Research Article

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On the critical fractional Schrödinger-Kirchhoff-Poisson equations with electromagnetic fields

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Abstract: This paper intend to study the following critical fractional Schrödinger-Kirchhoff-Poisson equations with electromagnetic fields in $\mathbb{R}^3$:

$$\varepsilon^{2s}\mathfrak{M}([u]_{s,A}^2)(-\Delta)_A^s u + V(x)u + (|x|^{2t-3} * |u|^2)u = f(x, |u|^2)u + |u|^{2_s^*-2}u, \quad x \in \mathbb{R}^3.$$

Under suitable assumptions, together with the concentration compactness principle and variational method, we prove that the existence and multiplicity of semiclassical solutions for above problem as $\varepsilon \to 0$.

Keywords: fractional Schrödinger-Kirchhoff-Poisson equations, fractional magnetic operator, critical nonlinearity, variational methods

MSC 2020: 35A15, 35B99, 35J60, 47G20

1 Introduction

This paper deals with the existence and multiplicity of solutions for the critical fractional Schrödinger-Kirchhoff-Poisson equations with electromagnetic fields in $\mathbb{R}^3$:

$$\varepsilon^{2s}\mathfrak{M}([u]_{s,A}^2)(-\Delta)_A^s u + V(x)u + (|x|^{2t-3} * |u|^2)u = f(x, |u|^2)u + |u|^{2_s^*-2}u, \quad x \in \mathbb{R}^3, \tag{1.1}$$

where $\varepsilon > 0$ is a positive parameter, $3/4 < s < 1$, $0 < t < 1$, $2_s^* = 6/(3-2s)$ is the usual Sobolev critical exponent, V is an electric potential, and $(-\Delta)_A^s$ and A are called the magnetic operator and magnetic potential, respectively. According to d'Avenia and Squassina in [1], the fractional operator $(-\Delta)_A^s$, which up to normalization constants, can be defined on smooth functions u as follows

$$(-\Delta)_A^s u(x) := 2\lim_{\varepsilon \to 0} \int\limits_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{u(x) - \mathrm{e}^{i(x-y)\cdot A\left(\frac{x+y}{2}\right)}u(y)}{|x-y|^{N+2s}}\mathrm{d}y, \quad x \in \mathbb{R}^N,$$

and magnetic potential A is given by

$$[u]_{s,A}^2 := \iint\limits_{\mathbb{R}^{2N}} \frac{|u(x) - \mathrm{e}^{i(x-y)\cdot A\left(\frac{x+y}{2}\right)}u(y)|^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y.$$

Throughout the paper, the electric potential V, Kirchhoff function $\mathfrak{M}$, and f satisfy the following assumptions:

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($\mathcal{V}$) $V(x) \in C(\mathbb{R}^3, \mathbb{R})$, $V(0) = \min_{x\in\mathbb{R}^3} V(x) = 0$, and there is $b > 0$ such that the set $V^b = \{x \in \mathbb{R}^N : V(x) < b\}$ has finite Lebesgue measure.

($\mathscr{M}$) (M_1) The Kirchhoff function $\mathfrak{M} : \mathbb{R}_0^+ \to \mathbb{R}^+$ is continuous, and there exists $\mathfrak{m}_0 > 0$ such that $\inf_{s\geq 0}\mathfrak{M}(s) = \mathfrak{m}_0$. ($M_2$) There exists $\sigma \in (2/2_s^*, 1]$ satisfying $\sigma\mathscr{M}(t) \geq \mathfrak{M}(t)t$ for all $t \geq 0$, where $\mathscr{M}(t) = \int_0^t \mathfrak{M}(s)\mathrm{d}s$.

($\mathscr{F}$) (f_1) $f \in C(\mathbb{R}^3 \times \mathbb{R}, \mathbb{R})$ and $f(x, t) = o(|t|)$ uniformly in x as $t \to 0$. (f_2) There exist $c_0 > 0$ and $q \in (2, 2_s^*)$ such that $|f(x, t)| \leq c_0\left(1 + t^{\frac{q-1}{2}}\right)$. ($f_3$) There exist $l_0 > 0$, $\max\{2/\sigma, 4\} < r$, and $\max\{2/\sigma, 4\} < \mu < 2_s^*$ such that $F(x, t) \geq l_0|t|^{r/2}$, and $\mu F(x, t) \leq 2f(x, t)t$ for all $(x, t) \in \mathbb{R}^3 \times \mathbb{R}$, where $F(x, t) = \int_0^t f(x, s)\mathrm{d}s$.

First, our motivation to study problem (1.1) mainly comes from the application of the fractional magnetic operator. We note that the equation with fractional magnetic operator often arises as a model for various physical phenomena, in particular in the study of the infinitesimal generators of Lévy stable diffusion processes [2]. Also, the number of literature on nonlocal operators and their applications has been studied, and hence, we refer interested readers to [3–7]. To further research this kind of equation by variational methods, many scholars have established the basic properties of fractional Sobolev spaces, readers are referred to [8,9].

Next, we note that some works that appeared in recent years concerning the follwing magnetic Schrödinger equation without Poisson term:

$$-(\nabla u - iA)^2 u + V(x)u = f(x, |u|)u, \tag{1.2}$$

where the magnetic operator in (1.2) is given by

$$-(\nabla u - iA)^2 u = -\Delta u + 2iA(x)\cdot\nabla u + |A(x)|^2 u + iu\,\mathrm{div}A(x).$$

As stated in the study by Squassina and Volzone [10], up to correcting the operator by the factor $(1 - s)$, it follows that $(-\Delta)_A^s u$ converges to $-(\nabla u - iA)^2 u$ as $s \to 1$. Thus, up to normalization, the nonlocal case can be seen as an approximation of the local one. Recently, many researchers have paid attention to the equations with fractional magnetic operator. In particular, Mingqi et al. [11] studied some existence results of Schrödinger-Kirchhoff type equation involving the fractional p-Laplacian and the magnetic operator:

$$M([u]_{s,A}^2)(-\Delta)_A^s u + V(x)u = f(x, |u|)u \quad \text{in } \mathbb{R}^N, \tag{1.3}$$

where f satisfies the subcritical growth condition. For the critical growth case, the authors in [12] first considered the following fractional Schrödinger equations:

$$\varepsilon^{2s}(-\Delta)_{A_\varepsilon}^s u + V(x)u = f(x, |u|)u + K(x)|u|^{2_\alpha^*-2}u \quad \text{in } \mathbb{R}^N. \tag{1.4}$$

They obtained the existence of ground state solution u_ε by using variational methods. Subsequently, Liang et al. [13] proved the existence and multiplicity of solutions to a class of Schrödinger-Kirchhoff type equation in the non-degenerate case. We draw the attention of the reader to the degenerate case involving the magnetic operator in the study by Liang et al. [14].

On the other hand, for case $A \equiv 0$ in problem (1.1), there have been numerous articles dedicated to the study of the fractional Schrödinger-Poisson system as it appears in an interesting physical context. For example, Giammetta in [15] first studied the local and global well-posedness of a fractional Schrödinger-Poisson system in one dimension. Zhang et al. in [16] obtained the existence of radial ground state solution to the fractional Schrödinger-Poisson system with a general subcritical or critical nonlinearity by using the perturbation approach. In [17], Murcia and Siciliano proved that the number of positive solutions for a class of doubly singularly perturbed fractional Schrödinger-Poisson system via the Ljusternick-Schnirelmann category. Liu in [18] concerned with the existence of multibump solutions for the fractional Schrödinger-Poisson system through the Lyapunov-Schmidt reduction method. Chen et al. in [19] admitted the existence of the Nehari-type ground state solutions for fractional Schrödinger-Poisson system by using the non-Nehari manifold approach. For more related results, we can cite the recent works [20–24] and the references therein.

Once we turn our attention to the Schrödinger-Kirchhoff-Poisson equations with electromagnetic fields, we immediately see that the literature is relatively scarce. In this case, we can cite the recent works [25,26]. We call attention to Ambrosio in [27] proved that the multiplicity and concentration results for a class of fractional Schrödinger-Poisson type equation with magnetic field and subcritical growth. For the critical growth case, Ambrosio in [28] also obtained the multiplicity and concentration of nontrivial solutions to the fractional Schrödinger-Poisson equation with the magnetic field. However, to the best of our knowledge, semiclassical solutions to fractional magnetic Schrödinger-Poisson equations problem (1.1) have not ever been considered until now.

Inspired by the previously mentioned works, our main objective is to study the critical fractional Schrödinger-Kirchhoff-Poisson equations with electromagnetic fields. The proof of these assertions is given by means of concentration compactness principle and variational method. For this purpose, we will use some minimax arguments. Moreover, due to the appearance of the critical term, the Sobolev embedding does not possess compactness. To this end, we need some technical estimations.

We are now in a position to state the existence result as follows.

Theorem 1.1. *Let $(\mathcal{V})$ and $(\mathcal{F})$ hold. If $\mathfrak{M}$ satisfies (M_1) and (M_2), then the following statements hold:*

(1) *For any $\kappa > 0$, there is $\mathcal{E}_\kappa > 0$ such that if $0 < \varepsilon < \mathcal{E}_\kappa$, then problem (1.1) has at least one solution u_ε satisfying*

$$\frac{\sigma\mu - 1}{2}\int_{\mathbb{R}^3} F(x, |u_\varepsilon|^2)\mathrm{d}x + \left(\frac{2}{\sigma} - \frac{1}{2_s^*}\right)\int_{\mathbb{R}^3} |u_\varepsilon|^{2_s^*}\mathrm{d}x \le \kappa\varepsilon^3, \tag{1.5}$$

$$\left(\frac{\sigma}{2} - \frac{1}{\mu}\right)\alpha_0\varepsilon^{2s}[u_\varepsilon]_{s,A}^2 + \left(\frac{1}{2} - \frac{1}{\mu}\right)\int_{\mathbb{R}^3} V(x)|u_\varepsilon|^2\mathrm{d}x \le \kappa\varepsilon^3. \tag{1.6}$$

Moreover, $u_\varepsilon \to 0$ in E as $\varepsilon \to 0$.

(2) *For any $m \in \mathbb{N}$ and $\kappa > 0$, there is $\mathcal{E}_{m\kappa} > 0$ such that if $0 < \varepsilon < \mathcal{E}_{m\kappa}$, then problem (1.1) has at least m pairs of solutions $u_{\varepsilon,i}$, $u_{\varepsilon,-i}$, $i = 1, 2, \ldots, m$ which satisfy the estimates (1.5) and (1.6). Moreover, $u_{\varepsilon,i} \to 0$ in E as $\varepsilon \to 0$, $i = 1, 2, \ldots, m$.*

The main feature of our consequence in the present paper is to establish the multiplicity result for problem (1.1) under the critical growth condition. There is no doubt that we encounter serious difficulties because of the lack of compactness. To overcome the challenge, we use the concentration-compactness principles for fractional Sobolev spaces according to [29–31] to prove the $(PS)_c$ condition at special levels c. On the other hand, we need to develop new techniques to construct sufficiently small minimax levels.

The rest of our paper is organized as follows. In Section 2, we briefly review some properties of the Sobolev spaces with fractional order. In Section 3, we prove the Palais-Smale condition at some special energy levels by using the concentration-compactness principles for fractional Sobolev spaces. Section 4 deals with the existence and multiplicity result for problem (1.1).

2 Preliminaries

In this section, we briefly review the definitions and list some basic properties of the Lebesgue spaces, which we use throughout this article.

For any $s \in (0, 1)$, fractional Sobolev space $H_A^s(\mathbb{R}^3, \mathbb{C})$ is defined by

$$H_A^s(\mathbb{R}^3, \mathbb{C}) = \{u \in L^2(\mathbb{R}^3, \mathbb{C}) : [u]_{s,A} < \infty\},$$

where $s \in (0,1)$ and $[u]_{s,A}$ denotes the so-called Gagliardo semi-norm, that is,

$$[u]_{s,A} = \left(\iint_{\mathbb{R}^6} \frac{|u(x) - e^{i(x-y)\cdot A\left(\frac{x+y}{2}\right)}u(y)|^2}{|x-y|^{N+2s}} dx dy \right)^{1/2},$$

and $H_A^s(\mathbb{R}^3, \mathbb{C})$ is endowed with the norm

$$\|u\|_{H_A^s(\mathbb{R}^3,\mathbb{C})} = ([u]_{s,A}^2 + \|u\|_{L^2}^2)^{\frac{1}{2}}.$$

For the reader's convenience, we will use the following embedding theorem, see Lemma 3.5 in [1].

Proposition 2.1. *The space $H_A^s(\mathbb{R}^3, \mathbb{C})$ is continuously embedded in $L^\vartheta(\mathbb{R}^3, \mathbb{C})$ for all $\vartheta \in [2, 2_s^*]$. Furthermore, the space $H_A^s(\mathbb{R}^3, \mathbb{C})$ is continuously compact embedded in $L^\vartheta(K, \mathbb{C})$ for all $\vartheta \in [2, 2_s^*]$ and any compact set $K \subset \mathbb{R}^3$.*

Next, we have the following diamagnetic inequality, and its proof can be found in the study by d'Avenia and Squassina [1].

Lemma 2.1. *Let $u \in H_A^s(\mathbb{R}^N)$, then $|u| \in H^s(\mathbb{R}^3)$. That is,*

$$\||u|\|_s \leq \|u\|_{s,A}.$$

From Proposition 3.6 in [2], for all $u \in H^s(\mathbb{R}^3)$, we have

$$[u]_s = \|(-\Delta)^{\frac{s}{2}}\|_{L^2(\mathbb{R}^3)},$$

i.e.

$$\iint_{\mathbb{R}^6} \frac{|u(x) - u(y)|^2}{|x-y|^{3+2s}} dx dy = \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u(x)|^2 dx.$$

Moreover,

$$\iint_{\mathbb{R}^6} \frac{(u(x) - u(y))(v(x) - v(y))}{|x-y|^{3+2s}} dx dy = \int_{\mathbb{R}^3} (-\Delta)^{\frac{s}{2}} u(x) \cdot (-\Delta)^{\frac{s}{2}} v(x) dx.$$

For problem (1.1), we will use the Banach space E defined by

$$E = \left\{ u \in H_A^s(\mathbb{R}^3, \mathbb{C}) : \int_{\mathbb{R}^3} V(x)|u|^2 dx < \infty \right\}$$

with the norm

$$\|u\|_E := \left([u]_{s,A}^2 + \int_{\mathbb{R}^3} V(x)|u|^2 dx \right)^{\frac{1}{2}}.$$

By the assumption $(\mathcal{V})$, we know that the embedding $E \hookrightarrow H_A^s(\mathbb{R}^3, \mathbb{C})$ is continuous. Note that the norm $\|\cdot\|_E$ is equivalent to the norm $\|\cdot\|_\varepsilon$ defined by

$$\|u\|_\varepsilon := \left([u]_{s,A}^2 + \varepsilon^{-2s} \int_{\mathbb{R}^3} V(x)|u|^2 dx \right)^{\frac{1}{2}}$$

for each $\varepsilon > 0$.

Obviously, for each $\theta \in [2, 2_s^*]$, there is $c_\theta > 0$ such that

$$|u|_\theta \le c_\theta \|u\|_E \le c_\theta \|u\|_\varepsilon, \tag{2.1}$$

where $0 < \varepsilon < 1$. Hereafter, we shortly denote by $\|\cdot\|_\nu$ the norm of Lebesgue space $L^\nu(\Omega)$ with $\nu \ge 1$.

Now, let $s, t \in (0,1)$ such that $4s + 2t \ge 3$, we can see that

$$H^s(\mathbb{R}^3, \mathbb{R}) \hookrightarrow L^{\frac{12}{3+2t}}(\mathbb{R}^3, \mathbb{R}). \tag{2.2}$$

Then, by (2.2), we have

$$\int_{\mathbb{R}^3} u^2 v \mathrm{d}x \le \|u\|^2_{L^{\frac{12}{3+2t}}(\mathbb{R}^3)} \|v\|_{2_t^*} \le C\|u\|^2_{H^s(\mathbb{R}^3,\mathbb{R})} \|v\|_{D^{t,2}(\mathbb{R}^3)}$$

for $u \in H^s(\mathbb{R}^3, \mathbb{R})$, where

$$\|v\|^2_{D^{t,2}(\mathbb{R}^3)} = \iint_{\mathbb{R}^6} \frac{|u(x) - u(y)|^2}{|x - y|^{3+2t}} \mathrm{d}x\mathrm{d}y.$$

Then, by the Lax-Milgram theorem, there exists a unique $\psi^t_{|u|}$ such that $\psi^t_{|u|} \in D^{t,2}(\mathbb{R}^3, \mathbb{R})$ such that

$$(-\Delta)^t \psi^t_{|u|} = |u|^2 \quad \text{in } \mathbb{R}^3. \tag{2.3}$$

Therefore, we obtain the following t-Riesz formula:

$$\psi^t_{|u|}(x) = \pi^{-\frac{3}{2}} 2^{-2t} \frac{\Gamma(3-2t)}{\Gamma(t)} \int_{\mathbb{R}^3} \frac{|u(y)|^2}{|x-y|^{3-2t}} \mathrm{d}y, \quad \forall x \in \mathbb{R}^3. \tag{2.4}$$

We note that the aforementioned integral is convergent at infinity since $|u|^2 \in L^{\frac{6}{3+2t}}(\mathbb{R}^3, \mathbb{R})$. Next we collect some properties of $\psi^t_{|u|}$, which will be used in this paper. The following proposition can be proved by using similar arguments as [27,28].

Proposition 2.2. *Assume that $4s + 2t \ge 3$ holds, for any $u \in E$, we have*

(i) $\psi^t_{|u|} : H^s(\mathbb{R}^3, \mathbb{R}) \to D^{t,2}(\mathbb{R}^3, \mathbb{R})$ *is continuous and maps bounded sets into bounded sets*;

(ii) *if $u_n \rightharpoonup u$ in E, then $\psi^t_{|u_n|} \rightharpoonup \psi^t_{|u|}$ in $D^{t,2}(\mathbb{R}^3, \mathbb{R})$*;

(iii) $\psi^t_{|\alpha u|} = \alpha^2 \psi^t_{|u|}$ *for any $\alpha \in \mathbb{R}$ and* $\psi^t_{|u(\cdot+y)|}(x) = \psi^t_{|u|}(x+y)$;

(iv) $\psi^t_{|u|} \ge 0$ *for all $u \in E$. Moreover,*

$$\|\psi^t_{|u|}\|_{D^{t,2}(\mathbb{R}^3,\mathbb{R})} \le C\|u\|^2_{L^{\frac{12}{3+2t}}(\mathbb{R}^3)} \le C\|u\|^2_\varepsilon$$

and

$$\int_{\mathbb{R}^3} \psi^t_{|u|} |u|^2 \mathrm{d}x \le C\|u\|^4_{L^{\frac{12}{3+2t}}(\mathbb{R}^3)} \le C\|u\|^4_\varepsilon.$$

3 Behavior of (PS) sequences

In this section, to overcome the lack of compactness caused by the critical exponents, we intend to employ the second concentration-compactness principle, see [29–31] for more details. Moreover, to obtain the solution of problem (1.1), we will use the following equivalent form:

$$\mathfrak{M}([u]^2_{s,A})(-\Delta)^s_A u + \varepsilon^{-2s} V(x) u + \psi^t_{|u|} u = \varepsilon^{-2s} f(x, |u|^2) u + \varepsilon^{-2s} |u|^{2_s^*-2} u, \tag{3.1}$$

for $x \in \mathbb{R}^3$. Now, let us consider the Euler-Lagrange functional $J_\varepsilon : E \to \mathbb{R}$ associated with (1.1), defined by

$$\begin{aligned}\mathcal{J}_\varepsilon(u) := \frac{1}{2}\mathscr{M}([u]_{s,A}^2) + \frac{\varepsilon^{-2s}}{2}\int_{\mathbb{R}^3} V(x)|u|^2\,\mathrm{d}x + \frac{\varepsilon^{-2s}}{4}\int_{\mathbb{R}^3}\psi_{|u|}^t|u|^2\,\mathrm{d}x \\ - \frac{\varepsilon^{-2s}}{2}\int_{\mathbb{R}^3} F(x,|u|^2)\mathrm{d}x - \frac{\varepsilon^{-2s}}{2_s^*}\int_{\mathbb{R}^3}|u|^{2_s^*}\mathrm{d}x.\end{aligned} \tag{3.2}$$

It is clear that $\mathcal{J}_\varepsilon$ is of class $C^1(E,\mathbb{R})$ under the assumptions $(\mathcal{F})$ (see [32]). Moreover, for all $u, v \in E$, the Fréchet derivative of $\mathcal{J}_\varepsilon$ is given by

$$\begin{aligned}\langle \mathcal{J}_\varepsilon'(u), v\rangle = \mathfrak{M}([u]_{s,A}^2)\,\mathrm{Re}\iint_{\mathbb{R}^6}\frac{(u(x) - \mathrm{e}^{i(x-y)\cdot A\left(\frac{x+y}{2}\right)}u(y))\overline{(v(x) - \mathrm{e}^{i(x-y)\cdot A\left(\frac{x+y}{2}\right)}v(y))}}{|x-y|^{3+2s}}\mathrm{d}x\mathrm{d}y \\ + \varepsilon^{-2s}\,\mathrm{Re}\int_{\mathbb{R}^3} V(x)u\bar{v}\mathrm{d}x + \varepsilon^{-2s}\,\mathrm{Re}\int_{\mathbb{R}^3}\psi_{|u|}^t u\bar{v}\mathrm{d}x - \varepsilon^{-2s}\,\mathrm{Re}\int_{\mathbb{R}^3}(|u|^{2_s^*-2}u + f(x,|u|^2)u)\bar{v}\mathrm{d}x.\end{aligned} \tag{3.3}$$

Thus, the weak solutions of (1.1) coincide with the critical points of $\mathcal{J}_\varepsilon$.

The main result of this section is the following compactness result.

Lemma 3.1. *Let $(\mathcal{V})$ and $(\mathcal{F})$ hold. If $\mathfrak{M}$ satisfies (M_1) and (M_2), then for any $0 < \varepsilon < 1$, $\mathcal{J}_\varepsilon$ satisfies $(PS)_c$ condition, for all $c \in (0, \sigma_0\varepsilon^{3-2s})$, where $\sigma_0 := \left(\frac{1}{\mu} - \frac{1}{2_s^*}\right)(\mathfrak{m}_0 S)^{\frac{3}{2s}}$, that is, any $(PS)_c$-sequence $\{u_n\}_n \subset E$ has a strongly convergent subsequence in E.*

Proof. Let $(u_n)_n$ be a $(PS)_{c_\lambda}$ sequence for $\mathcal{J}_\varepsilon$, we first claim that $(u_n)_n$ is bounded in E. In fact, by $\mathcal{J}_\varepsilon(u_n) \to c$ and $\mathcal{J}_\varepsilon'(u_n) \to 0$ in E', it follows from (M_2) and (f_3) that

$$\begin{aligned}c + o(1)\|u_n\|_\varepsilon = \mathcal{J}_\varepsilon(u_n) - \frac{1}{\mu}\langle\mathcal{J}_\varepsilon'(u_n), u_n\rangle &= \frac{1}{2}\mathscr{M}([u_n]_{s,A}^2) - \frac{1}{\mu}\mathfrak{M}([u_n]_{s,A}^2)[u_n]_{s,A}^2 \\ &+ \left(\frac{1}{2} - \frac{1}{\mu}\right)\varepsilon^{-2s}\int_{\mathbb{R}^3} V(x)|u_n|^2\,\mathrm{d}x + \left(\frac{1}{4} - \frac{1}{\mu}\right)\varepsilon^{-2s}\int_{\mathbb{R}^3}\psi_{|u_n|}^t|u_n|^2\,\mathrm{d}x \\ &+ \varepsilon^{-2s}\int_{\mathbb{R}^3}\left(\frac{1}{\mu}f(x,|u_n|^2)|u_n|^2 - \frac{1}{4}F(x,|u_n|^2)\right)\mathrm{d}x + \left(\frac{1}{\mu} - \frac{1}{2_s^*}\right)\varepsilon^{-2s}\int_{\mathbb{R}^3}|u_n|^{2_s^*}\mathrm{d}x \\ &\geq \left(\frac{1}{2\sigma} - \frac{1}{\mu}\right)M([u_n]_{s,A}^2)[u_n]_{s,A}^2 + \left(\frac{1}{2} - \frac{1}{\mu}\right)\varepsilon^{-2s}\int_{\mathbb{R}^3} V(x)|u_n|^2\,\mathrm{d}x \\ &\geq \left(\frac{\sigma}{2} - \frac{1}{\mu}\right)\mathfrak{m}_0[u_n]_{s,A}^2 + \left(\frac{1}{2} - \frac{1}{\mu}\right)\varepsilon^{-2s}\int_{\mathbb{R}^3} V(x)|u_n|^2\,\mathrm{d}x.\end{aligned} \tag{3.4}$$

We know that $\{u_n\}_n$ is bounded in E from $\max\{2/\sigma, 4\} < \mu < 2_s^*$. Furthermore, we can obtain $c \geq 0$ by passing to the limit in (3.4). Hence, by diamagnetic inequality, $\{|u_n|\}_n$ is bounded in $H^s(\mathbb{R}^3)$. Then, by using the fractional version of concentration compactness principle in the fractional Sobolev space (see [29–31]), up to a subsequence, we have

$$u_n \to u \quad \text{a.e. in } \mathbb{R}^3, \tag{3.5}$$

$$u_n \rightharpoonup u \quad \text{in } E,$$

$$|(-\Delta)^{\frac{s}{2}}u_n|^2 \rightharpoonup \mu = |(-\Delta)^{\frac{s}{2}}u|^2 + \tilde{\mu} + \sum_{j\in I}\delta_{x_j}\mu_j \quad \text{in the sense of measures in } \mathcal{M}(\mathbb{R}^3), \tag{3.6}$$

$$|u_n|^{2_s^*} \rightharpoonup \nu = |u|^{2_s^*} + \sum_{j\in I}\delta_{x_j}\nu_j \quad \text{in the sense of measures in } \mathcal{M}(\mathbb{R}^3), \tag{3.7}$$

$$\nu_j \le (S^{-1}\mu(\{x_j\}))^{\frac{2_s^*}{2}} \quad \text{for } j \in J, \tag{3.8}$$

where S is the best Sobolev constant, i.e.,

$$S = \inf_{u \in H^s(\mathbb{R}^3)\setminus\{0\}} \frac{\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 \,\mathrm{d}x}{\int_{\mathbb{R}^3} |u|^{2_s^*} \,\mathrm{d}x},$$

$x_j \in \mathbb{R}^N$, and δ_{x_j} are Dirac measures at x_j and μ_j, and ν_j are constants. Moreover, we have

$$\limsup_{n\to\infty} \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u_n|^2 \,\mathrm{d}x = \int_{\mathbb{R}^3} \mathrm{d}\mu + \mu_\infty, \tag{3.9}$$

$$\limsup_{n\to\infty} \int_{\mathbb{R}^3} |u_n|^{2_s^*} \,\mathrm{d}x = \int_{\mathbb{R}^3} \mathrm{d}\nu + \nu_\infty, \tag{3.10}$$

$$\nu_\infty \le (S^{-1}\nu_\infty)^{\frac{2_s^*}{2}}, \tag{3.11}$$

where

$$\mu_\infty = \lim_{R\to\infty} \limsup_{n\to\infty} \int_{\{x\in\mathbb{R}^3 : |x|>R\}} |(-\Delta)^{\frac{s}{2}} u_n|^2 \,\mathrm{d}x,$$

$$\nu_\infty = \lim_{R\to\infty} \limsup_{n\to\infty} \int_{\{x\in\mathbb{R}^3 : |x|>R\}} |u_n|^{2_s^*} \,\mathrm{d}x.$$

In the following, we shall prove that

$$J = \varnothing \quad \text{and} \quad \nu_\infty = 0.$$

Now, we suppose on the contrary that $J \ne \varnothing$. Then, we can construct a smooth cut-off function, take $\phi \in C_0^\infty(\mathbb{R}^3)$ such that $0 \le \phi \le 1$; $\phi \equiv 1$ in $B(x_j, \varepsilon)$, $\phi(x) = 0$ in $\mathbb{R}^3 \setminus B(x_j, 2\varepsilon)$. For any $\varepsilon > 0$, define $\phi_\varepsilon = \phi\left(\frac{x-x_j}{\varepsilon}\right)$, where $j \in J$. It is not difficult to see that $\{u_n\phi_\varepsilon\}_n$ is bounded in E. Then $\langle J_\varepsilon'(u_n), u_n\phi_\varepsilon\rangle \to 0$, which implies

$$\begin{aligned}
&\mathfrak{M}([u_n]_{s,A}^2) \iint_{\mathbb{R}^6} \frac{|u_n(x) - \mathrm{e}^{i(x-y)\cdot A\left(\frac{x+y}{2}\right)} u_n(y)|^2 \phi_\varepsilon(y)}{|x-y|^{3+2s}} \mathrm{d}x\mathrm{d}y + \varepsilon^{-2s} \int_{\mathbb{R}^3} V(x)|u_n|^2 \phi_\varepsilon(x)\mathrm{d}x \\
&= -\operatorname{Re}\left\{ \mathfrak{M}([u_n]_{s,A}^2) \iint_{\mathbb{R}^6} \frac{(u_n(x) - \mathrm{e}^{i(x-y)\cdot A\left(\frac{x+y}{2}\right)} u_n(y))\overline{u_n(x)(\phi_\varepsilon(x) - \phi_\varepsilon(y))}}{|x-y|^{3+2s}} \mathrm{d}x\mathrm{d}y \right\} \\
&\quad - \varepsilon^{-2s} \int_{\mathbb{R}^3} \psi_{|u_n|}^t |u_n|^2 \phi_\varepsilon \mathrm{d}x + \varepsilon^{-2s} \int_{\mathbb{R}^3} |u_n|^{2_s^*} \phi_\varepsilon \mathrm{d}x + \varepsilon^{-2s} \int_{\mathbb{R}^3} f(x, |u_n|^2)|u_n|^2 \phi_\varepsilon(x)\mathrm{d}x + o_n(1).
\end{aligned} \tag{3.12}$$

It is easy to verify that

$$\iint_{\mathbb{R}^6} \frac{||u_n(x)| - |u_n(y)||^2 \phi_\varepsilon(y)}{|x-y|^{3+2s}} \mathrm{d}x\mathrm{d}y \to \int_{\mathbb{R}^3} \phi_\rho \mathrm{d}\mu \quad \text{as} \quad n \to \infty$$

and

$$\int_{\mathbb{R}^3} \phi_\varepsilon \mathrm{d}\mu \to \mu(\{x_i\}) \quad \text{as } \rho \to 0.$$

Note that the Hölder inequality yields

$$\begin{aligned}
&\left|\operatorname{Re}\left\{\mathfrak{M}([u_n]_{s,A}^2)\int\int_{\mathbb{R}^6}\frac{(u_n(x)-e^{i(x-y)\cdot A\left(\frac{x+y}{2}\right)}u_n(y))\overline{u_n(x)(\phi_\varepsilon(x)-\phi_\varepsilon(y))}}{|x-y|^{3+2s}}\mathrm{d}x\mathrm{d}y\right\}\right| \\
&\le C\iint_{\mathbb{R}^6}\frac{|u_n(x)-e^{i(x-y)\cdot A\left(\frac{x+y}{2}\right)}u_n(y)|\cdot|\phi_\varepsilon(x)-\phi_\varepsilon(y)|\cdot|u_n(x)|}{|x-y|^{3+2s}}\mathrm{d}x\mathrm{d}y \qquad (3.13)\\
&\le C\left(\iint_{\mathbb{R}^6}\frac{|u_n(x)|^2|\phi_\varepsilon(x)-\phi_\varepsilon(y)|^2}{|x-y|^{3+2s}}\mathrm{d}x\mathrm{d}y\right)^{1/2}.
\end{aligned}$$

Lemma 3.4 in [33] gives that

$$\lim_{\varepsilon\to0}\lim_{n\to\infty}\iint_{\mathbb{R}^6}\frac{|u_n(x)|^2|\phi_\varepsilon(x)-\phi_\varepsilon(y)|^2}{|x-y|^{3+2s}}\mathrm{d}x\mathrm{d}y=0. \tag{3.14}$$

Due to the fact that f has the subcritical growth and ϕ_ε has the compact support, we have that

$$\lim_{\varepsilon\to0}\lim_{n\to\infty}\int_{\mathbb{R}^3}f(x,|u_n|^2)|u_n|^2\phi_\varepsilon\mathrm{d}x=\int_{\mathbb{R}^3}f(x,|u|^2)|u|^2\phi_\varepsilon\mathrm{d}x=0 \tag{3.15}$$

and

$$\lim_{\varepsilon\to0}\lim_{n\to\infty}\int_{\mathbb{R}^3}\psi_{|u_n|}^t|u_n|^2\phi_\varepsilon\mathrm{d}x=\int_{\mathbb{R}^3}\psi_{|u|}^t|u|^2\phi_\varepsilon\mathrm{d}x=0. \tag{3.16}$$

Since ϕ_ρ has compact support, so that, letting $n\to\infty$ in (3.12), we can deduce from (3.13)–(3.16) the diamagnetic inequality and (M_1) that

$$\mathfrak{m}_0\mu(\{x_j\})\le\varepsilon^{-2s}\nu_j.$$

Inserting this into (3.11), we obtain $\nu_j\ge(\mathfrak{m}_0S)^{\frac{3}{2s}}\varepsilon^3$. By $\mathcal{J}_\varepsilon(u_n)\to c_\lambda$ and $\mathcal{J}'_\varepsilon(u_n)\to0$ as $n\to\infty$, it follows from (3.4) that

$$\begin{aligned}
c&=\lim_{n\to\infty}\left(\mathcal{J}_\varepsilon(u_n)-\frac{1}{\mu}\langle\mathcal{J}'_\varepsilon(u_n),u_n\rangle\right)\\
&\ge\left(\frac{1}{\mu}-\frac{1}{2_s^*}\right)\varepsilon^{-2s}\int_{\mathbb{R}^3}|u_n|^{2_s^*}\mathrm{d}x\ge\left(\frac{1}{\mu}-\frac{1}{2_s^*}\right)\varepsilon^{-2s}\nu_\infty\ge\sigma_0\varepsilon^{3-2s},
\end{aligned}$$

where $\sigma_0=\left(\frac{1}{\mu}-\frac{1}{2_s^*}\right)(\alpha_0S)^{\frac{3}{2s}}$. This is an obvious contradiction. Hence, $J=\varnothing$.

Next, we prove that $\nu_\infty=0$. Suppose on the contrary that $\nu_\infty>0$. To obtain the possible concentration of mass at infinity, we similarly define a cut off function $\phi_R\in C_0^\infty(\mathbb{R}^N)$ such that $\phi_R(x)=0$ on $|x|<R$ and $\phi_R(x)=1$ on $|x|>R+1$. We can verify that $\{u_n\phi_R\}_n$ is bounded in E, and hence, $\langle\mathcal{J}'_\varepsilon(u_n),u_n\phi_R\rangle\to0$, and this implies that as $n\to\infty$

$$\begin{aligned}
&\mathfrak{M}([u_n]_{s,A}^2)\iint_{\mathbb{R}^6}\frac{|u_n(x)-e^{i(x-y)\cdot A\left(\frac{x+y}{2}\right)}u_n(y)|^2\phi_R(y)}{|x-y|^{3+2s}}\mathrm{d}x\mathrm{d}y+\varepsilon^{-2s}\int_{\mathbb{R}^3}V(x)|u_n|^2\phi_R(x)\mathrm{d}x\\
&=-\operatorname{Re}\left\{\mathfrak{M}([u_n]_{s,A}^2)\iint_{\mathbb{R}^6}\frac{(u_n(x)-e^{i(x-y)\cdot A\left(\frac{x+y}{2}\right)}u_n(y))\overline{u_n(x)(\phi_R(x)-\phi_R(y))}}{|x-y|^{3+2s}}\mathrm{d}x\mathrm{d}y\right\} \qquad (3.17)\\
&\quad-\varepsilon^{-2s}\int_{\mathbb{R}^3}\psi_{|u_n|}^t|u_n|^2\phi_\rho\mathrm{d}x+\varepsilon^{-2s}\int_{\mathbb{R}^3}|u_n|^{2_s^*}\phi_R\mathrm{d}x+\varepsilon^{-2s}\int_{\mathbb{R}^3}f(x,|u_n|^2)|u_n|^2\phi_R(x)\mathrm{d}x+o_n(1).
\end{aligned}$$

As mentioned earlier, we have

$$\limsup_{R\to\infty}\limsup_{n\to\infty}\iint_{\mathbb{R}^6}\frac{||u_n(x)|-|u_n(y)||^2\phi_R(y)}{|x-y|^{3+2s}}\mathrm{d}x\mathrm{d}y=\mu_\infty$$

and

$$\left|\operatorname{Re}\left\{\mathfrak{M}([u_n]_{s,A}^2)\iint_{\mathbb{R}^6}\frac{(u_n(x)-\mathrm{e}^{i(x-y)\cdot A\left(\frac{x+y}{2}\right)}u_n(y))\overline{u_n(x)(\phi_R(x)-\phi_R(y))}}{|x-y|^{3+2s}}\mathrm{d}x\mathrm{d}y\right\}\right|$$
$$\leq C\left(\iint_{\mathbb{R}^6}\frac{|u_n(x)|^2|\phi_R(x)-\phi_R(y)|^2}{|x-y|^{3+2s}}\mathrm{d}x\mathrm{d}y\right)^{1/2}.$$

Since

$$\limsup_{R\to\infty}\limsup_{n\to\infty}\iint_{\mathbb{R}^6}\frac{|u_n(x)|^2|\phi_R(x)-\phi_R(y)|^2}{|x-y|^{3+2s}}\mathrm{d}x\mathrm{d}y$$
$$=\limsup_{R\to\infty}\limsup_{n\to\infty}\iint_{\mathbb{R}^6}\frac{|u_n(x)|^2|(1-\phi_R(x))-(1-\phi_R(y))|^2}{|x-y|^{3+2s}}\mathrm{d}x\mathrm{d}y.$$

Similar to the proof of Lemma 3.4 in [33], we can show that

$$\limsup_{R\to\infty}\limsup_{n\to\infty}\iint_{\mathbb{R}^6}\frac{|u_n(x)|^2|(1-\phi_R(x))-(1-\phi_R(y))|^2}{|x-y|^{3+2s}}\mathrm{d}x\mathrm{d}y=0.$$

Moreover, we proceed as in (3.15) and (3.16) to obtain

$$\lim_{R\to\infty}\lim_{n\to\infty}\int_{\mathbb{R}^3}f(x,|u_n|^2)|u_n|^2\phi_R(x)\mathrm{d}x=0$$

and

$$\lim_{R\to\infty}\lim_{n\to\infty}\int_{\mathbb{R}^3}\psi_{|u_n|}^t|u_n|^2\phi_R\mathrm{d}x=\int_{\mathbb{R}^3}\psi_{|u|}^t|u|^2\phi_R\mathrm{d}x=0.$$

By (3.11) and letting $R\to\infty$ in (3.17), we obtain

$$\mathfrak{m}_0\mu_\infty\leq\varepsilon^{-2s}\nu_\infty.$$

By (3.17), we obtain $\nu_\infty\geq(\alpha_0 S)^{\frac{3}{2s}}\varepsilon^3$. Thus, we have

$$c\geq\left(\frac{1}{\mu}-\frac{1}{2_s^*}\right)\varepsilon^{-2s}\int_{\mathbb{R}^3}|u_n|^{2_s^*}\mathrm{d}x\geq\left(\frac{1}{\mu}-\frac{1}{2_s^*}\right)\varepsilon^{-2s}\nu_\infty\geq\sigma_0\varepsilon^{3-2s},$$

where $\sigma_0=\left(\frac{1}{\mu}-\frac{1}{2_s^*}\right)(\mathfrak{m}_0 S)^{\frac{3}{2s}}$. Thus,

$$\int_{\mathbb{R}^3}|u_n|^{2_s^*}\mathrm{d}x\to\int_{\mathbb{R}^3}|u|^{2_s^*}\mathrm{d}x. \tag{3.18}$$

From the Brézis–Lieb lemma, we obtain

$$u_n\to u\quad\text{in }L^{2_s^*}(\mathbb{R}^3)\quad\text{as }n\to\infty.$$

By the weak lower semicontinuity of the norm, condition (M_1), and the Brézis–Lieb lemma, we have

$$
\begin{aligned}
o(1)\|u_n\| = \langle \mathcal{J}'_\varepsilon(u_n), u_n\rangle &= \mathfrak{M}([u_n]_{s,A}^2)[u_n]_{s,A}^2 + \varepsilon^{-2s}\int_{\mathbb{R}^3} V(x)|u_n|^2\,dx \\
&\quad + \varepsilon^{-2s}\int_{\mathbb{R}^3}\psi^t_{|u_n|}|u_n|^2\,dx - \varepsilon^{-2s}\int_{\mathbb{R}^3}|u_n|^{2_s^*}\,dx - \varepsilon^{-2s}\int_{\mathbb{R}^3} f(x,|u_n|^2)|u_n|^2\,dx \\
&\geq \mathfrak{m}_0([u_n]_{s,A}^2 - [u]_{s,A}^2) + \varepsilon^{-2s}\int_{\mathbb{R}^3} V(x)(|u_n|^2 - |u|^2)dx + \mathfrak{M}([u]_{s,A}^2)[u]_{s,A}^2 \\
&\quad + \varepsilon^{-2s}\int_{\mathbb{R}^3} V(x)|u|^2\,dx + \varepsilon^{-2s}\int_{\mathbb{R}^3}\psi^t_{|u|}|u|^2\,dx - \varepsilon^{-2s}\int_{\mathbb{R}^3}|u|^{2_s^*}\,dx - \varepsilon^{-2s}\int_{\mathbb{R}^3} f(x,|u|^2)|u|^2\,dx \\
&\geq \min\{\mathfrak{m}_0, 1\}\|u_n - u\|_\varepsilon^2 + o(1)\|u\|_\varepsilon.
\end{aligned}
$$

Here, we use the fact that $\mathcal{J}'_\varepsilon(u) = 0$. This fact implies that $(u_n)_n$ strongly converges to u in E. Hence, the proof is complete. □

4 Proof of Theorem 1.1

To prove Theorem 1.1, let $0 < \varepsilon < 1$, and we first prove that functional $\mathcal{J}_\varepsilon(u)$ has the mountain pass geometry.

Lemma 4.1. *Let $(\mathcal{V})$ and $(\mathcal{F})$ hold. If $\mathfrak{M}$ satisfies (M_1) and (M_2), then*

(C_1) *there exist two positive constants $\beta_\varepsilon, \rho_\varepsilon > 0$ such that $\mathcal{J}_\varepsilon(u) > 0$ if $u \in B_{\rho_\varepsilon}\backslash\{0\}$ and $\mathcal{J}_\varepsilon(u) \geq \beta_\varepsilon$ if $u \in \partial B_{\rho_\varepsilon}$, where $B_{\rho_\varepsilon} = \{u \in E : \|u\|_\varepsilon \leq \rho_\varepsilon\}$;*

(C_2) *for any finite dimensional subspace $H \subset E$,*

$$\mathcal{J}_\varepsilon(u) \to -\infty \quad as \quad u \in H \quad with\ \|u\|_\varepsilon \to \infty.$$

Proof. From condition $(\mathcal{F})$, we can take $\varsigma \leq \left(2\min\left\{\frac{\sigma\mathfrak{m}_0}{2}, \frac{1}{2}\right\}c_2^2\right)^{-1}\varepsilon^{2s}$ and there exists $c_\varsigma > 0$ such that

$$\frac{1}{2_s^*}\int_{\mathbb{R}^3}|u|^{2_s^*}\,dx + \frac{1}{2}\int_{\mathbb{R}^3} F(x,|u|^2)dx \leq \varsigma|u|_2^2 + c_\varsigma|u|_{2_s^*}^{2_s^*},$$

where c_ς is the embedding constant given by (2.1). From (3.2), we obtain that

$$
\begin{aligned}
\mathcal{J}_\varepsilon(u) &\geq \min\left\{\frac{\sigma\mathfrak{m}_0}{2}, \frac{1}{2}\right\}\|u\|_\varepsilon^2 - \varepsilon^{-2s}\xi|u|_2^2 - \varepsilon^{-2s}c_\varsigma|u|_{2_s^*}^{2_s^*} \\
&\geq \frac{1}{2}\min\left\{\frac{\sigma\mathfrak{m}_0}{2}, \frac{1}{2}\right\}\|u\|_\varepsilon^2 - \varepsilon^{-2s}c_\varsigma c_{2_s^*}^{2_s^*}\|u\|_\varepsilon^{2_s^*}.
\end{aligned}
$$

This fact implies that the conclusion (C_1) in Lemma 4.1 holds true since $2_s^* > 2$.

Now we verify condition (C_2) of Lemma 4.1. We note that (M_2) implies that

$$\mathscr{M}(t) \leq \frac{\mathscr{M}(t_0)}{t_0^{1/\sigma}}t^{1/\sigma} = C_0 t^{1/\sigma} \quad \text{for all } t \geq t_0 > 0. \tag{4.1}$$

Thus, for all $u \in H$, we have

$$\mathcal{J}_\varepsilon(u) \leq \frac{C_0}{2}\|u\|_\varepsilon^{\frac{2}{\sigma}} + \frac{1}{2}\|u\|_\varepsilon^2 + \frac{\varepsilon^{-2s}}{4}C\|u\|_\varepsilon^4 - \frac{\varepsilon^{-2s}}{2_s^*}|u|_{2_s^*}^{2_s^*} - \varepsilon^{-2s}l_0|u|_r^r.$$

Since all norms in a finite-dimensional space are equivalent, $2 \leq 2/\sigma < 2_s^*$ and $4 < 2_s^*$ since $\frac{3}{4} < s < 1$, we obtain that (C_2) in Lemma 4.1 is valid. This completes the proof. □

Next, we will prove that $J_\varepsilon(u)$ satisfies $(PS)_c$ on the special finite-dimensional subspace. To do this, by assumption (V), we choose $x_0 \in \mathbb{R}^N$ such that $V(x_0) = \min_{x\in\mathbb{R}^N} V(x) = 0$. Without loss of generality, we can assume from now on that $x_0 = 0$.

Let $I_\varepsilon \in C^1(E, \mathbb{R})$ be defined by

$$I_\varepsilon(u) := \frac{1}{2}\mathscr{M}([u]_{s,A_\varepsilon}^2) + \frac{\varepsilon^{-2s}}{2}\int_{\mathbb{R}^3} V(x)|u|^2\,dx - l_0\varepsilon^{-2s}\int_{\mathbb{R}^3}|u|^r\,dx.$$

From (f_3), we can obtain $J_\varepsilon(u) \le I_\varepsilon(u)$ for all $u \in E$.

On the other hand, from Lemma 3.5 in [12], we know that

$$\inf\left\{\iint_{\mathbb{R}^6}\frac{|\phi(x)-\phi(y)|^2}{|x-y|^{3+2s}}dxdy : \phi \in \underset{0}{\overset{\infty}{C}}(\mathbb{R}^3),\ |\phi|_r = 1\right\} = 0.$$

Thus, for any $1 > \zeta > 0$, one can choose $\phi_\zeta \in C_0^\infty(\mathbb{R}^3)$ with $|\phi_\zeta|_r = 1$ and supp $\phi_\zeta \subset B_{r_\zeta}(0)$ so that

$$\iint_{\mathbb{R}^6}\frac{|\phi_\zeta(x)-\phi_\zeta(y)|^2}{|x-y|^{3+2s}}dxdy \le C\zeta^{\frac{6-(3-2s)r}{r}}.$$

Set

$$h_\zeta(x) = e^{iA(0)x}\phi_\zeta(x) \tag{4.2}$$

and

$$h_{\varepsilon,\zeta}(x) = h_\zeta(\varepsilon^{-1}x). \tag{4.3}$$

From condition (f_3), we have

$$\begin{aligned}
I_\varepsilon(th_{\varepsilon,\zeta}) \le{}& \frac{C_0}{2}t^{\frac{2}{\sigma}}\left(\iint_{\mathbb{R}^6}\frac{|h_{\varepsilon,\zeta}(x)-e^{i(x-y)\cdot A\left(\frac{x+y}{2}\right)}h_{\varepsilon,\zeta}(y)|^2}{|x-y|^{3+2s}}dxdy\right)^{1/\sigma}\\
&+\frac{t^2}{2}\varepsilon^{-2s}\int_{\mathbb{R}^3}V(x)|h_{\varepsilon,\zeta}|^2\,dx+\frac{t^4}{4}\varepsilon^{-2s}\int_{\mathbb{R}^3}|\psi^t_{|h_{\varepsilon,\zeta}|}||h_{\varepsilon,\zeta}|^2\,dx - t^r l_0\varepsilon^{-2s}\int_{\mathbb{R}^3}|h_{\varepsilon,\zeta}|^r\,dx\\
\le{}& \varepsilon^{3-2s}\left[\frac{C_0}{2}t^{\frac{2}{\sigma}}\left(\iint_{\mathbb{R}^6}\frac{|h_\zeta(x)-e^{i(x-y)\cdot A\left(\frac{\varepsilon x+\varepsilon y}{2}\right)}h_\zeta(y)|^2}{|x-y|^{3+2s}}dxdy\right)^{1/\sigma}\right.\\
&\left.+\frac{t^2}{2}\int_{\mathbb{R}^3}V(\varepsilon x)|h_\zeta|^2\,dx+\frac{t^4}{4}\varepsilon^{2t}\int_{\mathbb{R}^3}|\psi^t_{|h_\zeta|}||h_\zeta|^2\,dx - t^r l_0\int_{\mathbb{R}^3}|h_\zeta|^r\,dx\right]\\
\le{}& \varepsilon^{3-2s}\mathcal{I}_\varepsilon(th_\zeta),
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{I}_\varepsilon(u) :={}& \frac{C_0}{2}\left(\iint_{\mathbb{R}^6}\frac{|u(x)-e^{i(x-y)\cdot A\left(\frac{\varepsilon x+\varepsilon y}{2}\right)}u(y)|^2}{|x-y|^{3+2s}}dxdy\right)^{1/\sigma}\\
&+\frac{1}{2}\int_{\mathbb{R}^3}V(\varepsilon x)|u|^2\,dx+\frac{1}{4}\int_{\mathbb{R}^3}\psi^t_{|u|}|u|^2\,dx - l_0\int_{\mathbb{R}^3}|u|^r\,dx.
\end{aligned}$$

Since $r > 2/\sigma$, there exists a finite number $t_0 \in [0, +\infty)$ such that

$$\max_{t\geq 0} \mathcal{I}_\varepsilon(th_\zeta) \leq \frac{C_0}{2} t_0^{\frac{2}{\sigma}} \left(\iint_{\mathbb{R}^6} \frac{|h_\zeta(x) - e^{i(x-y)\cdot A\left(\frac{\varepsilon x+\varepsilon y}{2}\right)} h_\zeta(y)|^2}{|x-y|^{3+2s}} dx dy \right)^{1/\sigma} + \frac{t_0^2}{2} \int_{\mathbb{R}^3} V(\varepsilon x)|h_\zeta|^2 dx + \frac{t_0^4}{4} \int_{\mathbb{R}^3} \psi^{t_0}_{|h_\zeta|}|h_\zeta|^2 dx.$$

Let $\psi_\zeta(x) = e^{iA(0)x}\phi_\zeta(x)$, where $\phi_\zeta(x)$ is as defined earlier. From Lemma 3.6 in [12], we have the following lemma.

Lemma 4.2. *For any* $\zeta > 0$, *there exists* $\varepsilon_0 = \varepsilon_0(\zeta) > 0$ *such that*

$$\iint_{\mathbb{R}^6} \frac{|h_\zeta(x) - e^{i(x-y)\cdot A\left(\frac{\varepsilon x+\varepsilon y}{2}\right)} h_\zeta(y)|^2}{|x-y|^{3+2s}} dx dy \leq C\zeta^{\frac{6-(3-2s)r}{r}} + \frac{1}{1-s}\zeta^{2s} + \frac{4}{s}\zeta^{2s},$$

for all $0 < \varepsilon < \varepsilon_0$ *and some constant* $C > 0$ *depending only on* $[\phi]_{s,0}$.

On the one hand, since $V(0) = 0$ and note that $\operatorname{supp}\phi_\zeta \subset B_{r_\zeta}(0)$, there is $\varepsilon^* > 0$ such that

$$V(\varepsilon x) \leq \frac{\zeta}{|\phi_\zeta|_2^2} \quad \text{for all } |x| \leq r_\zeta \quad \text{and} \quad 0 < \varepsilon < \varepsilon^*.$$

This fact together with Proposition 2.2 imply that

$$\max_{t\geq 0} \mathcal{I}_\varepsilon(th_\zeta) \leq \mathcal{N}(\zeta), \tag{4.4}$$

where

$$\mathcal{N}(\zeta) := \frac{C_0}{2} t_0^{\frac{2}{\sigma}} \left(C\zeta^{\frac{6-(3-2s)r}{r}} + \frac{1}{1-s}\zeta^{2s} + \frac{4}{s}\zeta^{2s} \right)^{1/\sigma} + \frac{t_0^2}{2}\zeta + C\left(C\zeta^{\frac{6-(3-2s)r}{r}} + \frac{1}{1-s}\zeta^{2s} + \frac{4}{s}\zeta^{2s} + \zeta \right)^2. \tag{4.5}$$

Therefore, for all $0 < \varepsilon < \min\{\varepsilon_0, \varepsilon^*\}$, we have

$$\max_{t\geq 0} J_\varepsilon(th_\zeta) \leq \mathcal{N}(\zeta)\varepsilon^{3-2s}. \tag{4.6}$$

Thus, we have the following result.

Lemma 4.3. *Let* $(\mathcal{V})$ *and* $(\mathcal{F})$ *hold. If* $\mathfrak{M}$ *satisfies* (M_1) *and* (M_2), *for any* $\kappa > 0$, *there exists* $\mathcal{E}_\kappa > 0$ *such that for each* $0 < \varepsilon < \mathcal{E}_\kappa$, *there is* $\hat{e}_\varepsilon \in E$ *with* $\|\hat{e}_\varepsilon\| > \varrho_\varepsilon$, $J_\varepsilon(\hat{e}_\varepsilon) \leq 0$ *and*

$$\max_{t\in[0,1]} J_\varepsilon(t\hat{e}_\varepsilon) \leq \kappa\varepsilon^{3-2s}. \tag{4.7}$$

Proof. Let $\zeta > 0$ satisfies $\mathcal{N}(\zeta) \leq \kappa$. Set $\mathcal{E}_\kappa = \min\{\varepsilon_0, \varepsilon^*\}$ and $\hat{t}_\varepsilon > 0$ be such that $\hat{t}_\varepsilon\|\psi_{\varepsilon,\zeta}\|_\varepsilon > \varrho_\varepsilon$ and $J_\varepsilon(t\psi_{\varepsilon,\zeta}) \leq 0$ for all $t \geq \hat{t}_\varepsilon$. Choose $\hat{e}_\varepsilon = \hat{t}_\varepsilon\psi_{\varepsilon,\zeta}$, by (4.6), we know that the conclusion of Lemma 4.3 holds. □

To obtain the multiplicity of solutions, one can choose $m^* \in \mathbb{N}$ functions $\phi_\zeta^i \in C_0^\infty(\mathbb{R}^3)$ such that $\operatorname{supp}\phi_\zeta^i \cap \operatorname{supp}\phi_\zeta^k = \emptyset$, $i \neq k$, $|\phi_\zeta^i|_s = 1$ and

$$\iint_{\mathbb{R}^6} \frac{|\phi_\zeta^i(x) - \phi_\zeta^i(y)|^2}{|x-y|^{3+2s}} dx dy \leq C\zeta^{\frac{6-(3-2s)r}{r}}.$$

Let $r_\zeta^{m^*} > 0$ be such that supp $\phi_\zeta^i \subset B_{r_\zeta}^i(0)$ for $i = 1, 2, \ldots, m^*$. Set

$$h_\zeta^i(x) = \mathrm{e}^{iA(0)x}\phi_\zeta^i(x) \tag{4.8}$$

and

$$h_{\varepsilon,\zeta}^i(x) = h_\zeta^i(\varepsilon^{-1}x). \tag{4.9}$$

Denote

$$H_{\varepsilon\zeta}^{m^*} = \operatorname{span}\left\{h_{\varepsilon,\zeta}^1, h_{\varepsilon,\zeta}^2, \ldots, h_{\varepsilon,\zeta}^{m^*}\right\}.$$

Let $u = \sum_{i=1}^{m^*} c_i \psi_{\varepsilon,\zeta}^i \in H_{\varepsilon\zeta}^{m^*}$, thus

$$[u]_{s,A_\varepsilon}^2 \leq C\sum_{i=1}^{m^*}|c_i|^2\,[h_{\varepsilon,\zeta}^i]_{s,A_\varepsilon}^2,$$

$$\int_{\mathbb{R}^3} V(x)|u|^2\,\mathrm{d}x = \sum_{i=1}^{m^*}|c_i|^2 \int_{\mathbb{R}^3} V(x)|h_{\varepsilon,\zeta}^i|^2\,\mathrm{d}x$$

and

$$\frac{1}{2_s^*}\int_{\mathbb{R}^3}|u|^{2_s^*}\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^3}F(x,|u|^2)\mathrm{d}x = \sum_{i=1}^{m^*}\left(\frac{1}{2_s^*}\int_{\mathbb{R}^3}|c_i h_{\varepsilon,\zeta}^i|^{2_s^*}\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^3}F(x,|c_i h_{\varepsilon,\zeta}^i|^2)\mathrm{d}x\right).$$

Therefore,

$$J_\varepsilon(u) \leq C\sum_{i=1}^{m^*}J_\varepsilon(c_i h_{\varepsilon,\zeta}^i)$$

for some constant $C > 0$. By a similar argument as earlier, we know that

$$J_\varepsilon(c_i h_{\varepsilon,\zeta}^i) \leq \varepsilon^{3-2s}\mathcal{I}_\varepsilon(|c_i|h_\zeta^i).$$

As mentioned earlier, we can obtain the following estimate:

$$\max_{u\in H_{\varepsilon\delta}^{m^*}} J_\varepsilon(u) \leq Cm^*\mathcal{N}(\zeta)\varepsilon^{3-2s} \tag{4.10}$$

for all ζ small enough and some constant $C > 0$. Now, let $F_{\varepsilon m^*} = H_{\varepsilon\zeta}^{m^*} = \operatorname{span}\{h_{\varepsilon,\zeta}^1, h_{\varepsilon,\zeta}^2, \cdots, h_{\varepsilon,\zeta}^{m^*}\}$. From (4.10), we have the following lemma.

Lemma 4.4. *Let $(\mathcal{V})$ and $(\mathcal{F})$ hold. If $\mathfrak{M}$ satisfies (M_1) and (M_2), for any $m^* \in \mathbb{N}$ and $\kappa > 0$, there exists $\mathcal{E}_{m^*\kappa} > 0$ such that for each $0 < \varepsilon < \mathcal{E}_{m^*\kappa}$, there exists an m^*-dimensional subspace $F_{\varepsilon m^*}$ satisfying*

$$\max_{u\in F_{\lambda m^*}} J_\varepsilon(u) \leq \kappa\varepsilon^{3-2s}.$$

We now establish the existence and multiplicity results.

Proof of Theorem 3.1 (1). For any $0 < \kappa < \sigma_0$, we choose $\mathcal{E}_\kappa > 0$ and define for $0 < \varepsilon < \mathcal{E}_\kappa$, the minimax value

$$c_\varepsilon := \inf_{\gamma\in\Xi_\varepsilon}\max_{t\in[0,1]} J_\varepsilon(t\hat{e}_\varepsilon),$$

where

$$\Xi_\varepsilon := \{\gamma \in C([0,1],E) : \gamma(0) = 0 \quad \text{and} \quad \gamma(1) = \hat{e}_\varepsilon\}.$$

By Lemma 4.1, we have $\alpha_\varepsilon \le c_\varepsilon \le \kappa\varepsilon^{3-2s}$. From Lemma 3.1, we know that J_ε satisfies the $(PS)_{c_\varepsilon}$ condition, and there is $u_\varepsilon \in E$ such that $J'_\varepsilon(u_\varepsilon) = 0$ and $J_\varepsilon(u_\varepsilon) = c_\varepsilon$. Then u_ε is a nontrivial solution of problem (3.1). Moreover, it is well known that a mountain pass solution is a ground state solution of problem (3.1).

On the other hand, for $\tau \in [2, 2_s^*]$, we have

$$\begin{aligned}\kappa\varepsilon^{3-2s} &\ge J_\varepsilon(u_\varepsilon) = J_\varepsilon(u_\varepsilon) - \frac{1}{\tau}J'_\varepsilon(u_\varepsilon)u_\varepsilon \\ &\ge \left(\frac{\sigma}{2} - \frac{1}{\tau}\right)\mathfrak{m}_0[u_\varepsilon]^2_{s,A_\varepsilon} + \left(\frac{1}{2} - \frac{1}{\tau}\right)\varepsilon^{-2s}\int_{\mathbb{R}^3} V(x)|u_\varepsilon|^2\,dx \\ &\quad + \left(\frac{1}{\tau} - \frac{1}{2_s^*}\right)\varepsilon^{-2s}\int_{\mathbb{R}^3}|u_\varepsilon|^{2_s^*}\,dx + \left(\frac{\mu}{\tau} - \frac{1}{2}\right)\varepsilon^{-2s}\int_{\mathbb{R}^3} F(x, |u_\varepsilon|^2)\,dx.\end{aligned} \tag{4.11}$$

Taking $\tau = 2/\sigma$, we obtain the estimate (1.5), and taking $\tau = \mu$, we obtain the estimate (1.6). This completes the proof of Theorem 3.1 (1). □

Proof of Theorem 3.1 (2). Denote the set of all symmetric (in the sense that $-Z = Z$) and closed subsets of E by Σ, for each $Z \in \Sigma$. Let gen(Z) be the Krasnoselski genus and

$$j(Z) := \min_{\iota\in\Xi_{m^*}} \operatorname{gen}\left(\iota(Z) \cap \partial B_{\varrho_\varepsilon}\right),$$

where Ξ_{m^*} is the set of all odd homeomorphisms $\iota \in C(E, E)$ and ϱ_ε is the number from Lemma 4.1. Then j is a version of Benci's pseudoindex [34]. Let

$$c_{\varepsilon i} := \inf_{j(Z)\ge i}\sup_{u\in Z} J_\varepsilon(u), \quad 1 \le i \le m^*.$$

Since $J_\varepsilon(u) \ge \alpha_\varepsilon$ for all $u \in \partial B^+_{\varrho_\varepsilon}$ and since $j(F_{\varepsilon m^*}) = \dim F_{\varepsilon m^*} = m^*$, we obtain

$$\alpha_\varepsilon \le c_{\varepsilon 1} \le \cdots \le c_{\varepsilon m^*} \le \sup_{u\in H_{\varepsilon m^*}} J_\varepsilon(u) \le \kappa\varepsilon^{3-2s}.$$

It follows from Lemma 3.1 that J_ε satisfies the $(PS)_{c_\varepsilon}$ condition at all levels $c < \sigma_0\varepsilon^{3-2s}$. By the usual critical point theory, all $c_{\varepsilon i}$ are critical levels and J_ε has at least m^* pairs of nontrivial critical points satisfying

$$\alpha_\varepsilon \le J_\varepsilon(u_\varepsilon) \le \kappa\varepsilon^{3-2s}.$$

Hence, problem (3.1) has at least m^* pairs of solutions. Finally, as in the proof of Theorem 3.1, we see that these solutions satisfy the estimates (1.5) and (1.6). □

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