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Research Article

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Malmquist-type theorems on some complex differential-difference equations

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Abstract: This article is devoted to study the existence conditions of solutions to several complex differential-difference equations. We obtain some Malmquist theorems related to complex differential-difference equations with a more general form than the previous equations given by Zhang, Huang, and others. Moreover, some examples are provided to demonstrate why some restrictive conditions in some of our theorems cannot be removed.

Keywords: Malmquist-type, differential-difference equation, existence

MSC 2020: 39A13, 39B72, 30D35

1 Introduction and main results

The purpose of this article is to investigate the properties of the solutions of some differential-difference equations of Malmquist-type. We first assume that the reader is familiar with the basic notions of Nevanlinna value distribution theory (see [1–4]). Now, we recall the well-known Malmquist theorem about the existence of meromorphic solutions to a certain type of differential equation (see [5]) and obtain

Theorem A. (See [5].) *Let*

$$f'(z) = R(z, f(z)) = \frac{P(z, f(z))}{Q(z, f(z))} = \frac{\sum_{i=0}^{p} a_i(z) f(z)^i}{\sum_{j=0}^{q} b_j(z) f(z)^j}, \tag{1.1}$$

where $P(z, f(z))$ and $Q(z, f(z))$ are relatively prime polynomials in $f(z)$, and the coefficients $a_i(z)$ $(i = 1, \ldots, p)$ and $b_j(z)$ $(j = 1, \ldots, q)$ are rational functions. If equation (1.1) *admits a transcendental meromorphic solution, then $q = 0$ and $p \leq 2$.*

Theorem B. (See [5].) *Let $R(z, f)$ be a birational function. If a differential equation* (1.1) *admits a transcendental meromorphic solution, then the equation can be reduced to a Riccati differential equation*

$$f'(z) = a_0(z) + a_1(z) f(z) + a_2(z) f^2(z),$$

where $a_i(z)$ $(i = 0, 1, 2)$ are rational functions.

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We can find that Malmquist did not make use of Nevanlinna value distribution theory to prove Theorems A and B. In 1933, Yosida [6] proved Malmquist's theorem by applying the Nevanlinna value distribution theory. In the 1970s, Laine [7], Yang [8], and Hille [9] gave a generalization of Theorems A and B when the coefficients of $R(z,f)$ are meromorphic functions. In 1978, Steinmetz [10] considered the equation $(f')^n = R(z,f)$ and extended Malmquist's theorem by relying on a number of auxiliary functions; in 1980, Gackstatter and Laine [11] obtained a generalized result of Theorems A and B by using the properties of Valiron deficient values. In 2018, Zhang and Liao [12] further studied the existence of meromorphic solutions when f' is replaced by the differential polynomial in equation (1.1) and obtained

Theorem C. (See [12, Theorem 1.3].) *If the algebraic differential equation*

$$P(z,f,f',\ldots,f^{(n)}) = R(z,f) = \frac{P(z,f)}{Q(z,f)} = \frac{\sum_{i=0}^{p} a_i(z)f(z)^i}{\sum_{j=0}^{q} b_j(z)f(z)^j},$$

where

$$P(z,f,f',\ldots,f^{(n)}) = \sum_{\lambda\in I} \alpha_\lambda(z) f^{\lambda_0}(f')^{\lambda_1}\cdots(f^{(n)})^{\lambda_n}$$

is a differential polynomial in f *with meromorphic coefficients* $\alpha_\lambda(z)$ *and* $a_i(z)$ $(i = 0, 1, \ldots, p)$ *and* $b_j(z)$ $(j = 0, 1, \ldots, q)$ *are meromorphic functions, possesses an admissible meromorphic solution, then* $R(z,f)$ *is reduced to a polynomial in* f *of degree* $\le \Gamma_P$, *where* $\Gamma_P = \max_{\lambda\in I}\{\lambda_0 + 2\lambda_1 + \cdots + (n+1)\lambda_{n+1}\}$.

Remark 1.1. It should be pointed out that the proof of Theorem C is simpler than the proofs in [10,11].

In the past 20 years, there were a lot of results focusing on the growth of order and the existence of the solutions to Malmquist-type difference equations (see [13–19]), by applying the Nevanlinna theory for meromorphic functions. In particular, Heittokangas et al. [14] discussed the following difference equations:

$$\sum_{i=1}^{n} f(z+c_i) = R(z,f) = \frac{P(z,f)}{Q(z,f)} \quad \text{and} \tag{1.2}$$

$$\prod_{i=1}^{n} f(z+c_i) = R(z,f) = \frac{P(z,f)}{Q(z,f)}, \tag{1.3}$$

where $c_1,\ldots,c_n \in \mathbb{C}\{0\}$ and $R(z,f)$ is an irreducible rational function in $f(z)$ with meromorphic coefficients such that $a_p(z)b_q(z) \not\equiv 0$. They proved that $\max\{p,q\} \le n$ if equations (1.3) and (1.2) admit a transcendental meromorphic solution of finite order. After 4 years, Laine et al. [20] further studied the existence of solutions to the following complex difference equation:

$$\sum_{\{J\}} \alpha_J(z)\left(\prod_{j\in J} f(z+c_j)\right) = R(z,f(z)) = \frac{P(z,f(z))}{Q(z,f(z))}, \tag{1.4}$$

where $\{J\}$ is a collection of all subsets of $\{1, 2, \ldots, n\}$, P and Q are relatively prime polynomials in f over the field of rational functions, and the coefficients α_J are rational functions, and obtained that $\max\{p,q\} \le n$ if equation (1.2) admits a transcendental meromorphic solution of finite order. In 2010, Zhang and Liao [21] further studied the solutions of the difference equation with more general form than (1.2). Laine et al. [20], also proved the form of the solutions to complex difference equation (1.4) under some theoretical assumptions.

Theorem D. (See [20]). *Assume* $c_1, c_2, \ldots, c_n \in \mathbb{C}\setminus\{0\}$, $f(z)$ *is a transcendental meromorphic solution of the equation* (1.3), *and* $q := \deg_f Q > 0$. *If* $f(z)$ *has at most finitely many poles, it must be of the form*

$$f(z) = r(z)e^{g(z)} + s(z),$$

where $r(z)$ and $s(z)$ are rational functions and $g(z)$ is a transcendental entire function satisfying a difference equation of the form either

$$\sum_{j\in J} g(z+c_j) = (j_0 - q)g(z) + d$$

or

$$\sum_{j\in J} g(z+c_j) = \sum_{j\in I} g(z+c_j) + d,$$

were J and I are nonempty disjoint subsets of $\{1, 2, \ldots, n\}$, $j_0 \in \{0, 1, \ldots,\}$, $p := \deg_f P$, *and* $d \in \mathbb{C}$.

In view of the aforementioned theorems of Malmquist-type, a natural question is *whether similar results hold for more general difference equations of Malmquist-type*. For this question, the first aim of this article is to investigate the existence of solutions to the equation, when the left-hand sides of equations (1.1)–(1.4) are replaced by a differential-difference polynomial of f. To state our results, we first introduce the following definition.

Definition 1.1. (See [22, Definition 2.1] or [23].) A differential-difference polynomial in f is a finite sum of difference products of f, derivatives of f, and derivatives of their shifts, with all coefficients of these monomials being small functions of f.

Now, we give a differential-difference polynomial in f with the form

$$\begin{aligned} G(z,f) &= \sum_{\lambda\in I} \alpha_\lambda(z) f(z)^{\lambda_{0,0}} f'(z)^{\lambda_{0,1}} \cdots f^{(m)}(z)^{\lambda_{0,m}} f(z+c_1)^{\lambda_{1,0}} f'(z+c_1)^{\lambda_{1,1}} \cdots f^{(m)}(z+c_1)^{\lambda_{1,m}} \cdots \\ &\qquad f(z+c_n)^{\lambda_{n,0}} f'(z+c_n)^{\lambda_{n,1}} \cdots f^{(m)}(z+c_n)^{\lambda_{n,m}} \\ &= \sum_{\lambda\in I} \alpha_\lambda(z) \prod_{s=0}^{n}\prod_{t=0}^{m} f^{(t)}(z+c_s)^{\lambda_{s,t}}, \end{aligned} \tag{1.5}$$

where I is a finite set of multi-indices $\lambda = (\lambda_{0,0}, \ldots, \lambda_{0,m}, \lambda_{1,0}, \ldots, \lambda_{1,m}, \ldots, \lambda_{n,0}, \ldots, \lambda_{n,m})$, $\lambda_{s,t} \geq 0$ are not equal to 0, simultaneously, for $s = 0, 1, 2, \ldots, n$; $t = 0, 1, 2, \ldots, m$, and $c_0(=0), c_1, \ldots, c_n$ are distinct complex constants, and the meromorphic coefficients $\alpha_\lambda(z)(\not\equiv 0)$, $\lambda \in I$ are of growth $S(r,f)$. The degree of the monomial $\prod_{s=0}^{n}\prod_{t=0}^{m} f^{(t)}(z+c_s)^{\lambda_{s,t}}$ is defined by $d(\lambda) = \sum_{s=0}^{n}\sum_{t=0}^{m}\lambda_{s,t}$, and the degree of $G(z,f)$ is defined by

$$d(G) = \max_{\lambda\in I}\{d(\lambda)\}.$$

The first result of this article related to Malmquist theorem is obtained as follows.

Theorem 1.1. *Assume that $f(z)$ is a transcendental meromorphic solution of the differential-difference equation*:

$$G(z,f) = R(z,f) = \frac{P(z,f(z))}{Q(z,f(z))}, \tag{1.6}$$

where P and Q are relatively prime polynomials in f, the coefficients of $P(z,f)$ and $Q(z,f)$ are stated as in Theorem A, and the coefficients $\alpha_\lambda(z)(\not\equiv 0)$ are of growth $S(r,f)$. If f has at most finitely many poles and $q := \deg_f Q > 0$, then we have $\sigma(f) = \infty$, and it must take the form

$$f(z) = r(z)e^{g(z)} + s(z),$$

where $r(z)$ and $s(z)$ are rational functions, $g(z)$ *is a transcendental entire function, and there exist $n+1$ integers $k_0, k_1, \ldots, k_n$ and a constant η such that*

$$k_0 g(z) + k_1 g(z+c_1) + \cdots + k_n g(z+c_n) = \eta.$$

The following example shows that the condition that $f(z)$ has at most finitely many poles is necessary in Theorem 1.1.

Example 1.1. Let $c_1 = \frac{\pi}{4}$, $c_2 = \arctan 2$, $c_3 = \arctan 3$, and $f(z) = \tan z$. Then $f(z)$ satisfies the following equation:

$$f'(z)[f(z+c_1)f(z+c_2) + f(z+c_2)f(z+c_3)] = 4\frac{f^5 + 3f^4 + 2f^3 + f^2 + f - 2}{6f^3 - 11f^2 + 6f - 1},$$

but $f(z)$ is not of the form $f(z) = r(z)e^{g(z)} + s(z)$.

The following example shows that the conclusions in Theorem 1.1 can be realized.

Example 1.2. Let $f(z) = \frac{1}{z}e^{\sin(2\pi z)e^{\pi i z}}$ and $\alpha(z) = [-z^{-2} + (2\pi\cos(2\pi z) + \pi i\sin(2\pi z))e^{\pi i z}]^{-2}$. Then $f(z)$ satisfies the equation

$$\alpha(z)(f')^2[f(z+1)f(z+2) + f(z+2)f(z+4) + f(z+1)f(z+3)f(z+5)] = \frac{a_2f^5 + a_1f^3 + a_0}{f},$$

where $a_2 = \frac{z^4}{(z+2)(z+4)}$, $a_1 = \frac{z^2}{(z+1)(z+2)}$, and $a_0 = \frac{1}{z(z+1)(z+3)(z+5)}$. It can be seen that $g(z) = \sin(2\pi z)e^{\pi i z}$. Thus, it follows that $g(z)$ satisfies

$$g(z+1) + 2g(z+2) + g(z+3) = -e^{\pi i z}\sin(2\pi z) + 2e^{\pi i z}\sin(2\pi z) - e^{\pi i z}\sin(2\pi z) = 0.$$

Theorem 1.2. *Assume that $f(z)$ is a transcendental meromorphic solution with finite order of the differential-difference equation* (1.6), *where P and Q are relatively prime polynomials in f, the coefficients of $P(z,f)$* and *$Q(z,f)$ are stated as in* Theorem 1.1, *and the coefficients $\alpha_\lambda(z)$ of $G(z,f)$ are rational functions. If f is of finite order with at most finitely many poles, then $R(z,f)$ can be reduced to a polynomial in f of degree $p \le d(G)$.*

The following example shows that the condition "$f(z)$ has at most finitely many poles" cannot be omitted in Theorem 1.2.

Example 1.3. Let $c_1 = \arctan 1$, $c_2 = \arctan(-1)$, $c_3 = \arctan 2$, and $f(z) = \tan z$, then $f(z)$ satisfies the differential-difference equation:

$$f'(z)f(z+c_1)f(z+c_2)f(z+c_3) = \frac{f^3 + 2f^2 + f + 2}{2f - 1}.$$

It is easy to see that $p = 3 < d(G) = 4$ and $\sigma(f) = 1$. However, $R(z,f)$ is not a polynomial in f.

The following example shows that the restriction "f is finite order" also cannot be omitted in Theorem 1.2.

Example 1.4. Let $c_1 = \pi i$, $c_2 = 3\pi i$, and $f(z) = e^{e^z}$, then $f(z)$ satisfies the differential-difference equation

$$f'(z)[f'(z+c_1)]^2 f(z+c_2)^3 = \frac{e^{3z}}{f^4}.$$

The following example shows that the conclusion "$p \le d(G)$" is sharp to some extent in Theorem 1.2.

Example 1.5. Let $c_1 = \frac{\pi}{2}$, $c_2 = -\frac{\pi}{2}$, and $f(z) = \cos z$, then $f(z)$ satisfies the differential-difference equation

$$2f'(z)^2 - f(z+c_1)f(z+c_2) = 3(1 - f^2).$$

This shows that the equality in $p \le d(G)$ can be attained in Theorem 1.2.

The second purpose of this article is to study the existence and growth of solutions for a special differential-difference equation

$$\sum_{s=1}^{n}\alpha_s\prod_{t=1}^{m}[f^{(k_t)}(z)]^{\lambda_{s,t}}f(z+c_s)=R(z,f)=\frac{P(z,f(z))}{Q(z,f(z))}, \tag{1.7}$$

where $n \geq 2$, $m \geq 1$, $\lambda_{i,j} \geq 0$, $\alpha_s(z)(\not\equiv 0)$ are of growth $S(r,f)$, and $P(z,f)$ and $Q(z,f)$ are relatively prime polynomials in $f(z)$ with coefficients $a_i(i=0,\dots,p)$ and $b_j(j=0,\dots,q)$ such that $a_pb_q \not\equiv 0$. Denote

$$\Gamma=\max_{1\leq s\leq n}\left\{\sum_{t=1}^{m}\lambda_{s,t}(1+k_t)\right\}.$$

For the growth of solutions of equation (1.7), we obtain the following results.

Theorem 1.3. *Assume* $c_1, c_2, \dots, c_n \in \mathbb{C}\setminus\{0\}$, $f(z)$ *is a transcendental meromorphic solution of complex differential-difference equation of Malmquist-type* (1.7). *Let the coefficients* $a_i(i=0,\dots,p)$ *and* $b_j(j=0,\dots,q)$ *satisfy* $T(r,a_i)=S(r,f)$ *and* $T(r,b_j)=S(r,f)$. *If* $p > q+\Gamma+1$, *then* $\sigma(f)=\infty$.

Theorem 1.4. *Assume* $c_1, c_2, \dots, c_n \in \mathbb{C}\setminus\{0\}$, $f(z)$ *is a transcendental meromorphic solution of complex differential-difference equation of Malmquist-type* (1.7). *Let the coefficients* $a_i(i=0,\dots,p)$ *and* $b_j(j=0,\dots,q)$ *satisfy* $a_pb_q \not\equiv 0$. *If there is a dominant coefficient* a_μ *or* b_ν *satisfying*

$$\sigma_0=\sigma(a_\mu)=\mu(a_\mu)>\max\{\sigma(a_i),\sigma(b_j),\sigma(\alpha_s): 0\leq i\leq p, 0\leq j\leq q, 1\leq s\leq n, i\neq\mu\}$$

or

$$\sigma_0=\sigma(b_\nu)=\mu(b_\nu)>\max\{\sigma(a_i),\sigma(b_j),\sigma(\alpha_s): 0\leq i\leq p, 0\leq j\leq q, 1\leq s\leq n, j\neq\nu\},$$

then $\sigma_0 < \infty$. *If* $f(z)$ *is of finite regular growth and* $p > q+\Gamma+1$, *then* $\sigma(f)=\sigma_0$.

2 Proof of Theorem 1.1

Lemma 2.1. [24] *Let* $f(z)$ *be a meromorphic function with order* $\sigma=\sigma(f)<+\infty$ *and let* η *be a fixed nonzero complex number, then for each* $\varepsilon>0$, *we have*

$$T(r,f(z+\eta))=T(r,f)+O(r^{\sigma-1+\varepsilon})+O(\log r),$$

and if the exponent of convergence of poles $\lambda(\frac{1}{f})=\lambda<+\infty$, *then*

$$N(r,f(z+\eta))=N(r,f)+O(r^{\lambda-1+\varepsilon})+O(\log r)$$

and

$$m\left(r,\frac{f(z+\eta)}{f(z)}\right)+m\left(r,\frac{f(z)}{f(z+\eta)}\right)=O(r^{\lambda-1+\varepsilon}).$$

Lemma 2.2. (Valiron-Mohon'ko) [1]. *Let* $f(z)$ *be a meromorphic function. Then, for all irreducible rational functions in* $f(z)$,

$$R(z,f(z))=\frac{\sum_{i=0}^{m}a_i(z)f(z)^i}{\sum_{j=0}^{n}b_j(z)f(z)^j},$$

with meromorphic coefficients $a_i(z)$ *and* $b_j(z)$, *the characteristic function of* $R(z,f(z))$ *satisfies that*

$$T(r,R(z,f(z)))=dT(r,f)+O(\Psi(r)),$$

where $d=\max\{m,n\}$ *and* $\Psi(r)=\max_{i,j}\{T(r,a_i),T(r,b_j)\}$.

Lemma 2.3. [4]. *Suppose that* $f_j(z)$ $(j = 1, 2, \ldots, n)$ $(n \geq 2)$ *are meromorphic functions, and* $g_j(z)$ $(j = 1, 2, \ldots, n)$ *are entire functions satisfying the following conditions*:

(i) $\sum_{j=1}^{n} f_j(z)e^{g_j(z)} = 0$;

(ii) $g_j(z) - g_k(z)$ *are not constants for* $1 \leq j < k \leq n$;

(iii) *For* $1 \leq j \leq n$, $1 \leq h < k \leq n$,

$$T(r, f_j) = o\{T\big(r, e^{g_h - g_k}\big)\} \quad (r \to +\infty, r \in E),$$

where $E \subset (1, +\infty)$ *is the finite linear measure of finite logarithmic measure.*

Then $f_j(z) \equiv 0$ $(j = 1, 2, \ldots, n)$.

Lemma 2.4. [25]. *Let* $f(z)$ *be a meromorphic function and let* ϕ *be given by*

$$\phi = f^n + a_{n-1}f^{n-1} + \cdots + a_0,$$

where $a_t(i = 0, 1, \ldots, n-1)$ *are small meromorphic functions relative to* $f(z)$, *then either*

$$\phi = \left(f + \frac{a_{n-1}}{n}\right)^n$$

or

$$T(r, f) \leq \overline{N}\left(r, \frac{1}{\phi}\right) + \overline{N}(r, f) + S(r, f).$$

Proof of Theorem 1.1. Assume that $f(z)$ is a transcendental meromorphic solution of equation (1.6). Since the coefficients $a_i(z)$ $(i = 1, \ldots, p)$ and $b_j(z)$ $(j = 1, \ldots, q)$ are rational functions, in view of [20, p. 80], it follows that $P(z, f)$ and $Q(z, f)$ have only finitely many common zeros. Thus, by Lemma 2.3 in [20], it yields

$$N\left(r, \frac{1}{Q(z, f)}\right) \leq N\left(r, \frac{P(z, f)}{Q(z, f)}\right) + O(\log r). \tag{2.1}$$

Suppose that $f(z)$ has at most finitely many poles, then, in view of (1.6) and (2.1), we obtain

$$N\left(r, \frac{1}{Q(z, f)}\right) \leq N(r, G(z, f)) + O(\log r) = O(\log r). \tag{2.2}$$

By Lemma 2.4, it follows that either

$$Q(z, f) = (f(z) - s(z))^q,$$

where $s(z)$ is a rational function, or

$$T(r, f) \leq \overline{N}\left(r, \frac{1}{Q(z, f)}\right) + \overline{N}(r, f) + S(r, f). \tag{2.3}$$

Since $f(z)$ has at most finitely many poles, and in view of (2.2) and (2.3), we can deduce that $T(r, f) = S(r, f)$, which is a contradiction. Thus, $Q(z, f) = (f(z) - s(z))^q$ and $Q(z, f)$ has at most finitely many poles. In view of (2.1), we can see that $Q(z, f)$ has finitely many zeros. As a result, if there exist a rational function $r^*(z)$ and an entire function $g^*(z)$ such that

$$(f(z) - s(z))^q = r^*(z)e^{g^*(z)},$$

then we have

$$f(z) - s(z) = r(z)e^{g(z)},$$

where $r(z)$ is the qth root of $r^*(z)$ and $g(z) := \frac{g^*(z)}{q}$. Set $r_0(z) = r(z)$, $r_1(z) := r'(z) + r(z)g(z)$, $r_n(z) = r'(z) + r_{n-1}(z)g(z)$, and $s_n(z) := s^{(n)}(z)$, where $s_0(z) = s(z)$, then it follows that

$$\begin{aligned} f'(z) &= [r'(z)+r(z)g'(z)]e^{g(z)}+s'(z)=r_1(z)e^{g(z)}+s_1(z),\\ f''(z) &= [r'(z)+r_1(z)g(z)]e^{g(z)}+s''(z)=r_2(z)e^{g(z)}+s_2(z),\\ \cdots &,\\ f^{(j)}(z) &= r_j(z)e^{g(z)}+s_j(z),\\ \cdots &. \end{aligned} \tag{2.4}$$

Substituting $f(z)$ and (2.4) into (1.6), it yields

$$\begin{aligned} &\sum_{\lambda\in M}\alpha_\lambda(z)r^*(z)\prod_{s=0}^{n}\prod_{t=0}^{m}r_t(z+c_s)^{\lambda_{s,t}}e^{\sum_{s=0}^{n}\sum_{t=0}^{m}\lambda_{s,t}g(z+c_s)+qg(z)}\\ &+\sum_{\kappa\in J}\beta_\kappa(z)r^*(z)\prod_{s=0}^{n}\prod_{t=0}^{m}r_t(z+c_s)^{\kappa_{s,t}}e^{\sum_{s=0}^{n}\sum_{t=0}^{m}\kappa_{s,t}g(z+c_s)+qg(z)} = \sum_{i=0}^{p}p_i^*(z)e^{ig(z)}, \end{aligned} \tag{2.5}$$

where $M=\{\lambda=(\lambda_{0,0},\ldots,\lambda_{0,m},\lambda_{1,0},\ldots,\lambda_{1,m},\ldots,\lambda_{n,0},\ldots,\lambda_{n,m})\in I|\sum_{s=0}^{n}\sum_{t=0}^{m}\lambda_{s,t}=d(G)\}$, $J=\{\kappa=(\kappa_{0,0},\ldots,\kappa_{0,m},\kappa_{1,0},\ldots,\kappa_{1,m},\ldots,\kappa_{n,0},\ldots,\kappa_{n,m})\in I|\sum_{s=0}^{n}\sum_{t=0}^{m}\kappa_{s,t}\le d(G)-1,\ \kappa_{s,t}\in\mathbb{N}\cup\{0\}\}$, and $\alpha_\lambda(z)(\lambda\in M)$, $\beta_\kappa(z)(\kappa\in J)$, and $p_i^*(z)(i=0,1,\ldots,p)$ are rational functions.

If $g(z)$ is a nonconstant polynomial, let

$$g(z)=d_kz^k+d_{k-1}z^{k-1}+\cdots+d_0,\quad d_k\neq 0,$$

then, for any $c_i\neq 0$, it follows that

$$g(z+c_i)-g(z)=kd_kc_iz^{k-1}+o(z^{k-1}). \tag{2.6}$$

Thus, equation (1.6) can be represented as

$$\alpha_\lambda^*(z)e^{d(G)g(z)}+\sum_{\kappa=0}^{d(G)-1}\beta_\kappa^*(z)e^{\kappa g(z)}=\frac{\sum_{i=0}^{p}p_i^*(z)e^{ig(z)}}{r^*(z)e^{qg(z)}}, \tag{2.7}$$

where $\kappa=\sum_{s=0}^{n}\sum_{t=0}^{m}\kappa_{s,t}$,

$$\alpha_\lambda^*(z)=\sum_{\lambda\in M}\alpha_\lambda(z)\prod_{s=0}^{n}\prod_{t=0}^{m}r_t(z+c_s)^{\lambda_{s,t}}e^{\sum_{s=0}^{n}\sum_{t=0}^{m}\lambda_{s,t}[g(z+c_s)-g(z)]},$$

and

$$\beta_\kappa^*(z)=\beta_\kappa(z)\prod_{s=0}^{n}\prod_{t=0}^{m}r_t(z+c_s)^{\kappa_{s,t}}e^{\sum_{s=0}^{n}\sum_{t=0}^{m}\kappa_{s,t}[g(z+c_s)-g(z)]}.$$

Due to the small functions $\alpha_\lambda(z)$, the rational functions $r(z)$, $s(z)$, and (2.6), it follows that $T(r,\alpha_\lambda^*(z))=S(r,e^{g(z)})$ and $T(r,\beta_\kappa^*(z))=S(r,e^{g(z)})$ for any $\kappa=0,1,\cdots,d(G)-1$. Given the assumptions on $a_i(z)$ and $b_j(z)$, it follows from Lemma 2.2 and (2.7) that $d(G)=\max\{p,q\}$. In view of $q>0$, equation (2.7) can be represented as the form:

$$\alpha_\lambda^*(z)r(z)e^{(d(G)+q)g(z)}+\sum_{\kappa=0}^{d(G)-1}r(z)\beta_\kappa^*(z)e^{(\kappa+q)g(z)}=\sum_{i=0}^{p}p_i^*(z)e^{ig(z)},$$

which implies that $d(G)+q=p$, which is a contradiction. Thus, $g(z)$ is a transcendental entire function. In view of Lemma 2.3 and (2.5), there exists a constant $\eta\in\mathbb{C}$ such that

$$\sum_{s=0}^{n}\sum_{t=0}^{m}\lambda_{s,t}g(z+c_s)-\sum_{s=0}^{n}\sum_{t=0}^{m}\kappa_{s,t}g(z+c_s)=\eta$$

for some $\lambda\in M$ and $\kappa\in J$, or

$$\sum_{s=0}^{n}\sum_{t=0}^{m}\lambda_{s,t}g(z+c_s)+(q-j_0)g(z)=\eta$$

for some $\lambda \in M$ and $j_0 \in \{0, 1, \ldots, p\}$. As a result, there exist $n+1$ integers $k_0, k_1, \ldots, k_n$ and a constant η such that

$$k_0 g(z) + k_1 g(z + c_1) + \cdots + k_n g(z + c_n) = \eta.$$

Therefore, this completes the proof of Theorem 1.1. □

3 Proof of Theorem 1.2

Lemma 3.1. ([26, Lemma 7].) *Let $f(z)$ be a transcendental meromorphic function of hyper-order $\sigma_2(f) < 1$ and let $G(z, f)$ be a differential-difference polynomial of the form* (1.5), *then one has*

$$m(r, G(z, f)) \leq d(G)m(r, f) + S(r, f).$$

Proof of Theorem 1.2. Suppose that $f(z)$ is a transcendental meromorphic solution of finite order for the differential-difference equation (1.6) and has at most finitely many poles. If $q := \deg_f Q > 0$, then it follows from Theorem 1.1 that $\sigma(f) = \infty$, which is a contradiction with $f(z)$ being finite order. Thus, $q = 0$, implying that $R(z, f)$ is reduced to a polynomial. Hence, by Lemma 2.2, it follows that

$$T(r, R(z, f)) = T(r, P(z, f)) = pT(r, f) + S(r, f). \tag{3.1}$$

Since $f(z)$ has at most finitely many poles, then it yields $N(r, f) = O(\log r)$. In view of Lemma 3.1, we conclude that

$$T(r, G(z, f)) = m(r, G(z, f)) + N(r, G(z, f)) \leq d(G)m(r, f) + O(\log r) + S(r, f) \leq d(G)T(r, f) + S(r, f). \tag{3.2}$$

Thus, in view of (3.1) and (3.2), it yields

$$pT(r, f) \leq d(G)T(r, f) + S(r, f),$$

which leads to $p \leq d(G)$.

Therefore, this completes the proof of Theorem 1.2. □

4 Proof of Theorem 1.3

We recall some notations from [27]. If $f(z)$ has more than $S(r, f)$ poles of a certain type, then we claim that the integrated counting function of these poles is not of type $S(r, f)$. Thus, we use $\infty^k(0^k)$ to denote a pole (zero) of f with multiplicity k. The following lemma is also from [27, Lemma 3.1].

Lemma 4.1. ([27, Lemma 3.1].) *Let $f(z)$ be a meromorphic function having more than $S(r, f)$ poles and a_s $(s = 1, 2, \ldots, n)$ be small meromorphic functions with respect to f. Denote the maximum order of zeros and poles of the functions a_s at z_j by m_j. Then, for any $\varepsilon > 0$, there are at most $S(r, f)$ points z_j such that*

$$f(z_j) = \infty^{k_j},$$

where $m_j \geq \varepsilon k_j$.

Lemma 4.2. ([22, Lemma 2.2].) *Let f be a transcendental meromorphic solution of finite order ρ of a difference equation of the form*

$$f^n P(z, f) = Q(z, f),$$

where $P(z, f)$, and $Q(z, f)$ are differential-difference polynomials in f such that the total degree of $Q(z, f)$ is $\leq n$. Then, for any $\varepsilon > 0$,

$$m(r, P(z, f)) = O(r^{\sigma-1+\varepsilon}) + S(r, f),$$

possibly outside of an exceptional set of finite logarithmic measure.

Proof of Theorem 1.3. We will use the method in the proof of Proposition 5.4 in [28]. Let

$$F_L(z) := \sum_{s=1}^{n} \alpha_s \prod_{t=1}^{m} [f^{(k_t)}(z)]^{\lambda_{s,t}} f(z + c_s)$$

and

$$F_R(z) := \frac{a_p(z)f(z)^p + a_{p-1}(z)f(z)^{p-1} + \cdots + a_1(z)f(z) + a_0(z)}{b_q(z)f(z)^q + b_{q-1}(z)f(z)^{q-1} + \cdots + b_1(z)f(z) + b_0(z)}.$$

Suppose that $f(z)$ is a transcendental meromorphic solution with finite order of equation (1.7). Since $p > q + \Gamma + 1$, then, in view of Lemma 4.2, it yields that $m(r, f) = S(r, f)$, which implies that $f(z)$ has more than $S(r, f)$ poles, counting multiplicity. Thus, we can choose such poles sequence $\{z_i\}$ such that $f(z_i) = \infty^{k_i}$ and $m_i < \varepsilon k_i$, where m_i is the maximum order of poles or zeros of the coefficients a_i, b_j, and α_λ in (1.7) at z_i, and ε is any small constant. Taking the subsequence $\{z_{1,i}\}$ of poles as our starting point, and let $\varepsilon < \frac{1}{6}$, we may prove that

$$F_R(z_{1,i}) = \infty^{k'_{2,i}}, \quad k'_{2,i} \geq (p - \varepsilon)k_{1,i} - (q + \varepsilon)k_{1,i} \geq (2 - 2\varepsilon)k_{1,i}.$$

Analyzing the poles of F_L and considering $p > q + \Gamma + 1$, we can see that at least one of the points $z_{1,i} + c_1, \ldots, z_{1,i} + c_n$ is a pole of $f(z)$ with order $k_{2,i} \geq k'_{2,i} - (\Gamma + \varepsilon)k_{1,j}$. We first apply Lemma 4.2 to obtain that there are more than $S(r, f)$ such points $z_{2,i}$ with $f(z_{2,i}) = \infty^{k_{2,i}}$ and $m_{2,i} < \varepsilon k_{2,i}$. Then, we choose only one of these points and denote it by $z_{2,i}$. Thus, we have $F_R(z_{2,i}) = \infty^{k'_{3,i}}$ for each permitted $z_{2,i}$, and a pole $z_{3,i}$ satisfying $f(z_{3,i}) = \infty^{k_{3,i}}$, where

$$k_{3,i} \geq k'_{3,i} - (\Gamma + \varepsilon)k_{2,i} \geq (p - q - \Gamma - 3\varepsilon)k_{2,i} \geq (2 - 3\varepsilon)^2 k_{1,i}.$$

Continuing the above process, we can obtain a sequence $\{z_m\}$ satisfying $f(z_m) = \infty^{k_m}$ and $k_m \geq (2 - 3\varepsilon)^{m-1}k_1 \geq (2 - 3\varepsilon)^{m-1}$.

Next, we continue to give the estimate of the counting function $N(r, f)$. Set $C = \max\{c_1, c_2, \ldots, c_n\}$ and $r_m = |z_1| + (m - 1)C$. Thus, by a simple geometric observation, it is easy to see that

$$z_m \in B(z_1, (m - 1)C) \subset B(0, |z_1| + (m - 1)C) = B(0, r_m),$$

where $B(z_0, r)$ is an open disc of radius r centered at z_0. For sufficiently large m, it yields $r_m \leq 2(m - 1)C$, which leads to

$$n(r_m, f) \geq (2 - 3\varepsilon)^{m-1} \geq \left(\frac{3}{2}\right)^{m-1}.$$

Thus,

$$\begin{aligned}\lambda_2\left(\frac{1}{f}\right) &= \limsup_{r\to\infty} \frac{\log\log n(r, f)}{\log r} \\ &\geq \limsup_{m\to\infty} \frac{\log\log n(r_m, f)}{\log r_m} \\ &\geq \limsup_{m\to\infty} \frac{\log\log\left(\frac{3}{2}\right)^{m-1}}{\log r_m} = 1,\end{aligned}$$

implying that $\sigma_2(f) \geq \lambda_2\left(\frac{1}{f}\right) \geq 1$. This is a contradiction with the assumption of finite order $f(z)$. Thus, $\sigma(f) = \infty$. Therefore, this completes the proof of Theorem 1.3. □

5 Proof of Theorem 1.4

To prove Theorem 1.4, we need the following lemma.

Lemma 5.1. ([4, p. 37] or [3].) *Let $f(z)$ be a nonconstant meromorphic function in the complex plane and l be a positive integer. Then*

$$N(r, f^{(l)}) = N(r, f) + l\overline{N}(r, f) \quad \text{and} \quad T(r, f^{(l)}) \le T(r, f) + l\overline{N}(r, f) + S(r, f).$$

Proof of Theorem 1.4. Without losing generality, assume that

$$\sigma_0 = \sigma(a_p) > \max\{\sigma(a_i), \sigma(b_j), \sigma(\alpha_s) : 0 \le i \le p-1, 0 \le j \le q, 1 \le s \le n\}. \tag{5.1}$$

Now, two cases will be discussed below.

Case 1. If $\sigma(f) < \sigma(a_p) = \sigma_0 < \infty$, it then follows from (5.1) that

$$T(r, \alpha_s) = S(r, a_p), \quad T(r, a_i) = S(r, a_p), \quad T(r, b_j) = S(r, a_p), \quad \text{and} \quad T(r, f^l) = lT(r, f) = S(r, a_p),$$

for $s = 1, 2, \ldots, n$, $i = 0, 1, \ldots, p-1$, $j = 0, 1, \ldots, q$, and $l = 1, 2, \ldots, \max\{p, q\}$. So, in view of Lemma 2.2 and $p > q + \Gamma + 1$, combining the above equations yields

$$T\left(r, \frac{a_p(z)f(z)^p + a_{p-1}(z)f(z)^{p-1} + \cdots + a_1(z)f(z) + a_0(z)}{b_q(z)f(z)^q + b_{q-1}(z)f(z)^{q-1} + \cdots + b_1(z)f(z) + b_0(z)}\right) = pT(r, a_p) + S(r, a_p). \tag{5.2}$$

In addition, by Lemma 5.1, we have

$$T\big(r, f^{(k_t)}\big) = (k_t + 1)T(r, f) = S(r, a_p), \tag{5.3}$$

for $t = 0, 1, \ldots, m$. Thus, in view of Lemma 2.1 and (5.3), we can conclude that

$$T\left(r, \sum_{s=1}^{n} \alpha_s \prod_{t=1}^{m} [f^{(k_t)}(z)]^{\lambda_{s,t}} f(z + c_s)\right) \le \sum_{s=1}^{n}\left(\sum_{t=1}^{m}(k_t + 1)\lambda_{s,t} + 1\right)T(r, f) + \sum_{s=1}^{n} T(r, \alpha_s) + S(r, f). \tag{5.4}$$

Hence, from (1.7), (5.2), and (5.4), this leads to

$$pT(r, a_p) + S(r, a_p) \le \sum_{s=1}^{n}\left(\sum_{t=1}^{m}(k_t + 1)\lambda_{s,t} + 1\right)T(r, f) + \sum_{s=1}^{n} T(r, \alpha_s) + S(r, f) = S(r, a_p) + S(r, f).$$

This is a contradiction.

Case 2. If $\sigma_0 = \sigma(a_p) < \sigma(f) < \infty$, then it implies that all the coefficients a_i, b_j, α_s in (1.7) are small functions with respect to $f(z)$. With a view of $p > q + \Gamma + 1$, thus, it is easy to see that $\sigma(f) = \infty$ from Theorem 1.3, which is a contradiction.

Hence, we can conclude that $\sigma(f) = \sigma_0$. This completes the proof of Theorem 1.4. □

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