

Research Article

Miguel J. Vivas-Cortez, Hasan Kara, Hüseyin Budak, Muhammad Aamir Ali*,
and Saowaluck Chasreechai*

Generalized fractional Hermite-Hadamard type inclusions for co-ordinated convex interval-valued functions

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Abstract: In this article, we introduce the notions of generalized fractional integrals for the interval-valued functions (IVFs) of two variables. We establish Hermite-Hadamard (H-H) type inequalities and some related inequalities for co-ordinated convex IVFs by using the newly defined integrals. The fundamental benefit of these inequalities is that these can be turned into classical H-H inequalities and Riemann-Liouville fractional H-H inequalities, and new k -Riemann-Liouville fractional H-H inequalities can be obtained for co-ordinated convex IVFs without having to prove each one separately.

Keywords: H-H inclusion, IVFs, fractional integral, co-ordinated convex, integral inclusions

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1 Introduction

The interval-valued analysis is a particular case of set-valued analysis. In the 1950s and 1960s, some mathematicians focused on interval analysis to put bounds on rounding errors and measurement errors in mathematical computation, and thus, they developed numerical methods that yielded more effective results. To put it in a different way, this theory was improved as an attempt to eliminate the interval uncertainty that shows up in a great many mathematical and computer models of some deterministic problems. The main purpose of interval calculus is to determine the upper and lower endpoints for the interval of values of a mapping that has one or more variables. For example, one can make sure that the temperature is somewhere between 19 and 21° C degrees by using interval arithmetic, instead of measuring the temperature of the weather as 20° C by using standard arithmetic. We also note that the interval analysis is a special case of set-valued analysis, that is, the work of the sets that form the basis of mathematical analysis and general topology.

The first book on interval analysis was written by Moore, who is known as the first user of intervals in computational mathematics [1]. After this book, several researchers began to investigate the theory and

* **Corresponding author: Muhammad Aamir Ali**, Jiangsu Key Laboratory for NSLSCS, School of Mathematical Sciences, Nanjing Normal University, Nanjing, 210023, China, e-mail: mahr.muhammad.aamir@gmail.com

* **Corresponding author: Saowaluck Chasreechai**, Department of Mathematics, Faculty of Applied Sciences, King Mongkut's University of Technology North Bangkok 10800, Thailand, e-mail: saowaluck.c@sci.kmutnb.ac.th

Miguel J. Vivas-Cortez: Pontificia Universidad Católica del Ecuador, Facultad de Ciencias Exactas y Naturales, Escuela de Ciencias Matemáticas y Físicas, Av. 12 de octubre 1076, Apartado 17-01-2184, Quito, Ecuador, e-mail: mjevivas@puce.edu.ec

Hasan Kara: Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce, Turkey, e-mail: hasan64kara@gmail.com

Hüseyin Budak: Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce, Turkey, e-mail: hsyn.budak@gmail.com

applications of interval analysis. Recently, it has had many applications because of this, and interval analysis is a useful tool in various areas interested intensely in uncertain data.

What is more, several certain inequalities have been studied for interval-valued functions (IVFs) in recent years, such as Hermite-Hadamard (H-H), and Ostrowski. In [2,3], Chalco-Cano et al. established Ostrowski-type inequalities for IVFs by utilizing the Hukuhara derivative for IVFs. However, inequalities were studied for more general set-valued maps. For example, in [4–9], the authors gave the H-H inequalities.

In recent years, some inequalities based on IVFs have been worked on by mathematicians. For instance, Sadowska [9] established the following H-H inequality for IVFs by using convexity:

Theorem 1. [9] *If $\Phi : [\varrho, \varsigma] \rightarrow \mathbb{R}_I^+$ is an interval-valued convex function such that $\Phi(\vartheta) = [\underline{\Phi}(\vartheta), \overline{\Phi}(\vartheta)]$, then we have:*

$$\Phi\left(\frac{\varrho + \varsigma}{2}\right) \supseteq \frac{1}{\varsigma - \varrho} (\text{IR}) \int_{\varrho}^{\varsigma} \Phi(\xi) d\xi \supseteq \frac{\Phi(\varrho) + \Phi(\varsigma)}{2}. \quad (1)$$

It is obvious that if $\underline{\Phi}(\vartheta) = \overline{\Phi}(\vartheta)$ in inclusion (1), then we have the following H-H inequality for convex functions (see [10–14]):

$$\Phi\left(\frac{\varrho + \varsigma}{2}\right) \leq \frac{1}{\varsigma - \varrho} \int_{\varrho}^{\varsigma} \Phi(\xi) d\xi \leq \frac{\Phi(\varrho) + \Phi(\varsigma)}{2}.$$

What has more, Budak et al. derived fractional H-H type inequalities with the help of interval-valued Riemann-Liouville fractional integrals in [15]. In [16], Tunç proved some H-H type inequalities for fractional integrals of a IVFs with respect to the real-valued function. In [17], the authors gave some applications of fractional integrals. Zhao et al. first presented a new definition of interval-valued fractional integral that is called “interval-valued generalized fractional integrals (GFIs)” in [18]. Then the authors proved some results that generalize some well-known H-H type inequalities for IVFs. On the other hand, Zhao et al., in [19], introduced the concept of interval-valued co-ordinated convex functions, and they established some H-H inequalities for this kind of function on the rectangle in the plane. In [20], Riemann-Liouville fractional integrals of two variables IVFs are defined. They established fractional H-H and some related inequalities for interval-valued co-ordinated convex functions. In [21–24], Khan et al. established different and new variants of H-H inequalities for IVFs. Moreover, in [25], Kara et al. defined interval-valued left-sided and right-sided generalized Riemann-Liouville fractional double integrals and established inequalities of H-H type for co-ordinated interval-valued convex functions.

Inspired by the ongoing studies, we define GFIs for the IVFs of two variables to develop some new H-H type inequalities for co-ordinated interval-valued convex functions. The fundamental benefit of these inequalities is that these can be turned into classical H-H inequalities for co-ordinated convex IVFs [19], Riemann-Liouville fractional H-H inequalities for co-ordinated convex IVFs [20], and new k -Riemann-Liouville fractional H-H inequalities can be obtained for co-ordinated convex IVFs without having to prove each one separately.

The structure of this article is as follows: The principles of interval-valued calculus, as well as other relevant research in this discipline, are briefly discussed in Section 2. We use GFIs for IVFs of two variables to prove H-H type inequalities for co-ordinated convex IVFs in Section 3. In Section 4, we prove several H-H inequalities for the product of two convex IVFs that are co-ordinated. The relationship between the findings given here and similar findings in the literature is also taken into account. Section 5 presents some research suggestions for the future.

2 Fractional integral of IVFs

In this section, we recall some basics of interval-valued calculus and related inequalities.

Lupulescu defined the left-sided Riemann-Liouville fractional integral with interval values in [26].

Definition 1. For an IVF $\Phi : [\varrho, \varsigma] \rightarrow \mathbb{R}_I$, where $\Phi(\vartheta) = [\underline{\Phi}(\vartheta), \overline{\Phi}(\vartheta)]$ and let $\alpha > 0$. The left-sided Riemann-Liouville fractional integral for the IVF Φ is defined by

$$\mathcal{J}_{\varrho^+}^{\alpha} \Phi(\xi) = \frac{1}{\Gamma(\alpha)} (\text{IR}) \int_{\varrho}^{\xi} (\xi - \nu)^{\alpha-1} \Phi(\vartheta) d\vartheta, \quad \xi > \varrho,$$

where Γ is Euler Gamma function.

Budak et al. in [15] defined right-sided Riemann-Liouville fractional integral of IVF Φ as follows:

$$\mathcal{J}_{\varsigma^-}^{\alpha} \Phi(\xi) = \frac{1}{\Gamma(\alpha)} (\text{IR}) \int_{\xi}^{\varsigma} (\nu - \xi)^{\alpha-1} \Phi(\vartheta) d\vartheta, \quad \xi < \varsigma,$$

where Γ is the Euler Gamma function.

Theorem 2. If $\Phi : [\varrho, \varsigma] \rightarrow \mathbb{R}_I$ is an IVF such that $\Phi(\vartheta) = [\underline{\Phi}(\vartheta), \overline{\Phi}(\vartheta)]$, then we have

$$\mathcal{J}_{\varrho^+}^{\alpha} \Phi(\xi) = [I_{\varrho^+}^{\alpha} \underline{\Phi}(\xi), I_{\varrho^+}^{\alpha} \overline{\Phi}(\xi)]$$

and

$$\mathcal{J}_{\varsigma^-}^{\alpha} \Phi(\xi) = [I_{\varsigma^-}^{\alpha} \underline{\Phi}(\xi), I_{\varsigma^-}^{\alpha} \overline{\Phi}(\xi)].$$

Definition 2. [18] Let $\Phi : [\varrho, \varsigma] \rightarrow \mathbb{R}_I$ be an IVF such that $\Phi(\vartheta) = [\underline{\Phi}(\vartheta), \overline{\Phi}(\vartheta)]$ and $\Phi \in \text{IR}_{([\varrho, \varsigma])}$. Then, the interval-valued left-sided and right-sided GFIs of a function Φ , respectively, are given as follows:

$${}_{\varrho^+} I_{\varphi} \Phi(\xi) = \frac{1}{\Gamma(\alpha)} (\text{IR}) \int_{\varrho}^{\xi} \frac{\varphi(\xi - \vartheta)}{\xi - \vartheta} \Phi(\vartheta) d\vartheta, \quad \xi > \varrho$$

and

$${}_{\varsigma^-} I_{\varphi} \Phi(\xi) = \frac{1}{\Gamma(\alpha)} (\text{IR}) \int_{\xi}^{\varsigma} \frac{\varphi(\vartheta - \xi)}{\vartheta - \xi} \Phi(\vartheta) d\vartheta, \quad \xi < \varsigma.$$

Throughout this study, for clarity, we define

$$\Lambda(\xi) = (\text{IR}) \int_0^{\xi} \frac{\varphi((\varsigma - \varrho)\vartheta)}{\vartheta} d\vartheta, \quad \Delta(\eta) = (\text{IR}) \int_0^{\eta} \frac{\psi((1 - \zeta)\nu)}{\nu} d\nu.$$

Theorem 3. [18] If $\Phi : [\varrho, \varsigma] \rightarrow \mathbb{R}_I^+$ is a convex IVF such that $\Phi(\vartheta) = [\underline{\Phi}(\vartheta), \overline{\Phi}(\vartheta)]$, then we have the following inclusion for the GFIs:

$$\Phi\left(\frac{\varrho + \varsigma}{2}\right) \supseteq \frac{1}{2\Lambda(1)} [{}_{\varrho^+} I_{\varphi} \Phi(\varsigma) + {}_{\varrho^+} I_{\varphi} \Phi(\varrho)] \supseteq \frac{\Phi(\varrho) + \Phi(\varsigma)}{2}. \quad (2)$$

Theorem 4. [18] If $\Phi, \Omega : [\varrho, \varsigma] \rightarrow \mathbb{R}_I^+$ are two convex IVFs such that $\Phi(\vartheta) = [\underline{\Phi}(\vartheta), \overline{\Phi}(\vartheta)]$ and $\Omega(\vartheta) = [\underline{\Omega}(\vartheta), \overline{\Omega}(\vartheta)]$, then we have the following inclusion for the GFIs:

$$[{}_{\varrho^+} I_{\varphi} \Phi(\varsigma) \Omega(\varsigma) + {}_{\varsigma^-} I_{\varphi} \Phi(\varrho) \Omega(\varrho)] \supseteq J_1 \mathcal{A}(\varrho, \varsigma) + J_2 \mathcal{B}(\varrho, \varsigma), \quad (3)$$

where

$$J_1 = \int_0^1 \frac{\varphi((\varsigma - \varrho)\vartheta)}{\vartheta} (2t^2 - 2t + 1) d\vartheta,$$

$$J_2 = \int_0^1 \frac{\varphi((\varsigma - \varrho)\vartheta)}{\vartheta} (2t - 2t^2) d\vartheta,$$

and

$$\mathcal{A}(\varrho, \varsigma) = \Phi(\varrho)\Omega(\varrho) + \Phi(\varsigma)\Omega(\varsigma),$$

$$\mathcal{B}(\varrho, \varsigma) = \Phi(\varrho)\Omega(\varsigma) + \Phi(\varsigma)\Omega(\varrho).$$

Theorem 5. [18] If $\Phi, \Omega : [\varrho, \varsigma] \rightarrow \mathbb{R}_I^+$ are two convex IVFs such that $\Phi(\vartheta) = [\underline{\Phi}(\vartheta), \overline{\Phi}(\vartheta)]$ and $\Omega(\vartheta) = [\underline{\Omega}(\vartheta), \overline{\Omega}(\vartheta)]$, then we have the following inclusion for the GFIs:

$$2\Phi\left(\frac{\varrho + \varsigma}{2}\right)\Omega\left(\frac{\varrho + \varsigma}{2}\right) \supseteq \frac{1}{2\Lambda(1)}[\varrho^+ I_\varphi \Phi(\varsigma)\Omega(\varsigma) + \varsigma^- I_\varphi \Phi(\varrho)\Omega(\varrho)] + \frac{1}{2\Lambda(1)}[J_2 \mathcal{A}(\varrho, \varsigma) + J_1 \mathcal{B}(\varrho, \varsigma)], \quad (4)$$

where $J_1, J_2, \mathcal{A}(\varrho, \varsigma)$, and $\mathcal{B}(\varrho, \varsigma)$ are defined as in Theorem 4.

Now recall the interval-valued double integral notion introduced by Zhao et al. in [27].

Theorem 6. [27] Let $\Delta = [\varrho, \varsigma] \times [\zeta, \iota]$. If $\Phi : \Delta \rightarrow \mathbb{R}_I$ is ID-integrable on Δ , then we have

$$(\text{ID}) \iint_{\Delta} \Phi(\nu, \vartheta) dA = (\text{IR}) \int_{\varrho}^{\varsigma} (\text{IR}) \int_{\zeta}^{\iota} \Phi(\nu, \vartheta) ds dt.$$

By applying the concepts of Lupulescu [26] and Zhao et al. [27] about interval-valued integrals, the authors defined the following interval-valued Riemann-Liouville double fractional integrals of a function $\Phi(\xi, \eta)$:

Definition 3. [20] Let $\Phi \in L_1([\varrho, \varsigma] \times [\zeta, \iota])$. The Riemann-Liouville integrals $\mathcal{J}_{\varrho^+, \zeta^+}^{\alpha, \beta}, \mathcal{J}_{\varrho^+, \iota^-}^{\alpha, \beta}, \mathcal{J}_{\varsigma^-, \zeta^+}^{\alpha, \beta}$, and $\mathcal{J}_{\varsigma^-, \iota^-}^{\alpha, \beta}$ of order $\alpha, \beta > 0$ with $\varrho, \zeta \geq 0$ are defined by

$$\mathcal{J}_{\varrho^+, \zeta^+}^{\alpha, \beta} \Phi(\xi, \eta) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} (\text{IR}) \int_{\varrho}^{\xi} \int_{\zeta}^{\eta} (\xi - \vartheta)^{\alpha-1} (\eta - \nu)^{\beta-1} \Phi(\vartheta, \nu) ds dt, \quad \xi > \varrho, \eta > \zeta,$$

$$\mathcal{J}_{\varrho^+, \iota^-}^{\alpha, \beta} \Phi(\xi, \eta) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} (\text{IR}) \int_{\varrho}^{\xi} \int_{\eta}^{\iota} (\xi - \vartheta)^{\alpha-1} (\nu - \eta)^{\beta-1} \Phi(\vartheta, \nu) ds dt, \quad \xi > \varrho, \eta > \iota,$$

$$\mathcal{J}_{\varsigma^-, \zeta^+}^{\alpha, \beta} \Phi(\xi, \eta) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} (\text{IR}) \int_{\xi}^{\varsigma} \int_{\zeta}^{\eta} (\vartheta - \xi)^{\alpha-1} (\eta - \nu)^{\beta-1} \Phi(\vartheta, \nu) ds dt, \quad \xi < \varsigma, \eta > \zeta,$$

$$\mathcal{J}_{\varsigma^-, \iota^-}^{\alpha, \beta} \Phi(\xi, \eta) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} (\text{IR}) \int_{\xi}^{\varsigma} \int_{\eta}^{\iota} (\vartheta - \xi)^{\alpha-1} (\nu - \eta)^{\beta-1} \Phi(\vartheta, \nu) ds dt, \quad \xi < \varsigma, \eta < \iota,$$

respectively.

Definition 4. [19] A function $\Phi : \Delta \rightarrow \mathbb{R}_I^+$ is said to be co-ordinated convex IVF. If the following inclusion holds

$\Phi(tx + (1 - \vartheta)\eta, su + (1 - \nu)w) \geq ts\Phi(\xi, u) + \vartheta(1 - \nu)\Phi(\xi, w) + \nu(1 - \vartheta)\Phi(\eta, u) + (1 - \nu)(1 - \vartheta)\Phi(\eta, w)$,
for all $(\xi, \eta), (u, w) \in \Delta$, and $\nu, \vartheta \in [0, 1]$.

Theorem 7. [20] If $\Phi : \Delta \rightarrow \mathbb{R}_I^+$ is a co-ordinated convex IVF on Δ such that $\Phi(\vartheta) = [\underline{\Phi}(\vartheta), \overline{\Phi}(\vartheta)]$, then the following inclusions hold:

$$\begin{aligned} \Phi\left(\frac{\varrho + \varsigma}{2}, \frac{\zeta + \iota}{2}\right) &\supseteq \frac{\Gamma(\alpha + 1)}{4(\varsigma - \varrho)^\alpha} \left[\mathcal{J}_{\varrho^+}^\alpha \Phi\left(\varsigma, \frac{\zeta + \iota}{2}\right) + \mathcal{J}_{\varsigma^-}^\alpha \Phi\left(\varrho, \frac{\zeta + \iota}{2}\right) \right] \\ &\quad + \frac{\Gamma(\beta + 1)}{4(\iota - \zeta)^\beta} \left[\mathcal{J}_{\zeta^+}^\beta \Phi\left(\frac{\varrho + \varsigma}{2}, \iota\right) + \mathcal{J}_{\iota^-}^\beta \Phi\left(\frac{\varrho + \varsigma}{2}, \zeta\right) \right] \\ &\supseteq \frac{\Gamma(\alpha + 1)\Gamma(\beta + 1)}{4(\varsigma - \varrho)^\alpha(\iota - \zeta)^\beta} [\mathcal{J}_{\varrho^+, \zeta^+}^{\alpha, \beta} \Phi(\varsigma, \iota) + \mathcal{J}_{\varrho^+, \iota^-}^{\alpha, \beta} \Phi(\varsigma, \zeta) + \mathcal{J}_{\varsigma^-, \zeta^+}^{\alpha, \beta} \Phi(\varrho, \iota) + \mathcal{J}_{\varsigma^-, \iota^-}^{\alpha, \beta} \Phi(\varrho, \zeta)] \\ &\supseteq \frac{\Gamma(\alpha + 1)}{8(\varsigma - \varrho)^\alpha} [\mathcal{J}_{\varrho^+}^\alpha \Phi(\varsigma, \zeta) + \mathcal{J}_{\varrho^+}^\alpha \Phi(\varsigma, \iota) + \mathcal{J}_{\varsigma^-}^\alpha \Phi(\varrho, \zeta) + \mathcal{J}_{\varsigma^-}^\alpha \Phi(\varrho, \iota)] \\ &\quad + \frac{\Gamma(\beta + 1)}{4(\iota - \zeta)^\beta} [\mathcal{J}_{\zeta^+}^\beta \Phi(\varrho, \iota) + \mathcal{J}_{\zeta^+}^\beta \Phi(\varsigma, \iota) + \mathcal{J}_{\iota^-}^\beta \Phi(\varrho, \zeta) + \mathcal{J}_{\iota^-}^\beta \Phi(\varsigma, \zeta)] \\ &\supseteq \frac{\Phi(\varrho, \zeta) + \Phi(\varrho, \iota) + \Phi(\varsigma, \zeta) + \Phi(\varsigma, \iota)}{4}. \end{aligned} \quad (5)$$

Theorem 8. [20] Let $\Phi, \Omega : \Delta := [\varrho, \varsigma] \times [\zeta, \iota] \rightarrow \mathbb{R}_I^+$ be two co-ordinated convex IVFs such that $\Phi(\vartheta) = [\underline{\Phi}(\vartheta), \overline{\Phi}(\vartheta)]$ and $\Omega(\vartheta) = [\underline{\Omega}(\vartheta), \overline{\Omega}(\vartheta)]$, then we have following H-H type inclusions:

$$\begin{aligned} &\frac{\Gamma(\alpha + 1)\Gamma(\beta + 1)}{4(\varsigma - \varrho)^\alpha(\iota - \zeta)^\beta} [\mathcal{J}_{\varrho^+, \zeta^+}^{\alpha, \beta} \Phi(\varsigma, \iota)\Omega(\varsigma, \iota) + \mathcal{J}_{\varrho^+, \iota^-}^{\alpha, \beta} \Phi(\varsigma, \zeta)\Omega(\varsigma, \zeta) + \mathcal{J}_{\varsigma^-, \zeta^+}^{\alpha, \beta} \Phi(\varrho, \iota)\Omega(\varrho, \iota) + \mathcal{J}_{\varsigma^-, \iota^-}^{\alpha, \beta} \Phi(\varrho, \zeta)\Omega(\varrho, \zeta)] \\ &\supseteq \left(\frac{1}{2} - \frac{\beta}{(\beta + 1)(\beta + 2)} \right) \left(\frac{1}{2} - \frac{\alpha}{(\alpha + 1)(\alpha + 2)} \right) \mathbf{K}(\varrho, \varsigma, \zeta, \iota) \\ &\quad + \left(\frac{1}{2} - \frac{\beta}{(\beta + 1)(\beta + 2)} \right) \left(\frac{\alpha}{(\alpha + 1)(\alpha + 2)} \right) \mathbf{L}(\varrho, \varsigma, \zeta, \iota) \\ &\quad + \left(\frac{\beta}{(\beta + 1)(\beta + 2)} \right) \left(\frac{1}{2} - \frac{\alpha}{(\alpha + 1)(\alpha + 2)} \right) \mathbf{M}(\varrho, \varsigma, \zeta, \iota) \\ &\quad + \left(\frac{\beta}{(\beta + 1)(\beta + 2)} \right) \left(\frac{\alpha}{(\alpha + 1)(\alpha + 2)} \right) \mathbf{N}(\varrho, \varsigma, \zeta, \iota), \end{aligned} \quad (6)$$

and

$$\begin{aligned} &4\Phi\left(\frac{\varrho + \varsigma}{2}, \frac{\zeta + \iota}{2}\right)\Omega\left(\frac{\varrho + \varsigma}{2}, \frac{\zeta + \iota}{2}\right) \\ &\supseteq \frac{\Gamma(\alpha + 1)\Gamma(\beta + 1)}{4(\varsigma - \varrho)^\alpha(\iota - \zeta)^\beta} [\mathcal{J}_{\varrho^+, \zeta^+}^{\alpha, \beta} \Phi(\varsigma, \iota)\Omega(\varsigma, \iota) + \mathcal{J}_{\varrho^+, \iota^-}^{\alpha, \beta} \Phi(\varsigma, \zeta)\Omega(\varsigma, \zeta) \\ &\quad + \mathcal{J}_{\varsigma^-, \zeta^+}^{\alpha, \beta} \Phi(\varrho, \iota)\Omega(\varrho, \iota) + \mathcal{J}_{\varsigma^-, \iota^-}^{\alpha, \beta} \Phi(\varrho, \zeta)\Omega(\varrho, \zeta)] \\ &\quad + \left[\frac{\alpha}{2(\alpha + 1)(\alpha + 2)} + \frac{\beta}{(\beta + 1)(\beta + 2)} \left(\frac{1}{2} - \frac{\alpha}{(\alpha + 1)(\alpha + 2)} \right) \right] \mathbf{K}(\varrho, \varsigma, \zeta, \iota) \\ &\quad + \left[\frac{1}{2} \left(\frac{1}{2} - \frac{\alpha}{(\alpha + 1)(\alpha + 2)} \right) + \frac{\alpha}{(\alpha + 1)(\alpha + 2)} \frac{\beta}{(\beta + 1)(\beta + 2)} \right] \mathbf{L}(\varrho, \varsigma, \zeta, \iota) \\ &\quad + \left[\frac{1}{2} \left(\frac{1}{2} - \frac{\beta}{(\beta + 1)(\beta + 2)} \right) + \frac{\alpha}{(\alpha + 1)(\alpha + 2)} \frac{\beta}{(\beta + 1)(\beta + 2)} \right] \mathbf{M}(\varrho, \varsigma, \zeta, \iota) \\ &\quad + \left[\frac{1}{4} - \frac{\alpha}{(\alpha + 1)(\alpha + 2)} \frac{\beta}{(\beta + 1)(\beta + 2)} \right] \mathbf{N}(\varrho, \varsigma, \zeta, \iota), \end{aligned} \quad (7)$$

where

$$\begin{aligned} \mathbf{K}(\varrho, \varsigma, \zeta, \iota) &= \Phi(\varrho, \zeta)\Omega(\varrho, \zeta) + \Phi(\varsigma, \zeta)\Omega(\varsigma, \zeta) + \Phi(\varrho, \iota)\Omega(\varrho, \iota) + \Phi(\varsigma, \iota)\Omega(\varsigma, \iota), \\ \mathbf{L}(\varrho, \varsigma, \zeta, \iota) &= \Phi(\varrho, \zeta)\Omega(\varsigma, \zeta) + \Phi(\varsigma, \zeta)\Omega(\varrho, \zeta) + \Phi(\varrho, \iota)\Omega(\varsigma, \iota) + \Phi(\varsigma, \iota)\Omega(\varrho, \iota), \\ \mathbf{M}(\varrho, \varsigma, \zeta, \iota) &= \Phi(\varrho, \zeta)\Omega(\varrho, \iota) + \Phi(\varsigma, \zeta)\Omega(\varsigma, \iota) + \Phi(\varrho, \iota)\Omega(\varrho, \zeta) + \Phi(\varsigma, \iota)\Omega(\varsigma, \zeta), \end{aligned}$$

and

$$\mathbf{N}(\varrho, \varsigma, \zeta, \iota) = \Phi(\varrho, \zeta)\Omega(\varsigma, \iota) + \Phi(\varsigma, \zeta)\Omega(\varrho, \iota) + \Phi(\varrho, \iota)\Omega(\varsigma, \zeta) + \Phi(\varsigma, \iota)\Omega(\varrho, \zeta).$$

3 Generalized H-H type inclusions for co-ordinated convex IVFs

In this section, we present the definitions of GFIs for IVFs of two variables and prove H-H type inclusions for co-ordinated convex IVFs via newly defined integrals. For brevity, we use the notations $\Phi(\xi, \eta) = [\underline{\Phi}(\xi, \eta), \overline{\Phi}(\xi, \eta)]$ and $\Omega(\xi, \eta) = [\underline{\Omega}(\xi, \eta), \overline{\Omega}(\xi, \eta)]$ throughout the article:

Definition 5. Let $\Phi \in \mathcal{IR}_{([\varrho, \varsigma] \times [\zeta, \iota])}$. The interval-valued GFIs are defined by

$$\begin{aligned} {}_{\varrho^+, \zeta^+} I_{\varphi, \psi} \Phi(\xi, \eta) &= (\text{IR}) \int_{\varrho}^{\xi} \int_{\zeta}^{\eta} \frac{\varphi(\xi - \vartheta)}{\xi - \vartheta} \frac{\psi(\eta - \upsilon)}{\eta - \upsilon} \Phi(\vartheta, \upsilon) d\varsigma dt, \quad \xi > \varrho, \eta > \zeta, \\ {}_{\varrho^+, \iota^-} I_{\varphi, \psi} \Phi(\xi, \eta) &= (\text{IR}) \int_{\varrho}^{\xi} \int_{\eta}^{\iota} \frac{\varphi(\xi - \vartheta)}{\xi - \vartheta} \frac{\psi(\upsilon - \eta)}{\upsilon - \eta} \Phi(\vartheta, \upsilon) d\varsigma dt, \quad \xi > \varrho, \eta < \iota, \\ {}_{\varsigma^-, \zeta^+} I_{\varphi, \psi} \Phi(\xi, \eta) &= (\text{IR}) \int_{\xi}^{\varsigma} \int_{\zeta}^{\eta} \frac{\varphi(\vartheta - \xi)}{\vartheta - \xi} \frac{\psi(\eta - \upsilon)}{\eta - \upsilon} \Phi(\vartheta, \upsilon) d\varsigma dt, \quad \xi < \varsigma, \eta > \zeta, \end{aligned}$$

and

$${}_{\varsigma^-, \iota^-} I_{\varphi, \psi} \Phi(\xi, \eta) = (\text{IR}) \int_{\xi}^{\varsigma} \int_{\eta}^{\iota} \frac{\varphi(\vartheta - \xi)}{\vartheta - \xi} \frac{\psi(\upsilon - \eta)}{\upsilon - \eta} \Phi(\vartheta, \upsilon) d\varsigma dt, \quad \xi < \varsigma, \eta < \iota.$$

Theorem 9. Let $\Phi : \Delta := [\varrho, \varsigma] \times [\zeta, \iota] \rightarrow \mathbb{R}_I^+$ be co-ordinated convex IVF on Δ with $\varrho < \varsigma, \zeta < \iota$, and $\Phi \in \mathbf{L}_1(\Delta)$ such that $\Phi(\xi, \eta) = [\underline{\Phi}(\xi, \eta), \overline{\Phi}(\xi, \eta)]$. Then one has the following inclusions:

$$\begin{aligned} \Phi\left(\frac{\varrho + \varsigma}{2}, \frac{\zeta + \iota}{2}\right) &\supseteq \frac{1}{4\Lambda(1)} \left[{}_{\varrho^+} I_{\varphi} \Phi\left(\varsigma, \frac{\zeta + \iota}{2}\right) + {}_{\varsigma^-} I_{\varphi} \Phi\left(\varrho, \frac{\zeta + \iota}{2}\right) \right] \\ &\quad + \frac{1}{4\Delta(1)} \left[{}_{\zeta^+} I_{\varphi} \Phi\left(\frac{\varrho + \varsigma}{2}, \iota\right) + {}_{\iota^-} I_{\psi} \Phi\left(\frac{\varrho + \varsigma}{2}, \zeta\right) \right] \\ &\supseteq \frac{1}{\Lambda(1)\Delta(1)} [{}_{\varrho^+, \zeta^+} I_{\varphi, \psi} \Phi(\varsigma, \iota) + {}_{\varrho^+, \iota^-} I_{\varphi, \psi} \Phi(\varsigma, \zeta) + {}_{\varsigma^-, \zeta^+} I_{\varphi, \psi} \Phi(\varrho, \iota) + {}_{\varsigma^-, \iota^-} I_{\varphi, \psi} \Phi(\varrho, \zeta)] \\ &\supseteq \frac{1}{8\Lambda(1)} [{}_{\varrho^+} I_{\varphi} \Phi(\varsigma, \zeta) + {}_{\varrho^+} I_{\varphi} \Phi(\varsigma, \iota) + {}_{\varsigma^-} I_{\varphi} \Phi(\varrho, \zeta) + {}_{\varsigma^-} I_{\varphi} \Phi(\varrho, \iota)] \\ &\quad + \frac{1}{8\Delta(1)} [{}_{\zeta^+} I_{\psi} \Phi(\varrho, \iota) + {}_{\zeta^+} I_{\psi} \Phi(\varsigma, \iota) + {}_{\iota^-} I_{\psi} \Phi(\varrho, \zeta) + {}_{\iota^-} I_{\psi} \Phi(\varsigma, \zeta)] \\ &\supseteq \frac{\Phi(\varrho, \zeta) + \Phi(\varrho, \iota) + \Phi(\varsigma, \zeta) + \Phi(\varsigma, \iota)}{4}. \end{aligned} \tag{8}$$

Proof. Since $\Phi : \Delta \rightarrow \mathbb{R}_I^+$ is a convex IVF on co-ordinates, it follows that the mapping $\Omega_\xi : [\zeta, \iota] \rightarrow \mathbb{R}_I^+$, $\Omega_\xi(\eta) = \Phi(\xi, \eta)$, is convex on $[\zeta, \iota]$ for all $\xi \in [\zeta, \iota]$ for all $\xi \in [\varrho, \varsigma]$. Then by using inclusion (2), for all $\xi \in [\varrho, \varsigma]$, we can write

$$\Omega_\xi\left(\frac{\zeta + \iota}{2}\right) \supseteq \frac{1}{2\Delta(1)}[\zeta^+ I_\varphi \Omega_\xi(\iota) + \iota^- I_\varphi \Omega_\xi(\zeta)] \supseteq \frac{\Omega_\xi(\zeta) + \Omega_\xi(\iota)}{2}.$$

That is,

$$\Phi\left(\xi, \frac{\zeta + \iota}{2}\right) \supseteq \frac{1}{2\Delta(1)} \left[(\text{IR}) \int_{\zeta}^{\iota} \frac{\varphi(\iota - \eta)}{\iota - \eta} \Phi(\xi, \eta) d\eta + (\text{IR}) \int_{\zeta}^{\iota} \frac{\varphi(\eta - \zeta)}{\eta - \zeta} \Phi(\xi, \eta) d\eta \right] \frac{\Phi(\xi, \zeta) + \Phi(\xi, \iota)}{2} \quad (9)$$

for all $\xi \in [\varrho, \varsigma]$. Then by multiplying both sides of (9) by $\frac{\varphi(\varsigma - \xi)}{\varsigma - \xi}$ and $\frac{\varphi(\xi - \varrho)}{\xi - \varrho}$ and integrating with respect to ξ over $[\varrho, \varsigma]$, we have

$$\begin{aligned} & (\text{IR}) \int_{\varrho}^{\varsigma} \frac{\varphi(\varsigma - \xi)}{\varsigma - \xi} \Phi\left(\xi, \frac{\zeta + \iota}{2}\right) d\xi \\ & \supseteq \frac{1}{2\Delta(1)} \left[(\text{IR}) \int_{\varrho}^{\varsigma} \int_{\zeta}^{\iota} \frac{\varphi(\varsigma - \xi)}{\varsigma - \xi} \frac{\varphi(\iota - \eta)}{\iota - \eta} \Phi(\xi, \eta) d\xi d\eta + (\text{IR}) \int_{\varrho}^{\varsigma} \int_{\zeta}^{\iota} \frac{\varphi(\varsigma - \xi)}{\varsigma - \xi} \frac{\varphi(\eta - \zeta)}{\eta - \zeta} \Phi(\xi, \eta) d\xi d\eta \right] \\ & \supseteq \frac{1}{2} \left[(\text{IR}) \int_{\varrho}^{\varsigma} \frac{\varphi(\varsigma - \xi)}{\varsigma - \xi} \Phi(\xi, \zeta) d\xi + (\text{IR}) \int_{\varrho}^{\varsigma} \frac{\varphi(\varsigma - \xi)}{\varsigma - \xi} \Phi(\xi, \iota) d\xi \right] \end{aligned} \quad (10)$$

and

$$\begin{aligned} & (\text{IR}) \int_{\varrho}^{\varsigma} \frac{\varphi(\xi - \varrho)}{\xi - \varrho} \Phi\left(\xi, \frac{\zeta + \iota}{2}\right) d\xi \\ & \supseteq \frac{1}{2\Delta(1)} \left[(\text{IR}) \int_{\varrho}^{\varsigma} \int_{\zeta}^{\iota} \frac{\varphi(\xi - \varrho)}{\xi - \varrho} \frac{\varphi(\iota - \eta)}{\iota - \eta} \Phi(\xi, \eta) d\xi d\eta + (\text{IR}) \int_{\varrho}^{\varsigma} \int_{\zeta}^{\iota} \frac{\varphi(\xi - \varrho)}{\xi - \varrho} \frac{\varphi(\eta - \zeta)}{\eta - \zeta} \Phi(\xi, \eta) d\xi d\eta \right] \\ & \supseteq \frac{1}{2} \left[(\text{IR}) \int_{\varrho}^{\varsigma} \frac{\varphi(\xi - \varrho)}{\xi - \varrho} \Phi(\xi, \zeta) d\xi + (\text{IR}) \int_{\varrho}^{\varsigma} \frac{\varphi(\xi - \varrho)}{\xi - \varrho} \Phi(\xi, \iota) d\xi \right]. \end{aligned} \quad (11)$$

By the similar argument applied for the mapping $\Omega_\eta : [\varrho, \varsigma] \rightarrow \mathbb{R}_I^+$, $\Omega_\eta : \Phi(\xi, \eta)$, we have

$$\begin{aligned} & (\text{IR}) \int_{\zeta}^{\iota} \frac{\varphi(\iota - \eta)}{\iota - \eta} \Phi\left(\frac{\varrho + \varsigma}{2}, \eta\right) d\eta \\ & \supseteq \frac{1}{2\Delta(1)} \left[(\text{IR}) \int_{\varrho}^{\varsigma} \int_{\zeta}^{\iota} \frac{\varphi(\varsigma - \xi)}{\varsigma - \xi} \frac{\varphi(\iota - \eta)}{\iota - \eta} \Phi(\xi, \eta) d\xi d\eta + (\text{IR}) \int_{\varrho}^{\varsigma} \int_{\zeta}^{\iota} \frac{\varphi(\xi - \varrho)}{\xi - \varrho} \frac{\varphi(\iota - \eta)}{\iota - \eta} \Phi(\xi, \eta) d\xi d\eta \right] \\ & \supseteq \frac{1}{2} \left[(\text{IR}) \int_{\zeta}^{\iota} \frac{\varphi(\iota - \eta)}{\iota - \eta} \Phi(\varrho, \eta) d\eta + (\text{IR}) \int_{\zeta}^{\iota} \frac{\varphi(\iota - \eta)}{\iota - \eta} \Phi(\varsigma, \eta) d\eta \right] \end{aligned} \quad (12)$$

and

$$\begin{aligned}
& (\text{IR}) \int_{\zeta}^{\iota} \frac{\varphi(\eta - \zeta)}{\eta - \zeta} \Phi\left(\frac{\varrho + \varsigma}{2}, \eta\right) d\eta \\
& \geq \frac{1}{2\Lambda(1)} \left[(\text{IR}) \int_{\varrho}^{\varsigma} \int_{\zeta}^{\iota} \frac{\varphi(\varsigma - \xi)}{\varsigma - \xi} \frac{\varphi(\eta - \zeta)}{\eta - \zeta} \Phi(\xi, \eta) d\xi d\eta + (\text{IR}) \int_{\varrho}^{\varsigma} \int_{\zeta}^{\iota} \frac{\varphi(\eta - \zeta)}{\eta - \zeta} \frac{\varphi(\iota - \eta)}{\iota - \eta} \Phi(\xi, \eta) d\xi d\eta \right] \\
& \geq \frac{1}{2} \left[(\text{IR}) \int_{\varrho}^{\varsigma} \frac{\varphi(\eta - \zeta)}{\eta - \zeta} \Phi(\varrho, \eta) d\eta + (\text{IR}) \int_{\varrho}^{\varsigma} \frac{\varphi(\eta - \zeta)}{\eta - \zeta} \Phi(\varsigma, \eta) d\eta \right].
\end{aligned} \tag{13}$$

By adding inclusions (10)–(13), we have

$$\begin{aligned}
& \frac{1}{\Lambda(1)} \left[{}_{\varrho^+} I_{\varphi} \Phi\left(\varsigma, \frac{\zeta + \iota}{2}\right) + {}_{\varsigma^-} I_{\varphi} \Phi\left(\varrho, \frac{\zeta + \iota}{2}\right) \right] + \frac{1}{\Lambda(1)} \left[{}_{\varsigma^+} I_{\varphi} \Phi\left(\frac{\varrho + \varsigma}{2}, \iota\right) + {}_{\iota^-} I_{\psi} \Phi\left(\frac{\varrho + \varsigma}{2}, \zeta\right) \right] (\text{IR}) \int_0^1 \frac{\varphi((\varsigma - \varrho)\vartheta)}{\vartheta} d\vartheta \\
& \geq \frac{1}{\Lambda(1)\Delta(1)} [{}_{\varrho^+, \varsigma^+} I_{\varphi, \psi} \Phi(\varsigma, \iota) + {}_{\varrho^+, \iota^-} I_{\varphi, \psi} \Phi(\varsigma, \zeta) + {}_{\varsigma^-, \zeta^+} I_{\varphi, \psi} \Phi(\varrho, \iota) + {}_{\varsigma^-, \iota^-} I_{\varphi, \psi} \Phi(\varrho, \zeta)] \\
& \geq \frac{1}{2\Lambda(1)} [{}_{\varrho^+} I_{\varphi} \Phi(\varsigma, \zeta) + {}_{\varrho^+} I_{\varphi} \Phi(\varsigma, \iota) + {}_{\varsigma^-} I_{\varphi} \Phi(\varrho, \zeta) + {}_{\varsigma^-} I_{\varphi} \Phi(\varrho, \iota)] \\
& \quad + \frac{1}{2\Lambda(1)} [{}_{\varsigma^+} I_{\varphi} \Phi(\varrho, \iota) + {}_{\varsigma^+} I_{\varphi} \Phi(\varsigma, \iota) + {}_{\iota^-} I_{\varphi} \Phi(\varrho, \zeta) + {}_{\iota^-} I_{\varphi} \Phi(\varsigma, \zeta)],
\end{aligned}$$

which gives the second and the third inclusions in (8).

Now, by using the first inclusion in (2), we also have

$$\Phi\left(\frac{\varrho + \varsigma}{2}, \frac{\zeta + \iota}{2}\right) \geq \frac{1}{2\Lambda(1)} \left[(\text{IR}) \int_{\varrho}^{\varsigma} \frac{\varphi(\varsigma - \xi)}{\varsigma - \xi} \Phi\left(\xi, \frac{\zeta + \iota}{2}\right) d\xi + (\text{IR}) \int_{\varrho}^{\varsigma} \frac{\varphi(\xi - \varrho)}{\xi - \varrho} \Phi\left(\xi, \frac{\zeta + \iota}{2}\right) d\xi \right]$$

and

$$\Phi\left(\frac{\varrho + \varsigma}{2}, \frac{\zeta + \iota}{2}\right) \geq \frac{1}{2\Lambda(1)} \left[(\text{IR}) \int_{\zeta}^{\iota} \frac{\varphi(\iota - \eta)}{\iota - \eta} \Phi\left(\frac{\varrho + \varsigma}{2}, \eta\right) d\eta + (\text{IR}) \int_{\zeta}^{\iota} \frac{\varphi(\eta - \zeta)}{\eta - \zeta} \Phi\left(\frac{\varrho + \varsigma}{2}, \eta\right) d\eta \right].$$

Adding the aforementioned inclusions, we have

$$\Phi\left(\frac{\varrho + \varsigma}{2}, \frac{\zeta + \iota}{2}\right) \geq \frac{1}{4\Lambda(1)} \left[{}_{\varrho^+} I_{\varphi} \Phi\left(\varsigma, \frac{\zeta + \iota}{2}\right) + {}_{\varsigma^-} I_{\varphi} \Phi\left(\varrho, \frac{\zeta + \iota}{2}\right) \right] + \frac{1}{4\Lambda(1)} \left[{}_{\varsigma^+} I_{\varphi} \Phi\left(\frac{\varrho + \varsigma}{2}, \iota\right) + {}_{\iota^-} I_{\psi} \Phi\left(\frac{\varrho + \varsigma}{2}, \zeta\right) \right],$$

which gives the first inclusion in (8).

Finally, by using the second inclusion in (2), we can also state,

$$\begin{aligned}
& \frac{1}{2\Lambda(1)} \left[(\text{IR}) \int_{\varrho}^{\varsigma} \frac{\varphi(\varsigma - \xi)}{\varsigma - \xi} \Phi(\xi, \zeta) d\xi + (\text{IR}) \int_{\varrho}^{\varsigma} \frac{\varphi(\xi - \varrho)}{\xi - \varrho} \Phi(\xi, \zeta) d\xi \right] \geq \frac{\Phi(\varrho, \zeta) + \Phi(\varsigma, \zeta)}{2} \\
& \frac{1}{2\Lambda(1)} \left[(\text{IR}) \int_{\varrho}^{\varsigma} \frac{\varphi(\varsigma - \xi)}{\varsigma - \xi} \Phi(\xi, \iota) d\xi + (\text{IR}) \int_{\varrho}^{\varsigma} \frac{\varphi(\xi - \varrho)}{\xi - \varrho} \Phi(\xi, \iota) d\xi \right] \geq \frac{\Phi(\varrho, \iota) + \Phi(\varsigma, \iota)}{2} \\
& \frac{1}{2\Lambda(1)} \left[(\text{IR}) \int_{\zeta}^{\iota} \frac{\varphi(\iota - \eta)}{\iota - \eta} \Phi(\varrho, \eta) d\eta + (\text{IR}) \int_{\zeta}^{\iota} \frac{\varphi(\eta - \zeta)}{\eta - \zeta} \Phi(\varrho, \eta) d\eta \right] \geq \frac{\Phi(\varrho, \zeta) + \Phi(\varrho, \iota)}{2}
\end{aligned}$$

and

$$\frac{1}{2\Delta(1)} \left[(\text{IR}) \int_{\zeta}^{\mathfrak{t}} \frac{\varphi(\mathfrak{t} - \eta)}{\mathfrak{t} - \eta} \Phi(\zeta, \eta) d\eta + (\text{IR}) \int_{\zeta}^{\mathfrak{t}} \frac{\varphi(\eta - \zeta)}{\eta - \zeta} \Phi(\zeta, \eta) d\eta \right] \geq \frac{\Phi(\zeta, \zeta) + \Phi(\zeta, \mathfrak{t})}{2}.$$

By adding the last four inclusions, we obtain

$$\begin{aligned} & \frac{1}{8\Delta(1)} [\mathfrak{e}^+ I_{\varphi} \Phi(\zeta, \zeta) + {}^{\vartheta} \mathfrak{e}^+ \Phi(\zeta, \mathfrak{t}) + {}_{\zeta}^- I_{\varphi} \Phi(\mathfrak{e}, \zeta) + {}_{\zeta}^- I_{\varphi} \Phi(\mathfrak{e}, \mathfrak{t})] \\ & + \frac{1}{8\Delta(1)} [\zeta^+ I_{\varphi} \Phi(\mathfrak{e}, \mathfrak{t}) + \zeta^+ I_{\varphi} \Phi(\zeta, \mathfrak{t}) + {}_{\mathfrak{t}}^- I_{\varphi} \Phi(\mathfrak{e}, \zeta) \\ & + {}_{\mathfrak{t}}^- I_{\varphi} \Phi(\zeta, \zeta)] \geq \frac{\Phi(\mathfrak{e}, \zeta) + \Phi(\mathfrak{e}, \mathfrak{t}) + \Phi(\zeta, \zeta) + \Phi(\zeta, \mathfrak{t})}{4}, \end{aligned} \quad (14)$$

and the proof is ended. $\square$

Remark 1. Under the assumption of Theorem 9 with $\varphi(\vartheta) = \vartheta$ and $\psi(\nu) = \nu$, Theorem 9 reduces to [19, Theorem 7].

Remark 2. Under the assumption of Theorem 9 with $\varphi(\vartheta) = \frac{\vartheta^{\alpha}}{\Gamma(\alpha)}$ and $\psi(\nu) = \frac{\nu^{\beta}}{\Gamma(\beta)}$, the inclusion (8) reduces to inclusion (5).

4 Generalized fractional H-H type inclusions for product of co-ordinated convex IVFs

In this section, we establish H-H type inclusions for the product of co-ordinated convex IVFs via the GFIs. Throughout this section, we suppose I_1 and I_2 as follows:

$$I_1 = \int_0^1 \frac{\psi((\mathfrak{t} - \zeta)\nu)}{\nu} (2s^2 - 2s + 1) d\nu$$

and

$$I_2 = \int_0^1 \frac{\psi((\mathfrak{t} - \zeta)\nu)}{\nu} (2s - 2s^2) d\nu.$$

Theorem 10. Let $\Phi, \Omega : \Delta := [\mathfrak{e}, \zeta] \times [\zeta, \mathfrak{t}] \rightarrow \mathbb{R}_I^+$ be two co-ordinated convex IVFs on Δ , then we have the following H-H type inclusion for the GFIs:

$$\begin{aligned} & [\mathfrak{e}^+, \zeta^+ I_{\varphi, \psi} \Phi(\zeta, \mathfrak{t}) \Omega(\zeta, \mathfrak{t}) + \mathfrak{e}^+, {}_{\mathfrak{t}}^- I_{\varphi, \psi} \Phi(\zeta, \zeta) \Omega(\zeta, \zeta)] + [\zeta^-, \zeta^+ I_{\varphi, \psi} \Phi(\mathfrak{e}, \mathfrak{t}) \Omega(\mathfrak{e}, \mathfrak{t}) + {}_{\zeta}^-, {}_{\mathfrak{t}}^- I_{\varphi, \psi} \Phi(\mathfrak{e}, \zeta) \Omega(\mathfrak{e}, \zeta)] \\ & \geq I_1 J_1 \mathbf{K}(\mathfrak{e}, \zeta, \zeta, \mathfrak{t}) + I_1 J_2 \mathbf{L}(\mathfrak{e}, \zeta, \zeta, \mathfrak{t}) + I_2 J_1 \mathbf{M}(\mathfrak{e}, \zeta, \zeta, \mathfrak{t}) + I_2 J_2 \mathbf{N}(\mathfrak{e}, \zeta, \zeta, \mathfrak{t}), \end{aligned}$$

where $\mathbf{K}(\mathfrak{e}, \zeta, \zeta, \mathfrak{t})$, $\mathbf{L}(\mathfrak{e}, \zeta, \zeta, \mathfrak{t})$, $\mathbf{M}(\mathfrak{e}, \zeta, \zeta, \mathfrak{t})$ and $\mathbf{N}(\mathfrak{e}, \zeta, \zeta, \mathfrak{t})$ are the same as in Theorem 8, and J_1 and J_2 are defined in Theorem 4.

Proof. Since Φ and Ω are co-ordinated convex IVFs on Δ , if we define the mappings $\Phi_{\xi} : [\zeta, \mathfrak{t}] \rightarrow \mathbb{R}_I^+$, $\Phi_{\xi}(\eta) = \Phi(\xi, \eta)$, and $\Omega_{\xi} : [\zeta, \mathfrak{t}] \rightarrow \mathbb{R}_I^+$, $\Omega_{\xi}(\eta) = \Omega(\xi, \eta)$, then $\Phi_{\xi}(\eta)$ and $\Omega_{\xi}(\eta)$ are convex on $[\zeta, \mathfrak{t}]$ for all $\xi \in [\mathfrak{e}, \zeta]$. If we apply the inclusion (3) for the convex functions $\Phi_{\xi}(\eta)$ and $\Omega_{\xi}(\eta)$, then we have

$$[\zeta^+ I_{\psi} \Phi_{\xi}(\mathfrak{t}) \Omega_{\xi}(\mathfrak{t}) + {}_{\mathfrak{t}}^- I_{\psi} \Phi_{\xi}(\zeta) \Omega_{\xi}(\zeta)] \geq I_1 [\Phi_{\xi}(\zeta) \Omega_{\xi}(\zeta) + \Phi_{\xi}(\mathfrak{t}) \Omega_{\xi}(\mathfrak{t})] + I_2 [\Phi_{\xi}(\zeta) \Omega_{\xi}(\mathfrak{t}) + \Phi_{\xi}(\mathfrak{t}) \Omega_{\xi}(\zeta)]. \quad (15)$$

That is,

$$\left[(\text{IR}) \int_{\zeta}^{\iota} \frac{\psi((\iota - \eta)v)}{v} \Phi(\xi, \eta) \Omega(\xi, \eta) d\eta + (\text{IR}) \int_{\zeta}^{\iota} \frac{\psi((\eta - \zeta)v)}{v} \Phi(\xi, \eta) \Omega(\xi, \eta) d\eta \right] \quad (16)$$

$$\geq I_1[\Phi(\xi, \zeta) \Omega(\xi, \zeta) + \Phi(\xi, \iota) \Omega(\xi, \iota)] + I_2[\Phi(\xi, \zeta) \Omega(\xi, \iota) + \Phi(\xi, \iota) \Omega(\xi, \zeta)].$$

By multiplying inclusion (16) by $\frac{\varphi((\zeta - \xi)\vartheta)}{g}$ and integrating the resulting inclusion with respect to ξ from ϱ to ς , we obtain

$$[\varrho^+, \varsigma^+ I_{\varphi, \psi} \Phi(\varsigma, \iota) \Omega(\varsigma, \iota) + \varrho^+, \iota^- I_{\varphi, \psi} \Phi(\varsigma, \zeta) \Omega(\varsigma, \zeta)] \geq I_1[\varrho^+ I_{\varphi} \Phi(\varsigma, \zeta) \Omega(\varsigma, \zeta) + \varrho^+ I_{\varphi} \Phi(\varsigma, \iota) \Omega(\varsigma, \iota)] \quad (17)$$

$$+ I_2[\varrho^+ I_{\varphi} \Phi(\varsigma, \zeta) \Omega(\varsigma, \iota) + \varrho^+ I_{\varphi} \Phi(\varsigma, \iota) \Omega(\varsigma, \zeta)].$$

Similarly, by multiplying inclusion (16) by $\frac{\varphi(\xi - \varrho)}{\xi - \varrho}$ and integrating the resulting inclusion with respect to ξ on $[\varrho, \varsigma]$, we have

$$[\varsigma^-, \varrho^+ I_{\varphi, \psi} \Phi(\varrho, \iota) \Omega(\varrho, \iota) + \varsigma^-, \iota^- I_{\varphi, \psi} \Phi(\varrho, \zeta) \Omega(\varrho, \zeta)] \geq I_1[\varsigma^- I_{\varphi} \Phi(\varrho, \zeta) \Omega(\varsigma, \zeta) + \varsigma^- I_{\varphi} \Phi(\varrho, \iota) \Omega(\varrho, \iota)] \quad (18)$$

$$+ I_2[\varsigma^- I_{\varphi} \Phi(\varrho, \zeta) \Omega(\varrho, \iota) + \varsigma^- I_{\varphi} \Phi(\varrho, \iota) \Omega(\varrho, \zeta)].$$

From inclusions (17) and (18), we obtain

$$[\varrho^+, \varsigma^+ I_{\varphi, \psi} \Phi(\varsigma, \iota) \Omega(\varsigma, \iota) + \varrho^+, \iota^- I_{\varphi, \psi} \Phi(\varsigma, \zeta) \Omega(\varsigma, \zeta) + \varsigma^-, \varrho^+ I_{\varphi, \psi} \Phi(\varrho, \iota) \Omega(\varrho, \iota) + \varsigma^-, \iota^- I_{\varphi, \psi} \Phi(\varrho, \zeta) \Omega(\varrho, \zeta)] \quad (19)$$

$$\geq I_1[\varrho^+ I_{\varphi} \Phi(\varsigma, \zeta) \Omega(\varsigma, \zeta) + \varrho^+ I_{\varphi} \Phi(\varsigma, \iota) \Omega(\varsigma, \iota)] + I_1[\varsigma^- I_{\varphi} \Phi(\varrho, \zeta) \Omega(\varsigma, \zeta) + \varsigma^- I_{\varphi} \Phi(\varrho, \iota) \Omega(\varrho, \iota)]$$

$$+ I_2[\varrho^+ I_{\varphi} \Phi(\varsigma, \zeta) \Omega(\varsigma, \iota) + \varrho^+ I_{\varphi} \Phi(\varsigma, \iota) \Omega(\varsigma, \zeta)] + I_2[\varsigma^- I_{\varphi} \Phi(\varrho, \zeta) \Omega(\varrho, \iota) + \varsigma^- I_{\varphi} \Phi(\varrho, \iota) \Omega(\varrho, \zeta)].$$

For each term of the right-hand side of (19), by inclusion (3), we have

$$[\varrho^+ I_{\varphi} \Phi(\varsigma, \zeta) \Omega(\varsigma, \zeta) + \varrho^+ I_{\varphi} \Phi(\varsigma, \iota) \Omega(\varsigma, \iota)] \quad (20)$$

$$\geq J_1[\Phi(\varrho, \zeta) \Omega(\varrho, \zeta) + \Phi(\varsigma, \zeta) \Omega(\varsigma, \zeta)] + J_2[\Phi(\varrho, \zeta) \Omega(\varsigma, \zeta) + \Phi(\varsigma, \zeta) \Omega(\varrho, \zeta)],$$

$$[\varsigma^- I_{\varphi} \Phi(\varrho, \zeta) \Omega(\varsigma, \zeta) + \varsigma^- I_{\varphi} \Phi(\varrho, \iota) \Omega(\varrho, \iota)] \quad (21)$$

$$\geq J_1[\Phi(\varrho, \iota) \Omega(\varrho, \iota) + \Phi(\varsigma, \iota) \Omega(\varsigma, \iota)] + J_2[\Phi(\varrho, \iota) \Omega(\varsigma, \iota) + \Phi(\varsigma, \zeta) \Omega(\varrho, \iota)],$$

$$[\varrho^+ I_{\varphi} \Phi(\varsigma, \zeta) \Omega(\varsigma, \iota) + \varrho^+ I_{\varphi} \Phi(\varsigma, \iota) \Omega(\varsigma, \zeta)] \quad (22)$$

$$\geq J_1[\Phi(\varrho, \zeta) \Omega(\varrho, \iota) + \Phi(\varsigma, \zeta) \Omega(\varsigma, \varsigma)] + J_2[\Phi(\varrho, \zeta) \Omega(\varsigma, \iota) + \Phi(\varsigma, \zeta) \Omega(\varrho, \iota)],$$

and

$$[\varsigma^- I_{\varphi} \Phi(\varrho, \zeta) \Omega(\varrho, \iota) + \varsigma^- I_{\varphi} \Phi(\varrho, \iota) \Omega(\varrho, \zeta)] \quad (23)$$

$$\geq J_1[\Phi(\varrho, \iota) \Omega(\varrho, \zeta) + \Phi(\varsigma, \iota) \Omega(\varsigma, \zeta)] + J_2[\Phi(\varrho, \iota) \Omega(\varsigma, \zeta) + \Phi(\varsigma, \iota) \Omega(\varrho, \zeta)].$$

If we substitute the inclusions (20)–(23) in (19), we obtain the desired inclusion (10). $\square$

Remark 3. Under the assumption of Theorem 10 with $\varphi(\vartheta) = \vartheta$ and $\psi(v) = v$, Theorem 10 reduces to [19, Theorem 8].

Remark 4. Under the assumption of Theorem 10 with $\varphi(\vartheta) = \frac{\vartheta^\alpha}{\Gamma(\alpha)}$ and $\psi(v) = \frac{v^\beta}{\Gamma(\beta)}$, inclusion (10) reduces to inclusion (6).

Remark 5. If we choose $\Omega(\xi, \eta) = 1$ for all $(\xi, \eta) \in \Delta$ in Theorem 10, then we have the following inclusion:

$$[\varrho^+, \varsigma^+ I_{\varphi, \psi} \Phi(\varsigma, \iota) + \varrho^+, \iota^- I_{\varphi, \psi} \Phi(\varsigma, \zeta) + \varsigma^-, \varrho^+ I_{\varphi, \psi} \Phi(\varrho, \iota) + \varsigma^-, \iota^- I_{\varphi, \psi} \Phi(\varrho, \zeta)]$$

$$\geq (\Phi(\varrho, \zeta) + \Phi(\varrho, \iota) + \Phi(\varsigma, \zeta) + \Phi(\varsigma, \iota)) [I_1 J_1 + I_2 J_1 + I_1 J_2 + I_2 J_2].$$

Proof. For $\Omega(\xi, \eta) = 1$, we have

$$\mathbf{K}(\varrho, \varsigma, \zeta, \iota) = \mathbf{L}(\varrho, \varsigma, \zeta, \iota) = \mathbf{M}(\varrho, \varsigma, \zeta, \iota) = \mathbf{N}(\varrho, \varsigma, \zeta, \iota) = \Phi(\varrho, \zeta) + \Phi(\varrho, \iota) + \Phi(\varsigma, \zeta) + \Phi(\varsigma, \iota).$$

It follows that

$$\begin{aligned} & I_1 J_1 \mathbf{K}(\varrho, \varsigma, \zeta, \iota) + I_1 J_2 \mathbf{L}(\varrho, \varsigma, \zeta, \iota) + I_2 J_1 \mathbf{M}(\varrho, \varsigma, \zeta, \iota) + I_2 J_2 \mathbf{N}(\varrho, \varsigma, \zeta, \iota) \\ &= (\Phi(\varrho, \zeta) + \Phi(\varrho, \iota) + \Phi(\varsigma, \zeta) + \Phi(\varsigma, \iota)) [I_1 J_1 + I_2 J_1 + I_1 J_2 + I_2 J_2], \end{aligned}$$

which completes the proof. $\square$

Theorem 11. Let $\Phi, \Omega : \Delta := [\varrho, \varsigma] \times [\zeta, \iota] \rightarrow \mathbb{R}_I^+$ be two co-ordinated convex IVFs on Δ , then we have the following H-H type inclusion for GFIs:

$$\begin{aligned} & 4\Phi\left(\frac{\varrho + \varsigma}{2}, \frac{\zeta + \iota}{2}\right)\Omega\left(\frac{\varrho + \varsigma}{2}, \frac{\zeta + \iota}{2}\right) \\ & \supseteq \frac{1}{4\Lambda(1)\Delta(1)} [\varrho^+, \varsigma^+ I_{\varphi, \psi} \Phi(\varsigma, \iota) \Omega(\varsigma, \iota) + \varrho^+, \iota^- I_{\varphi, \psi} \Phi(\varsigma, \zeta) \Omega(\varsigma, \zeta) + \varsigma^-, \zeta^+ I_{\varphi, \psi} \Phi(\varrho, \iota) \Omega(\varrho, \iota) \\ & \quad + \varsigma^-, \iota^- I_{\varphi, \psi} \Phi(\varrho, \zeta) \Omega(\varrho, \zeta)] + \frac{1}{4\Lambda(1)\Delta(1)} \{ [J_2 I_1 + I_2 J_1 + J_2 I_2] \mathbf{K}(\varrho, \varsigma, \zeta, \iota) + [J_1 I_1 + J_2 I_2 \\ & \quad + J_1 I_2] \mathbf{L}(\varrho, \varsigma, \zeta, \iota) + [J_2 I_2 + J_1 I_1 + J_2 I_1] \mathbf{M}(\varrho, \varsigma, \zeta, \iota) + [J_1 I_2 + J_2 I_1 + J_1 I_1] \mathbf{N}(\varrho, \varsigma, \zeta, \iota) \}, \end{aligned} \quad (24)$$

where $\mathbf{K}(\varrho, \varsigma, \zeta, \iota)$, $\mathbf{L}(\varrho, \varsigma, \zeta, \iota)$, $\mathbf{M}(\varrho, \varsigma, \zeta, \iota)$, and $\mathbf{N}(\varrho, \varsigma, \zeta, \iota)$ are defined as shown in Theorem 8.

Proof. Since Φ and Ω are co-ordinated convex IVFs on Δ , by the inclusion (4), we have

$$\begin{aligned} & 2\Phi\left(\frac{\varrho + \varsigma}{2}, \frac{\zeta + \iota}{2}\right)\Omega\left(\frac{\varrho + \varsigma}{2}, \frac{\zeta + \iota}{2}\right) \\ & \supseteq \frac{1}{2\Lambda(1)} \left[(\text{IR}) \int_{\varrho}^{\varsigma} \frac{\varphi(\varsigma - \xi)}{\varsigma - \xi} \Phi\left(\xi, \frac{\zeta + \iota}{2}\right) \Omega\left(\xi, \frac{\zeta + \iota}{2}\right) d\xi + (\text{IR}) \int_{\varrho}^{\varsigma} \frac{\varphi(\xi - \varrho)}{\xi - \varrho} \Phi\left(\xi, \frac{\zeta + \iota}{2}\right) \Omega\left(\xi, \frac{\zeta + \iota}{2}\right) d\xi \right] \\ & \quad + \frac{J_2}{2\Lambda(1)} \left[\Phi\left(\varrho, \frac{\zeta + \iota}{2}\right) \Omega\left(\varrho, \frac{\zeta + \iota}{2}\right) + \Phi\left(\varsigma, \frac{\zeta + \iota}{2}\right) \Omega\left(\varsigma, \frac{\zeta + \iota}{2}\right) \right] \\ & \quad + \frac{J_1}{2\Lambda(1)} \left[\Phi\left(\varrho, \frac{\zeta + \iota}{2}\right) \Omega\left(\varsigma, \frac{\zeta + \iota}{2}\right) + \Phi\left(\varsigma, \frac{\zeta + \iota}{2}\right) \Omega\left(\varrho, \frac{\zeta + \iota}{2}\right) \right] \end{aligned} \quad (25)$$

and

$$\begin{aligned} & 2\Phi\left(\frac{\varrho + \varsigma}{2}, \frac{\zeta + \iota}{2}\right)\Omega\left(\frac{\varrho + \varsigma}{2}, \frac{\zeta + \iota}{2}\right) \\ & \supseteq \frac{1}{2\Delta(1)} \left[(\text{IR}) \int_{\zeta}^{\iota} \frac{\psi(\iota - \eta)}{\iota - \eta} \Phi\left(\frac{\varrho + \varsigma}{2}, \eta\right) \Omega\left(\frac{\varrho + \varsigma}{2}, \eta\right) d\eta + (\text{IR}) \int_{\zeta}^{\iota} \frac{\psi(\eta - \zeta)}{\eta - \zeta} \Phi\left(\frac{\varrho + \varsigma}{2}, \eta\right) \Omega\left(\frac{\varrho + \varsigma}{2}, \eta\right) d\eta \right] \\ & \quad + \frac{I_2}{2\Delta(1)} \left[\Phi\left(\frac{\varrho + \varsigma}{2}, \zeta\right) \Omega\left(\frac{\varrho + \varsigma}{2}, \zeta\right) + \Phi\left(\frac{\varrho + \varsigma}{2}, \iota\right) \Omega\left(\frac{\varrho + \varsigma}{2}, \iota\right) \right] \\ & \quad + \frac{I_1}{2\Delta(1)} \left[\Phi\left(\frac{\varrho + \varsigma}{2}, \zeta\right) \Omega\left(\frac{\varrho + \varsigma}{2}, \iota\right) + \Phi\left(\frac{\varrho + \varsigma}{2}, \iota\right) \Omega\left(\frac{\varrho + \varsigma}{2}, \zeta\right) \right]. \end{aligned} \quad (26)$$

From inclusions (25) and (26), we obtain the following inclusion:

$$\begin{aligned}
& 8\Phi\left(\frac{\varrho + \varsigma}{2}, \frac{\zeta + \iota}{2}\right)\Omega\left(\frac{\varrho + \varsigma}{2}, \frac{\zeta + \iota}{2}\right) \\
& \supseteq \frac{1}{2\Lambda(1)} \left[(\text{IR}) \int_{\varrho}^{\varsigma} \frac{\varphi(\varsigma - \xi)}{\varsigma - \xi} 2\Phi\left(\xi, \frac{\zeta + \iota}{2}\right)\Omega\left(\xi, \frac{\zeta + \iota}{2}\right) d\xi + (\text{IR}) \int_{\varrho}^{\varsigma} \frac{\varphi(\xi - \varrho)}{\xi - \varrho} 2\Phi\left(\xi, \frac{\zeta + \iota}{2}\right)\Omega\left(\xi, \frac{\zeta + \iota}{2}\right) d\xi \right] \\
& \quad \times \frac{1}{2\Lambda(1)} \left[(\text{IR}) \int_{\zeta}^{\iota} \frac{\psi(\iota - \eta)}{\iota - \eta} 2\Phi\left(\frac{\varrho + \varsigma}{2}, \eta\right)\Omega\left(\frac{\varrho + \varsigma}{2}, \eta\right) d\eta \right. \\
& \quad \left. + (\text{IR}) \int_{\zeta}^{\iota} \frac{\psi(\eta - \zeta)}{\eta - \zeta} 2\Phi\left(\frac{\varrho + \varsigma}{2}, \eta\right)\Omega\left(\frac{\varrho + \varsigma}{2}, \eta\right) d\eta \right] \\
& \quad + \frac{J_2}{2\Lambda(1)} \left[2\Phi\left(\varrho, \frac{\zeta + \iota}{2}\right)\Omega\left(\varrho, \frac{\zeta + \iota}{2}\right) + 2\Phi\left(\varsigma, \frac{\zeta + \iota}{2}\right)\Omega\left(\varsigma, \frac{\zeta + \iota}{2}\right) \right] \\
& \quad + \frac{J_1}{2\Lambda(1)} \left[2\Phi\left(\varrho, \frac{\zeta + \iota}{2}\right)\Omega\left(\varsigma, \frac{\zeta + \iota}{2}\right) + 2\Phi\left(\varsigma, \frac{\zeta + \iota}{2}\right)\Omega\left(\varrho, \frac{\zeta + \iota}{2}\right) \right] \\
& \quad + \frac{I_2}{2\Lambda(1)} \left[2\Phi\left(\frac{\varrho + \varsigma}{2}, \zeta\right)\Omega\left(\frac{\varrho + \varsigma}{2}, \zeta\right) + 2\Phi\left(\frac{\varrho + \varsigma}{2}, \iota\right)\Omega\left(\frac{\varrho + \varsigma}{2}, \iota\right) \right] \\
& \quad + \frac{I_1}{2\Lambda(1)} \left[2\Phi\left(\frac{\varrho + \varsigma}{2}, \zeta\right)\Omega\left(\frac{\varrho + \varsigma}{2}, \iota\right) + 2\Phi\left(\frac{\varrho + \varsigma}{2}, \iota\right)\Omega\left(\frac{\varrho + \varsigma}{2}, \zeta\right) \right].
\end{aligned} \tag{27}$$

Since the mappings $\Phi_{\xi} : [\zeta, \iota] \rightarrow \mathbb{R}$, $\Phi_{\xi}(\eta) = \Phi(\xi, \eta)$, and $\Omega_{\xi} : [\zeta, \iota] \rightarrow \mathbb{R}$, $\Omega_{\xi}(\eta) = \Omega(\xi, \eta)$, by applying inclusion (4), we have

$$\begin{aligned}
& 2\Phi\left(\varrho, \frac{\zeta + \iota}{2}\right)\Omega\left(\varrho, \frac{\zeta + \iota}{2}\right) \\
& \supseteq \frac{1}{2\Lambda(1)} [\zeta^+ I_{\psi} \Phi(\varrho, \iota) \Omega(\varrho, \iota) + \iota^- I_{\psi} \Phi(\varrho, \zeta) \Omega(\varrho, \zeta)] + \frac{I_2}{2\Lambda(1)} [\Phi(\varrho, \zeta) \Omega(\varrho, \zeta) + \Phi(\varrho, \iota) \Omega(\varrho, \iota)] \\
& \quad + \frac{I_1}{2\Lambda(1)} [\Phi(\varrho, \zeta) \Omega(\varrho, \iota) + \Phi(\varrho, \iota) \Omega(\varrho, \zeta)],
\end{aligned} \tag{28}$$

$$\begin{aligned}
& 2\Phi\left(\varsigma, \frac{\zeta + \iota}{2}\right)\Omega\left(\varsigma, \frac{\zeta + \iota}{2}\right) \\
& \supseteq \frac{1}{2\Lambda(1)} [\zeta^+ I_{\psi} \Phi(\varsigma, \iota) \Omega(\varsigma, \iota) + \iota^- I_{\psi} \Phi(\varsigma, \zeta) \Omega(\varsigma, \zeta)] + \frac{I_2}{2\Lambda(1)} [\Phi(\varsigma, \zeta) \Omega(\varsigma, \zeta) + \Phi(\varsigma, \iota) \Omega(\varsigma, \iota)] \\
& \quad + \frac{I_1}{2\Lambda(1)} [\Phi(\varsigma, \zeta) \Omega(\varsigma, \iota) + \Phi(\varsigma, \iota) \Omega(\varsigma, \zeta)],
\end{aligned} \tag{29}$$

$$\begin{aligned}
& 2\Phi\left(\varrho, \frac{\zeta + \iota}{2}\right)\Omega\left(\varsigma, \frac{\zeta + \iota}{2}\right) \\
& \supseteq \frac{1}{2\Lambda(1)} [\zeta^+ I_{\psi} \Phi(\varrho, \iota) \Omega(\varsigma, \iota) + \iota^- I_{\psi} \Phi(\varrho, \zeta) \Omega(\varsigma, \zeta)] \\
& \quad + \frac{I_2}{2\Lambda(1)} [\Phi(\varrho, \zeta) \Omega(\varsigma, \zeta) + \Phi(\varrho, \iota) \Omega(\varsigma, \iota)] + \frac{I_1}{2\Lambda(1)} [\Phi(\varrho, \zeta) \Omega(\varsigma, \iota) + \Phi(\varrho, \iota) \Omega(\varsigma, \zeta)],
\end{aligned} \tag{30}$$

and

$$\begin{aligned}
& 2\Phi\left(\varsigma, \frac{\zeta + \iota}{2}\right)\Omega\left(\varrho, \frac{\zeta + \iota}{2}\right) \\
& \supseteq \frac{1}{2\Lambda(1)} [\zeta^+ I_{\psi} \Phi(\varsigma, \iota) \Omega(\varrho, \iota) + \iota^- I_{\psi} \Phi(\varsigma, \zeta) \Omega(\varrho, \zeta)] + \frac{I_2}{2\Lambda(1)} [\Phi(\varsigma, \zeta) \Omega(\varrho, \zeta) + \Phi(\varsigma, \iota) \Omega(\varrho, \iota)] \\
& \quad + \frac{I_1}{2\Lambda(1)} [\Phi(\varsigma, \zeta) \Omega(\varrho, \iota) + \Phi(\varsigma, \iota) \Omega(\varrho, \zeta)].
\end{aligned} \tag{31}$$

Similarly, since the mappings $\Phi_\eta : [\varrho, \varsigma] \rightarrow \mathbb{R}$, $\Phi_\eta(\xi) = \Phi(\xi, \eta)$ and $\Omega_\eta : [\varrho, \varsigma] \rightarrow \mathbb{R}$, $\Omega_\eta(\xi) = \Omega(\xi, \eta)$, by applying inclusion (4), we have

$$\begin{aligned} & 2\Phi\left(\frac{\varrho + \varsigma}{2}, \zeta\right)\Omega\left(\frac{\varrho + \varsigma}{2}, \zeta\right) \\ & \supseteq \frac{1}{2\Lambda(1)}[{}_q^- I_\varphi \Phi(\varsigma, \zeta)\Omega(\varsigma, \zeta) + {}_\varsigma^- I_\varphi \Phi(\varrho, \zeta)\Omega(\varrho, \zeta)] + \frac{J_2}{2\Lambda(1)}[\Phi(\varrho, \zeta)\Omega(\varrho, \zeta) + \Phi(\varsigma, \zeta)\Omega(\varsigma, \zeta)] \\ & \quad + \frac{J_1}{2\Lambda(1)}[\Phi(\varrho, \zeta)\Omega(\varsigma, \zeta) + \Phi(\varsigma, \zeta)\Omega(\varrho, \zeta)], \end{aligned} \quad (32)$$

$$\begin{aligned} & 2\Phi\left(\frac{\varrho + \varsigma}{2}, \iota\right)\Omega\left(\frac{\varrho + \varsigma}{2}, \iota\right) \\ & \supseteq \frac{1}{2\Lambda(1)}[{}_q^- I_\varphi \Phi(\varsigma, \iota)\Omega(\varsigma, \iota) + {}_\varsigma^- I_\varphi \Phi(\varrho, \iota)\Omega(\varrho, \iota)] + \frac{J_2}{2\Lambda(1)}[\Phi(\varrho, \iota)\Omega(\varrho, \iota) + \Phi(\varsigma, \iota)\Omega(\varsigma, \iota)] \\ & \quad + \frac{J_1}{2\Lambda(1)}[\Phi(\varrho, \iota)\Omega(\varsigma, \iota) + \Phi(\varsigma, \iota)\Omega(\varrho, \iota)], \end{aligned} \quad (33)$$

$$\begin{aligned} & 2\Phi\left(\frac{\varrho + \varsigma}{2}, \zeta\right)\Omega\left(\frac{\varrho + \varsigma}{2}, \iota\right) \\ & \supseteq \frac{1}{2\Lambda(1)}[{}_q^- I_\varphi \Phi(\varsigma, \zeta)\Omega(\varsigma, \iota) + {}_\varsigma^- I_\varphi \Phi(\varrho, \zeta)\Omega(\varrho, \iota)] + \frac{J_2}{2\Lambda(1)}[\Phi(\varrho, \zeta)\Omega(\varrho, \iota) + \Phi(\varsigma, \zeta)\Omega(\varsigma, \iota)] \\ & \quad + \frac{J_1}{2\Lambda(1)}[\Phi(\varrho, \zeta)\Omega(\varsigma, \iota) + \Phi(\varsigma, \zeta)\Omega(\varrho, \iota)], \end{aligned} \quad (34)$$

and

$$\begin{aligned} & 2\Phi\left(\frac{\varrho + \varsigma}{2}, \iota\right)\Omega\left(\frac{\varrho + \varsigma}{2}, \zeta\right) \\ & \supseteq \frac{1}{2\Lambda(1)}[{}_q^- I_\varphi \Phi(\varsigma, \iota)\Omega(\varsigma, \zeta) + {}_\varsigma^- I_\varphi \Phi(\varrho, \iota)\Omega(\varrho, \zeta)] + \frac{J_2}{2\Lambda(1)}[\Phi(\varrho, \iota)\Omega(\varrho, \zeta) + \Phi(\varsigma, \iota)\Omega(\varsigma, \zeta)] \\ & \quad + \frac{J_1}{2\Lambda(1)}[\Phi(\varrho, \iota)\Omega(\varsigma, \zeta) + \Phi(\varsigma, \iota)\Omega(\varrho, \zeta)]. \end{aligned} \quad (35)$$

On the other hand, by applying inclusion (4), we obtain

$$\begin{aligned} & \frac{1}{2\Lambda(1)}(\text{IR}) \int_{\varrho}^{\varsigma} \frac{\varphi(\varsigma - \xi)}{\varsigma - \xi} 2\Phi\left(\xi, \frac{\zeta + \iota}{2}\right)\Omega\left(\xi, \frac{\zeta + \iota}{2}\right) d\xi \\ & \supseteq \frac{1}{4\Lambda(1)\Delta(1)} \left[(\text{IR}) \int_{\varrho}^{\varsigma} \int_{\zeta}^{\iota} \frac{\varphi(\varsigma - \xi)}{\varsigma - \xi} \frac{\psi(\iota - \eta)}{\iota - \eta} \Phi(\xi, \eta)\Omega(\xi, \eta) d\xi d\eta \right. \\ & \quad \left. + (\text{IR}) \int_{\varrho}^{\varsigma} \int_{\zeta}^{\iota} \frac{\varphi(\varsigma - \xi)}{\varsigma - \xi} \frac{\psi(\eta - \zeta)}{\eta - \zeta} \Phi(\xi, \eta)\Omega(\xi, \eta) d\xi d\eta \right] \\ & \quad + \frac{I_2}{4\Lambda(1)\Delta(1)} (\text{IR}) \int_{\varrho}^{\varsigma} \frac{\varphi(\varsigma - \xi)}{\varsigma - \xi} [\Phi(\xi, \zeta)\Omega(\xi, \zeta) + \Phi(\xi, \iota)\Omega(\xi, \iota)] d\xi \\ & \quad + \frac{I_1}{4\Lambda(1)\Delta(1)} (\text{IR}) \int_{\varrho}^{\varsigma} \frac{\varphi(\varsigma - \xi)}{\varsigma - \xi} [\Phi(\xi, \zeta)\Omega(\xi, \iota) + \Phi(\xi, \iota)\Omega(\xi, \zeta)] d\xi \end{aligned} \quad (36)$$

$$\begin{aligned}
&= \frac{1}{4\Lambda(1)\Delta(1)} [{}_{\varrho^+}I_{\varphi,\psi}\Phi(\varsigma, \mathfrak{v})\Omega(\varsigma, \mathfrak{v}) + {}_{\varrho^+,\varsigma^-}I_{\varphi,\psi}\Phi(\varsigma, \zeta)\Omega(\varsigma, \zeta)] \\
&\quad + \frac{I_2}{4\Lambda(1)\Delta(1)} [{}_{\varrho^+}I_{\varphi}\Phi(\varsigma, \zeta)\Omega(\varsigma, \zeta) + {}_{\varrho^+}I_{\varphi}\Phi(\varsigma, \mathfrak{v})\Omega(\varsigma, \mathfrak{v})] \\
&\quad + \frac{I_1}{4\Lambda(1)\Delta(1)} [{}_{\varrho^+}I_{\varphi}\Phi(\varsigma, \zeta)\Omega(\varsigma, \mathfrak{v}) + {}_{\varrho^+}I_{\varphi}\Phi(\varsigma, \mathfrak{v})\Omega(\varsigma, \zeta)].
\end{aligned}$$

Similarly, we also have

$$\begin{aligned}
&\frac{1}{2\Lambda(1)}(\text{IR}) \int_{\varrho}^{\varsigma} \frac{\varphi(\xi - \varrho)}{\xi - \varrho} 2\Phi\left(\xi, \frac{\zeta + \mathfrak{v}}{2}\right) \Omega\left(\xi, \frac{\zeta + \mathfrak{v}}{2}\right) d\xi \\
&\geq \frac{1}{4\Lambda(1)\Delta(1)} [{}_{\varsigma^-, \zeta^+}I_{\varphi}\Phi(\varrho, \mathfrak{v})\Omega(\varrho, \mathfrak{v}) + {}_{\varsigma^-, \mathfrak{v}^-}I_{\varphi}\Phi(\varrho, \zeta)\Omega(\varrho, \zeta)] \\
&\quad + \frac{I_2}{4\Lambda(1)\Delta(1)} [{}_{\varsigma^-}I_{\varphi}\Phi(\varrho, \zeta)\Omega(\varrho, \zeta) + {}_{\varsigma^-}I_{\varphi}\Phi(\varrho, \mathfrak{v})\Omega(\varrho, \mathfrak{v})] \\
&\quad + \frac{I_1}{4\Lambda(1)\Delta(1)} [{}_{\varsigma^-}I_{\varphi}\Phi(\varrho, \zeta)\Omega(\varrho, \mathfrak{v}) + {}_{\varsigma^-}I_{\varphi}\Phi(\varrho, \mathfrak{v})\Omega(\varrho, \zeta)],
\end{aligned} \tag{37}$$

$$\begin{aligned}
&\frac{1}{2\Delta(1)}(\text{IR}) \int_{\zeta}^{\mathfrak{v}} \frac{\psi(\mathfrak{v} - \eta)}{\mathfrak{v} - \eta} 2\Phi\left(\frac{\varrho + \varsigma}{2}, \eta\right) \Omega\left(\frac{\varrho + \varsigma}{2}, \eta\right) d\eta \\
&\geq \frac{1}{4\Lambda(1)\Delta(1)} [{}_{\varrho^+, \zeta^+}I_{\varphi,\psi}\Phi(\varsigma, \mathfrak{v})\Omega(\varsigma, \mathfrak{v}) + {}_{\varsigma^-, \zeta^+}I_{\varphi,\psi}\Phi(\varrho, \mathfrak{v})\Omega(\varrho, \mathfrak{v})] \\
&\quad + \frac{J_2}{4\Lambda(1)\Delta(1)} [{}_{\zeta^+}I_{\psi}\Phi(\varrho, \mathfrak{v})\Omega(\varrho, \mathfrak{v}) + {}_{\zeta^+}I_{\psi}\Phi(\varsigma, \mathfrak{v})\Omega(\varsigma, \mathfrak{v})] \\
&\quad + \frac{J_1}{4\Lambda(1)\Delta(1)} [{}_{\zeta^+}I_{\psi}\Phi(\varrho, \mathfrak{v})\Omega(\varsigma, \mathfrak{v}) + {}_{\zeta^+}I_{\psi}\Phi(\varsigma, \mathfrak{v})\Omega(\varrho, \mathfrak{v})],
\end{aligned} \tag{38}$$

and

$$\begin{aligned}
&\frac{1}{2\Delta(1)}(\text{IR}) \int_{\zeta}^{\mathfrak{v}} \frac{\psi(\eta - \zeta)}{\eta - \zeta} 2\Phi\left(\frac{\varrho + \varsigma}{2}, \eta\right) \Omega\left(\frac{\varrho + \varsigma}{2}, \eta\right) d\eta \\
&\geq \frac{1}{4\Lambda(1)\Delta(1)} [{}_{\varrho^+, \varsigma^-}I_{\varphi,\psi}\Phi(\varsigma, \zeta)\Omega(\varsigma, \zeta) + {}_{\varsigma^-, \mathfrak{v}^-}I_{\varphi,\psi}\Phi(\varrho, \zeta)\Omega(\varrho, \zeta)] \\
&\quad + \frac{J_2}{4\Lambda(1)\Delta(1)} [{}_{\mathfrak{v}^-}I_{\psi}\Phi(\varrho, \zeta)\Omega(\varrho, \zeta) + {}_{\mathfrak{v}^-}I_{\psi}\Phi(\varsigma, \zeta)\Omega(\varsigma, \zeta)] \\
&\quad + \frac{J_1}{4\Lambda(1)\Delta(1)} [{}_{\mathfrak{v}^-}I_{\psi}\Phi(\varrho, \zeta)\Omega(\varsigma, \zeta) + {}_{\mathfrak{v}^-}I_{\psi}\Phi(\varsigma, \zeta)\Omega(\varrho, \zeta)].
\end{aligned} \tag{39}$$

By substituting inclusions (28)–(39) in (27), we obtain the following inclusion:

$$\begin{aligned}
&8\Phi\left(\frac{\varrho + \varsigma}{2}, \frac{\zeta + \mathfrak{v}}{2}\right) \Omega\left(\frac{\varrho + \varsigma}{2}, \frac{\zeta + \mathfrak{v}}{2}\right) \\
&\geq \frac{1}{4\Lambda(1)\Delta(1)} [{}_{\varrho^+, \zeta^+}I_{\varphi,\psi}\Phi(\varsigma, \mathfrak{v})\Omega(\varsigma, \mathfrak{v}) + {}_{\varrho^+, \varsigma^-}I_{\varphi,\psi}\Phi(\varsigma, \zeta)\Omega(\varsigma, \zeta) + {}_{\varsigma^-, \zeta^+}I_{\varphi,\psi}\Phi(\varrho, \mathfrak{v})\Omega(\varrho, \mathfrak{v}) \\
&\quad + {}_{\varsigma^-, \mathfrak{v}^-}I_{\varphi,\psi}\Phi(\varrho, \zeta)\Omega(\varrho, \zeta)] \\
&\quad + \frac{J_2}{4\Lambda(1)\Delta(1)} \{[{}_{\zeta^+}I_{\psi}\Phi(\varrho, \mathfrak{v})\Omega(\varrho, \mathfrak{v}) + {}_{\zeta^+}I_{\psi}\Phi(\varsigma, \mathfrak{v})\Omega(\varsigma, \mathfrak{v})] + [{}_{\mathfrak{v}^-}I_{\psi}\Phi(\varrho, \zeta)\Omega(\varrho, \zeta) + {}_{\mathfrak{v}^-}I_{\psi}\Phi(\varsigma, \zeta)\Omega(\varsigma, \zeta)]\} \\
&\quad + \frac{J_1}{4\Lambda(1)\Delta(1)} \{[{}_{\zeta^+}I_{\psi}\Phi(\varrho, \mathfrak{v})\Omega(\varsigma, \mathfrak{v}) + {}_{\zeta^+}I_{\psi}\Phi(\varsigma, \mathfrak{v})\Omega(\varrho, \mathfrak{v})] + [{}_{\mathfrak{v}^-}I_{\psi}\Phi(\varrho, \zeta)\Omega(\varsigma, \zeta) + {}_{\mathfrak{v}^-}I_{\psi}\Phi(\varsigma, \zeta)\Omega(\varrho, \zeta)]\}
\end{aligned} \tag{40}$$

$$\begin{aligned}
& + \frac{I_2}{4\Lambda(1)\Delta(1)} \{ [\varrho^+ I_\varphi \Phi(\varsigma, \zeta) \Omega(\varsigma, \zeta) + \varrho^+ I_\varphi \Phi(\varsigma, \iota) \Omega(\varsigma, \iota)] + [\varsigma^- I_\varphi \Phi(\varrho, \zeta) \Omega(\varrho, \zeta) + \varsigma^- I_\varphi \Phi(\varrho, \iota) \Omega(\varrho, \iota)] \} \\
& + \frac{I_1}{4\Lambda(1)\Delta(1)} \{ [\varrho^+ I_\varphi \Phi(\varsigma, \zeta) \Omega(\varsigma, \iota) + \varrho^+ I_\varphi \Phi(\varsigma, \iota) \Omega(\varsigma, \zeta)] + [\varsigma^- I_\varphi \Phi(\varrho, \zeta) \Omega(\varrho, \iota) + \varsigma^- I_\varphi \Phi(\varrho, \iota) \Omega(\varrho, \zeta)] \} \\
& + \frac{J_2 I_2}{2\Lambda(1)\Delta(1)} \mathbf{K}(\varrho, \varsigma, \zeta, \iota) + \frac{J_2 I_1}{2\Lambda(1)\Delta(1)} \mathbf{M}(\varrho, \varsigma, \zeta, \iota) + \frac{J_1 I_2}{2\Lambda(1)\Delta(1)} \mathbf{L}(\varrho, \varsigma, \zeta, \iota) + \frac{J_1 I_1}{2\Lambda(1)\Delta(1)} \mathbf{N}(\varrho, \varsigma, \zeta, \iota).
\end{aligned}$$

By using inclusion (3), we have the following inclusions:

$$\begin{aligned}
& [\zeta^+ I_\psi \Phi(\varrho, \iota) \Omega(\varrho, \iota) + \zeta^+ I_\psi \Phi(\varsigma, \iota) \Omega(\varsigma, \iota)] + [\iota^- I_\psi \Phi(\varrho, \zeta) \Omega(\varrho, \zeta) + \iota^- I_\psi \Phi(\varsigma, \zeta) \Omega(\varsigma, \zeta)] \\
& \supseteq I_1 \mathbf{K}(\varrho, \varsigma, \zeta, \iota) + I_2 \mathbf{M}(\varrho, \varsigma, \zeta, \iota),
\end{aligned} \tag{41}$$

$$\begin{aligned}
& [\zeta^+ I_\psi \Phi(\varrho, \iota) \Omega(\varsigma, \iota) + \zeta^+ I_\psi \Phi(\varsigma, \iota) \Omega(\varrho, \iota)] + [\iota^- I_\psi \Phi(\varrho, \zeta) \Omega(\varsigma, \zeta) + \iota^- I_\psi \Phi(\varsigma, \zeta) \Omega(\varrho, \zeta)] \\
& \supseteq I_1 \mathbf{L}(\varrho, \varsigma, \zeta, \iota) + I_2 \mathbf{N}(\varrho, \varsigma, \zeta, \iota),
\end{aligned} \tag{42}$$

$$\begin{aligned}
& [\varrho^+ I_\varphi \Phi(\varsigma, \zeta) \Omega(\varsigma, \zeta) + \varrho^+ I_\varphi \Phi(\varsigma, \iota) \Omega(\varsigma, \iota)] + [\varsigma^- I_\varphi \Phi(\varrho, \zeta) \Omega(\varrho, \zeta) + \varsigma^- I_\varphi \Phi(\varrho, \iota) \Omega(\varrho, \iota)] \\
& \supseteq J_1 \mathbf{K}(\varrho, \varsigma, \zeta, \iota) + J_2 \mathbf{L}(\varrho, \varsigma, \zeta, \iota),
\end{aligned} \tag{43}$$

and

$$\begin{aligned}
& [\varrho^+ I_\varphi \Phi(\varsigma, \zeta) \Omega(\varsigma, \iota) + \varrho^+ I_\varphi \Phi(\varsigma, \iota) \Omega(\varsigma, \zeta)] + [\varsigma^- I_\varphi \Phi(\varrho, \zeta) \Omega(\varrho, \iota) + \varsigma^- I_\varphi \Phi(\varrho, \iota) \Omega(\varrho, \zeta)] \\
& \supseteq J_1 \mathbf{M}(\varrho, \varsigma, \zeta, \iota) + J_2 \mathbf{N}(\varrho, \varsigma, \zeta, \iota).
\end{aligned} \tag{44}$$

If we substitute inclusions (41)–(44) in (40) and divide the resulting inclusion by 2, then we obtain the desired result (24). This completes the proof. $\square$

Remark 6. Under the assumption of Theorem 11 with $\varphi(\vartheta) = \vartheta$ and $\psi(\nu) = \nu$, Theorem 11 reduces to [19, Theorem 9].

Remark 7. Under the assumption of Theorem 10 with $\varphi(\vartheta) = \frac{\vartheta^\alpha}{\Gamma(\alpha)}$ and $\psi(\nu) = \frac{\nu^\beta}{\Gamma(\beta)}$, inclusion (10) reduces to inclusion (7).

Corollary 1. If we choose $\Omega(\xi, \eta) = 1$ for all $(\xi, \eta) \in \Delta$ in Theorem 10, then we have the following inclusion:

$$\begin{aligned}
4\Phi\left(\frac{\varrho + \varsigma}{2}, \frac{\zeta + \iota}{2}\right) & \supseteq \frac{1}{4\Lambda(1)\Delta(1)} [\varrho^+, \zeta^+ I_{\varphi, \psi} \Phi(\varsigma, \iota) + \varrho^+, \iota^- I_{\varphi, \psi} \Phi(\varsigma, \zeta) + \varsigma^-, \zeta^+ I_{\varphi, \psi} \Phi(\varrho, \iota) + \varsigma^-, \iota^- I_{\varphi, \psi} \Phi(\varrho, \zeta)] \\
& + \frac{3[\Phi(\varrho, \zeta) + \Phi(\varrho, \iota) + \Phi(\varsigma, \zeta) + \Phi(\varsigma, \iota)]}{4\Lambda(1)\Delta(1)} [J_1 I_1 + J_1 I_2 + J_2 I_1 + J_2 I_2].
\end{aligned}$$

Proof. For $\Omega(\xi, \eta) = 1$, we have

$$\mathbf{K}(\varrho, \varsigma, \zeta, \iota) = \mathbf{L}(\varrho, \varsigma, \zeta, \iota) = \mathbf{M}(\varrho, \varsigma, \zeta, \iota) = \mathbf{N}(\varrho, \varsigma, \zeta, \iota) = \Phi(\varrho, \zeta) + \Phi(\varrho, \iota) + \Phi(\varsigma, \zeta) + \Phi(\varsigma, \iota).$$

It follows that

$$\begin{aligned}
& [J_2 I_1 + J_2 I_2 + J_1 I_1] \mathbf{K}(\varrho, \varsigma, \zeta, \iota) + [J_1 I_1 + J_2 I_2 + J_1 I_2] \mathbf{L}(\varrho, \varsigma, \zeta, \iota) + [J_2 I_2 + J_1 I_1 + J_2 I_1] \mathbf{M}(\varrho, \varsigma, \zeta, \iota) \\
& + [J_1 I_2 + J_2 I_1 + J_1 I_1] \mathbf{N}(\varrho, \varsigma, \zeta, \iota) \\
& = \frac{[\Phi(\varrho, \zeta) + \Phi(\varrho, \iota) + \Phi(\varsigma, \zeta) + \Phi(\varsigma, \iota)]}{4\Lambda(1)\Delta(1)} \{ [J_2 I_1 + J_2 I_2 + J_1 I_1] + [J_1 I_1 + J_2 I_2 + J_1 I_2] + [J_2 I_2 + J_1 I_1 + J_2 I_1] \\
& + [J_1 I_2 + J_2 I_1 + J_1 I_1] \},
\end{aligned}$$

which completes the proof. $\square$

5 Concluding remarks

In this study, we presented a new GFIs for co-ordinated IVFs and utilizing this new integral, and we established H-H type inclusions for co-ordinated convex IVFs. It is also proved that the newly established inequalities can be converted into classical H-H inequalities for interval-valued co-ordinated convex functions, Riemann-Liouville fractional H-H inequalities interval-valued co-ordinated convex functions, and k -Riemann-Liouville fractional H-H inequalities interval-valued co-ordinated convex functions. Interested readers can find more new integral inclusions by using our newly defined integral, and they can study other type convexity of IVFs.

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