Research Article

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Orbital stability and Zhukovskiĭ quasi-stability in impulsive dynamical systems

https://doi.org/10.1515/math-2022-0478
received February 25, 2022; accepted June 23, 2022

Abstract: In this article, we deal with orbital stability and Zhukovskiĭ quasi-stability of periodic or recurrent orbits in an impulsive dynamical system defined in the n-dimensional Euclidean space $\mathbb{R}^n$. We show that for a periodic orbit of an impulsive system, its asymptotically orbital stability is equivalent to the asymptotically Zhukovskiĭ quasi-stability, and for a recurrent orbit, the orbital stability is equivalent to the Zhukovskiĭ quasi-stability.

Keywords: periodicity, recurrence, orbital stability, Zhukovskiĭ quasi-stability

MSC 2020: 37B25, 34D20

1 Introduction

Impulsive dynamical systems are a generalization of classical dynamical systems. They describe the evolution of systems where the continuous development of a process is interrupted by abrupt perturbations. The behavior of an impulsive system is much richer than that of the corresponding continuous dynamical system. In particular, the theory of impulsive dynamical systems represents a natural framework for the mathematical modeling of many real-world phenomena. Recently, the theory of impulsive dynamical systems has been intensively investigated. For the elementary results in this field, we refer readers to [1–3].

The research of impulsive semidynamical systems in a metric space was started by Kaul [4–6] and Rozhko [7,8]. Specifically, Rozhko dealt with a class of almost periodic motions in pulsed systems and the stability theory in terms of Lyapunov for impulsive systems. Later on, Kaul continued the study for impulsive semidynamical systems and established a list of important results about the structure of limit sets, periodicity and recurrence of an orbit, minimality and stabilities of closed subsets, etc. Also, Ciesielski presented many fundamental results in this field; for example, he applied his section theory of semidynamical systems to obtain the continuity of an impulsive time function [9–11]. Recently, in [12–15], Bonotto and his research group developed a list of significant results on impulsive semidynamical systems, which include many counterparts of basic properties in classical dynamical systems. In addition, the authors of this article also established some interesting results on the limit sets and limit set maps [16,17], Lyapunov quasi-stability [18], and Zhukovskiĭ quasi-stability [19].

Poincaré (orbital) stability and Zhukovskiĭ stability are two different important stabilities of solutions of differential equations. Since Zhukovskiĭ stability admits a time lag, it is a more suitable concept for the study of impulsive dynamical systems. The Zhukovskiĭ quasi-stability in impulsive dynamical systems was first introduced in [18], where the author proved that the limit set of a uniformly asymptotically Zhukovskiĭ

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quasi-stability orbit is composed of a rest point or a periodic orbit. Clearly, the structure of limit sets can be determined by the stability of orbits. Furthermore, the inverse problem was considered in [19]. Actually, it was shown in [19] that (i) if the positive limit set of an orbit for a planar system is an asymptotically stable limit cycle, then it is a uniformly asymptotically Zhukovskiĭ quasi-stable orbit; (ii) if an orbit is not eventually periodic and its positive limit set is a periodic orbit, then it is an asymptotically Zhukovskiĭ quasi-stable orbit.

In this article, we deal with the recurrence, orbital stability, and Zhukovskiĭ quasi-stability of orbits in an impulsive dynamical system defined in $\mathbb{R}^n$. First, for a periodic orbit (or eventually periodic orbit), we show that the asymptotically orbital stability is equivalent to the asymptotically Zhukovskiĭ quasi-stability. Second, it is shown that for a recurrent orbit, the orbital stability is also equivalent to the Zhukovskiĭ quasi-stability.

2 Definitions and notations

We consider the system of differential equations

$$X' = F(X), \tag{1}$$

where $F : \mathbb{R}^n \to \mathbb{R}^n$. Obviously, the map $F : \mathbb{R}^n \to \mathbb{R}^n$ defines a vector field $\mathbf{F}$ of system (1) on $\mathbb{R}^n$. Assume that the vector field $\mathbf{F}$ defines a flow φ on $\mathbb{R}^n$; i.e., $\varphi : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is continuous such that $\varphi(p, 0) = p$ for all $p \in \mathbb{R}^n$ and $\varphi(\varphi(p, t), s) = \varphi(p, t + s)$ for all $p \in \mathbb{R}^n, t, s \in \mathbb{R}$. If $A \subset \mathbb{R}^n$ and $J \subset \mathbb{R}$, we write $\varphi(A \times J) = A \cdot J$, in particular, $\varphi(p, t) = p \cdot t$. For a point $p \in \mathbb{R}^n$, the orbit of p is the set $\gamma(p) = p \cdot \mathbb{R}$. The positive and negative semi-orbits are the sets $\gamma^+(p) = p \cdot \mathbb{R}^+$ and $\gamma^-(p) = p \cdot \mathbb{R}^-$, respectively.

Let $M = \{p \in U | G(p) = 0\}$ be a simple smooth surface in an open subset U of $\mathbb{R}^n$, where $G : U \to \mathbb{R}$ is a smooth function with $G'(p) = (\partial G/\partial x_1, \ldots, \partial G/\partial x_n) \neq (0, \cdots, 0)$ for $p = (x_1, x_2, \cdots, x_n) \in U$. The surface M is said to be transversal to the vector field $\mathbf{F}$ if the inner product $G'(p) \cdot \mathbf{F}(p) \neq 0$ for all $p \in M$, it is also called a contact-free surface [20]. Now, denote $\Omega = \mathbb{R}^n \backslash M$. Let $I : M \to \Omega$ be a continuous function and $N = I(M)$. If $p \in M$, we shall denote $I(p)$ by p^+ and say p jumps to p^+. Meanwhile, M is said to be an *impulsive set* and I is called an *impulsive function*. For each $p \in \Omega$, by $M^+(p)$, we mean the set $\gamma^+(p) \cap M$. We can define a function $\psi : \Omega \to \mathbb{R}^+ \cup \{+\infty\}$ (the space of extended positive reals) by

$$\psi(p) = \begin{cases} s, & \text{if } p \cdot s \in M \quad \text{and} \quad p \cdot t \notin M \quad \text{for } t \in [0, s), \\ +\infty, & \text{if } M^+(p) = \varnothing. \end{cases}$$

In general, $\psi : \Omega \to \mathbb{R}^+ \cup \{+\infty\}$ is not continuous. Fortunately, some easy applicable conditions given by Ciesielski in [9] guarantee the continuity of ψ. Throughout this article, we always assume that ψ is a continuous function on Ω.

Now, we define an impulsive system $(\Omega, \tilde{\varphi})$ by portraying the trajectory of each point in Ω. Let $p \in \Omega$, the impulsive trajectory of p is an Ω-valued function $\tilde{\varphi}_p$ defined on a subset of $\mathbb{R}^+$. If $M^+(p) = \varnothing$, then $\psi(p) = +\infty$, and we set $\tilde{\varphi}_p(t) = p \cdot t$ for all $t \in \mathbb{R}^+$. If $M^+(p) \neq \varnothing$, it is easy to see that there is a positive number t_0 such that $p \cdot t_0 = p_1 \in M$ and $p \cdot t \notin M$ for all $t \in [0, t_0)$. Thus, we define $\tilde{\varphi}_p(t)$ on $[0, t_0]$ by

$$\tilde{\varphi}_p(t) = \begin{cases} p \cdot t, & 0 \leq t < t_0, \\ p_1^+, & t = t_0, \end{cases}$$

where $\psi(p) = t_0$ and $p_1^+ = I(p_1) \in \Omega$.

Since $t_0 < +\infty$, we continue the process by starting with p_1^+. Similarly, if $M^+(p_1^+) = \varnothing$, i.e., $\psi(p_1^+) = +\infty$, we define $\tilde{\varphi}_p(t) = p_1^+ \cdot (t - t_0)$ for $t_0 < t < +\infty$. Otherwise, let $\psi(p_1^+) = t_1$, where $p_1^+ \cdot t_1 = p_2 \in M$, and $p_1^+ \cdot t \notin M$ for any $t \in [0, t_1)$, then we define $\tilde{\varphi}_p(t)$ on $[t_0, t_0 + t_1]$ by

$$\tilde{\varphi}_p(t) = \begin{cases} p_1^+ \cdot (t - t_0), & t_0 \leq t < t_0 + t_1, \\ p_2^+, & t = t_0 + t_1, \end{cases}$$

where $p_2^+ = I(p_2) \in \Omega$.

Thus, continuing inductively, the aforementioned process either ends after a finite number of steps, whenever $M^+(p_n^+) = \varnothing$ for some n, or it continues infinitely, if $M^+(p_n^+) \neq \varnothing$ for $n = 0, 1, 2, \dots$, and $\tilde{\varphi}_p$ is defined on the interval $[0, t_p)$, where $t_p = \sum_{i=0}^{+\infty} t_i$. We call $\{t_i\}$ the *impulsive intervals* of $\tilde{\varphi}_p$ and $\{t_p(k) = \sum_{i=0}^{k} t_i : k = 0, 1, 2, \dots\}$ the *impulsive times* of $\tilde{\varphi}_p$. After setting each trajectory $\tilde{\varphi}_p$ for every point $p \in \Omega$, we let $\tilde{\varphi}(p, t) = \tilde{\varphi}_p(t)$ for $p \in \Omega$ and $t \in [0, t_p)$, then we obtain a discontinuous system $(\Omega, \tilde{\varphi})$ satisfying the following properties:

(i) $\tilde{\varphi}(p, 0) = p$ for all $p \in \Omega$, and
(ii) $\tilde{\varphi}(\tilde{\varphi}(p, s), t) = \tilde{\varphi}(p, s + t)$ for all $p \in \Omega$ and $s, \quad t \in [0, t_p)$, such that $s + t \in [0, t_p)$.

We call $(\Omega, \tilde{\varphi})$, with $\tilde{\varphi}$ as defined earlier, an *impulsive dynamical system* associated with (Ω, φ). Also for simplicity of exposition, we denote $\tilde{\varphi}(p, t)$ by $p * t$. Thus, (ii) reads $(p * s) * t = p * (s + t)$. Similarly, if $A \subset \Omega$ and $J \subset \mathbb{R}^+$, we denote $A * J = \{p * t | p \in A \quad \text{and} \quad t \in J\}$. In particular, if $J = \{t\}$, we let $A * t = A * \{t\} = \tilde{\varphi}_t(A)$. For $p \in \Omega$ the mapping $\tilde{\varphi}_p : \mathbb{R}^+ \to \Omega$ defined by $t \to p * t$ and for a $t \in \mathbb{R}^+$ the mapping $\tilde{\varphi}_t : \Omega \to \Omega$ defined by $p \to p * t$ may not be continuous. However, $\tilde{\varphi}_p$ is continuous from the right hand for any $p \in \Omega$.

For an impulsive dynamical system $(\Omega, \tilde{\varphi})$, the trajectories that are of interest are those with an infinite number of discontinuities and with $[0, +\infty)$ as the interval of definition. Following Kaul in [4], the trajectories are called infinite impulsive trajectories. Furthermore, for an impulsive dynamical system, Ciesielski used time reparametrization to obtain an isomorphic system whose impulsive trajectories are global, i.e., the resulting dynamics is defined for all positive times, [11]. Hence, from now on, we always assume $t_p = +\infty$ for any $p \in \Omega$.

In the following, for a point $p \in \Omega$, let $B_\delta(p) = \{q \in \Omega | d(p, q) < \delta\}$ be the open ball in Ω with center p and radius $\delta > 0$, where d is the Euclidean metric on $\mathbb{R}^n$, and the closed ball $\overline{B}_\delta(p) = \{q \in \Omega | d(p, q) \leq \delta\}$. In addition, for $S \subset \Omega$, the r-neighborhood of S in Ω is denoted by $U(S, r) = \{q \in \Omega | d(q, S) < r\}$ for $r > 0$, where $d(q, S) = \inf\{d(q, p) | p \in S\}$. Here, with no confusion, we also use d for the distance between a point and a set. The orbit of p in $(\Omega, \tilde{\varphi})$ is the set $\tilde{\gamma}(p) = p * \mathbb{R}$. The positive and negative semi-orbits of p are the sets $\tilde{\gamma}^+(p) = p * \mathbb{R}^+$ and $\tilde{\gamma}^+(p) = p * \mathbb{R}^-$, respectively. A subset S of Ω is said to be positively invariant if $\tilde{\gamma}^+(p) \subset S$ for any $p \in S$; furthermore, it is said to be invariant if it is positively invariant, and for any $p \in S$ and $t \in \mathbb{R}^+$, there exist a $q \in S$ such that $q * t = p$.

Now, we introduce several definitions that will be used in the sequel.

Definition 2.1. Let $p \in \Omega$. The positive semi-orbit $\tilde{\gamma}^+(p) = p * \mathbb{R}^+$ is said to be orbitally stable if, given an $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon) > 0$ such that for any $q \in B_\delta(p)$, then we have that $q * \mathbb{R}^+ \subset U(p * \mathbb{R}^+, \varepsilon)$. Moreover, if there is a $\eta > 0$ such that if $q \in B_\eta(p)$ implies $d(q * t, p * \mathbb{R}^+) \to 0$ as $t \to +\infty$, then the positive semi-orbit $\tilde{\gamma}^+(p)$ is asymptotically orbitally stable.

Next, we give the following concept of $\tilde{\varphi}$-recurrence in impulsive dynamical systems, which was introduced first in [21].

Definition 2.2. A point $p \in \Omega$ is said to be $\tilde{\varphi}$-recurrent if for every $\varepsilon > 0$, there exists a $T = T(\varepsilon) > 0$, such that for any $t, s \in \mathbb{R}^+$, the interval $[0, T]$ contains a real number $\tau > 0$ such that

$$d(p * t, p * (s + \tau)) < \varepsilon.$$

A positive semi-orbit $\tilde{\gamma}^+(p)$ is said to be $\tilde{\varphi}$-recurrent if p is $\tilde{\varphi}$-recurrent.

Obviously, $p \in \Omega$ is $\tilde{\varphi}$-recurrent means for any $\varepsilon > 0$; there exists a $T = T(\varepsilon) > 0$ such that $\tilde{\gamma}^+(p) \subset U(p * [s, s + T], \varepsilon)$ holds for every $s \geq 0$.

The idea of a time reparametrization is useful in our discussion about Zhukovskiĭ quasi-stabilities, see [16,18].

Definition 2.3. A time reparametrization is a homeomorphism ρ from $\mathbb{R}^+$ onto $\mathbb{R}^+$ with $\rho(0) = 0$. Furthermore, for a $\sigma > 0$, by a time σ-reparametrization, we mean a homeomorphism ρ from $\mathbb{R}^+$ onto $\mathbb{R}^+$ with $\rho(0) = 0$ such that $|\rho(t) - t| < \sigma$ for all $t \geq 0$.

Now, we recall the concepts of Zhukovskiĭ quasi-stabilities, which were first introduced for impulsive dynamical systems in [16].

Definition 2.4. Let $p \in \Omega$. The positive semi-orbit $\tilde{\gamma}^+(p) = p * \mathbb{R}^+$ is Zhukovskiĭ quasi-stable provided that given any $\varepsilon > 0$, there exists a $\delta = \delta(p, \varepsilon) > 0$ such that if $q \in B_\delta(p)$, then one can find a time reparametrization ρ_q such that $d(p * t, q * \rho_q(t)) < \varepsilon$ holds for all $t \geq 0$. Moreover, if there is a $\lambda > 0$ such that if $q \in B_\lambda(p)$, then $d(p * t, q * \rho_q(t)) \to 0$ as $t \to +\infty$, and then the orbit $p * \mathbb{R}^+$ is said to be asymptotically Zhukovskiĭ quasi-stable.

Furthermore, the property of Zhukovskiĭ quasi-stability can be strengthened to that of uniformly asymptotically Zhukovskiĭ quasi-stability as follows:

Definition 2.5. The positive semi-orbit $\tilde{\gamma}^+(p) = p * \mathbb{R}^+$ of $p \in \Omega$ is uniformly asymptotically Zhukovskiĭ quasi-stable provided that given any $\varepsilon > 0$; there exists a $\delta > 0$ such that for each $s > 0$ and $q \in B_\delta(p * s)$, one can find a time reparametrization ρ_q such that $d(p * (s + t), q * \rho_q(t)) < \varepsilon$ holds for all $t \geq 0$, and also, $d(p * (s + t), q * \rho_q(t)) \to 0$ as $t \to +\infty$.

The definition of an impulsive periodic orbit was first presented by Kaul in [4] as follows:

Definition 2.6. Let $p \in \Omega$. The positive semi-orbit $\tilde{\gamma}^+(p) = p * \mathbb{R}^+$ is said to be (impulsive) periodic of period τ and order k if $\tilde{\gamma}^+(p)$ has k components and τ is the least positive number such that $p * \tau = p$. Thus the point p is called an (impulsive) periodic point of order k.

A periodic orbit of an impulsive dynamical system $(\Omega, \tilde{\varphi})$ is an invariant closed set in Ω, and it is not connected if $k \neq 1$. If $\tilde{\gamma}^+(p)$ is not a periodic orbit, but there exists a $t > 0$ such that $\tilde{\gamma}^+(p * t)$ is a periodic orbit, then $\tilde{\gamma}^+(p)$ is said to be *eventually periodic*. Clearly, a periodic orbit is eventually periodic, but easy examples can be constructed to show that the converse may not be true.

The continuous dependence on the initial conditions is a fundamental property in the theory of dynamical systems. Fortunately, in [16], the author establishes a counterpart of the continuous dependence for impulsive dynamical systems, which is crucial for the study of impulsive dynamical systems. It is called Quasi-Continuous Dependence and is presented as follows.

Quasi-continuous dependence: Let $(\Omega, \tilde{\varphi})$ be an impulsive dynamical system and $p \in \Omega$. For any $\varepsilon > 0$, $\sigma > 0$, and a positive number τ, there exists a $\delta > 0$ such that if $q \in B_\delta(p)$, then the inequality $d(p * t, q * \rho_q(t)) < \varepsilon$ holds for all $t \in [0, \tau]$, where ρ_q is a time σ-reparametrization.

From the aforementioned definition, it is obvious that the quasi-continuous dependence property is a natural generalization of the standard continuous dependence on the initial conditions. For simplicity, we denote the quasi-continuous dependence by QCD property in the sequel. In [16], a crucial proposition was established by the author. It is shown that for impulsive dynamical systems, the QCD property is equivalent to the continuity of ψ. Hence, in this article the QCD property holds for our impulsive system $(\Omega, \tilde{\varphi})$ by previous assumption that ψ is continuous on Ω.

3 Main results

The concepts of asymptotically orbital stability and asymptotically Zhukovskiĭ stability are different in their dynamical properties. However, for an impulsive periodic orbit, the following result holds.

Theorem 3.1. *Let* Γ *be a periodic orbit in an impulsive dynamical system* $(\Omega, \tilde{\varphi})$*. Then,* Γ *is asymptotically orbitally stable if and only if* Γ *is asymptotically Zhukovskiĭ quasi-stable.*

Proof. Assume that $\Gamma = p_1 * \mathbb{R}^+$ with period τ and order k, where $p_1 \in N$. Then, Γ can be written as

$$\Gamma = p_1 * [0, t_1) \cup p_2 * [0, t_2) \cup \cdots \cup p_k * [0, t_k),$$

where $p_i \in N$, $\psi(p_i) = t_i$ $(i = 1, 2, \ldots, k)$, and $\sum_{i=1}^{k} t_i = \tau$. Clearly, we have $I(p_i * t_i) = p_{i+1}$ for $i = 1, 2, \ldots, k-1$ and $I(p_k * t_k) = p_1$; the solution segments $\{p_i * [0, t_i) : i = 1, 2, \ldots, k\}$ are pairwise disjoint.

For a given $\varepsilon > 0$, by the quasi-continuous dependence, there exists a $\theta \in (0, \varepsilon)$ such that if $q \in B_\theta(p_1)$; then $d(p_1 * t, q * \rho_q(t)) < \varepsilon$ holds for all $t \in [0, \tau]$, where ρ_q is a time reparametrization. Now, assume that $\Gamma = p_1 * \mathbb{R}^+$ is asymptotically orbitally stable; then for the aforementioned θ, there is a $\delta \in (0, \theta)$ such that if $q \in B_\delta(p_1)$, we have $q * \mathbb{R}^+ \subset U(\Gamma, \theta)$ and $d(q * t, \Gamma) \to 0$ as $t \to +\infty$. Without loss of generality, let ε be sufficiently small so that $U(\Gamma, \varepsilon)$ is composed of k pairwise disjoint components, i.e., k disjoint tubes.

Let $q \in B_\delta(p_1)$, and we define a time reparametrization τ_q as follows. Write $\mathfrak{L} = N \cap B_\theta(p_1)$, then we have $q_1 = q * \rho_q(\tau) \in \mathfrak{L}$. Clearly, it follows from $q_1 * \mathbb{R}^+ \subset U(\Gamma, \theta)$ that there exists a time reparametrization ρ_{q_1} so that $d(p_1 * t, q_1 * \rho_{q_1}(t)) < \theta$ holds for all $t \in [0, \tau]$. Thus, inductively, there exist two sequences $\{q_n\}$ and $\{\rho_{q_n}\}$ satisfying $d(p_1 * t, q_n * \rho_{q_n}(t)) < \theta$ and $q_n * \rho_{q_n}(\tau) = q_{n+1}$ for $n = 1, 2, \ldots$, where $\{\rho_{q_n}\}$ are time reparametrizations. Let $q_0 = q$ and $\rho_{q_0} = \rho_q$. Then, we define $\tau_q(t) = \rho_{q_n}(t - n\tau)$ for $t \in [n\tau, (n+1)\tau](n = 0, 1, 2, \ldots)$, and it is easy to see that τ_q is a time reparametrization. Now, we obtain $d(p_1 * t, q * \rho_q(t)) < \theta < \varepsilon$, and certainly $d(p_1 * t, q * \rho_q(t)) \to 0$ as $t \to +\infty$. So, $\Gamma = p_1 * \mathbb{R}^+$ is asymptotically Zhukovskiĭ quasi-stable.

Conversely, from definitions, it is easy to see that if Γ is asymptotically Zhukovskiĭ quasi-stable, then Γ is asymptotically orbitally stable. Thus, the proof is completed. ☐

From the proof of Theorem 3.1, it is easy to see that the following statement is true.

Corollary 3.2. *Let* $\Gamma = p * \mathbb{R}^+$ *be a eventually periodic orbit in an impulsive dynamical system* $(\Omega, \tilde{\varphi})$*, Then,* Γ *is asymptotically orbitally stable if and only if* Γ *is asymptotically Zhukovskiĭ quasi-stable.*

Proof. We just need to show the necessity of the statement since the sufficiency is obvious by the definitions of asymptotically orbital stability and asymptotically Zhukovskiĭ quasi-stability.

Let Γ be eventually periodic. That means there exists a $t_0 > 0$ such that $\Gamma_1 = p_1 * \mathbb{R}^+$ is periodic, where $p_1 = p * t_0$. Assume that Γ is asymptotically orbitally stable, so is Γ_1. Thus, by Theorem 3.1, we have Γ_1 which is asymptotically Zhukovskiĭ quasi-stable. That is, for every $\varepsilon > 0$, there is a $\theta \in (0, \varepsilon)$ such that if $q_1 \in B_\theta(p_1)$, then $d(p_1 * t, q_1 * \rho_1(t)) < \varepsilon$ for each $t \in \mathbb{R}^+$ and $d(p_1 * t, q_1 * \rho_1(t)) \to 0$ as $t \to +\infty$, where ρ_1 is a time reparametrization. For the aforementioned $t_0 > 0$ and $\theta > 0$, by quasi continuous dependence of Γ, there exists a $\delta > 0$ such that if $q \in B_\delta(p)$, then $d(p * t, q * \rho_0(t)) < \theta$ holds for $t \in [0, t_0]$, where ρ_0 is a time reparametrization. Set $q_1 = q * \tau_0(t_0)$, and it is clear that $q_1 \in B_\theta(p_1)$. Given a $q \in B_\delta(p)$, we define a time reparametrization as follows. Let $\rho_q(t) = \tau_0(t)$ if $t \in [0, t_0]$ and $\rho_q(t) = \rho_1(t - t_0)$ if $t \in [t_0, +\infty)$. It is easy to verified that ρ_q is a time reparametrization. Certainly, we have $d(p * t, q * \rho_q(t)) < \varepsilon$ for all $t \in \mathbb{R}^+$ and $d(p * t, q * \rho_q(t)) \to 0$ as $t \to +\infty$. Thus Γ is asymptotically Zhukovskiĭ quasi-stable.

Now, we turn to consider orbitally stability, recurrence, and Zhukovskiĭ quasi-stability in a planar impulsive dynamical system as follows. ☐

Theorem 3.3. *Let* $p_0 \in \Omega$, $\Gamma = p_0 * \mathbb{R}^+$ *be the positive orbit of* p_0*. If* Γ *is* $\tilde{\varphi}$*-recurrent, then* Γ *is orbitally stable if and only if* Γ *is Zhukovskiĭ quasi-stable.*

Proof. Suppose p_0 is $\tilde{\varphi}$-recurrent, then for any $\varepsilon > 0$, there exists a $T > 0$ such that $\gamma^+(p_0) \subset U(p_0 * [0, T], \varepsilon)$. By the quasi-continuous dependence, for the $\varepsilon > 0$ and $T > 0$ earlier, there exists a $\theta \in (0, \varepsilon)$ such that for every $q \in B_\theta(p_0)$, we have $d(p_0 * t, q * \rho_q(t)) < \varepsilon$ for each $t \in [0, T]$, where ρ_q is a time reparametrization. Furthermore, we can find a $\delta_1 > 0$, such that if $q \in B_{\delta_1}(p_0)$, then $d(p_0 * t, q * \rho_q(t)) < \theta$ holds for each

$t \in [0, T]$. By the orbitally stability of Γ, for the θ above, there is a $\delta_2 > 0$ such that if $q \in B_{\delta_2}(p_0)$, then $q * \mathbb{R}^+ \subset U(\Gamma, \theta)$.

Setting $\delta = \min\{\delta_1, \delta_2\}$. For any $q \in B_\delta(p_0)$, we define a time reparametrization τ_q as follows. Let $p_1 = p_0 * T$ and $q_1 = q * \rho_q(T)$; thus, we have $d(p_1, a_1) < \theta$. Clearly, $q_1 * \mathbb{R}^+ \subset U(\Gamma, \theta)$ implies there exists a time reparametrization ρ_1 such that $d(p_1 * t, q_1 * \rho_1(t)) < \theta$ holds for all $t \in [0, T]$. Thus, inductively there exist three sequences $\{p_n\}$, $\{q_n\}$, and $\{\rho_n\}$ such that $d(p_n * t, q_n * \rho_n(t)) < \theta$ for each $t \in [0, T]$ and $p_{n+1} = p_n * T$, $q_{n+1} = q_n * \rho_n(T)$ for $n = 1, 2, \ldots$, where $\{\rho_n\}$ are time reparametrizations. Let $q_0 = q$ and $\rho_0 = \rho_q$. Then, we define $\tau_q(t) = \rho_n(t - nT)$ for $t \in [nT, (n+1)T](n = 0, 1, 2, \ldots)$, and it is easy to see that τ_q is a time reparametrization and $d(p_0 * t, q * \tau_q(t)) < \varepsilon$ for every $t \in \mathbb{R}^+$; then, $\Gamma = p_0 * \mathbb{R}^+$ is Zhukovskiĭ quasi-stable.

Conversely, assume that Γ is a Zhukovskiĭ quasi-stable orbit. That is, for any $\varepsilon > 0$, there exists a $\delta > 0$ such that, if $q \in B_\delta(p_0)$, $d(p_0 * t, q * \tau_q(t)) < \varepsilon$ for all $t \in \mathbb{R}^+$, where τ_q is a time reparametrization. That means $q * \mathbb{R}^+ \subset U(\Gamma, \varepsilon)$, so Γ is orbitally stable and the proof is completed. □

Acknowledgements: The authors thank the anonymous referees for their suggestions which led to improvements in the exposition.

Funding information: This study was funded by The National Natural Science Foundation of China (Grant Nos. 12061037, and 41801219) and the High-level Personnel of Special Support Program of Xiamen University of Technology (No. 4010520009).

Author contributions: The authors read and approve the final manuscript.

Conflict of interest: The authors state no conflict of interest.

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