Research Article

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Minimal period problem for second-order Hamiltonian systems with asymptotically linear nonlinearities

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Abstract: By applying the combination of discrete variational method and approximation, developed in a previous study [J. Kuang, W. Chen, and Z. Guo, *Periodic solutions with prescribed minimal period for second-order even Hamiltonian systems*, Commun. Pure Appl. Anal. **21** (2022), no. 1, 47–59], we overcome some difficulties in the absence of Ambrosetti-Rabinowitz condition and obtain new sufficient conditions for the existence of periodic solutions with prescribed minimal period for second-order Hamiltonian systems with asymptotically linear nonlinearities.

Keywords: minimal period, second-order Hamiltonian system, discrete variational method, approximation

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1 Introduction and main result

In the pioneering work [1] of 1978, Rabinowitz conjectured that the first- and second-order Hamiltonian systems have nonconstant solutions with any prescribed minimal period. In 1985, Ekeland and Hofer [2] made important progress and confirmed the conjecture for a first-order Hamiltonian system with strictly convex assumptions. Since then, the minimal periodic problem of the Hamiltonian system has been extensively studied in the literature. The reader may refer to [3–21]. In particular, Kuang et al. [8] recently introduced a discrete variational method and approximation to study the minimal period problem for second-order even Hamiltonian systems under the assumption of the Ambrosetti-Rabinowitz condition. Note that discrete variational methods are very effective tools for difference equations, to mention a few, see [22–37].

Fan and Zhang [3] dealt with the minimal periodic problem for a first-order Hamiltonian system under the assumption of asymptotically linear nonlinearities. However, little work has been done that has referred to the minimal periodic problem for second-order Hamiltonian systems with such assumptions. Therefore, in this article, we mainly consider the minimal periodic problem for the classical second-order Hamiltonian system

$$x'' + f(x) = 0, \tag{1}$$

where $x \in \mathbb{R}$ and $f(x)$ is asymptotically linear at infinity.

Now, we state our main result.

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Theorem 1.1. *Assume that the following conditions hold*:
(f_1) $f \in C(\mathbb{R}, \mathbb{R})$ *and* $f(-x) = -f(x)$ *for all* $x \in \mathbb{R}$.
(f_2)

$$\lim_{x\to\infty} \frac{f(x)}{x} = d_* > 0 \quad \textit{and} \quad \lim_{x\to 0} \frac{f(x)}{x} = 0.$$

(f_3) *There exist two positive constants* η_* *and* β *such that*

$$f(x)x - 2F(x) \geq \beta f(x) \tag{2}$$

for $x \geq \eta_*$, *where* $F(x) = \int_0^x f(s)\mathrm{d}s$ *for* $x \in \mathbb{R}$.

Then, for each $T_* > 2\pi/\sqrt{d_*}$, *problem* (1) *has at least one nonconstant periodic solution* x_* *with minimal period* T_*.

Remark 1.1. The famous Ambrosetti-Rabinowitz condition in the case when x is scalar, there exist two positive constants η and $\mu > 2$ such that

$$xf(x) \geq \mu F(x) > 0 \quad \text{for all } |x| \geq \eta,$$

which implies

$$xf(x) - 2F(x) = \left(1 - \frac{2}{\mu}\right)xf(x) + \frac{2}{\mu}[xf(x) - \mu F(x)] \geq \left(1 - \frac{2}{\mu}\right)\eta f(x)$$

for $x \geq \eta$, that is, (f_3) in Theorem 1.1 is weaker than that in the Ambrosetti-Rabinowitz condition.

Corollary 1.1. *Assume that conditions* (f_1)–(f_3) *hold and* $d_* = +\infty$. *Then, for each* $T_* > 0$, *problem* (1) *has at least one nonconstant periodic solution* x_* *with minimal period* T_*.

Example 1.1. Let $f(x)$ be odd with

$$f(x) = \begin{cases} c_1 x - c_2 x^m, & \text{for } x \geq 1, \\ (c_1 - c_2)x^r, & \text{for } 0 \leq x \leq 1, \end{cases} \tag{3}$$

where $c_1 > c_2 > 0, r > 1$ and $0 < m < 1$. It is easy to check that (f_1)–(f_3) are satisfied; however, the Ambrosetti-Rabinowitz condition is not fulfilled. Then, for each $T_* > 2\pi/\sqrt{c_1}$, problem (1) has at least one nonconstant periodic solution x_* with minimal period T_* by Theorem 1.1.

The rest of this article is organized as follows. In Section 2, we give some preliminary results. In Section 3, we prove Theorem 1.1 by using discrete variational method and approximation.

2 Preliminary results

Let $\mathbb{Z}^+ = \{k > 0 : k \in \mathbb{Z}\}$ and $\mathbb{R}_* = \{x \geq 0 : x \in \mathbb{R}\}$. For each $T_* > 0$ and $x \in C([0, T_*/2], \mathbb{R}_*)$, we define x_* by

$$x_*(t) = \begin{cases} x(t - kT_*), & \text{for } t \in [kT_*, kT_* + T_*/2], \\ -x(kT_* + T_* - t), & \text{for } t \in [kT_* + T_*/2, kT_* + T_*] \end{cases} \tag{4}$$

for all $k \in \mathbb{Z}$, provided that $x \in C([0, T_*/2], \mathbb{R}_*)$ is a nontrivial solution of the following system:

$$\begin{cases} x'' + f(x) = 0, \\ x(0) = x(T_*/2) = 0. \end{cases} \tag{5}$$

Then, it follows from (f_1) that x_* is a nonconstant solution with minimal period T_* of problem (1).

For each fixed positive integer n, we denote by $h_n = \frac{T_*}{4n}$ and $u(k) = x(kh_n)$ for all $k \in [0, 2n]_{\mathbb{Z}}$, where $[0, 2n]_{\mathbb{Z}}$ denotes the discrete interval $[0, 1, \cdots, 2n]$. The following system can be regarded as a discrete analog of (5)

$$\begin{cases} -\Delta^2 u(k-1) = h_n^2 f(u(k)) & \text{for } k \in [1, 2n-1]_{\mathbb{Z}}, \\ u(0) = u(2n) = 0, \end{cases} \tag{6}$$

where Δ is the forward difference operator defined by $\Delta u(k) = u(k+1) - u(k)$ and $\Delta^2 u(k) = \Delta(\Delta u(k))$. Define

$$K_{n,j} = \frac{u_n(j+1) - u_n(j)}{h_n} \quad \text{for } 0 \le j \le 2n-1. \tag{7}$$

Then, (6) can be written as

$$\begin{cases} K_{n,0} - K_{n,1} = h_n f(u_n(1)), \\ K_{n,1} - K_{n,2} = h_n f(u_n(2)), \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots , \\ K_{n,2n-2} - K_{n,2n-1} = h_n f(u_n(2n-1)), \end{cases} \tag{8}$$

where $u_n(0) = u_n(2n) = 0$.

Now, we present the variational framework. Let $E_n = \{u = \{u(k)\}_{k=0}^{2n} : u(0) = u(2n) = 0\}$ and $E_n^* = \{u \in E_n : u(k) \ge 0 \quad \text{for } 0 \le k \le 2n\}$. So, E_n is isomorphic to $\mathbb{R}^{2n-1}$. Then, E_n can be equipped with the inner product $\langle \cdot, \cdot \rangle_n$ and norm $\|\cdot\|_n$ as

$$\langle u, v \rangle_n = \sum_{k=1}^{2n-1} u(k)v(k), u, v \in E_n$$

and

$$\|u\|_n = \left(\sum_{k=1}^{2n-1} |u(k)|^2 \right)^{1/2},$$

respectively. For the convenience of notations, we will identify $u = (0, u(1), \ldots, u(2n-1), 0)$ with $(u(1), \ldots, u(2n-1)) \in \mathbb{R}^{(2n-1)}$.

Consider the functional I_n defined by

$$I_n(u) = \sum_{k=0}^{2n-1} \frac{|u(k+1) - u(k)|^2}{2} - h_n^2 \sum_{k=1}^{2n-1} F(u(k)). \tag{9}$$

Then, we find that I_n is Fréchet differentiable in E_n, and its Fréchet derivative is given by

$$\langle I_n'(u), v \rangle = \sum_{k=1}^{2n-1} [-\Delta^2 u(k-1) - h_n^2 f(u(k))] v(k) \tag{10}$$

for u and $v \in E_n$. In view of (9) and (10), it is easy to obtain that a critical point of the functional I_n in E_n is a solution of problem (6).

In order to construct an equivalent form for the functional I_n, define the matrix B by

$$B = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 2 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 2 \end{pmatrix}.$$

Then, it is easy to see that

$$I_n(u) = \frac{1}{2}uBu^T - h_n^2 \sum_{k=1}^{2n-1} F(u(k)) \tag{11}$$

and

$$\begin{pmatrix} -\Delta^2 u(0) \\ -\Delta^2 u(1) \\ \cdots \\ -\Delta^2 u(2n-2) \end{pmatrix} = B \begin{pmatrix} u(1) \\ u(2) \\ \cdots \\ u(2n-1) \end{pmatrix}, \tag{12}$$

for $u \in E_n$. By direct computations, we obtain that the eigenvalues of B are given by

$$\lambda_k = 4\sin^2 \frac{k\pi}{4n}, \quad k = 1, 2, \ldots, 2n-1,$$

and the eigenvector ζ_k corresponding to λ_k is given by

$$\zeta_k = \left(\sin\frac{k\pi}{2n}, \sin\frac{2k\pi}{2n}, \ldots, \sin\frac{(2n-1)k\pi}{2n}\right)^T.$$

Now, we recall the definition of the Palais-Smale (PS) condition and the classical mountain pass lemma needed in the proof of our Theorem 1.1. As usual, let H be a real Banach space, we denote by B_r the open ball in H with radius r and center 0, and ∂B_r its boundary.

Definition 2.1. Let H be a real Banach space. A functional $I \in C^1(H, R)$ is said to satisfy the PS condition if every sequence $\{x_j\}$ in H, such that $\{I(x_j)\}$ is bounded and $I'(x_j) \to 0$ as $j \to +\infty$, has a convergent subsequence.

Lemma 2.1. (Mountain pass lemma) *Let H be a real Banach space and $I \in C^1(H, R)$ satisfies the PS condition. Assume that $I(0) = 0$ and the following two conditions hold.*
(J_1) *There exist constants $a > 0$ and $\rho > 0$ such that $I_{\partial B_\rho} \geq a$;*
(J_2) *There exists $e \in H \backslash B_\rho$ such that $I(e) \leq 0$.*

Then I possesses a critical value $\alpha \geq a$. Moreover, α can be characterized as

$$\alpha = \inf_{\varphi\in\Gamma} \max_{s\in[0,1]} I(\varphi(s)),$$

where

$$\Gamma = \{\varphi \in C([0,1], H) : \varphi(0) = 0, \varphi(1) = e\}.$$

3 Proof of Theorem 1.1

In order to prove Theorem 1.1, we need the following lemma.

Lemma 3.1. *Define $E_n^* = \{u \in E_n : u(k) \geq 0 \quad \text{for all } 0 \leq k \leq 2n\}$. If conditions ($f_1$)–($f_3$) are satisfied, then* (6) *has at least a nontrivial solution u_n in E_n^*. Moreover, there exist positive numbers M_* and n_* such that*

$$|u_n(k)| \leq T_* M_*$$

for $n \geq n_*$ and $1 \leq k \leq 2n-1$, where M_* is independent of n.

Proof. We divide the proof of Lemma 3.1 into three steps.

Step 1. We prove that problem (6) has a nonzero solution. For this, we first prove that I_n satisfies the PS condition. Let $\{I_n(u_j)\}$ be a bounded sequence and $I_n'(u_j) \to 0$ as $j \to +\infty$, where $\{u_j\}$ is a sequence in E_n. Then, there exists a positive constant M_n such that $|I_n(u_j)| \le M_n$ for all $j \in Z^+$. By (f_2), we have

$$\lim_{u\to\infty} \frac{F(u)}{u^2} = \frac{1}{2}\lim_{u\to\infty} \frac{f(u)}{u} = \frac{1}{2} d_* \ (d_* = +\infty \quad \text{is also valid}).$$

Since $T_* > 2\pi / \sqrt{d_*}$, we can choose $L > 0$ such that $d_* > L > 4\pi^2 / T_*^2$. Then,

$$\lim_{u\to\infty} \frac{F(u)}{u^2} = \frac{1}{2} d_* > \frac{1}{2} L \quad \text{and} \quad \lim_{u\to\infty} \frac{f(u)}{u} = d_* > L \ (d_* = +\infty \ \text{is also valid}).$$

Hence, there exists $\eta_0 > 0$ such that

$$F(u) > \frac{L}{2} u^2 \quad \text{and} \quad |f(u)| > L|u| \quad \text{for all } |u| > \eta_0. \tag{13}$$

Let

$$\bar{\eta} = \max\{\eta_*, \eta_0\}$$

and

$$M_0 = \max\left\{ \left| F(u) - \frac{L}{2} u^2 \right| : 0 \le |u| \le \bar{\eta} \right\}.$$

Then, we have

$$F(u) \ge \frac{L}{2} u^2 - M_0 \quad \text{for all } u \in \mathbb{R}. \tag{14}$$

By the continuity of f, there exist two positive constants σ and P_* such that

$$|f(u)| \le \sigma \quad \text{and} \quad |f(u)u - 2F(u)| \le P_* \tag{15}$$

for $|u| \le \bar{\eta}$, which, combined with (f_1), (f_3), and (13), produces

$$\begin{aligned} \sum_{k=1}^{2n-1} [f(u_j(k))u_j(k) - 2F(u_j(k))] &= \left(\sum_{|u_j(k)|\le\bar{\eta}} + \sum_{|u_j(k)|>\bar{\eta}} \right) [f(u_j(k))u_j(k) - 2F(u_j(k))] \\ &\ge -(2n-1)P_* + \sum_{|u_j(k)|>\bar{\eta}} [f(|u_j(k)|)|u_j(k)| - 2F(|u_j(k)|)] \\ &\ge -(2n-1)P_* + \beta \sum_{|u_j(k)|>\bar{\eta}} f(|u_j(k)|) \\ &\ge -(2n-1)P_* + L\beta \sum_{|u_j(k)|>\bar{\eta}} |u_j(k)| \\ &\ge -2nP_* + L\beta \sum_{|u_j(k)|>\bar{\eta}} |u_j(k)|, \end{aligned} \tag{16}$$

for each $j \in \mathbb{Z}^+$.

Denote $u_{j*} = \max\{|u_j(k)| : 1 \le k \le 2n-1\}$, then $u_{j*} \ge \|u_j\|_n / \sqrt{2n-1}$. Now, we claim that $\|u_j\|_n$ is bounded. Arguing by the contradiction, assume up to a subsequence which we still denote by $\{u_j\}$, that $\|u_j\|_n \to +\infty$ as $j \to +\infty$. By $I_n'(u_j) \to 0$ as $j \to +\infty$, we can choose j large enough such that

$$\|I_n'(u_j)\| < \varepsilon < \frac{Lh_n^2\beta}{\sqrt{2n-1}},$$

which, together with (16), produces

$$
\begin{aligned}
2M_n + \varepsilon\|u_j\|_n &\geq 2I_n(u_j) - \langle I_n'(u_j), u_j\rangle \\
&= h_n^2\sum_{k=1}^{2n-1}[f(u_j(k))u_j(k) - 2F(u_j(k))] \\
&\geq -2nh_n^2P_* + L\beta h_n^2\sum_{|u_j(k)|>\overline{\eta}}|u_j(k)| \\
&= -2nh_n^2P_* + L\beta h_n^2\left[\sum_{|u_j(k)|>\overline{\eta}} + \sum_{|u_j(k)|\leq\overline{\eta}}\right]|u_j(k)| - L\beta h_n^2\sum_{|u_j(k)|\leq\overline{\eta}}|u_j(k)| \\
&\geq -2nh_n^2P_* + L\beta h_n^2 u_{j*} - 2L\beta h_n^2 n\overline{\eta} \\
&\geq -2nh_n^2P_* + \frac{Lh_n^2\beta}{\sqrt{2n-1}}\|u_j\|_n - 2L\beta h_n^2 n\overline{\eta},
\end{aligned} \tag{17}
$$

which contradicts the fact that $\|u_j\|_n \to +\infty$ as $j \to +\infty$. Hence, I_n satisfies the PS condition for each $n \in \mathbb{Z}^+$.

Next, we show that I_n satisfies (J_1) in Lemma 2.1. It follows from (f_2) that there exist two positive constants $b_0 < \pi^2/T_*^2$ and ρ such that

$$
|f(x)| \leq b_0|x| \quad \text{and} \quad F(x) \leq b_0|x|^2 \quad \text{for all } |x| \leq \rho. \tag{18}
$$

By

$$
\lim_{n\to\infty}\frac{\sin\frac{\pi}{4n}}{h_n} = \frac{\pi}{T_*},
$$

there exists a positive integer n_* such that

$$
\frac{\pi^2}{2T_*^2} \leq \left(\frac{\sin\frac{\pi}{4n}}{h_n}\right)^2 \leq \frac{\pi^2}{T_*^2} \quad \text{for } n \geq n_*. \tag{19}
$$

Hence, for all $u \in \partial B_\rho$ and $n \geq n_*$,

$$
I_n(u) \geq 2\sin^2\frac{\pi}{4n}\|u\|_n^2 - b_0h_n^2\|u\|_n^2 \geq \left(\frac{\pi^2}{T_*^2} - b_0\right)h_n^2\rho^2 > 0. \tag{20}
$$

Thus, I_n satisfies (J_1) in Lemma 2.1.

Finally, we prove that I_n satisfies (J_2) of Lemma 2.1 for $n \geq n_*$. Let $e_n \in E_n$ with

$$
e_n = d\left(\sin\frac{\pi}{2n}, \sin\frac{2\pi}{2n}, \ldots, \sin\frac{(2n-1)\pi}{2n}\right)^T = d\zeta_1. \tag{21}
$$

We can choose d large enough such that $d > \sqrt{M_0}\,/\sqrt{\frac{L}{4} - \frac{\pi^2}{T_*^2}}$, where the fixed number d is independent of n. By (14), we have

$$
I_n(e_n) \leq 2d^2\sin^2\frac{\pi}{4n}\sum_{k=1}^{2n-1}\sin^2\frac{k\pi}{2n} - h_n^2\sum_{k=1}^{2n-1}\left(\frac{L}{2}\left(\mathrm{d}\sin\frac{k\pi}{2n}\right)^2 - M_0\right). \tag{22}
$$

In view of

$$
\begin{aligned}
\sum_{k=1}^{2n-1}\sin^2\frac{k\pi}{2n} &= \frac{1}{2}\sum_{k=1}^{2n-1}\left[1 - \cos\frac{k\pi}{n}\right] \\
&= \frac{2n-1}{2} - \frac{2\cos\frac{\pi}{n}\sin\frac{\pi}{2n} + 2\cos\frac{2\pi}{n}\sin\frac{\pi}{2n} + \cdots + 2\cos\frac{2n\pi-\pi}{n}\sin\frac{\pi}{2n}}{4\sin\frac{\pi}{2n}} \\
&= n,
\end{aligned} \tag{23}
$$

which, combined with (22), gives us

$$I_n(e_n) \le \frac{2nd^2h_n^2\pi^2}{T_*^2} - \frac{Lnd^2h_n^2}{2} + h_n^2 2nM_0 \le 2nh_n^2\left[-d^2\left(\frac{L}{4} - \frac{\pi^2}{T_*^2}\right) + M_0\right]. \tag{24}$$

Then, it follows from $d > \sqrt{M_0}\,/\sqrt{\frac{L}{4} - \frac{\pi^2}{T_*^2}}$ that $I_n(e_n) < 0$. Hence, we have verified all assumptions of Lemma 2.1. For $n \ge n_*$, we know that I_n possesses a critical value α_n, where

$$\alpha_n = \inf_{\varphi\in\Gamma}\max_{s\in[0,1]} I_n(\varphi(s)),\ \Gamma = \{\varphi \in C([0,1], E_n) : \varphi(0) = 0, \varphi(1) = e_n\}.$$

Thus, (6) has at least a nontrivial solution in E_n.

Step 2. We prove that problem (6) has a nonzero solution $u_n \in E_n^*$ for any $n \ge n_*$. For each $\varphi \in \Gamma$ with

$$\varphi(t) = (\varphi_1(t), \varphi_2(t), \ldots, \varphi_{2n-1}(t))$$

satisfying $\varphi(0) = 0$ and $\varphi(1) = e_n$, where e_n is given in (21), we denote by φ^*,

$$\varphi^*(t) = (|\varphi_1(t)|, |\varphi_2(t)|, \ldots, |\varphi_{2n-1}(t)|)$$

for $t \in [0,1]$. It is easy to check that $\varphi^* \in \Gamma^*$, where

$$\Gamma^* = \{\varphi \in C([0,1], E_n^*) : \varphi(0) = 0, \varphi(1) = e_n\} \subset \Gamma.$$

By (f_1), it is easy to obtain that

$$\max_{s\in[0,1]} I_n(\varphi(s)) \ge \max_{s\in[0,1]} I_n(\varphi^*(s))$$

and

$$\alpha_n = \inf_{\varphi\in\Gamma}\max_{s\in[0,1]} I_n(\varphi(s)) = \inf_{\varphi\in\Gamma^*}\max_{s\in[0,1]} I_n(\varphi(s)).$$

Thus, there exists a nonzero critical point $u_n \in E_n^*$ and $I_n(u_n) = \alpha_n > 0$ for $n \ge n_*$.

Step 3. For $n \ge n_*$, we claim that there exists a positive constant M_*, independent of n, such that $0 \le u_n(k) \le M_*$ for all $0 \le k \le 2n$. Choose

$$\overline{\varphi}(t) = \left(\mathrm{d}t\sin\frac{\pi}{2n}, \mathrm{d}t\sin\frac{2\pi}{2n}, \ldots, \mathrm{d}t\sin\frac{(2n-1)\pi}{2n}\right)^T \in \Gamma,$$

using the similar argument as (24), we have

$$\alpha_n \le \max_{s\in[0,1]} I_n(\overline{\varphi}(s)) \le 2nh_n^2 M_0 = \frac{1}{2}h_n T_* M_0. \tag{25}$$

On the other hand, it follows from (9) and (10) that

$$\alpha_n = I_n(u_n) - \frac{1}{2}\langle I_n'(u_n), u_n\rangle = h_n^2\sum_{k=1}^{2n-1}\left(\frac{1}{2}f(u_n(k))u_n(k) - F(u_n(k))\right). \tag{26}$$

Combining (f_3), (15), (25), and (26) together, we obtain that

$$\begin{aligned}\left|h_n\sum_{k=1}^{j} f(u_n(k))\right| &\le h_n\sum_{|u_n(k)|\le\overline{\eta}} |f(u_n(k))| + h_n\sum_{|u_n(k)|>\overline{\eta}} |f(u_n(k))|\\ &\le \frac{1}{2}\sigma T_* + \frac{2}{\beta}h_n\sum_{|u_n(k)|>\overline{\eta}}\left(\frac{1}{2}f(u_n(k))u_n(k) - F(u_n(k))\right)\\ &\le \frac{1}{2}\sigma T_* + \frac{2}{\beta}h_n\sum_{|u_n(k)|>\overline{\eta}}\left(\frac{1}{2}f(u_n(k))u_n(k) - F(u_n(k))\right)\\ &\quad + \frac{2}{\beta}h_n\sum_{|u_n(k)|\le\overline{\eta}}\left(\frac{1}{2}f(u_n(k))u_n(k) - F(u_n(k))\right) + \frac{1}{\beta}h_n\sum_{|u_n(k)|\le\overline{\eta}} P_*\\ &\le \frac{1}{2}\sigma T_* + \frac{2\alpha_n}{\beta h_n} + \frac{1}{2\beta}T_*P_* \le \frac{1}{2}\sigma T_* + \frac{M_0 T_*}{\beta} + \frac{1}{2\beta}T_*P_*,\end{aligned} \tag{27}$$

for each $1 \le j \le 2n-1$. Let

$$M_* = \frac{1}{2}\sigma T_* + \frac{M_0 T_*}{\beta} + \frac{1}{2\beta}T_* P_*, \tag{28}$$

then, it is easy to check that M_* is independent of n. By Step 2, $u_n \in E_n^*$ is a nonzero solution to (6), then

$$0 \le K_{n,0} \le M_* \quad \text{and} \quad 0 \ge K_{n,4n-1} \ge -M_*, \tag{29}$$

which, combined with (8) and (27), gives us

$$|K_{n,j}| = \left|K_{n,0} - h_n \sum_{k=1}^{j} f(u_n(k))\right| \le |K_{n,0}| + \left|h_n \sum_{k=1}^{j} f(u_n(k))\right| \le 2M_* \tag{30}$$

for all $1 \le j \le 2n-1$. Hence,

$$\left|\frac{u_n(k)}{h_n}\right| = \left|\frac{(u_n(k) - u_n(k-1)) + \cdots + (u_n(1) - u_n(0))}{h_n}\right| \le 4nM_*$$

and

$$|u_n(k)| \le 4nM_* h_n = T_* M_*$$

for all $n \ge n_*$ and $0 < k \le 2n$, where T_* and M_* are independent of n, n_* is given in (19). This completes the proof of the lemma. □

By Lemma 3.1, there exists a nonzero solution $u_n \in E_n^*$ to (6) for $n \ge n_*$. We define x_n by

$$x_n(t) = \begin{cases} tK_{n,0}, & \text{for } 0 \le t \le h_n, \\ tK_{n,1} - K_{n,1}h_n + u_n(1), & \text{for } h_n \le t \le 2h_n, \\ tK_{n,2} - 2K_{n,2}h_n + u_n(2), & \text{for } 2h_n \le t \le 3h_n, \\ \cdots\cdots\cdots\cdots , & \\ tK_{n,2n-1} - (2n-1)h_n K_{n,2n-1} + u_n(2n-1), & \text{for } (2n-1)h_n \le t \le T_*/2. \end{cases} \tag{31}$$

Then, $x_n \in C([0, T_*/2], \mathbb{R}_*)$ with $x_n(0) = x_n(T_*/2) = 0$ for $n \ge n_*$.

Now, we are in a position to prove Theorem 1.1.

Proof. First, we prove that there exists a subsequence of $\{x_n\}_{n=n_*}^{+\infty}$, which converges uniformly on $[0, T_*/2]$. For any s and t with $0 \le s \le t \le T_*/2$, there exist two integers j_1 and j_2 such that

$$j_1 h_n \le s \le t \le j_2 h_n,$$

satisfying $0 \le s - j_1 h_n < h_n$ and $0 \le j_2 h_n - t < h_n$. Then, by (29) and (30), we have

$$|x_n(t) - x_n(s)| = \begin{cases} 0 \le 2M_*(t-s), & \text{for } j_2 = j_1, \\ |K_{n,j_1}(t-s)| \le 2M_*(t-s), & \text{for } j_2 = j_1 + 1, \\ |x_n(t) - x_n(j_1 h_n + h_n) + x_n(j_1 h_n + h_n) - x_n(s)| \le 2M_*(t-s), & \text{for } j_2 = j_1 + 2, \\ |x_n(t) - x_n(j_2 h_n - h_n) + \cdots + x_n(j_1 h_n + h_n) - x_n(s)| \le 2M_*(t-s), & \text{for } j_2 \ge j_1 + 3. \end{cases} \tag{32}$$

Moreover,

$$|x_n(t)| = |x_n(t) - x_n(0)| \le 2M_* t \le M_* T_* \tag{33}$$

for $0 \le t \le T_*/2$ and $n \ge n_*$. By Arzela-Ascoli theorem, we can choose a subsequence, still denoted by $\{x_n\}_{n=n_*}^{+\infty}$, such that $\{x_n(t)\}_{n=n_*}^{+\infty}$ converges uniformly to $x(t)$ on $[0, T_*/2]$, where $x \in C([0, T_*/2], \mathbb{R}_*)$ with $x(0) = x(T_*/2) = 0$.

Next, we claim that, for each $n \ge n_*$, there exists a positive integer $k_* \in [1, 2n-1]_{\mathbb{Z}}$ such that

$$|u_n(k_*)| > \rho. \tag{34}$$

Suppose by contrary that $|u_n(k)| \le \rho$ for all $k \in [1, 2n-1]_{\mathbb{Z}}$. By (18), we have

$$\sum_{k=1}^{2n-1} f(u_n(k))u_n(k) \le b_0\|u_n\|_n^2,$$

which, together with (19), produces

$$\langle I_n'(u_n), u_n\rangle \ge 4\sin^2\frac{\pi}{4n}\|u_n\|_n^2 - h_n^2\sum_{k=1}^{2n-1} f(u_n(k))u_n(k) \ge \left(\frac{2\pi^2}{T_*^2} - b_0\right)h_n^2\|u_n\|_n^2 > 0 \tag{35}$$

for $n \ge n_*$, which contradicts the fact that $\langle I_n'(u_n), u_n\rangle = 0$. Thus, our claim is proved.

Now, we claim that x is a nonzero function. Arguing by the contradiction, assume that $x = 0$. Since $\{x_n\}_{n=n_*}^{+\infty}$ converges uniformly to x on $[0, T_*/2]$ for $n \ge n_*$, there exists a positive integer $\bar{n} > n_*$ such that

$$|x_n(t)| \le \rho \quad \text{for all } n \ge \bar{n},\ 0 \le t \le T_*/2, \tag{36}$$

which contradicts (34). Hence, x is a nonzero function.

Finally, we show that $x \in C([0, T_*/2], \mathbb{R}_*)$ is a nonzero solution of problem (5) and x_* is a nonconstant periodic solution with prescribed minimal period T_* of problem (1), where x_* is given in (4). It follows from (29) that there exists a subsequence of $\{x_n\}_{n=n_*}^{+\infty}$, still denoted by $\{x_n\}_{n=n_*}^{+\infty}$, such that

$$K_{n,0} \to K_* \quad \text{as } n \to +\infty.$$

It is easy to check that $K_* \ge 0$. By the definition of $\{x_n\}_{n=n_*}^{+\infty}$, we have that $x_n(t)$ is left differentiable on $(0, T_*/2]$ for $n \ge n_*$. For every $t \in (0, T_*/2]$, there exists $j \in [0, 2n-1]_{\mathbb{Z}}$ such that $jh_n < t \le (j+1)h_n$. Then, the left derivative $x_{n-}'(t)$ is given by

$$x_{n-}'(t) = K_{n,j} = K_{n,0} - h_n\sum_{k=1}^{j} f(u_n(k)) = K_{n,0} - \int_0^t f(x_n(s))\mathrm{d}s + \tau_n(t), \tag{37}$$

where

$$\tau_n(t) = \int_0^t f(x_n(s))\mathrm{d}s - h_n\sum_{k=1}^{j} f(u_n(k)) = \int_0^{jh_n} f(x_n(s))\mathrm{d}s - h_n\sum_{k=1}^{j} f(x_n(kh_n)) + \int_{jh_n}^{t} f(x_n(s))\mathrm{d}s. \tag{38}$$

By (33) and the continuity of f, there exists a positive constant $\overline{M}$ such that

$$-\overline{M} \le f(x_n(t)) \le \overline{M},$$

then,

$$-\overline{M}h_n \le \int_{jh_n}^{t} f(x_n(s))\mathrm{d}s \le \overline{M}h_n \tag{39}$$

for all $t \in [0, T_*/2]$ and $n \ge n_*$. Since $\{x_n(t)\}_{n=n_*}^{+\infty}$ converges uniformly to $x(t)$ on $[0, T_*/2]$, we have

$$\begin{aligned}
&\int_0^{jh_n} f(x_n(s))\mathrm{d}s - h_n\sum_{k=1}^{j} f(x_n(kh_n))\\
&\quad \le -\int_0^{jh_n} f(x(s))\mathrm{d}s + \int_0^{jh_n} f(x_n(s))\mathrm{d}s + h_n\sum_{k=1}^{j} f(x(kh_n)) - h_n\sum_{k=1}^{j} f(x_n(kh_n)) + \int_0^{jh_n} f(x(s))\mathrm{d}s\\
&\qquad - h_n\sum_{k=1}^{j} f(x(kh_n)) \to 0 \quad \text{as } n \to +\infty,
\end{aligned} \tag{40}$$

which, together with (39), produces

$$\tau_n(t) \to 0 \quad \text{as } n \to +\infty. \tag{41}$$

Similarly, we can prove that $x_n(t)$ is right differentiable on $[0, T_*/2)$, and the right derivative is given by

$$x'_{n-}(t) = K_{n,0} - \int_0^t f(x_n(s))\mathrm{d}s + \gamma_n(t), \tag{42}$$

where $\gamma_n(t) \to 0$ as $n \to \infty$. Given $t > 0$ and $\delta < 0$, for any s, satisfying $0 \le t + \delta < s \le t \le T_*/2$, there exist two integers j_3 and j_4 such that

$$j_3 h_n < s \le \cdots \le t \le j_4 h_n,$$

satisfying $0 < s - j_3 h_n \le h_n$ and $0 \le j_4 h_n - t < h_n$. By (8) and (30), it is easy to see that, $j_4 h_n - j_3 h_n \le t - s + 2h_n \le -\delta + 2h_n$ and

$$|x'_{n-}(t) - x'_{n-}(s)| = |x'_{n-}(j_4 h_n) - x'_{n-}((j_3+1)h_n)| = |K_{n,j_4-1} - K_{n,j_3}| \le 2(j_4 - j_3 - 1)h_n M_* \le 2M_*(-\delta + h_n),$$

then,

$$x'_{n-}(t) - 2M_*(-\delta + h_n) \le x'_{n-}(s) \le x'_{n-}(t) + 2M_*(-\delta + h_n). \tag{43}$$

Similarly, for any $t > 0$ and $\delta < 0$, satisfying $0 \le t + \delta < t \le T_*/2$, there exist two integers j_5 and j_6 such that

$$t + \delta < j_5 h_n \le \cdots \le j_6 h_n \le t,$$

satisfying $0 < j_5 h_n - (t+\delta) \le h_n$ and $0 \le t - j_6 h_n < h_n$. By direct computations, we have

$$\begin{aligned}\frac{x_n(t+\delta) - x_n(t)}{\delta} &= \frac{x_n(t+\delta) - x_n(j_5 h_n)}{\delta} + \cdots + \frac{x_n(j_6 h_n) - x_n(t)}{\delta}\\ &= \frac{x'_{n-}(j_5 h_n)\cdot(t + \delta - j_6 h_n)}{\delta} + \cdots + \frac{x'_{n-}(t)\cdot(j_6 h_n - t)}{\delta},\end{aligned}$$

which, combined with (43), gives us

$$x'_{n-}(t) - 2M_*(-\delta + h_n) \le \frac{x_n(t+\delta) - x_n(t)}{\delta} \le x'_{n-}(t) + 2M_*(-\delta + h_n). \tag{44}$$

Taking the limit $n \to +\infty$ implies

$$K_* - \int_0^t f(x(s))\mathrm{d}s + 2M_*\delta \le \frac{x(t+\delta) - x(t)}{\delta} \le K_* - \int_0^t f(x(s))\mathrm{d}s - 2M_*\delta. \tag{45}$$

Then,

$$\lim_{\delta \to 0^-} \frac{x(t+\delta) - x(t)}{\delta} = K_* - \int_0^t f(x(s))\mathrm{d}s \tag{46}$$

for $0 < t \le T_*/2$. Using a similar argument as above, we have

$$\lim_{\delta \to 0^+} \frac{x(t+\delta) - x(t)}{\delta} = K_* - \int_0^t f(x(s))\mathrm{d}s \tag{47}$$

for $0 \le t < T_*/2$. Thus, $x \in C^1([0, T_*/2], \mathbb{R}_*)$, $x(0) = x(T_*/2) = 0$, and

$$x'(t) - K_* = -\int_0^t f(x(s))\mathrm{d}s \tag{48}$$

for $0 \le t \le T_*/2$. Hence, $x \in C([0, T_*/2], \mathbb{R}_*)$ is a nontrivial solution of (5). By (f_1), it is easy to obtain that x_* is a nonconstant periodic solution with prescribed minimal period T_* of problem (1), where x_* is given in (4). This completes the proof of Theorem 1.1. □

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