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Research Article

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The exact solutions of generalized Davey-Stewartson equations with arbitrary power nonlinearities using the dynamical system and the first integral methods

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Abstract: The exact traveling wave solutions of generalized Davey-Stewartson equations with arbitrary power nonlinearities are studied using the dynamical system and the first integral methods. Taking different parameter conditions, we obtain periodic wave solutions, exact solitary wave solutions, kink wave solutions, and anti-kink wave solutions.

Keywords: generalized Davey-Stewartson equation, the dynamical system method, the first integral method, solitary wave solution, kink wave solution, periodic wave solution

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1 Introduction

The study of traveling wave solutions to nonlinear wave equations plays a major role in the fields of plasma, elastic media, optical fibers, etc. Mathematicians and physicists have made significant progress toward determining the exact traveling wave solutions. Various methods have been presented, such as the Darboux transformation, the inverse scattering method, the Hirota bilinear method, the tanh method, the homogeneous balance method, and the Jacobi elliptic function expansion method (see [1–28]). To obtain exact solutions for nonlinear partial differential equations (PDEs), Jibin Li employed the dynamical system method to study nonlinear PDEs (see [29]). In addition, based on the theory of rings of commutative algebra, Feng introduced the first integral method to solve nonlinear PDEs and obtain many exact solutions (see [33]). It is worth noting that using the bifurcation theory of dynamical systems method to calculate exact solutions is a new and effective method that can be utilized to study fractional PDEs in a conformal sense (see [34–38]).

In this article, we discuss the exact solutions to the following generalized Davey-Stewartson equations with arbitrary power nonlinearities:

$$\begin{cases} iu_t + u_{xx} + u_{yy} + \gamma|u|^p u + \alpha uv + \delta|u|^{2p} u = 0, \\ v_{xx} + v_{yy} - \beta(|u|^p)_{xx} = 0, \end{cases} \tag{1.1}$$

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where α, β, γ, and δ are real parameters, p is a positive integer, v is a real function, and u is a complex function. Davey-Stewartson equations are significant for describing the motion of long and short waves in shallow water. Many researchers have used various methodologies to solve these problems. For example, Mirzazadeh applied the trial equation method and the ansatz approach to establish solitary waves solitons, dark solitons, and singular solitary-wave soliton solutions to Davey-Stewartson equations (see [41]). Song and Miswas studied Davey-Stewartson equations with power-law nonlinearity and carried out several different solutions for bifurcation analysis (see [42]). Zinati and Manafian used He's semi-inverse variational principle method, the improved $\tan\frac{\phi}{2}$-expansion method and the generalized $\left(\frac{G'}{G}\right)$-expansion method to find more exact solutions to Davey-Stewartson equations (see [43]). The generalized Kudryashov method is introduced to obtain new soliton solutions from the Davey-Stewartson equations with power-law nonlinearity (see [44]). Aghdaei and Adibi applied the generalized $\tan\frac{\phi}{2}$ and He's semi-inverse variational methods to find the exact solitary wave solutions of Davey-Stewartson equations with power-law nonlinearity (see [45]). As a result, a more comprehensive study is necessary for equation (1.1).

This article is structured as follows: Sections 2 and 3 introduce the dynamical system and the first integral methods. Sections 4 and 5 address equation (1.1) by implementing these two proposed methods. Briefly, Section 6 outlines some of the findings.

2 Description of the dynamical system method

In this section, we consider the following nonlinear PDEs:

$$P\Big(t, x_i, u_t, u_{x_i}, u_{x_i x_i}, u_{x_i x_j}, u_{tt}, \ldots\Big), \tag{2.1}$$

where $i, j = 1, 2, \ldots, n$. P is a polynomial in $u(x, t)$ and its partial derivatives, in which nonlinear terms and the highest order derivatives are involved. $u(x, t)$ is an unknown function.

The main steps of the dynamical system method (see [30–32]) in this document are as follows:

Step 1. Transform

$$u(t, x_1, u_2, \ldots, u_n) = \phi(\xi), \ \xi = \sum_{i=1}^{n} k_i x_i - ct. \tag{2.2}$$

Equation (2.1) can be reduced to the following nonlinear ordinary differential equations (ODEs):

$$D(\xi, \phi_\xi, \phi_{\xi\xi}, \phi_{\xi\xi\xi}, \ldots) = 0, \tag{2.3}$$

where k_i are nonzero constants and c is the wave speed. Multiple integrations are performed for equation (2.3), if equation (2.3) can be reduced to the following the second-order nonlinear ODEs:

$$E(\xi, \phi_\xi, \phi_{\xi\xi}) = 0. \tag{2.4}$$

Then, assuming $\phi_\xi = \frac{\mathrm{d}\phi}{\mathrm{d}\xi} = y$, equation (2.4) can be reduced to the following two-dimensional dynamical system:

$$\frac{\mathrm{d}\phi}{\mathrm{d}\xi} = y, \quad \frac{\mathrm{d}y}{\mathrm{d}\xi} = f(\phi, y), \tag{2.5}$$

where $f(\phi, y)$ is an integral expression or a fraction. If $f(\phi, y)$ is a fraction, we make $f(\phi, y) = \frac{F(\phi, y)}{g(\phi)}$, when $g(\phi_s) = 0$, $\phi = \phi_s$, $\frac{\mathrm{d}y}{\mathrm{d}\xi}$ does not exist. Then, if we transform $\mathrm{d}\zeta = g(\phi)\mathrm{d}\xi$, equation (2.5) can be rewritten as

$$\frac{\mathrm{d}\phi}{\mathrm{d}\zeta} = g(\phi)y, \quad \frac{\mathrm{d}y}{\mathrm{d}\zeta} = F(\phi, y), \tag{2.6}$$

where ζ is a parameter. If it is possible to reduce equation (2.1) to equation (2.5) (or equation (2.6)), then we can proceed to the next step.

Step 2. If equation (2.5) is an integrable system, then we can reduce equation (2.5) (or equation (2.6)) for subsequent differential equations as follows:

$$\frac{\mathrm{d}y}{\mathrm{d}\phi} = \frac{f(\phi, y)}{y}, \quad \frac{\mathrm{d}y}{\mathrm{d}\phi} = \frac{F(\phi, y)}{g(\phi)y} = \frac{f(\phi, y)}{y}. \tag{2.7}$$

Then equations (2.5) and (2.6) have the same first integral (i.e., Hamiltonian) as follows:

$$H(\phi, y) = h, \tag{2.8}$$

where h is an integral constant, and we can obtain all kinds of phase portraits in the parametric space using the first integral. Since the phase orbit of the vector field that defines equation (2.5) (or equation (2.6)) determines all the traveling wave solutions of equation (2.1), we study the bifurcation of the phase diagram of equation (2.5) (or equation (2.6)) to find the traveling wave solutions of equation (2.1). As a rule, a periodic orbit corresponds to a periodic wave solution, a homoclinic orbit corresponds to a solitary wave solution, and a heteroclinic orbit (or the so-called connected orbit) corresponds to a kink (or anti-kink) wave solutions. Once we obtain all the phase orbits, we can derive the value of h or its range.

Step 3. If h is determined, then we can obtain the following relationship from equation (2.8):

$$y = y(\phi, y), \tag{2.9}$$

that is, $\frac{\mathrm{d}\phi}{\mathrm{d}\xi} = y(\phi, y)$. When equation (2.9) is an integrable expression, we can substitute it into the first term of equation (2.5) and integrate it to obtain

$$\int_{\phi_0}^{\phi} \frac{\mathrm{d}\phi}{y(\phi, h)} = \int_0^{\xi} \xi \mathrm{d}\xi, \tag{2.10}$$

where ϕ_0 and 0 are initial constants. Usually, the initial constants can be taken by a root of equation (2.9) or infection points of the traveling waves. Making proper initial constants and integrating equation (2.10), we obtain the exact traveling wave solutions of equation (2.1) through the elliptic Jacobian functions.

3 Description of the first integral method

The main steps of the first integral method (see [39,40]), summarized as follows:

Step 1. The simplification of equation (2.1) gives

$$u(x, t) = u(\xi), \tag{3.1}$$

where $\xi = x - ct$, and c is the wave speed. Then

$$\frac{\partial}{\partial t}(\cdot) = -c\frac{\partial}{\partial \xi}(\cdot), \quad \frac{\partial}{\partial x}(\cdot) = \frac{\partial}{\partial \xi}(\cdot), \quad \frac{\partial^2}{\partial t^2}(\cdot) = c^2\frac{\partial^2}{\partial \xi^2}(\cdot), \ldots\ldots \tag{3.2}$$

Step 2. Depending on the transformation, equation (3.1) shows the following nonlinear ODEs:

$$Q(u, u_\xi, u_{\xi\xi}, \ldots\ldots) = 0. \tag{3.3}$$

Step 3. Assuming X and Y as new independent variables

$$X(\xi) = u(\xi), \quad Y(\xi) = u_\xi(\xi), \tag{3.4}$$

we have the following system of ODEs:

$$\begin{cases} X_\xi(\xi) = Y(\xi), \\ Y_\xi(\xi) = F(X(\xi), Y(\xi)). \end{cases} \tag{3.5}$$

Step 4. The general solutions are found if we obtain the first integrals to equation (3.7) under the same conditions. However, no systematic theory tells us how to find its first integrals. Thankfully, we apply the division theorem to reduce equation (3.4) to a first-order integrable ODE. Thus, by solving these equations, we arrive at the exact solutions to equation (3.1).

4 Applications of the dynamical system method

Consider the solutions to equation (1.1) with the following form:

$$u(x,y,t) = \phi^{\frac{1}{2p}}(\xi)e^{i\eta}, \quad v(x,y,t) = \psi(\xi), \tag{4.1}$$

where $\xi = x + y - 2(k+\lambda)t$, $\eta = kx + \lambda y - wt$, λ, k, and w are constants.

Substituting equation (4.1) into equation (1.1), we obtain that

$$\psi = \frac{\beta}{2}\phi \tag{4.2}$$

and

$$(w - \lambda^2 - k^2) + \frac{2}{p}\left(\frac{1}{p} - 1\right)\phi^{-2}\phi'\phi' + \frac{2}{p}\phi^{-1}\phi'' + \gamma\phi + \alpha\psi + \delta\phi^2 = 0 \tag{4.3}$$

where "$'$" is the derivative as regards ξ.

Separating the real and imaginary parts in equation (4.3), respectively, it is possible to obtain

$$\phi'' = \frac{a\phi'^2 + \phi^2(A\phi^2 + B\phi + C)}{\phi}, \tag{4.4}$$

where $1 - \frac{1}{p} = a < 1$, $A = -\frac{p\delta}{2}$, $B = -\frac{p}{2}\left(\gamma + \frac{\alpha\beta}{2}\right)$, and $C = \frac{p}{2}(\lambda^2 + k^2 - w)$. Equation (4.4) refers to the planar dynamical system

$$\frac{d\phi}{d\xi} = y, \quad \frac{dy}{d\xi} = \frac{ay^2 + \phi^2(A\phi^2 + B\phi + C)}{\phi}, \tag{4.5}$$

where equation (4.5) is a singular nonlinear traveling wave system with the singular straight line $\phi = 0$ in phase plane (ϕ, y). Equation (4.5) has an associated regular system

$$\frac{d\phi}{d\zeta} = \phi y, \quad \frac{dy}{d\zeta} = ay^2 + \phi^2(A\phi^2 + B\phi + C), \tag{4.6}$$

where $d\xi = \phi d\zeta$. Equations (4.5) and (4.6) have the same first integrals as follows:

$$H = y^2\phi^{-2a} + \phi^{-2a}\left(\frac{A}{a-2}\phi^4 + \frac{2B}{2a-3}\phi^3 + \frac{C}{a-1}\phi^2\right) = h. \tag{4.7}$$

4.1 Bifurcations of phase portraits of equation (4.6)

Suppose that $AB \neq 0$ and $a \neq 1, 2, \frac{3}{2}$. Write that $f(\phi) = A\phi^2 + B\phi + C$, and $\Delta = B^2 - 4AC$. Clearly, when $\Delta < 0$, equation (4.6) receives only one equilibrium point $E_0(0,0)$. When $\Delta > 0$, equation (4.6) receives three equilibrium points $E_0(0,0)$, $E_1(\phi_1, 0)$, and $E_2(\phi_2, 0)$, where $\phi_1 = \frac{-B-\sqrt{\Delta}}{2A}$ and $\phi_2 = \frac{-B+\sqrt{\Delta}}{2A}$. When $\Delta = 0$, equation (4.6) receives an equilibrium point $E_0(0,0)$ and a double equilibrium point $E_d(\phi_d, 0)$, where $\phi_d = -\frac{B}{2A}$.

Making $M(\phi_j, 0)$ the coefficient matrix of the linearized system of equation (4.6) at an equilibrium point E_j, we obtain that

$$M = \begin{pmatrix} 0 & \phi_j \\ \phi_j^2 f'(\phi_j) & 0 \end{pmatrix}.$$

Then, we obtain

$$J(0,0) = \det\hat{M}(0,0) = 0, \quad J(\phi_d, 0) = \det M(\phi_d, 0) = 0,$$
$$J(\phi_{1,2}, 0) = \det M(\phi_{1,2}, 0) = -\phi_{1,2}^3 f'(\phi_{1,2}).$$

By the theory of planar dynamical systems and an equilibrium point of a planar integrable system, we obtain that if $J < 0$, then the equilibrium point is a saddle point; if $J > 0$ and $(\mathrm{trace}M)^2 - 4J < 0(or > 0)$, then it is a center point (a node point); if $J = 0$ and the Poincaré index of the equilibrium point is 0, then the equilibrium point is cusped. Thus, we note that the equilibrium point $E_0(0, 0)$ is a high-order singular point.

Depending on the change of parameter pair (A, B) for $a = \frac{1}{2}$ and a fixed $C < 0$, we obtain the bifurcations of phase portraits of equation (4.6) as shown in Figure 1.

Similarly, we receive the bifurcations of phase portraits of equation (4.6) when $C > 0$ or $a \neq \frac{1}{2}$.

4.2 Exact solutions of equation (4.5)

When $a = \frac{1}{2}$, we study the exact solutions of equation (4.5) in this section to obtain some solutions to equation (1.1). Then, only the case $\phi > 0$ is considered. When $a = \frac{1}{2}$, we can deduce from equation (4.7) that

$$H(\phi, y) = \frac{y^2}{\phi} - \frac{2}{3}\phi\left(3C + \frac{3}{2}B\phi + A\phi^2\right) = h, \tag{4.8}$$

$$h_1 = H(\phi_1, 0) = \frac{1}{6A}\phi_1(-8AC + B^2 + B\sqrt{\Delta}), \text{ and}$$

$$h_2 = H(\phi_2, 0) = \frac{1}{6A}\phi_2(-8AC + B^2 - B\sqrt{\Delta}).$$

This shows that

$$y^2 = \frac{2}{3}A\phi\left(\frac{3h}{2A} + \frac{3C}{A}\phi + \frac{3B}{2A}\phi^2 + \phi^3\right), \quad \text{for } A > 0 \text{ and}$$
$$y^2 = \frac{2}{3}|A|\phi\left(\frac{3h}{2|A|} + \frac{3C}{|A|}\phi + \frac{3B}{2|A|}\phi^2 - \phi^3\right), \quad \text{for } A < 0.$$

Then, using the first equation of equation (4.5), we know that for $A > 0$, the following function holds

$$\sqrt{\frac{3}{2A}}\xi = \int_{\phi_0}^{\phi} \frac{d\phi}{\sqrt{\phi\left(\frac{3h}{2A} + \frac{3C}{A}\phi + \frac{3B}{2A}\phi^2 + \phi^3\right)}}, \tag{4.9}$$

or for $A < 0$, the following function holds

$$\sqrt{\frac{3}{2A}}\xi = \int_{\phi_0}^{\phi} \frac{d\phi}{\sqrt{\phi\left(\frac{3h}{2|A|} + \frac{3C}{|A|}\phi + \frac{3B}{2|A|}\phi^2 - \phi^3\right)}}. \tag{4.10}$$

Case 1. $A > 0$ and $B > 0$ (Figure 1). If so, we obtain $\phi_1 < 0 < \phi_2$, $h_1 < 0 < h_2$.

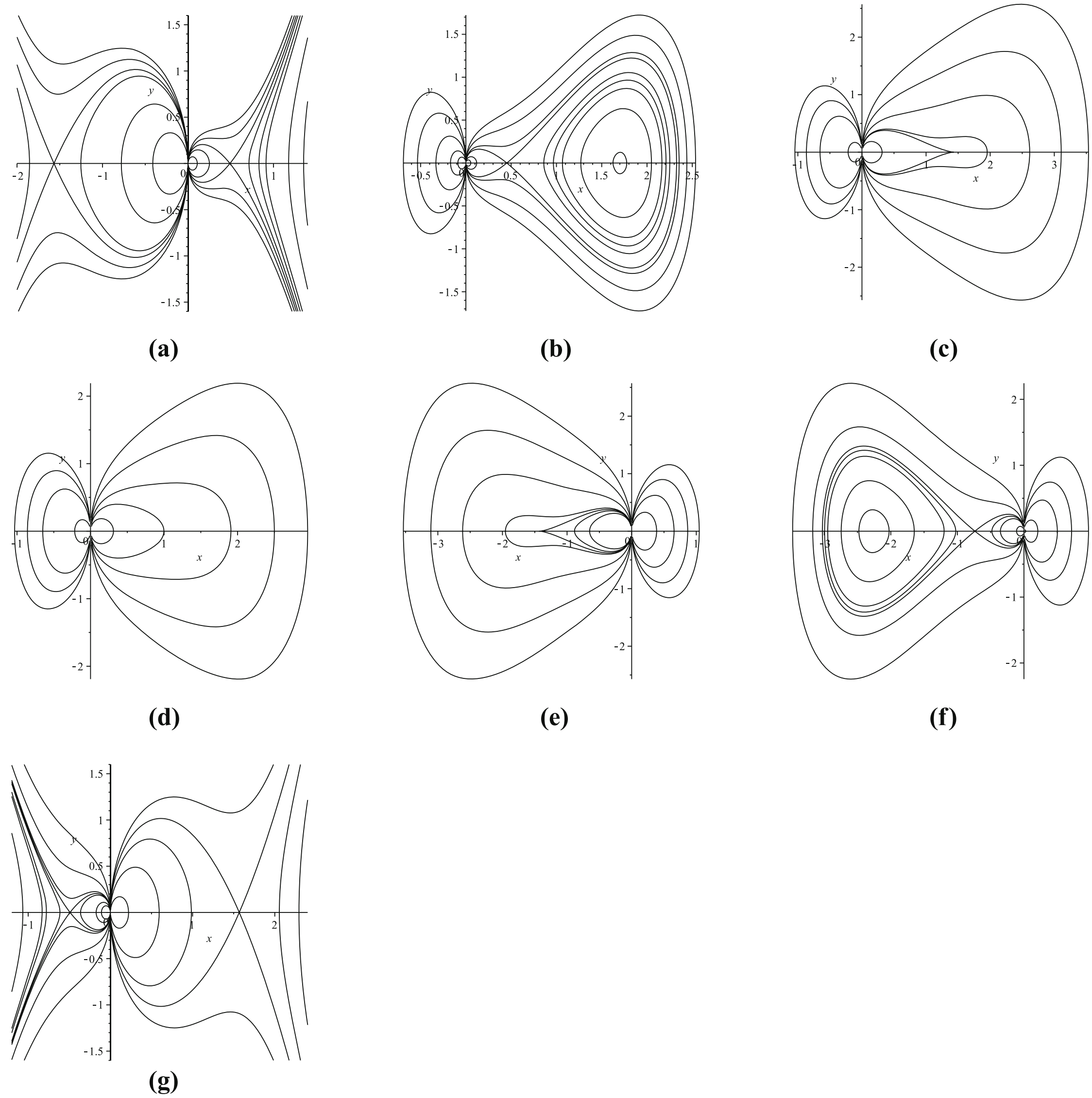

Figure 1: Bifurcations of phase portraits of equation (4.6) when $C < 0$. (a) $A > 0$ and $B > 0$, (b) $\Delta > 0, A < 0$, and $B > 0$, (c) $\Delta = 0, A < 0$, and $B > 0$, (d) $\Delta < 0$ and $A < 0$, (e) $\Delta = 0, A < 0, B < 0$ (f) $\Delta > 0, A < 0$, and $B < 0$, (g) $A > 0, B > 0$.

(i) For every $h \in (0, h_2)$, the level curves defined by $H_{a=\frac{1}{2}}(\phi, y) = h$ contain two open branches passing through the points $(\phi_L, 0)$ and $(\phi_l, 0)$ $\quad (\phi_l < 0 < \phi_M < \phi_2 < \phi_L)$, respectively. A close branch contacts to the singular straight line $\phi = 0$ at the equilibrium point $E_0(0, 0)$. Based on the finite-time interval theorem, we have the family of close branches that causes a family of periodic solutions to equation (4.5). Then, $y^2 = \frac{2}{3}A(\phi_L - \phi)(\phi_M - \phi)\phi(\phi - \phi_l)$. Thus, by equation (4.9), we receive the parametric representation of periodic solutions to equation (4.5) as follows:

$$\phi(\xi) = \phi_M - \frac{\phi_M(\phi_L - \phi_M)sn^2(\Omega_1\xi, k)}{\phi_L - \phi_M sn^2(\Omega_1\xi, k)}, \tag{4.11}$$

where $k^2 = \frac{\phi_M(\phi_L - \phi_l)}{\phi_L(\phi_M - \phi_l)}$, $\Omega_1 = \frac{1}{2}\sqrt{\frac{3\phi_L(\phi_M - \phi_l)}{2A}}$. Equation (4.11) shows the following exact periodic wave solutions to equation (1.1):

$$\begin{cases} u(x,t) = \left(\phi_M - \dfrac{\phi_M(\phi_L - \phi_M)sn^2(\Omega_1\xi, k)}{\phi_L - \phi_M sn^2(\Omega_1\xi, k)}\right)^{\frac{1}{2p}} e^{i\eta}, \\ v(x,t) = \dfrac{1}{3}\left(\phi_M - \dfrac{\phi_M(\phi_L - \phi_M)sn^2(\Omega_1\xi, k)}{\phi_L - \phi_M sn^2(\Omega_1\xi, k)}\right). \end{cases} \tag{4.12}$$

(ii) Corresponding to the level curves defined by $H_{a=\frac{1}{2}}(\phi, y) = h_2$, there are two heteroclinic orbits of equation (4.5) for $y^2 = \frac{2}{3}A(\phi_2 - \phi)^2\phi(\phi - \phi_l)$. Hence, we obtain the parametric representation of the kink wave and anti-kink wave solutions to equation (4.5) as follows:

$$\phi(\xi) = \Phi_2 - \frac{4A_1P_1}{P_1^2e^{\pm\omega_1\xi} + \Phi_l^2e^{\mp\omega_1\xi} - 2B_1P_1}, \tag{4.13}$$

where $0 < \phi_0 < \phi_2$, $A_1 = \phi_2(\phi_2 - \phi_l)$, $B_1 = -(2\phi_2 - \phi_l)$, $\omega_1 = \sqrt{\frac{3A_1}{2A}}$, and $P_1 = \frac{2\sqrt{A_1(A_1 + B_1\phi_0 + \phi_0^2)} + B_1\phi_0 + 2A_1}{\phi_0}$.

Then, we draw the exact solutions to equation (1.1) as follows:

$$\begin{cases} u(x,t) = \left(\phi_2 - \dfrac{4A_1P_1}{P_1^2e^{\pm\omega_1\xi} + \phi_l^2e^{\mp\omega_1\xi} - 2B_1P_1}\right)^{\frac{1}{2p}} e^{i\eta}, \\ v(x,t) = \dfrac{1}{3}\left(\phi_2 - \dfrac{4A_1P_1}{P_1^2e^{\pm\omega_1\xi} + \phi_l^2e^{\mp\omega_1\xi} - 2B_1P_1}\right). \end{cases} \tag{4.14}$$

Case 2. $A < 0$, $B > 0$, and $\Delta > 0$ (Figure 1(b)). If so, we obtain $0 < \phi_2 < \phi_1$, $h_1 < 0 < h_2$.

(i) When $h \in (h_1, 0)$, the level curves defined by $H_{a=\frac{1}{2}}(\phi, y) = h$ contain a close branch enclosing the equilibrium point $E_1(\phi_1, 0)$, for which we have

$$y^2 = \frac{2}{3}|A|\phi\left(-\phi^3 + \frac{3B}{2|A|}\phi^2 + \frac{3C}{|A|} + \frac{3h}{2|A|}\right) = \frac{2}{3}|A|\phi(r_1 - \phi)(\phi - r_2)(\phi - r_3).$$

Thus, we know from equation (4.10) that the family of periodic orbits has the parametric representation

$$\phi(\xi) = \frac{r_2}{1 - \tilde{A}_1^2 sn^2(\Omega_2\xi, k)}, \tag{4.15}$$

where $\Omega_2 = \frac{1}{2}\sqrt{\frac{3r_1(r_2 - r_3)}{2|A|}}$, $k^2 = \frac{(r_2 - r_1)r_3}{(r_2 - r_3)r_1}$, and $\tilde{A}_1^2 = \frac{r_1 - r_2}{r_1}$.

Equation (4.15) indicates the following exact solutions to equation (1.1):

$$\begin{cases} u(x,t) = \left(\dfrac{r_2}{1 - \tilde{A}_1^2 sn^2(\Omega_2\xi, k)}\right)^{\frac{1}{2p}} e^{i\eta}, \\ v(x,t) = \dfrac{1}{3}\left(\dfrac{r_2}{1 - \tilde{A}_1^2 sn^2(\Omega_2\xi, k)}\right). \end{cases} \tag{4.16}$$

(ii) When $h \in (0, h_2)$, we are aware that the level curves defined by $H_{a=\frac{1}{2}}(\phi, y) = h$ contain two close branches, for which one family encloses the equilibrium point $E_1(\phi_1, 0)$, while the other contacts to singular straight line $\phi = 0$ at the equilibrium point $E_0(0, 0)$. We have $y^2 = \frac{2}{3}|A|\phi(r_1 - \phi)(\phi - r_2)(\phi - r_3)$ and $y^2 = \frac{2}{3}|A|\phi(r_1 - \phi)(r_2 - \phi)(r_3 - \phi)$, respectively . Thus, the left family of periodic orbits has the parametric representation

$$\phi(\xi) = r_1 - \frac{r_1}{1 - \tilde{A}_3^2 sn^2(\Omega_3\xi, k)}, \tag{4.17}$$

and the right family of periodic orbits has the parametric representation

$$\phi(\xi) = r_3 + \frac{r_2 - r_3}{1 - \tilde{A}_2^{\,2} sn^2(\Omega_3\xi, k)}, \tag{4.18}$$

where $\Omega_3 = \frac{1}{2}\sqrt{\frac{3r_2(r_1 - r_3)}{2|A|}}$, $k^2 = \frac{(r_1 - r_2)r_3}{(r_1 - r_3)r_2}$, $\tilde{A}_2^{\,2} = \frac{r_1 - r_2}{r_1 - r_3}$, and $\tilde{A}_3^{\,2} = \frac{-r_2}{r_1 - r_3}$.

From equations (4.17) and (4.18), we are able to find the following two families of exact solutions:

$$\begin{cases} u(x,t) = \left(r_3 + \dfrac{r_2 - r_3}{1 - \tilde{A}_2^{\,2} sn^2(\Omega_3\xi, k)}\right)^{\frac{1}{2p}} e^{i\eta}, \\ v(x,t) = \dfrac{1}{3}\left(r_3 + \dfrac{r_2 - r_3}{1 - \tilde{A}_2^{\,2} sn^2(\Omega_3\xi, k)}\right). \end{cases} \tag{4.19}$$

$$\begin{cases} u(x,t) = \left(r_1 - \dfrac{r_1}{1 - \tilde{A}_2^{\,2} sn^2(\Omega_3\xi, k)}\right)^{\frac{1}{2p}} e^{i\eta}, \\ v(x,t) = \dfrac{1}{3}\left(r_1 - \dfrac{r_1}{1 - \tilde{A}_2^{\,2} sn^2(\Omega_3\xi, k)}\right). \end{cases} \tag{4.20}$$

(iii) The level curves defined by $H_{a=\frac{1}{2}}(\phi, y) = h_2$ contain a homoclinic orbit enclosing the equilibrium points $E_1(\phi_1, 0)$ and two heteroclinic orbits connecting the equilibrium points: $E_2(\phi_2, 0)$ and $E_0(0, 0)$. We receive that $y^2 = \frac{2}{3}|A|(\phi_M - \phi)(\phi - \phi_2)^2\phi$.

In addition, matching to the homoclinic orbit, we sketch the next solitary wave solutions

$$\phi(\xi) = \phi_2 + \frac{2\phi_2(\phi_M - \phi_2)}{\phi_M \cosh(\Omega_0\xi) - (\phi_M - 2\phi_2)}, \tag{4.21}$$

where $\Omega_0 = \sqrt{\frac{3\phi_2(\phi_M - \phi_2)}{2A}}$.

Writing to the two heteroclinic orbits, we gain the following kink and anti-kink wave solutions:

$$\phi(\xi) = \phi_2 - \frac{4A_2P_2}{P_2^2 e^{\pm\omega_2\xi} + \phi_M^2 e^{\mp\omega_2\xi} - 2B_2P_2}, \tag{4.22}$$

where $0 < \phi_0 < \phi_2$, $\omega_2 = \sqrt{\frac{3A_2}{2A}}$, $A_2 = \phi_2(\phi_M - \phi_2)$, $B_2 = 2\phi_2 - \phi_M$, and $P_2 = \frac{2\sqrt{A_2(A_2 + B_2\phi_0 + \phi_0^2)} + B_2\phi_0 + 2A_2}{\phi_0}$.

From equations (4.21) and (4.22), we obtain the exact solutions to equation (1.1) as follows:

$$\begin{cases} u(x,t) = \left(\phi_2 - \dfrac{4A_2P_2}{P_2^2 e^{\pm\omega_2\xi} + \phi_M^2 e^{\mp\omega_2\xi} - 2B_2P_2}\right)^{\frac{1}{2p}} e^{i\eta}, \\ v(x,t) = \dfrac{1}{3}\left(\phi_2 - \dfrac{4A_2P_2}{P_2^2 e^{\pm\omega_2\xi} + \phi_M^2 e^{\mp\omega_2\xi} - 2B_2P_2}\right). \end{cases} \tag{4.23}$$

$$\begin{cases} u(x,t) = \left(\phi_2 + \dfrac{2\phi_2(\phi_M - \phi_2)}{\phi_M \cosh(\Omega_0\xi) - (\phi_M - 2\phi_2)}\right)^{\frac{1}{2p}} e^{i\eta}, \\ v(x,t) = \dfrac{1}{3}\left(\phi_2 + \dfrac{2\phi_2(\phi_M - \phi_2)}{\phi_M \cosh(\Omega_0\xi) - (\phi_M - 2\phi_2)}\right). \end{cases} \tag{4.24}$$

(iv) When $h \in (h_2, \infty)$, we know that the level curves defined by $H_{a=\frac{1}{2}}(\phi, y) = h$ are a global family of close orbits of equation (4.5). At the same time, it encloses the equilibrium points $E_1(\phi_1, 0)$ and $E_2(\phi_2, 0)$. Besides, it also contacts the singular straight line $\phi = 0$ at $E_0(0, 0)$. Next, we can obtain $y^2 = \frac{2}{3}|A|(\phi_M - \phi)[(\phi - b_1)^2 + a_1^2]\phi$. Thus, equation (4.10) exists the following periodic solutions:

$$\phi(\xi) = \frac{\phi_M B_3(1 - cn(\Omega_4\xi, k))}{(A_3 + B_3) - (A_3 + B_3)cn(\Omega_4\xi, k)}, \tag{4.25}$$

where $\Omega_4 = \frac{1}{2}\sqrt{\frac{3A_3B_3}{2|A|}}$, $k^2 = \frac{\phi_M^2 - (A_3 - B_3)^2}{4A_3B_3}$, $A_3^2 = (\phi_M - b_1)^2 + a_1^2$, and $B_3^2 = a_1^2 + b_1^2$.

Hence, equation (4.25) can gain the exact solutions to equation (1.1) as follows:

$$\begin{cases} u(x,t) = \left(\frac{\phi_M B_3(1 - cn(\Omega_4\xi, k))}{(A_3 + B_3) - (A_3 + B_3)cn(\Omega_4\xi, k)}\right)^{\frac{1}{2p}} e^{i\eta}, \\ v(x,t) = \frac{1}{3}\left(\frac{\phi_M B_3(1 - cn(\Omega_4\xi, k))}{(A_3 + B_3) - (A_3 + B_3)cn(\Omega_4\xi, k)}\right). \end{cases} \tag{4.26}$$

Case 3. $A < 0, B > 0, \Delta = 0$ (Figure 1(c)). If so, we obtain $\phi_1 = \phi_2 = \frac{B}{2|A|}$ and $h_1 = h_2 = \frac{B^2}{12A^2}$.

(i) When $h \in (0, h_2)$ and $h \in (h_2, \infty)$, the level curves defined by $H_{a=\frac{1}{2}}(\phi, y) = h$ are two families of periodic orbits of equation (4.5). Then, we recognize that they can obtain the same representation of the parameters as in equation (4.25). Thus, equation (1.1) has the same exact solutions as equation (4.26).

(ii) The level curves defined by $H_{a=\frac{1}{2}}(\phi, y) = h_2$ are two heteroclinic orbits. Then, we know that $y^2 = \frac{2}{3}|A|(\phi_2 - \phi)^3\phi$. So, we can hold the following kink and anti-kink wave solutions:

$$\phi(\xi) = \frac{3\phi_2^3\xi^2}{8|A| + 3\phi_2^2\xi^2} = \frac{3B^3\xi^2}{64A^4 + 6|A|B^2\xi^2}. \tag{4.27}$$

Equation (4.26) receives the exact solutions to equation (1.1) as follows:

$$\begin{cases} u(x,t) = \left(\frac{3B^3\xi^2}{64A^4 + 6|A|B^2\xi^2}\right)^{\frac{1}{2p}} e^{i\eta}, \\ v(x,t) = \frac{1}{3}\left(\frac{3B^3\xi^2}{64A^4 + 6|A|B^2\xi^2}\right). \end{cases} \tag{4.28}$$

Case 4. $A < 0$ and $\Delta < 0$ (Figure 1(d)).

When $h \in (0, \infty)$, the level curves defined by $H_{a=\frac{1}{2}}(\phi, y) = h$ are a family of periodic orbits of equation (4.5) contacting to the singular straight line $\phi = 0$ at $E_0(0, 0)$. So, we can see that it has the same parametric representations as equation (4.26). From here, we can deduce that equation (1.1) has the same exact solutions as equation (4.27).

Then, we can make a similar discussion for the cases in Figure 1(e–g). We omit them.

5 Applications of the first integral method

Accordingly, we use the first integral method to solve equation (1.1). Assume that

$$u(x, y, t) = f(\xi)e^{i\eta}, v(x, y, t) = f_0(\xi), \tag{5.1}$$

where $\xi = x + y - 2(k + \lambda)t$ and $\eta = kx + \lambda y - wt$. λ, k, and w are real parameters, $f(\xi)$ is a real function.

Putting equation (5.1) into equation (1.1), then making real and imaginary part be zero, respectively,

$$\begin{cases} (w - k^2 - \lambda^2)f + 2f'' + \gamma f^{p+1} + \alpha f f_0 + \delta f^{2p+1} = 0, \\ f_0 = \frac{\beta}{2}f^p, \end{cases} \tag{5.2}$$

we gain

$$f'' = -\frac{1}{2}\delta f^{2p+1} - \frac{1}{2}\left(\gamma + \frac{\alpha\beta}{2}\right)f^{p+1} + \frac{1}{2}(\lambda^2 + k^2 - w)f. \tag{5.3}$$

Having simplified, we arrive at the following equation:

$$f'' = Af^{2p+1} + Bf^{p+1} + Cf, \tag{5.4}$$

where $-\frac{1}{2}\delta = A(A \neq 0)$, $B = -\frac{1}{2}\left(\gamma + \frac{\alpha\beta}{2}\right)$, and $C = \frac{1}{2}(\lambda^2 + k^2 - w)$.

Substituting equation (3.6) into equation (5.4), we are able to obtain

$$\begin{cases} X' = Y, \\ Y' = AX^{2p+1} + BX^{p+1} + CX. \end{cases} \tag{5.5}$$

According to the first integral method, we assume that $X = X(\xi)$ and $Y = Y(\xi)$ are the nontrivial solutions of equation (5.5), and

$$P(X, Y) = \sum_{i=0}^{m} a_i(X)Y^i \tag{5.6}$$

is an irreducible polynomial in the complex domain $C[X, Y]$. Then, we make

$$P(X(\xi), Y(\xi)) = \sum_{i=0}^{m} a_i(X(\xi))Y(\xi)^i = 0, \tag{5.7}$$

where $a_i(X)(i = 0, 1, \ldots, m)$ are polynomials of X and $a_m(X) \neq 0$. After that, equation (5.2) is called the first integral of equation (5.5). Note that $P(X(\xi), Y(\xi))$ is a polynomial in X and Y, and $\frac{dP}{d\xi}$ implies $\left.\frac{dP}{d\xi}\right|_{(5.6)} = 0$.

There exists a polynomial $H(X, Y) = h(X) + g(X)Y$ in $C(X, Y)$. Thus,

$$\left.\frac{dP}{d\xi}\right|_{(3.22)} = \left.\left(\frac{dP}{dX}\frac{dX}{d\xi} + \frac{dP}{dY}\frac{dY}{d\xi}\right)\right|_{(3.22)} = (h(X) + g(X)Y)\left(\sum_{i=0}^{m} a_i(X)Y^i\right). \tag{5.8}$$

Case 1. Let us assume $m = 1$ in equation (5.2). From equation (5.3), we have

$$\sum_{i=0}^{1} a_i'(X)Y^{i+1} + \sum_{i=0}^{1} ia_i(X)Y^{i-1}(Y'(\xi)) = (h(X) + g(X)Y)\left(\sum_{i=0}^{1} a_i(X)Y^i\right), \tag{5.9}$$

where the prime denotes differentiating with respect to the variable X. Comparing to the coefficient of $Y^i(i = 2, 1, 0)$ on equation (5.4), we receive

$$a_1'(X) = a_1(X)h(X), \tag{5.10}$$

$$a_0'(X) = a_1(X)g(X) + a_0(X)h(X), \quad \text{and} \tag{5.11}$$

$$a_0(X)g(X) = a_1(X)(AX^{2p+1} + BX^{p+1} + CX). \tag{5.12}$$

Since $a_i(X)(i = 0, 1)$ are polynomials, we obtain that $a_1(X)$ is a constant from equation (5.3) and make $a_1(X) = 1$. According to equations (5.11) and (5.12), we obtain

$$\deg[a_0(x)] + \deg[g(X)] = 2p + 1,$$

$$\deg[g(X)] = p,$$

$$\deg[a_0(X)] = p + 1.$$

Making it

$$g(X) = \sum_{i=0}^{p} A_iX^i, \tag{5.13}$$

where $A_p \neq 0$, from equation (5.11), we obtain

$$a_0(X) = \sum_{i=0}^{p} \frac{1}{i+1} A_i X^{i+1} + B_0, \tag{5.14}$$

where B_0 is an arbitrary integration constant. By entering $a_0(X)$, $a_1(X)$, and $g(X)$ into equation (5.12), the coefficients of $X^i (i = 0, 1, \ldots, 8)$ can be compared and obtained as follows:

$$\begin{aligned}
&X^0 : A_0 B_0 = 0,\\
&X^1 : A_0^2 + A_1 B_0 = C,\\
&X^2 : \frac{1}{2} A_0 A_1 + A_0 A_0 + A_2 B_0 = 0,\\
&\vdots\\
&X^p : \frac{1}{p} A_0 A_{p-1} + \frac{1}{p-1} A_1 A_{p-2} + \cdots + \frac{1}{2} A_{p-2} A_1 + A_{p-1} A_0 + A_p B_0 = 0,\\
&X^{p+1} : \frac{1}{p+1} A_0 A_p + \frac{1}{p} A_1 A_{p-1} + \cdots + \frac{1}{2} A_{p-1} A_1 + A_p A_0 + A_p B_0 = B,\\
&X^{p+2} : \frac{1}{p+1} A_1 A_p + \frac{1}{p} A_2 A_{p-1} + \cdots + \frac{1}{3} A_{p-1} A_2 + \frac{1}{2} A_p A_1 = 0,\\
&\vdots\\
&X^{p+n} : \frac{1}{p+1} A_{n-1} A_p + \frac{1}{p} A_n A_{p-1} + \cdots + \frac{1}{n} A_p A_{n-1} = 0,\\
&\vdots\\
&X^{2p} : \frac{1}{p+1} A_{p-1} A_p + \frac{1}{p} A_p A_{p-1} = 0,\\
&X^{2p+1} : \frac{1}{p+1} A_p^2 = A.
\end{aligned}$$

Substituting $a_0(X)$, $a_1(X)$, and $h(X)$ into equation (5.12) and making all the coefficients of X to be zero, we receive a system of nonlinear algebraic equations and use Maple to solve them. Therefore, we obtain

$$A_p = \pm\sqrt{(p+1)A}, \quad A_{p-1} = A_{p-2} = \cdots = A_1 = 0, \quad A_0 = \pm\sqrt{C}, \quad B_0 = 0, \tag{5.15}$$

where $A, C > 0$, and $(p+1)B^2 = (p+2)^2 AC$.

Next, we obtain $a_0(X) = A_0 X + \frac{1}{p+1} A_p X^{p+1}$. Take it into equation (5.8) and obtain

$$A_0 X + \frac{1}{p+1} A_p X^{p+1} + Y = 0. \tag{5.16}$$

From equation (5.16), we have

$$X' = -A_0 X - \frac{1}{p+1} A_p X^{p+1}. \tag{5.17}$$

(i) When $A_p = -\sqrt{(p+1)A}$, and $A_0 = -\sqrt{C}$, we hold $X' = \sqrt{C} X + \frac{1}{p+1}\sqrt{(p+1)A}\, X^{p+1}$. Let us have

$$X = \left(-\frac{\sqrt{(p+1)C}\, e^{p\sqrt{C}(\xi+\xi_{01})}}{\sqrt{A}\,(e^{p\sqrt{C}(\xi+\xi_{01})} \pm 1)} \right)^{\frac{1}{p}}. \tag{5.18}$$

Moreover, one can obtain the exact solutions to equation (1.1) as follows

$$\begin{cases} u(x,t) = \left(-\dfrac{\sqrt{(p+1)C}\, e^{p\sqrt{C}(\xi+\xi_{01})}}{\sqrt{A}\,(e^{p\sqrt{C}(\xi+\xi_{01})} \pm 1)} \right)^{\frac{1}{p}} e^{i\eta}, \\ v(x,t) = \dfrac{B}{2}\left(-\dfrac{\sqrt{(p+1)C}\, e^{p\sqrt{C}(\xi+\xi_{01})}}{\sqrt{A}\,(e^{p\sqrt{C}(\xi+\xi_{01})} \pm 1)} \right), \end{cases} \tag{5.19}$$

where ξ_{01} is an arbitrary constant.

(ii) When $A_p = \sqrt{(p+1)A}$ and $A_0 = -\sqrt{C}$, we hold $X' = \sqrt{C}X - \frac{1}{p+1}\sqrt{(p+1)A}X^{p+1}$. Let us have

$$X = \left(\frac{\sqrt{(p+1)C}\,e^{p\sqrt{C}(\xi+\xi_{02})}}{\sqrt{A}\,(e^{p\sqrt{C}(\xi+\xi_{02})} \pm 1)}\right)^{\frac{1}{p}}. \tag{5.20}$$

Moreover, one can obtain the exact solutions to equation (1.1) as follows:

$$\begin{cases} u(x,t) = \left(\frac{\sqrt{(p+1)C}\,e^{p\sqrt{C}(\xi+\xi_{02})}}{\sqrt{A}\,(e^{p\sqrt{C}(\xi+\xi_{02})} \pm 1)}\right)^{\frac{1}{p}} e^{i\eta}, \\ v(x,t) = \frac{B}{2}\left(\frac{\sqrt{(p+1)C}\,e^{p\sqrt{C}(\xi+\xi_{02})}}{\sqrt{A}\,(e^{p\sqrt{C}(\xi+\xi_{02})} \pm 1)}\right), \end{cases} \tag{5.21}$$

where ξ_{02} is an arbitrary constant.

(iii) When $A_p = \sqrt{(p+1)A}$ and $A_0 = \sqrt{C}$, we hold $X' = -\sqrt{C}X - \frac{1}{p+1}\sqrt{(p+1)A}X^{p+1}$. Let us have

$$X = \left(-\frac{\sqrt{(p+1)C}}{\sqrt{A}\,(1 \pm e^{p\sqrt{C}(\xi+\xi_{03})})}\right)^{\frac{1}{p}}. \tag{5.22}$$

Moreover, one can obtain the exact solutions to equation (1.1) as follows:

$$\begin{cases} u(x,t) = \left(-\frac{\sqrt{(p+1)C}}{\sqrt{A}\,(1 \pm e^{p\sqrt{C}(\xi+\xi_{03})})}\right)^{\frac{1}{p}} e^{i\eta}, \\ v(x,t) = \frac{B}{2}\left(-\frac{\sqrt{(p+1)C}}{\sqrt{A}\,(1 \pm e^{p\sqrt{C}(\xi+\xi_{03})})}\right), \end{cases} \tag{5.23}$$

where ξ_{03} is an arbitrary constant.

(iv) When $A_p = -\sqrt{(p+1)A}$ and $A_0 = \sqrt{C}$, we hold $X' = -\sqrt{C}X + \frac{1}{p+1}\sqrt{(p+1)A}X^{p+1}$. Let us have

$$X = \left(\frac{\sqrt{(p+1)C}}{\sqrt{A}\,(1 \pm e^{p\sqrt{C}(\xi+\xi_{04})})}\right)^{\frac{1}{p}}. \tag{5.24}$$

Moreover, one can obtain the exact solutions to equation (1.1) as follows:

$$\begin{cases} u(x,t) = \left(\frac{\sqrt{(p+1)C}}{\sqrt{A}\,(1 \pm e^{p\sqrt{C}(\xi+\xi_{04})})}\right)^{\frac{1}{p}} e^{i\eta}, \\ v(x,t) = \frac{B}{2}\left(\frac{\sqrt{(p+1)C}}{\sqrt{A}\,(1 \pm e^{p\sqrt{C}(\xi+\xi_{04})})}\right), \end{cases} \tag{5.25}$$

where ξ_{04} is an arbitrary constant.

Case 2. Suppose that $m = 2$. Compared to the coefficient of $Y^i (i = 3, 2, 1, 0)$ on equation (5.3), we have

$$a_2'(X) = a_2(X)h(X), \tag{5.26}$$

$$a_1'(X) = a_2(X)g(X) + a_1(X)h(X), \tag{5.27}$$

$$a_0'(X) + 2a_2(X)Y' = a_1(X)g(X) + a_0(X)h(X), \tag{5.28}$$

$$a_1(X)Y' = a_0(X)g(X) = a_1(X)(AX^{2p+1} + BX^{p+1} + CX). \tag{5.29}$$

Since $a_i(X)(i = 0, 1, 2) = 0$ are polynomials, we obtain $\deg[a_2(X)] = 0$, $h(X) = 0$, $\deg[a_0(X)] = 2p + 2$. Then, we make $a_2(X) = 1$.

Next, let us discuss $a_1(X)$ and $g(X)$ in two cases.

(**I**) When $a_1(X) = 0$ and $g(X) = 0$, we obtain

$$a_0(X) = -\frac{1}{p+1}AX^{2p+2} - \frac{2}{p+2}BX^{p+2} - CX^2.$$

Then bringing $a_0(X)$, $a_1(X)$, and $a_2(X)$ into equation (5.8), we obtain

$$X' = \pm\sqrt{\frac{1}{p+1}AX^2\left[(X^p)^2 + \left(2 - \frac{2}{p+2}\right)\frac{B}{A}X^p + (p+1)\frac{C}{A}\right]}. \tag{5.30}$$

The simultaneous integration of equation (5.30) yields

$$X = \left(\frac{\sqrt{D}}{e^{\sqrt{\frac{AD}{p+1}}p(\xi+\xi_{05})} - 1}\right)^{\frac{1}{p}} e^{\sqrt{\frac{AD}{p+1}}(\xi+\xi_{05})},$$

$$X = \left(\frac{\sqrt{D}}{e^{\sqrt{\frac{AD}{p+1}}p(\xi+\xi_{06})} + 1}\right)^{\frac{1}{p}}, \tag{5.31}$$

where $A > 0$, $C > 0$, $B_2 = \frac{(p+2)^2}{p+1}AC$, and $D = \sqrt{\frac{C(p+1)}{A}}$. Next, we have the exact solutions to equation (1.1) as follows:

$$\begin{cases} u(x,t) = \left(\frac{\sqrt{D}}{e^{\sqrt{\frac{AD}{p+1}}p(\xi+\xi_{05})} - 1}\right)^{\frac{1}{p}} e^{\sqrt{\frac{AD}{p+1}}(\xi+\xi_{05})} e^{i\eta}, \\ v(x,t) = \frac{B}{2}\left(\frac{\sqrt{D}}{e^{\sqrt{\frac{AD}{p+1}}p(\xi+\xi_{05})} - 1}\right) e^{p\sqrt{\frac{AD}{p+1}}(\xi+\xi_{05})}. \end{cases} \tag{5.32}$$

$$\begin{cases} u(x,t) = \left(\frac{\sqrt{D}}{e^{\sqrt{\frac{AD}{p+1}}p(\xi+\xi_{06})} + 1}\right)^{\frac{1}{p}} e^{i\eta}, \\ v(x,t) = \frac{B}{2}\left(\frac{\sqrt{D}}{e^{\sqrt{\frac{AD}{p+1}}p(\xi+\xi_{06})} + 1}\right). \end{cases} \tag{5.33}$$

(**II**) When $\deg[a_1(X)] = \deg[g(X)] + 1$, we make it to be two cases as follows:

When $\deg[g(X)] = n (n < p)$, $\deg[a_1(X)] = n + 1$, it is contradictory.

When $\deg[g(X)] = p$ and $\deg[a_1(X)] = p + 1$, we make $g(X) = \sum_{i=0}^{p} A_i X^i (A_p \neq 0)$. Then $a_1(X) = \sum_{i=0}^{p} \frac{1}{i+1} A_i X^{i+1} + B_0$, where B_0 is a constant. Similar ones can be launched.

(i) When

$$A_p = \pm 2\sqrt{(p+1)A}, \quad A_{p-1} = \cdots = A_1 = A_0 = B_0 = 0, \tag{5.34}$$

we have $a_1(X) = \pm 2\sqrt{\frac{A}{p+1}}X^{p+1}$ and $a_0(X) = \frac{A}{p+1}X^{2p+2}$, where $a_0(X)$ is a constant, $C_0 = 0$ and $B = C = 0$. Bringing it into equation (5.8), we obtain $X' = \pm\sqrt{\frac{A}{p+1}}X^{p+1}$.

Then, it is to obtain,

$$X = \pm\left(\sqrt{\frac{p+1}{A}}\frac{1}{p(\xi+\xi_{07})}\right)^{\frac{1}{p}}, \tag{5.35}$$

where ξ_{07} is a constant. We have the exact solutions to equation (1.1) as follows:

$$\begin{cases} u(x,t) = \pm\left(\sqrt{\frac{p+1}{A}}\frac{1}{p(\xi+\xi_{07})}\right)^{\frac{1}{p}} e^{i\eta}, \\ v(x,t) = \pm\frac{B}{2}\sqrt{\frac{p+1}{A}}\frac{1}{p(\xi+\xi_{07})}. \end{cases} \tag{5.36}$$

(ii) When

$$A_p = 2\sqrt{(p+1)A}, \quad A_0 = 2\sqrt{C}, \quad A_{p-1} = \cdots = A_1 = B_0 = 0, \tag{5.37}$$

we have $a_1(X) = 2X\left(\sqrt{\frac{A}{p+1}}X^p + \sqrt{C}\right)$ and $a_0(X) = X^2\left(\sqrt{\frac{A}{p+1}}X^p + \sqrt{C}\right)^2$, where $a_0(X)$ is a constant, $C_0 = 0$, $(p+1)B^2 = (p+2)^2AC$, and $B > 0$. Taking it to equation (5.8), we obtain $X' = -X\left(\sqrt{\frac{A}{p+1}}X^p + \sqrt{C}\right)$.

Then, it is to obtain

$$X = \left(\frac{\sqrt{C}}{e^{p\sqrt{C}(\xi+\xi_{08})} - \sqrt{\frac{A}{p+1}}}\right)^{\frac{1}{p}}, \tag{5.38}$$

where ξ_{08} is a constant. We have the exact solutions to equation (1.1) as follows:

$$\begin{cases} u(x,t) = \left(\dfrac{\sqrt{C}}{e^{p\sqrt{C}(\xi+\xi_{08})} - \sqrt{\frac{A}{p+1}}}\right)^{\frac{1}{p}} e^{i\eta}, \\ v(x,t) = \dfrac{B}{2}\left(\dfrac{\sqrt{C}}{e^{p\sqrt{C}(\xi+\xi_{08})} - \sqrt{\frac{A}{p+1}}}\right). \end{cases} \tag{5.39}$$

(iii) When

$$A_p = 2\sqrt{(p+1)A}, \quad A_0 = -2\sqrt{C}, \quad A_{p-1} = \cdots = A_1 = B_0 = 0, \tag{5.40}$$

we have $a_1(X) = 2X\left(\sqrt{\frac{A}{p+1}}X^p - \sqrt{C}\right)$ and $a_0(X) = X^2\left(\sqrt{\frac{A}{p+1}}X^p - \sqrt{C}\right)^2$, where $a_0(X)$ is a constant, $C_0 = 0$, $(p+1)B^2 = (p+2)^2AC$, and $B < 0$. Bringing it into equation (5.8), we obtain $X' = -X\left(\sqrt{\frac{A}{p+1}}X^p - \sqrt{C}\right)$.

Then, it is to obtain

$$X = \left(\frac{\sqrt{C}}{\sqrt{\frac{A}{p+1}}e^{p\sqrt{C}(\xi+\xi_{09})} - 1}\right)^{\frac{1}{p}} e^{\sqrt{C}(\xi+\xi_{09})}, \tag{5.41}$$

where ξ_{09} is a constant. We have the exact solutions to equation (1.1) as follows:

$$\begin{cases} u(x,t) = \left(\dfrac{\sqrt{C}}{\sqrt{\frac{A}{p+1}}e^{p\sqrt{C}(\xi+\xi_{09})} - 1}\right)^{\frac{1}{p}} e^{\sqrt{C}(\xi+\xi_{09})}e^{i\eta}, \\ v(x,t) = \dfrac{B}{2}\left(\dfrac{\sqrt{C}}{\sqrt{\frac{A}{p+1}}e^{p\sqrt{C}(\xi+\xi_{09})} - 1}\right)e^{p\sqrt{C}(\xi+\xi_{09})}. \end{cases} \tag{5.42}$$

(iv) When

$$A_p = -2\sqrt{(p+1)A}, \quad A_0 = 2\sqrt{C}, \quad A_{p-1} = \cdots = A_1 = B_0 = 0, \tag{5.43}$$

we have $a_1(X) = -2X\left(\sqrt{\frac{A}{p+1}}X^p - \sqrt{C}\right)$, and $a_0(X) = X^2\left(\sqrt{\frac{A}{p+1}}X^p - \sqrt{C}\right)^2$, where $a_0(X)$ is a constant, $C_0 = 0$, $(p+1)B^2 = (p+2)^2AC$, and $B < 0$.

Taking it into equation (5.8), we obtain $X' = X\left(\sqrt{\frac{A}{p+1}}X^p - \sqrt{C}\right)$.

Then, it is to obtain

$$X = \left(\frac{\sqrt{C}}{-e^{p\sqrt{C}(\xi+\xi_{10})} + \sqrt{\frac{A}{p+1}}}\right)^{\frac{1}{p}}, \tag{5.44}$$

where ξ_{10} is a constant. We have the exact solutions to equation (1.1) as follows:

$$\begin{cases} u(x,t) = \left(\dfrac{\sqrt{C}}{-e^{p\sqrt{C}(\xi+\xi_{10})} + \sqrt{\frac{A}{p+1}}}\right)^{\frac{1}{p}} e^{i\eta}, \\ v(x,t) = \dfrac{B}{2}\left(\dfrac{\sqrt{C}}{-e^{p\sqrt{C}(\xi+\xi_{10})} + \sqrt{\frac{A}{p+1}}}\right). \end{cases} \tag{5.45}$$

(v) When

$$A_p = -2\sqrt{(p+1)A}, \quad A_0 = -2\sqrt{C}, \quad A_{p-1} = \cdots = A_1 = B_0 = 0, \tag{5.46}$$

we have $a_1(X) = -2X\left(\sqrt{\frac{A}{p+1}}X^p + \sqrt{C}\right)$ and $a_0(X) = X^2\left(\sqrt{\frac{A}{p+1}}X^p - \sqrt{C}\right)^2$, where $a_0(X)$ is a constant, $C_0 = 0$, $(p+1)B^2 = (p+2)^2AC$, and $B > 0$. Bringing it into equation (5.8), we obtain $X' = X\left(\sqrt{\frac{A}{p+1}}X^p + \sqrt{C}\right)$.

Then, it is to obtain

$$X = \left(\frac{\sqrt{C}}{1 - \sqrt{\frac{A}{p+1}}e^{p\sqrt{C}(\xi+\xi_{11})}}\right)^{\frac{1}{p}} e^{\sqrt{C}(\xi+\xi_{11})}, \tag{5.47}$$

where ξ_{11} is a constant. We have the exact solutions to equation (1.1) as follows:

$$\begin{cases} u(x,t) = \left(\dfrac{\sqrt{C}}{1 - \sqrt{\frac{A}{p+1}}e^{p\sqrt{C}(\xi+\xi_{11})}}\right)^{\frac{1}{p}} e^{\sqrt{C}(\xi+\xi_{11})}e^{i\eta}, \\ v(x,t) = \dfrac{B}{2}\left(\dfrac{\sqrt{C}}{1 - \sqrt{\frac{A}{p+1}}e^{p\sqrt{C}(\xi+\xi_{11})}}\right)e^{p\sqrt{C}(\xi+\xi_{11})}. \end{cases} \tag{5.48}$$

6 Conclusion

This article obtains many new exact solutions to generalized Davey-Stewartson equations with arbitrary power nonlinearity by the dynamical system and the first integral methods. These exact solutions include periodic wave solutions, exact solitary wave solutions, kink wave solutions, anti-kink wave solutions, etc. We conclude that our results are novel and abundant by comparing the results with [41–45] using the software Maple. Hence, the two methods can also be extended to solve other nonlinear PDEs and obtain more exact solutions. Nevertheless, one can note that the integer-order derivative equations considered in our dissertation are, in fact, only time-integers. Would the traveling wave solutions be consistent with our results or more for time and spatial-fractional order derivatives regarding the generalized Davey-Stewartson equations? We leave this topic for future analysis.

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