Research Article

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Ambrosetti-Prodi-type results for a class of difference equations with nonlinearities indefinite in sign

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Abstract: In this article, we are concerned with the periodic solutions of first-order difference equation

$$\Delta u(t-1) = f(t, u(t)) - s, \quad t \in \mathbb{Z}, \qquad (P)$$

where $s \in \mathbb{R}$, $f : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}$ is continuous with respect to $u \in \mathbb{R}$, $f(t, u) = f(t + T, u)$, $T > 1$ is an integer, $\Delta u(t-1) = u(t) - u(t-1)$. We prove a result of Ambrosetti-Prodi-type for (P) by using the method of lower and upper solutions and topological degree. We relax the coercivity assumption on f in Bereanu and Mawhin [1] and obtain Ambrosetti-Prodi-type results.

Keywords: periodic solutions, Ambrosetti-Prodi-type results, lower and upper solutions, topological degree

MSC 2020: 39A12, 39A23

1 Introduction

Let $T > 1$ be an integer, $[1, T]_{\mathbb{Z}} := \{1, 2, \ldots, T\}$. In this article, we establish Ambrosetti-Prodi-type results of first-order difference equation

$$\Delta u(t-1) = f(t, u(t)) - s, \quad t \in \mathbb{Z}, \tag{1.1}$$

where $s \in \mathbb{R}$, $f : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}$ is continuous with respect to $u \in \mathbb{R}$, $f(t, u) = f(t + T, u)$, $t \in \mathbb{Z}$.

The Ambrosetti-Prodi problem for an equation of the form

$$F(u) = s \tag{1.2}$$

consists of determining how varying the parameter s affects the number of solutions u. Usually, an Ambrosetti-Prodi-type result yields the existence of a number s_0 such that (1.2) has zero, at least one or two solutions according to $s < s_0$, $s = s_0$ or $s > s_0$.

The founding work is in the study by Ambrosetti and Prodi [2], which received immediate attention from several authors. In 1975, Fucik [3] was concerned with the weak solvability of the elliptic equation and obtained Ambrosetti-Prodi-type results. In 1980, Hess [4] studied Ambrosetti-Prodi-type results of elliptic equation, he extended the works of Ambrosetti and Prodi [2] and Kazdan and Warner [5]. After that, several studies have sprung up [1,7–11, 13–19,22,24].

Most of the aforementioned literature is about differential equations. Periodic problems for differential equations were studied in [12, 20, 21] Zhou [25, 26] studied periodic solutions of difference equations. Since

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there are many essential differences between difference equations and differential equations, such as in the continuous case, the minimum or maximum points t_0 satisfy $u'(t_0) = 0$, but in discrete case, the minimum or maximum points t_0 do not necessarily satisfy $\Delta u(t_0) = 0$, and the definition of generalized zeros in difference is complex, and chaotic behaviors in Strogatz [23]; there are few researches on Ambrosetti-Prodi-type results of difference equations. Through searching for an analogue for Ambrosetti-Prodi-type results of difference equations, in 2006, Bereanu and Mawhin [1] were concerned with the first-order difference equation

$$\Delta x(t-1) + f(t, x(t)) = s, \quad t \in \mathbb{Z}. \tag{1.3}$$

They obtained the following:

Theorem A. [1, Theorem 6] *Assume $f : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}$ is continuous, with T-periodicity in the t variable, $s \in \mathbb{R}$. If*

$$\lim_{|x|\to\infty} f(t, x) = +\infty, t \in [1, T]_{\mathbb{Z}}. \tag{1.4}$$

Then there exists an $s_0 \in \mathbb{R}$ such that
- *if $s < s_0$, there is no T-periodic solution of equation* (1.3),
- *if $s = s_0$, there is at least one T-periodic solution of equation* (1.3),
- *if $s > s_0$, there are at least two T-periodic solutions of equation* (1.3).

Nonlinearity f in [1] *satisfies the coercivity condition, under the coercivity condition, the periodic Ambrosetti-Prodi problem has been investigated by several authors* [1,13,15,16,17,21]. *Inspired by Obersnel and Omari* [15], *in this short note, we want to push further into the direction of relaxing the coercivity assumption on f. We assume*:

(H1) *$f : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}$ is continuous upon $u \in \mathbb{R}$, $f(t, u) = f(t + T, u)$.*
(H2) *There exist $a, b : [1, T]_{\mathbb{Z}} \to \mathbb{R}$, $p \in (0, 1]$, such that $f(t, u) \geqslant a(t)|u|^p + b(t)$, $t \in [1, T]_{\mathbb{Z}}$, for all $u \in \mathbb{R}$.*
(H3) $\sum_{t=1}^{T} a(t) > 0$.

Theorem 1.1. *Assume* (H1)–(H3) *hold, there exists $s_0 \in \mathbb{R}$, such that*
- *if $s < s_0$, there is no T-periodic solution of equation* (1.1),
- *if $s = s_0$, there is at least one T-periodic solution of equation* (1.1),
- *if $s > s_0$, there are at least two T-periodic solutions of equation* (1.1).

Remark 1.2. Obersnel and Omari [15] investigated an Ambrosetti-Prodi-type result of first-order differential equation; they studied the existence and multiplicity of solutions when the parameter s exceeds a constant s_0 using normal-order upper and lower solutions and reverse-order upper and lower solutions. However, for first-order difference equations, reverse order upper and lower solutions cannot be used; in addition, lower solutions must be smaller than the upper solutions to make the method conclusive, and relevant conclusions can be found in [6]. Hence, the multiplicity of solutions when the parameter s exceeds a constant is the difficulty in this article.

Remark 1.3. In [6], Bereanu and Mawhin showed counterexamples when $T \geq 2$ is odd, $T > 2$ is even and $T = 2$, respectively. These counterexamples show that first-order difference equations have no solution when lower solutions are larger than upper solutions.

Example 1.4. First-order difference equation

$$\Delta u(t-1) = (\sin t + 1/2)|\sqrt{u(t)} + 1| + \cos t - s, \quad t \in \mathbb{Z}. \tag{1.5}$$

We take $f(t, u) = (\sin t + 1/2)|\sqrt{u} + 1| + \cos t$, $f(t + T, u) = f(t, u)$, and $T = 2\pi$; hence, (H1) holds. There exist $a(t) = \sin t + 1/3$, $b(t) = \cos t - 1/2$ and $p = 1/3$ such that $f(t, u) \geqslant a(t)|u|^p + b(t)$, $t \in [1, T]_{\mathbb{Z}}$, for all $u \in \mathbb{R}$; hence, (H2) holds. Obviously, $\sum_{t=1}^{T} a(t) > 0$, and hence, (H3) holds. According to Theorem 1.1, we can obtain $s_0 \in \mathbb{R}$ such that

(i) if $s < s_0$, there is no T-periodic solution of equation (1.5);
(ii) if $s = s_0$, there is at least one T-periodic solution of equation (1.5);
(iii) if $s > s_0$, there are at least two T-periodic solutions of equation (1.5).

2 Preliminary results

Let $X = \{u | u : [1, T]_{\mathbb{Z}} \to \mathbb{R}, u(0) = u(T)\}$ be a Banach space under norm

$$\|u\| = \max_{t \in [1,T]_{\mathbb{Z}}} |u(t)|.$$

For convenience, we only need to consider the first-order periodic boundary value problem

$$\begin{cases} \Delta u(t-1) = f(t, u(t)) - s, & t \in [1, T]_{\mathbb{Z}}, \\ u(0) = u(T). \end{cases} \tag{2.1}$$

The definition of the upper and lower solutions of problem (2.1) is given as follows:

Definition 2.1. $\alpha : [1, T]_{\mathbb{Z}} \to \mathbb{R}$ is a lower solution of problem (2.1), referring to α satisfies

$$\begin{cases} \Delta \alpha(t-1) \leqslant f(t, \alpha(t)) - s, & t \in [1, T]_{\mathbb{Z}}, \\ \alpha(0) < \alpha(T). \end{cases}$$

$\beta : [1, T]_{\mathbb{Z}} \to \mathbb{R}$ is an upper solution of problem (2.1), referring to β satisfies

$$\begin{cases} \Delta \beta(t-1) \geqslant f(t, \beta(t)) - s, & t \in [1, T]_{\mathbb{Z}}, \\ \beta(0) > \beta(T). \end{cases}$$

$\alpha : [1, T]_{\mathbb{Z}} \to \mathbb{R}$ is a strict lower solution of problem (2.1), referring to α satisfies

$$\begin{cases} \Delta \alpha(t-1) < f(t, \alpha(t)) - s, & t \in [1, T]_{\mathbb{Z}}, \\ \alpha(0) < \alpha(T). \end{cases}$$

$\beta : [1, T]_{\mathbb{Z}} \to \mathbb{R}$ is a strict upper solution of problem (2.1), referring to β satisfies

$$\begin{cases} \Delta \beta(t-1) > f(t, \beta(t)) - s, & t \in [1, T]_{\mathbb{Z}}, \\ \beta(0) > \beta(T). \end{cases}$$

Lemma 2.2. *Problem* (2.1) *has a lower solution α and an upper solution β, such that $\alpha(t) \leqslant \beta(t)$, $t \in [1, T]_{\mathbb{Z}}$, then problem* (2.1) *has at least one solution $u(t)$, such that $\alpha(t) \leqslant u(t) \leqslant \beta(t)$, $t \in [1, T]_{\mathbb{Z}}$.*

Proof. Construct auxiliary function $\gamma : [1, T]_{\mathbb{Z}} \times \mathbb{R} \to \mathbb{R}$ by

$$\gamma(t, u(t)) = \begin{cases} \beta(t), u(t) > \beta(t), \\ u(t), \alpha(t) \leqslant u(t) \leqslant \beta(t), \\ \alpha(t), u(t) < \alpha(t). \end{cases}$$

Consider the modified problem

$$\begin{cases} \Delta u(t-1) - f(t, \gamma(t, u(t))) + s + u(t) - \gamma(t, u(t)) = 0, & t \in [1, T]_{\mathbb{Z}}, \\ u(0) = u(T). \end{cases} \tag{2.2}$$

Using Brouwer fixed point theorem, at least one solution can be obtained for problem (2.2) in X, whose elements can be characterized by the coordinates $u(1), \dots, u(T)$. Indeed, the operator L is given by

$$Lu(1) = 2u(1) - u(0), \dots, Lu(T-1) = 2u(T-1) - u(T-2), Lu(T) = 2u(0) - u(T-1)$$

which is one to one, hence invertible, and (2.2) is equivalent to the fixed point problem

$$u(t) = L^{-1}(f(t, \gamma(t,u)) - s + \gamma(t,u)), \quad t \in [1, T]_{\mathbb{Z}}$$

in X. It remains to show that if $u(t)$ is a solution of (2.2), $t \in [1, T]_{\mathbb{Z}}$, then $\alpha(t) \leqslant u(t) \leqslant \beta(t)$, so that $u(t)$ is a solution of (2.1), $t \in [1, T]_{\mathbb{Z}}$. Suppose by contradiction that there exists a $\tau \in [1, T]_{\mathbb{Z}}$, such that $\alpha(\tau) - u(\tau) > 0$, then

$$\alpha(\tau - 1) - u(\tau - 1) \leqslant 0 < \alpha(\tau) - u(\tau),$$

we can obtain

$$\Delta\alpha(\tau - 1) - f(\tau, \alpha(\tau)) + s \geqslant \Delta u(\tau - 1) - f(\tau, \gamma(\tau, u)) + s = -u(\tau) + \alpha(\tau) > 0,$$

which contradicts with the definition of the lower solution.

Thus, $\alpha(t) \leqslant u(t)$. Similarly, $u(t) \leqslant \beta(t)$ can be proved. Then problem (2.1) has at least one solution $u(t)$, such that $\alpha(t) \leqslant u(t) \leqslant \beta(t)$, $t \in [1, T]_{\mathbb{Z}}$. □

Remark 2.3. Assume that α is a strict lower solution of (2.1), β is the strict upper solution of (2.1), then the problem (2.1) admits at least one solution u such that $\alpha < u < \beta$. Define the open set $\Omega_{\alpha,\beta} = \{u | u \in X, \alpha < u < \beta\}$ and the open ball B_ρ with the radius of ρ. The mapping $\Phi : \mathbb{R} \times X \to \mathbb{R}$ is defined by $\Phi(s, u(t)) = \Delta u(t-1) - f(t, u(t)) + s$, $t \in [1, T]_{\mathbb{Z}}$. If ρ is large enough, using the additivity-excision property of Brouwer degree, we have

$$|\deg[\Phi, \Omega_{\alpha,\beta}, 0]| = |\deg[\Phi, B_\rho, 0]| = 1.$$

3 Proof of the main result

Proof of Theorem 1.1. Step 1. We verify that for every $s \in \mathbb{R}$, there is $\xi_0 \in \mathbb{R}$, such that, for all $\xi \leqslant \xi_0$, any solution u of the Cauchy problem

$$\begin{cases} \Delta u(t-1) = a(t)|u(t)|^p + b(t) - s, & t \in [1, T]_{\mathbb{Z}}, \\ u(0) = \xi \end{cases} \tag{3.1}$$

is a strict lower solution of the T-periodic problem

$$\begin{cases} \Delta u(t-1) = a(t)|u(t)|^p + b(t) - s, & t \in [1, T]_{\mathbb{Z}}, \\ u(0) = u(T). \end{cases} \tag{3.2}$$

Hence, by (H2), u is a strict lower solution of problem (2.1).

We consider the case $p \in (0, 1)$ and prove the following claim first.

Claim For any $m \in \mathbb{R}$, there is $\xi_m \leqslant m$ such that, for every $\xi \leqslant \xi_m$, any solution u of (3.1) satisfies $\max_{t\in[1,T]_{\mathbb{Z}}} u(t) < m$.

Assume, by contradiction, that there exists $m_0 \in \mathbb{R}$ such that, for every $n \in \mathbb{Z}^-$, with $n < -|m_0|$, there is a solution u_n of problem (3.1) satisfying $u_n(0) \leqslant n$ and $\max_{t\in[1,T]_{\mathbb{Z}}} u_n(t) \geqslant m_0$. Let $s_n, t_n \in [1, T]_{\mathbb{Z}}$ be such that $s_n + 1 < t_n$ on $[s_n, t_n]_{\mathbb{Z}}$, $[s_n, t_n]_{\mathbb{Z}} := \{s_n, s_n + 1, \dots, t_n - 1, t_n\}$, $n \leqslant u_n(t) \leqslant m_0$, $t \in [s_n, t_n]_{\mathbb{Z}}$, $u_n(s_n) = n$ and $u_n(t_n) = m_0$, then

$$\begin{aligned} m_0 - n = u_n(t_n) - u_n(s_n) &= \sum_{t=s_n+1}^{t_n} \Delta u_n(t-1) \\ &\leqslant \sum_{t=s_n+1}^{t_n} |a(t)||u_n(t)|^p + \sum_{t=s_n+1}^{t_n} |b(t) - s| \\ &\leqslant |m_0|^p \sum_{t=1}^{T} |a(t)| + \sum_{t=1}^{T} |b(t) - s|. \end{aligned}$$

For fixed s, we obtain a contradiction if $n \to -\infty$; thus, our claim is proved.

In the case of $p \in (0, 1)$, suppose that there is a sequence $(\xi_n)_n \in \mathbb{R}$, with $\lim_{n\to-\infty}\xi_n = -\infty$ and the solution $(u_n)_n$ of problem (3.1) with $\xi = \xi_n$, for any $n \in \mathbb{Z}^-$, satisfies $u_n(T) \leqslant u_n(0)$. By the claim above, we can assume that $\max_{t\in[1,T]_{\mathbb{Z}}} u_n(t) \leqslant n$. Thus,

$$0 \geqslant \sum_{t=1}^{T} \frac{\Delta u_n(t-1)}{|u_n(t)|^p} = \sum_{t=1}^{T} a(t) + \sum_{t=1}^{T} \frac{b(t)-s}{|u_n(t)|^p}.$$

We obtain the contradiction $0 \geqslant \sum_{t=1}^{T} a(t) > 0$ when $n \to -\infty$. Hence, we have $u(T) > u(0)$, and u is a solution of (3.1).

The validity of step 1 when $p = 1$ can be verified by a direct inspection is obtained as follows:

$$u(t) = \xi\left(\prod_{s=1}^{t} \frac{1}{1-a(s)}\right)\left(\sum_{t=1}^{T} \frac{b(t)-s}{\xi \prod_{s=1}^{t+1} \frac{1}{1-a(s)}} + C\right), \ t \in [1, T]_{\mathbb{Z}},$$

where C is an arbitrary constant, choose

$$\xi < (1 - a(T+1))(b(T) - s) + C.$$

Then, we have $u(0) < u(T)$ and $u(t)$ is a solution of (3.1).

Step 2. We show that there exists s^* such that, for all $s > s^*$, equation (1.1) has at least one T-periodic solution. Indeed, it is easily verified that there exists $s^* \in \mathbb{R}$ such that, for all $s > s^*$, the constant $\beta \in \mathbb{R}$, $\sup_{t\in[1,T]_{\mathbb{Z}}} f(t, \beta) < +\infty$, β is a strict upper solution of problem (2.1). Furthermore, by the results proved in Step 1, problem (2.1) admits one strict lower solution α_1 satisfying $\alpha_1(t) \leqslant \beta$ for all $t \in [1, T]_{\mathbb{Z}}$. Therefore, equation (1.1) has at least one T-periodic solution u_1, satisfying $\alpha_1(t) \leqslant u_1(t) \leqslant \beta$ for all $t \in [1, T]_{\mathbb{Z}}$, $u_1 \neq \alpha_1, \beta$.

Step 3. We prove that the set of the parameters s for which equation (1.1) has at least one T-periodic solution is bounded from below. Define the set

$$\Psi = \{s \in \mathbb{R} : \text{ equation (1.1) has at least one } T\text{-periodic solution}\}.$$

We prove there exists $s_0 \in \mathbb{R}$, such that $s_0 = \inf\Psi$. Assume, by contradiction, that $\inf\Psi = -\infty$. Then, there exists a sequence $(s_n)_n \in \mathbb{R}$ with $\lim_{n\to+\infty} s_n = -\infty$, and a sequence $(u_n)_n$ of T-periodic solutions of equation (1.1) with $s = s_n$. We claim that $\lim_{n\to+\infty}\|u_n\| = +\infty$, otherwise, we would obtain

$$0 = \sum_{t=1}^{T} \Delta u_n(t-1) = \sum_{t=1}^{T} f(t, u_n(t)) - s_n T.$$

There would exist a function $\varphi : [1, T]_{\mathbb{Z}} \to \mathbb{R}$, such that

$$|s_n T| = \left|\sum_{t=1}^{T} f(t, u_n(t))\right| \leqslant \sum_{t=1}^{T} \varphi(t) < +\infty,$$

which is a contradiction. Moreover, by (H2) we have

$$\Delta u_n(t-1) = f(t, u_n(t)) - s_n \geqslant f(t, u_n(t)) \geqslant a(t)|u_n(t)|^p + b(t), \quad t \in [1, T]_{\mathbb{Z}}.$$

Thus, we obtain

$$0 = \sum_{t=1}^{T} \frac{\Delta u_n(t-1)}{|u_n(t)|^p} \geqslant \sum_{t=1}^{T} a(t) + \sum_{t=1}^{T} \frac{b(t)}{|u_n(t)|^p}.$$

Let $n \to +\infty$, and using (H3) yields the contradiction $0 \geqslant \sum_{t=1}^{T} a(t) > 0$.

Step 4. We show the existence of at least one T-periodic solution of equation (1.1) for $s = s_0$. Let $(s_n)_n$ be a sequence in Ψ converging to s_0 and let $(u_n)_n$ be the corresponding sequence of T-periodic solutions of equation (1.1) with $s = s_n$. Let us verify that there is $R > 0$, such that $\|u_n\| \leqslant R$ for all $n \in \mathbb{N}$. Indeed, otherwise, we can find a subsequence of $(u_n)_n$, we still denote by $(u_n)_n$, such that $\lim_{n\to+\infty}(u_n)_n = +\infty$.

Arguing similarly as in the proof of Step 3, thus easily leading to a contradiction as above. Therefore, $(u_n)_n$ is bounded in X; according to Weierstrass concentration theorem, we can obtain $\lim_{n\to+\infty}u_n(t) = u_0(t)$, $t \in [1, T]_{\mathbb{Z}}$. Besides, $\lim_{n\to+\infty}f(t, u_n(t)) = f(t, u_0(t))$, $t \in [1, T]_{\mathbb{Z}}$, and when n is large enough, $|f(t, u_n(t))| \leqslant \varphi(t)$. Sequence $(f(\cdot,u_n) - s_n)_n$, i.e., $(\Delta u_n)_n$, convergence to $f(\cdot,u_0) - s_0$ in X, with $\Delta u_0(t-1) = f(t, u_0(t)) - s_0$, $u_0(T) = u_0(0)$, $t \in [1, T]_{\mathbb{Z}}$, u_0 is a T-periodic solution of equation (1.1) for $s = s_0$.

Step 5. We show that for all $s > s_0$, equation (1.1) has at least two T-periodic solutions.

Claim For any constant $c \in \mathbb{R}$, there exists $\rho > 0$, such that, for all $s \leqslant c$, all possible periodic solutions u of equation (1.1) belong to open ball B_ρ.

For every $s \leqslant c$, we have

$$\sum_{t=1}^{T}\Delta u(t-1) = \sum_{t=1}^{T} f(t, u(t)) - Ts,$$
$$u(T) - u(0) = \sum_{t=1}^{T} f(t, u(t)) - Ts,$$
$$\sum_{t=1}^{T} f(t, u(t)) = Ts.$$

We need to show there exists a constant c_1, such that

$$\sum_{t=1}^{T}|\Delta u(t-1)| \leqslant c_1.$$

By (H2), we can obtain $f(t, u(t)) \geqslant a(t)|u(t)|^p + b(t)$, then

$$\begin{aligned}|f(t, u(t))| - |a(t)||u(t)|^p - |b(t)| &\leqslant |f(t, u(t)) - a(t)|u(t)|^p - b(t)| \\ &= f(t, u(t)) - a(t)|u(t)|^p - b(t) \\ &\leqslant f(t, u(t)) + |a(t)||u(t)|^p + |b(t)|.\end{aligned}$$

Thus,

$$|f(t, u(t))| \leqslant f(t, u(t)) + 2|a(t)||u(t)|^p + 2|b(t)|,$$

$$\begin{aligned}\sum_{t=1}^{T}|f(t, u(t))| &\leqslant \sum_{t=1}^{T} f(t, u(t)) + 2\sum_{t=1}^{T}|a(t)||u(t)|^p + 2\sum_{t=1}^{T}|b(t)| \\ &\leqslant Ts + 2T\|a\|\|u\|^p + 2T\|b\| =: c_1.\end{aligned}$$

Hence, all possible solutions of problem (2.1) belong to open ball B_ρ.

Using the Brouwer degree theory, obviously, $u(t)$ is a solution of problem (2.1) if and only if $u(t)$ is a zero of $\Phi(s, \cdot)$, $t \in [1, T]_{\mathbb{Z}}$. Let $s_2 < s_0 < s_1$, according to the claim above, we can find the corresponding ρ such that, for all $s \in [s_2, s_1]$, every possible zero points u of $\Phi(s, \cdot)$ satisfy $u \in B_\rho$. Consequently, the Brouwer degree $\deg[\Phi(s, \cdot), B_\rho, 0]$ is well defined and does not depend upon s. Using the conclusion of step 3, for $u \in X$, $u - \Phi(s_2, \cdot) \neq 0$. This implies that $\deg[\Phi(s_2, \cdot), B_\rho, 0] = 0$, so that $\deg[\Phi(s_1, \cdot), B_\rho, 0] = 0$. By excision property, $\deg[\Phi(s_1, \cdot), B_{\rho'}, 0] = 0$ if $\rho' > \rho$.

Let $\hat{u}$ be a solution of (2.1) with $s \in (s_0, s_1)$, then $\hat{u}$ is a strict upper solution of problem (2.1) with $s = s_1$. From Step 1, α_1 is a strict lower solution of problem (2.1). Consequently, using Remark 2.3, (2.1) with $s = s_1$ has a solution in $\Omega_{\alpha_1,\hat{u}}$, and

$$\left|\deg\left[\Phi(s_1, \cdot), \Omega_{\alpha_1,\hat{u}}, 0\right]\right| = 1.$$

Taking ρ' sufficiently large, we deduce from the additivity property of Brouwer degree that

$$\begin{aligned}\left|\deg\left[\Phi(s_1, \cdot), B_{\rho'}\backslash\Omega_{\alpha_1,\hat{u}}, 0\right]\right| &= \left|\deg[\Phi(s_1, \cdot), B_{\rho'}, 0] - \deg\left[\Phi(s_1, \cdot), \Omega_{\alpha_1,\hat{u}}, 0\right]\right| \\ &= \left|\deg\left[\Phi(s_1, \cdot), \Omega_{\alpha_1,\hat{u}}, 0\right]\right| = 1.\end{aligned}$$

When $s = s_1$, (2.1) has the second solution in $B_{\rho'}\backslash\Omega_{\alpha_1,\hat{u}}$. □

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