varnothing$. An application of Birkhoff's transitivity theorem [8] shows that hypercyclicty and topologically transitivity are equivalent. Likewise, the authors [6] showed that when the space X is a Fréchet space, the notions of disjoint hypercyclicity and disjoint topologically transitivity coincide.

Proposition 2.3. *Let $N \geq 2$ and $(T_{1,n})_{n=1}^\infty, \ldots, (T_{N,n})_{n=1}^\infty$ be sequences of operators in $L(X)$. Then the following are equivalent*:

(*i*) *$(T_{1,n})_{n=1}^\infty, \ldots, (T_{N,n})_{n=1}^\infty$ are disjoint disk-topologically transitive.*

(*ii*) *The set of disjoint disk-universal vectors for $(T_{1,n})_{n=1}^\infty, \ldots, (T_{N,n})_{n=1}^\infty$ is a dense G_δ set.*

Proof. (i) follows immediately from (ii).

(*i*) $\Rightarrow$ (*ii*). We assume $\{A_j : j \in \mathbb{N}\}$ be a basis for the topology of X. Then (i) implies that $\bigcup_{m \geq k} \bigcup_{\alpha_m \in \mathbb{C}, |\alpha_m| \geq 1} (T_{1,m}^{-1}(\alpha_m A_{j_1}) \cap \cdots \cap T_{N,m}^{-1}(\alpha_m A_{j_N}))$ is both open and dense in the Fréchet space X for every $J = (j_1, \ldots, j_N) \in \mathbb{N}^N$. On the other hand, the set of disjoint disk-universal vectors for $(T_{1,n})_{n=1}^\infty, \ldots, (T_{N,n})_{n=1}^\infty$ is

$$\bigcap_{J \in \mathbb{N}^N} \bigcap_{k \in \mathbb{N}} \bigcup_{m \geq k} \bigcup_{\alpha_m \in \mathbb{C}, |\alpha_m| \geq 1} \Big(T_{1,m}^{-1}\big(\alpha_m A_{j_1}\big) \cap \cdots \cap T_{N,m}^{-1}\big(\alpha_m A_{j_N}\big)\Big).$$

It follows that the set of disjoint disk-universal vectors for $(T_{1,n})_{n=1}^\infty, \ldots, (T_{N,n})_{n=1}^\infty$ is a dense G_δ set. □

Definition 2.4. We say that $N \geq 2$ sequences of operators $(T_{1,j})_{j=1}^{\infty}, \dots, (T_{N,j})_{j=1}^{\infty}$ in $L(X)$ satisfy the disjoint disk blow-up/collapse property provided for any nonempty open neighborhood W of zero of X and nonempty open subsets $V_0, V_1, \dots, V_N \subset X$, there exist $m \in \mathbb{N}$, $\alpha_m \in \mathbb{C}$ with $|\alpha_m| \geq 1$ so that

$$\varnothing \neq W \cap T_{1,m}^{-1}(\alpha_m V_1) \cap \cdots \cap T_{N,m}^{-1}(\alpha_m V_N),$$

$$\varnothing \neq V_0 \cap T_{1,m}^{-1}(\alpha_m W) \cap \cdots \cap T_{N,m}^{-1}(\alpha_m W).$$

We say that the operators $T_1, \dots, T_N$ in $L(X)$ satisfy the disjoint disk blow up/collapse property if their corresponding sequences of iterations $(T_1^j)_{j=1}^{\infty}, \dots, (T_N^j)_{j=1}^{\infty}$ do.

The following disjoint disk blow-up/collapse property is a sufficient condition for the disjointness of diskcyclic operators.

Proposition 2.5. *Let $N \geq 2$ and $(T_{1,n})_{n=1}^{\infty}, \dots, (T_{N,n})_{n=1}^{\infty}$ be sequences of operators in $L(X)$. If $(T_{1,n})_{n=1}^{\infty}, \dots, (T_{N,n})_{n=1}^{\infty}$ satisfy the disjoint disk blow-up/collapse property, then they are disjoint disk-topologically transitive.*

Our aim is to provide another sufficient condition called disjoint diskcyclic criterion for the disjoint diskcyclicity.

Definition 2.6. Let $(n_k)_{k=1}^{\infty}$ be a strictly increasing sequence of positive integers. We say that $T_1, \dots, T_N \in L(X)$ satisfy the disjoint diskcyclic criterion with respect to $(n_k)_{k=1}^{\infty}$ provided there exist dense subsets $X_0, X_1, \dots, X_N$ of X and mappings $S_l : X_l \to X (1 \leq l \leq N)$, so that for $1 \leq i \leq N$

(i) $S_l^{n_k} \underset{k\to\infty}{\to} 0$ pointwise on X_l,

(ii) $(T_l^{n_k} S_i^{n_k} - \delta_{i,l} Id_{X_i}) \underset{k\to\infty}{\to} 0$ pointwise on X_i,

(iii) $\lim_{k\to\infty} \|T_l^{n_k} x\| \cdot \|\sum_{i=1}^{N} S_i^{n_k} y_i\| = 0$ for any $x \in X_0$ and any $y_i \in X_i$.

In general, we say that $T_1, \dots, T_N$ satisfy the disjoint diskcyclic criterion, if there exists some sequence $(n_k)_{k=1}^{\infty}$ for which is satisfied the aforementioned three conditions.

Proposition 2.7. *Let $N \geq 2$ and $T_1, \dots, T_N \in L(X)$ satisfy the disjoint diskcyclicity criterion. Then $T_1, \dots, T_N$ have a residual set of disjoint diskcyclic vectors.*

Proof. Let $\varepsilon > 0$ and $e, f_1, \dots, f_N \in X$. Given $x \in X_0$ and $0 \neq y_i \in X_i$ such that $\|e - x\| < \frac{\varepsilon}{2}$, $\|f_i - y_i\| < \frac{\varepsilon}{4}$ for $1 \leq i \leq N$. Pick $k \in \mathbb{N}$, by conditions (*iii*) and (*ii*) of Definition 2.6, it follows that for $1 \leq l \leq N$

$$\|T_l^{n_k} x\| \cdot \left\| \sum_{i=1}^{N} S_i^{n_k} y_i \right\| < \frac{\varepsilon^2}{4},$$

$$\left\| f_l - \sum_{i=1}^{N} T_l^{n_k} S_i^{n_k} y_i \right\| < \frac{\varepsilon}{2}.$$

If $\|\sum_{i=1}^{N} S_i^{n_k} y_i\| = 0$, by condition (*ii*) of Definition 2.6, $y_i = 0$, which contradicts with $0 \neq y_i \in X_i$. So we can choose $0 < \alpha = \frac{2}{\varepsilon} \|\sum_{i=1}^{N} S_i^{n_k} y_i\| \leq 1$, since condition (*i*) of Definition 2.6. Indeed, let $\hat{x} = x + \frac{1}{\alpha} \sum_{i=1}^{N} S_i^{n_k} y_i$. Then we have $\|e - \hat{x}\| \leq \|e - x\| + \frac{1}{\alpha} \|\sum_{i=1}^{N} S_i^{n_k} y_i\| < \varepsilon$. It follows that

$$\begin{aligned}
\|f_l - \alpha T_l^{n_k} \hat{x}\| &= \left\| f_l - \alpha T_l^{n_k} x - \sum_{i=1}^{N} T_l^{n_k} S_i^{n_k} y_i \right\| \\
&\leq \left\| f_l - \sum_{i=1}^{N} T_l^{n_k} S_i^{n_k} y_i \right\| + \alpha \|T_l^{n_k} x\| \\
&< \frac{\varepsilon}{2} + \frac{2}{\varepsilon} \left\| \sum_{i=1}^{N} S_i^{n_k} y_i \right\| \|T_l^{n_k} x\| \\
&< \varepsilon.
\end{aligned}$$

Hence, $T_1, \dots, T_N$ are disjoint disk-topologically transitive. Proposition 2.3 implies that $T_1, \dots, T_N$ have a residual set of disjoint diskcyclic vectors. □

3 Powers of disjoint diskcyclic weighted shifts

In this section, we extend some results of Bès and Peris [6] and Martin [7] to the setting of disjoint diskcyclicity. Moreover, we will show an example to introduce that Disjoint diskcyclicity $\not\Rightarrow$ Disjoint hypercyclicity.

3.1 Case for weighted bilateral shifts

Theorem 3.1. *Let $X = c_0(\mathbb{Z})$ or $l^p(\mathbb{Z})(1 \le p < \infty)$. For $N \ge 2$ and $l = 1, \dots, N$, let $w_l = (w_{l,j})_{j\in\mathbb{Z}}$ be a bounded bilateral sequence of nonzero scalars, F_{w_l} be the associated forward shift on X given by $F_{w_l}e_k = w_{l,k}e_{k+1}$. For integers $1 \le r_1 < r_2 < \dots < r_N$, the following are equivalent:*

(i) $F_{w_1}^{r_1}, \dots, F_{w_N}^{r_N}$ have a dense set of disjoint diskcyclic vectors.

(ii) For each $\varepsilon > 0$ and $q \in \mathbb{N}$, there exists $m > 2q$ so that for $|j|, |k| \le q$, we have: If $1 \le s, l \le N$,

$$\left|\prod_{i=j-r_l m}^{j-1} w_{l,i}\right| > \frac{1}{\varepsilon}, \tag{3.1}$$

$$\left|\prod_{i=j}^{j+r_l m-1} w_{l,i}\right| < \varepsilon \left|\prod_{i=k-r_s m}^{k-1} w_{s,i}\right|. \tag{3.2}$$

If $1 \le s < l \le N$,

$$\left|\prod_{i=j-r_l m}^{j-1} w_{l,i}\right| > \frac{1}{\varepsilon}\left|\prod_{i=j-r_l m}^{j-(r_l-r_s)m-1} w_{s,i}\right|, \tag{3.3}$$

$$\left|\prod_{i=j-r_s m}^{j+(r_l-r_s)m-1} w_{l,i}\right| < \varepsilon \left|\prod_{i=j-r_s m}^{j-1} w_{s,i}\right|. \tag{3.4}$$

(iii) $F_{w_1}^{r_1}, \dots, F_{w_N}^{r_N}$ satisfy disjoint diskcyclic criterion.

Proof. $(i) \Rightarrow (ii)$. Suppose $F_{w_1}^{r_1}, \dots, F_{w_N}^{r_N}$ have a dense set of disjoint diskcyclic vectors. Let $0 < \delta < \frac{1}{2}$ with $\frac{\delta}{1-\delta} < \varepsilon$. We can find a disjoint diskcyclic vector $x = \sum_{-\infty}^{\infty} x_k e_k$, $0 \ne \alpha \in \mathbb{C}$ with $|\alpha| \le 1$ and $m > 2q$ such that

$$\left\|x - \sum_{|j|\le q} e_j\right\| < \delta, \tag{3.5}$$

$$\left\|\alpha F_{w_l}^{r_l m} x - \sum_{|j|\le q} e_j\right\| < \delta. \tag{3.6}$$

It follows from (3.5) that

$$|x_j - 1| < \delta \quad \text{if } |j| \le q, \tag{3.7}$$

$$|x_k| < \delta \quad \text{if } |k| > q. \tag{3.8}$$

Moreover, by (3.6),

$$\left|\alpha\left(\prod_{i=j-r_l m}^{j-1} w_{l,i}\right)x_{j-r_l m}-1\right|<\delta \quad \text{if } |j|\le q, \tag{3.9}$$

$$\left|\alpha\left(\prod_{i=k-r_l m}^{k-1} w_{l,i}\right)x_{k-r_l m}\right|<\delta \quad \text{if } |k|> q. \tag{3.10}$$

Now, fix $|j|\le q$. Since $r_l\ge 1$ and $m>2q$, $k=j-r_l m<-q$. By (3.8) and (3.9),

$$\left|\alpha\prod_{i=j-r_l m}^{j-1} w_{l,i}\right|>\frac{1-\delta}{\delta}>\frac{1}{\varepsilon}.$$

Since $0<|\alpha|\le 1$, (3.1) holds.

Fix $|j|, |k|\le q$ and $1\le s,l\le N$. By (3.7), $\frac{1}{2}<1-\delta<|x_j|<\delta+1$. Since $j-r_l m\le -q$, (3.10) yields that

$$\frac{1}{2}\left|\alpha\prod_{i=j}^{j+r_l m-1} w_{l,i}\right|\le\left|\alpha\prod_{i=j}^{j+r_l m-1} w_{l,i}\right||x_j|<\delta.$$

For $|k|\le q$, by (3.8), $|x_{k-r_s m}|<\delta$. So (3.9) implies

$$\frac{1}{2}<1-\left|\alpha\left(\prod_{i=k-r_s m}^{k-1} w_{s,i}\right)x_{k-r_s m}-1\right|<\left|\alpha\prod_{i=k-r_s m}^{k-1} w_{s,i}\right|\delta.$$

Combining the aforementioned inequalities, we obtain

$$\left|\prod_{i=j}^{j+r_l m-1} w_{l,i}\right|<4\delta^2\left|\prod_{i=k-r_s m}^{k-1} w_{s,i}\right|.$$

For $1\le s<l\le N$ and $1\le r_1<r_2<\cdots<r_N$ and $m>2q$, if $k=j-(r_l-r_s)m$, we have $k<-q$. Then by (3.9) and (3.10), we conclude that

$$\frac{|\prod_{i=j-r_l m}^{j-1} w_{l,i}|}{|\prod_{i=j-r_l m}^{j-(r_l-r_s)m-1} w_{s,i}|}=\frac{|\alpha(\prod_{i=j-r_l m}^{j-1} w_{l,i})x_{j-r_l m}|}{|\alpha(\prod_{i=j-r_l m}^{j-(r_l-r_s)m-1} w_{s,i})x_{j-r_l m}|}>\frac{1-\delta}{\delta}>\frac{1}{\varepsilon}.$$

Similarly, if $k=j+(r_l-r_s)m$, we have $k>q$. So

$$\frac{|\prod_{i=j-r_s m}^{j+(r_l-r_s)m-1} w_{l,i}|}{|\prod_{i=j-r_s m}^{j-1} w_{s,i}|}=\frac{|\alpha(\prod_{i=j-r_s m}^{j+(r_l-r_s)m-1} w_{l,i})x_{j-r_s m}|}{|\alpha(\prod_{i=j-r_s m}^{j-1} w_{s,i})x_{j-r_s m}|}<\frac{\delta}{1-\delta}<\varepsilon.$$

We obtain (3.3) and (3.4).

(*ii*) ⇒ (*iii*). By (*ii*), there exist integers $1\le m_1<m_2<\cdots$ so that for $|j|\le q$, we have:

If $1\le s<l\le N$,

$$\left|\prod_{i=j-r_l m}^{j-1} w_{l,i}\right|>q\left|\prod_{i=j-r_l m}^{j-(r_l-r_s)m-1} w_{s,i}\right|, \tag{3.11}$$

$$\left|\prod_{i=j-r_s m}^{j+(r_l-r_s)m-1} w_{l,i}\right|<\frac{1}{q}\left|\prod_{i=j-r_s m}^{j-1} w_{s,i}\right|. \tag{3.12}$$

If $1\le s,l\le N$,

$$\left|\prod_{i=j-r_l m}^{j-1} w_{l,i}\right|>q, \tag{3.13}$$

$$\left|\prod_{i=j}^{j+r_l m-1} w_{l,i}\right| < \frac{1}{q}\left|\prod_{i=k-r_s m}^{k-1} w_{s,i}\right|. \tag{3.14}$$

Let $X_0 = X_1 = \cdots = X_N = \operatorname{span}\{e_k : k \in \mathbb{Z}\}$. Define $B_l e_k = \frac{e_{k-1}}{w_{l,k-1}}(1 \le l \le N)$ on X. Thus, a simple calculation shows that

$$B_l^{r_l n_q} e_k = \frac{e_{k-r_l n_q}}{w_{l,k-1}\cdots w_{l,k-r_l n_q}}.$$

By (3.13), we can easily obtain $B_l^{r_l n_q} \underset{q\to\infty}{\to} 0$ pointwise on X_0. Since $X_0 = X_1 = \cdots = X_N$, $B_l^{r_l n_q} \underset{q\to\infty}{\to} 0$ pointwise on X_l for $1 \le l \le N$.

On the other hand, we can easily obtain that $B_l F_{w_l} = Id_{X_l}$. Moreover, by (3.11) and (3.12), if $1 \le s < l \le N$, we have

$$\|F_{w_l}^{r_l m_q} B_s^{r_s m_q} e_k\| = \frac{|\prod_{i=k-r_s m_q}^{k-(r_s-r_l)m_q-1} w_{l,i}|}{|\prod_{i=k-r_s m_q}^{k-1} w_{s,i}|} < \frac{1}{q},$$

$$\|F_{w_s}^{r_s m_q} B_l^{r_l m_q} e_k\| = \frac{|\prod_{i=k-r_l m_q}^{k-(r_l-r_s)m_q-1} w_{s,i}|}{|\prod_{i=k-r_l m_q}^{k-1} w_{l,i}|} < \frac{1}{q}.$$

So $(F_{w_l}^{r_l m_q} B_i^{r_i m_q} - \delta_{i,l} Id_{X_i}) \underset{q\to\infty}{\to} 0$ pointwise on X_i.

Finally, let $y_0, y_1, \ldots, y_N \in span\{e_k : k \in \mathbb{Z}\}$ and $C := \max\{\|y_k\| : 0 \le k \le N\}$. Pick $y_i = \sum_{|j|\le q_0} y_{i,j} e_j$ for q_0 sufficiently large and $1 \le i \le N$. Then for $q > q_0$, (3.14) implies that

$$\|F_{w_l}^{r_l m_q} y_0\| \left\|\sum_{k=1}^{N} B_k^{r_k m_q} y_k\right\| \le C\left(\sum_{|j|\le q_0}\left|\prod_{i=j}^{j+r_l m_q-1} w_{l,i}\right|\right)\left|\sum_{k=1}^{N}\sum_{|j|\le q_0}\left(\prod_{i=j-r_k m_q}^{j-1} w_{k,i}\right)^{-1}\right| \underset{q\to\infty}{\to} 0.$$

Hence, $F_{w_1}^{r_1}, \ldots, F_{w_N}^{r_N}$ satisfy disjoint diskcyclic criterion.

$(iii) \Rightarrow (i)$. This is the result of Proposition 2.7. □

If the shifts on Theorem 3.1 are invertible, this leads to the following result.

Corollary 3.2. *Let $X = c_0(\mathbb{Z})$ or $l^p(\mathbb{Z})(1 \le p < \infty)$. For $N \ge 2$ and $l = 1, \ldots, N$, let $F_{w_l} e_k = w_{l,j} e_{k+1}$ be an invertible bilateral weighted forward shift on X, with weighted sequence of nonzero scalars $w_l = (w_{l,j})_{j\in\mathbb{Z}}$. For any integers $1 \le r_1 < r_2 < \cdots < r_N$, the following are equivalent:*

(i) $F_{w_1}^{r_1}, \ldots, F_{w_N}^{r_N}$ have a dense set of disjoint diskcyclic vectors.

(ii) $F_{w_1}^{r_1}, \ldots, F_{w_N}^{r_N}$ satisfy disjoint diskcyclic criterion.

(iii) There exist integers $1 \le n_1 < n_2 < \cdots$ so that we have:
For $j \in \mathbb{N}$, if $1 \le s < l \le N$,

$$\lim_{q\to\infty} \frac{|\prod_{i=j-r_l n_q}^{j-1} w_{l,i}|}{|\prod_{i=j-r_l n_q}^{j-(r_l-r_s)n_q-1} w_{s,i}|} = \infty,$$

$$\lim_{q\to\infty} \frac{|\prod_{i=j-r_s n_q}^{j+(r_l-r_s)n_q-1} w_{l,i}|}{|\prod_{i=j-r_s n_q}^{j-1} w_{s,i}|} = 0.$$

If $1 \le s, l \le N$,

$$\lim_{q\to\infty}\left|\prod_{i=-r_l n_q}^{1} w_{l,i}\right| = \infty,$$

$$\lim_{q\to\infty} \max\left\{\frac{|\prod_{i=-1}^{r_l n_q} w_{l,i}|}{\left|\prod_{i=-r_s n_q}^{-1} w_{s,i}\right|}\right\} = 0.$$

As an application, we will show that disjoint diskcyclicity is not equivalent to disjoint hypercyclicity in the following example.

Example 3.3. Let F_a be a hypercyclic bilateral forward weighted shift with the associated weight sequence $a = (a_k)_{k\in\mathbb{Z}}$ where

$$a_k = \begin{cases} \frac{1}{2}, & \text{if } k \in \{2^n - n, \dots, 2^n - 1\} \quad \text{for some odd } n \in \mathbb{N}; \\ 2, & \text{if } k \in \{2^n, \dots, 2^n + n - 1\} \quad \text{or } k = -2^n \quad \text{for some odd } n \in \mathbb{N}; \\ 1, & \text{otherwise.} \end{cases}$$

By the definition of a_k, a looks like

$$\left(\dots, 1, 2, 1, \dots, 1, 2, 1, [1], \frac{1}{2}, 2, 1, 1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 2, 2, 2, 1, \dots\right),$$

where [.] denotes the 0th coefficient. Applying (iii) of Corollary 3.2, we observe that F_a, F_a^2 are disjoint diskcyclic. On the other hand, by Remark 4.10 of [9], since

$$\prod_{i=1}^{2m-1} a_i = 1 \quad \text{when} \quad \prod_{i=1}^{2m-1} a_{-i} > 1 \quad \text{for all } m \in \mathbb{N},$$

F_a, F_a^2 are not disjoint hypercyclic.

3.2 Case for weighted unilateral shifts

Theorem 3.4. *Let $X = c_0(\mathbb{N})$ or $l_p(\mathbb{N})(1 \le p < \infty)$. For $N \ge 2$ and $l = 1, \dots, N$, let $w_l = (w_{l,j})_{j=N}^{\infty}$ be a bounded sequence of nonzero scalars, $B_{w_l} : x = (x_0, x_1, \dots) \mapsto (w_{l,1}x_1, w_{l,2}x_2, \dots)$ be the associated backward shift on X. For any integers $1 \le r_1 < r_2 < \dots < r_N$, the following are equivalent:*

(*i*) $B_{w_1}^{r_1}, \dots, B_{w_N}^{r_N}$ *have a dense set of disjoint diskcyclic vectors.*

(*ii*) *For each $\varepsilon > 0$ and $q \in \mathbb{N}$, there exists $m \in \mathbb{N}$ so that for each $0 \le j \le q$, we have:*

$$\left|\prod_{i=j+1}^{j+r_l m} w_{l,i}\right| > \frac{1}{\varepsilon} \quad (1 \le l \le N) \tag{3.15}$$

and

$$\frac{|\prod_{i=j+1}^{j+r_l m} w_{l,i}|}{|\prod_{i=j+(r_l-r_s)m+1}^{j+r_l m} w_{s,i}|} > \frac{1}{\varepsilon} \quad (1 \le s < l \le N). \tag{3.16}$$

(*iii*) $B_{w_1}^{r_1}, \dots, B_{w_N}^{r_N}$ *satisfy disjoint diskcyclic criterion.*

(*iv*) $B_{w_1}^{r_1}, \dots, B_{w_N}^{r_N}$ *have a dense set of disjoint hypercyclic vectors.*

(*v*) $B_{w_1}^{r_1}, \dots, B_{w_N}^{r_N}$ *satisfy disjoint hypercyclic criterion.*

Proof. Conditions (*ii*), (*iv*), and (*v*) are equivalent, see [6, Theorem 4.1]. It is easy to see that the implication (*i*) ⇒ (*ii*) is similar to (*iv*) ⇒ (*ii*). By Proposition 2.7, we obtain (*iii*) ⇒ (*i*). For (*ii*) ⇒ (*iii*), the proof is similar to the discussion in Theorem 2.1 in [10], so we omit it. □

4 The Disjoint diskcyclic weighted shifts

In this section, we will show that disjoint diskcyclicity can coincide with disjoint hypercyclicity in special case.

Theorem 4.1. *Let $X = l^2(\mathbb{Z})$ and $\{e_i : i \in \mathbb{Z}\}$ be the standard orthonormal basis of X. For $N \geq 2$ and $m = 1, \ldots, N$, suppose F_m is a bilateral weighted forward shift on X given by $F_m e_i = w_i^{(m)} e_{i+1}$, where $w_i^{(m)}$ is the weight sequence for $i \in \mathbb{Z}$. Then the following are equivalent:*

(*i*) *$F_1, \ldots, F_N$ are disjoint diskcyclic.*

(*ii*) *There exists a strictly increasing sequence $(n_k)_{k=0}^{\infty}$ of positive integers such that for each integer i and integer m with $1 \leq m \leq N$, we have*

$$\left|\prod_{j=1}^{n_k} w_{i-j}^{(1)}\right| \to \infty \tag{4.1}$$

and

$$\left|\prod_{j=0}^{n_k-1} w_{i+j}^{(m)}\right| \to 0 \tag{4.2}$$

as $k \to \infty$.

Otherwise, $\{(\ldots, \lambda_{-1,n_k}^{(2)}, \ldots, \lambda_{-1,n_k}^{(N)}, \lambda_{0,n_k}^{(2)}, \ldots, \lambda_{0,n_k}^{(N)}, \lambda_{1,n_k}^{(2)}, \ldots, \lambda_{1,n_k}^{(N)}, \ldots) : k \geq 0\}$ is dense in $\mathbb{K}^{\mathbb{Z}}$ with respect to the product topology, where $\lambda_{i,n}^{(m)} = \prod_{j=1}^{n} \frac{w_{i-j}^{(m)}}{w_{i-j}^{(1)}}$.

(*iii*) *$F_1, \ldots, F_N$ satisfy the disjoint disk blow-up/collapse property.*

Proof. The proof is similar to the discussion in Theorem 2.1 in [11], so we omit it. □

The bilateral and unilateral weighted backward shift cases of the characterization are similar to the bilateral weighted forward shift case in Theorem 4.1, and so we omit the details.

Remark 4.2. Though disjoint diskcyclicity is not equal to disjoint hypercyclicity, sometimes, disjoint diskcyclicity can coincide with disjoint hypercyclicity in special case. By comparing Theorem 4.1 with the construction of [11], we can add another two equivalent conditions:

(iv) $F_1, \ldots, F_N$ are disjoint hypercyclic.

(v) $F_1, \ldots, F_N$ satisfy the disjoint blow-up/collapse property.

Finally, we end this article with two questions.

Question 1: Suppose $T_1, \ldots, T_N$ are disjoint diskcyclic, is the set of disjoint diskcyclic vectors dense in X?

Question 2: Suppose $T_1, \ldots, T_N$ are disjoint diskcyclic and invertible, are $T_1^{-1}, \ldots, T_N^{-1}$ disjoint diskcyclic?

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References

[1] G. Godefroy and J. H. Shapiro, *Operators with dense, invariant, cyclic vector manifolds*, J. Funct. Anal. **98** (1991), no. 2, 229–269, DOI: https://doi.org/10.1016/0022-1236(91)90078-J.
[2] C. Kitai, *Invariant Closed Sets for Linear Operators*, Ph.D. thesis, University of Toronto, 1982.
[3] Z. J. Zeana, *Cyclic phenomena of operators on Hilbert Space*, PhD thesis, University of Baghdad, 2002.
[4] M. Amouch and O. Benchiheb, *Diskcyclicity of sets of operators and applications*, Acta Math. Sin. (Engl. Ser.) **36** (2020), no. 11, 1203–1220, DOI: https://doi.org/10.1007/s10114-020-9307-3.
[5] N. Bamerni, A. Kılıçman, and M. S. M. Noorani, *A review of some works in the theory of diskcyclic operators*, Bull. Malays. Math. Sci. Soc. **39** (2016), no. 2, 723–739, DOI: https://doi.org/10.1007/s40840-015-0137-x.
[6] J. Bès and A. Peris, *Disjointness in hypercyclicity*, J. Math. Anal. Appl. **336** (2007), no. 1, 297–315, DOI: https://doi.org/10.1016/j.jmaa.2007.02.043.
[7] Ö. Martin, *Disjoint Hypercyclic and Supercyclic Composition Operators*, PhD thesis, Bowling Green State University, 2010.
[8] K. G. Grosse-Erdmann and A. Peris, *Linear Chaos, Universitext: Tracts in Mathematics*, Springer, New York, 2011.
[9] L. Bernal-González, *Disjoint hypercyclic operators*, Studia Math. **182** (2007), no. 2, 113–131, DOI: https://doi.org/10.4064/sm182-2-2.
[10] L. Zhang, H. Q. Lu, X. M. Fu, and Z. H. Zhou, *Disjoint hypercyclic powers of weighted translations on groups*, Czechoslovak Math. J. **67** (2017), no. 142, 839–853, DOI: https://doi.org/10.21136/CMJ.2017.0204-16.
[11] J. Bès, Ö. Martin, and R. Sanders, *Weighted shifts and disjoint hypercyclicity*, J. Operator Theory **72** (2014), no. 1, 15–40, DOI: https://doi.org/10.7900/jot.2012aug20.1970.