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Research Article

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Global existence and blow up of the solution for nonlinear Klein-Gordon equation with variable coefficient nonlinear source term

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Abstract: This article is devoted to study the existence of global solutions and finite time blow-up of local solution for nonlinear Klein-Gordon equation with variable coefficient nonlinear source term. By applying the potential well and energy estimation method, in low initial energy and critical initial energy, we derive some sufficient conditions which are global existence and explosion of the solutions for this type of Klein-Gordon equation.

Keywords: Klein-Gordon equation, global existence of the solutions, blow up of the solutions, potential well method, energy estimation method

MSC 2020: 35J05, 35J09, 35J10

1 Introduction

It is well known that the Klein-Gordon equation is an important wave equation that arises in relativistic quantum mechanics and quantum fields [1,2]. It is used to model many physics phenomena, for example, the motion of electric charges in an electric or magnetic field. Particularly, if an electric field was generated by multiple charges with different signs, each charge will be subjected to force from these field sources. So the following question naturally comes to our mind, which is what happens to the properties of the solution for this Klein-Gordon equation with multiple nonlinear source terms. Based on this concern, for simplicity, this article is devoted to investing the existence of global solutions and finite time blow-up of local solutions for the initial-boundary value problem of the nonlinear Klein-Gordon equation with two variable coefficient nonlinear source terms, which has the following form:

$$\begin{cases} u_{tt} - \Delta u + u = a(y)|u|^{p-2}u - b(y)|u|^{q-2}u, x \in \Omega, t \in [0, T], y \in \mathbb{R}, \\ u(x, 0) = u_0(x), x \in \Omega, \\ u_t(x, 0) = u_1(x), x \in \Omega, \\ u(x, t)|_{\partial\Omega} = 0, x \in \partial\Omega, t \in (0, T], \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth boundary $\partial\Omega$ domain, $u = u(x, t)$, Δ denotes the Laplace operator on Ω, $u_0 \in H_0^1(\Omega)$, $u_1 \in L^2(\Omega)$. $a(y), b(y) \in C(\mathbb{R}, L^2(\mathbb{R}))$. Moreover, $a(y) > 0$, $b(y) > 0$. Here,

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$$\begin{cases} 2 < p < q < \infty, & \text{if } N = 1, 2, \\ 2 < p < q \leq \dfrac{N+2}{N-2}, & \text{if } N \geq 3. \end{cases} \tag{2}$$

In recent decades, a number of researchers are interested in the theory of existence and nonexistence of global solutions of the Klein-Gordon equation with nonlinear source term, and many important results have been obtained [3–8]. For a special case, in [9], Li and Zhang studied the global existence for the solutions of the following Cauchy problem:

$$\begin{cases} u_{tt} - \Delta u + u = u^2 + u^3, \;\; x \in \mathbb{R}^n, \;\; t \geq 0, \\ u(x, 0) = u_0(x), \;\; x \in \mathbb{R}^n, \\ u_t(x, 0) = u_1(x), \;\; x \in \mathbb{R}^n. \end{cases}$$

By applying the variational method, they obtain the necessary and sufficient conditions of the existence of global solutions for $E(0) = \frac{1}{2}(\|u_1\|_{L^2}^2 + \|u_0\|_{H^1}^2) - \int_{\mathbb{R}^n}\int_0^{u_0}(s^2 + s^3)\mathrm{d}s\mathrm{d}x < d$ with $n \leq 3$.

Besides, in [10], Gan et al. considered the following system:

$$\begin{cases} u_{tt} - \Delta u + u = f(u), \;\; x \in \mathbb{R}^2, \;\; t \geq 0, \\ u(x, 0) = \varphi(x), \;\; x \in \mathbb{R}^2, \\ u_t(x, 0) = \psi(x), \;\; x \in \mathbb{R}^2, \end{cases}$$

where $f(u) = |u|^2 u$. They established a sharp threshold about the solution global existence and explosion by introducing a cross-constrained variational method.

Ginibre and Velo [11] studied the Cauchy problem for the nonlinear Klein-Gordon equation with the following type:

$$\begin{cases} u_{tt} - \Delta u + f(u) = 0, \;\; x \in \mathbb{R}^n, \;\; t \geq 0, \\ u(x, t_0) = u_0(x), \;\; x \in \mathbb{R}^n, \\ u_t(x, t_0) = u_1(x), \;\; x \in \mathbb{R}^n, \end{cases} \tag{3}$$

where $f(u) = \sum_{j=1,2}\lambda_j|\varphi|^{p_j-1}\varphi$, $1 \leq p_1 \leq p_2 < \infty$, $\lambda_j \geq 0$. They proved the uniqueness of weak solutions and the existence and uniqueness of global strongly continuous solutions with nonlinear Klein-Gordon equation in the energy space under $p_2 < 1 + \frac{4}{n-2}$, for details see [11].

In [12], Lu and Miao consider the following Cauchy problem for the nonlinear combined Klein-Gordon equation:

$$\begin{cases} u_{tt} - \Delta u + u = f(u), \;\; (x, t) \in \mathbb{R}^d \times \mathbb{R}, \;\; d \geq 3, \\ u(x, t_0) = u_0(x) \in H^1(\mathbb{R}^d), \\ u_t(x, t_0) = u_1(x) \in L^2(\mathbb{R}^d), \end{cases}$$

where $f(u) = |u|^{p-1}u - |u|^{q-1}u$. They give a threshold of blow-up and global well-posedness by a modified variational approach in energy space $H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$. Here, $1 + \frac{4}{d} < q < p \leq 1 + \frac{4}{d-2}$.

In [13], Wang considered nonexistence of global solutions for the following system:

$$\begin{cases} u_{tt} - \Delta u + m^2 u = f(u), \;\; x \in \mathbb{R}^n, \;\; t \in [0, T), \\ u(x, 0) = u_0(x), \;\; x \in \mathbb{R}^n, \\ u_t(x, 0) = u_1(x), \;\; x \in \mathbb{R}^n. \end{cases}$$

The nonlinear term $f(u)$, which satisfies the condition that there is a real number $\varepsilon > 0$ subject to any $s \in \mathbb{R}$,

$$f(s)s \geq (2 + \varepsilon)F(s),$$

where $F(s) = \int_0^s f(\xi)\mathrm{d}\xi$. He applied the concavity method to obtain sufficient condition of this system local solutions blow-up when the initial energy is arbitrarily high.

Furthermore, in [14], Xu considered the Cauchy problem of the nonlinear Klein-Gordon equation with dissipative term and nonlinear source, which has the following form:

$$\begin{cases} u_{tt} - \Delta u + u + \gamma u_t = |u|^{p-1}u, \quad \gamma \geq 0, x \in \mathbb{R}^n, \ t > 0, \\ u(x,0) = u_0(x), \ x \in \mathbb{R}^n, \\ u_t(x,0) = u_1(x), \ x \in \mathbb{R}^n, \end{cases} \tag{4}$$

where p satisfies the following condition:

$$\text{if } n \geq 3, 1 < p < \frac{n+2}{n-2}; \quad \text{if } n = 2, 1 < p < \infty.$$

There exists a damping term γu_t, so that classical convexity method of [15] cannot be directly applied to derive the finite time blow-up of solutions. He successfully introduced a family of potential wells and proved the global existence, finite time blow up as well as the asymptotic behavior of the solutions for system (4).

Gazzola and Squassina [16] studied the behavior of solutions of the superlinear hyperbolic equation with (possibly strong) linear damping, which has the following form:

$$\begin{cases} u_{tt} - \triangle u - \omega \Delta u_t + \mu u_t = |u|^{p-2}u, \ (x,t) \in \Omega \times [0,T], \\ u(x,0) = u_0(x), \ x \in \Omega, \\ u_t(x,0) = u_1(x), \ x \in \Omega, \\ u(x,t) = 0, \ (x,t) \in \partial\Omega \times [0,T], \end{cases} \tag{5}$$

where Ω is an open bounded Lipschitz subset of $\mathbb{R}^n (n \geq 1)$, $T > 0$, $u_0 \in H_0^1(\Omega)$, $u_1 \in L^2(\Omega)$, $\omega \geq 0$, $\mu > -\omega\lambda_1$, λ_1 denotes homogeneous Dirichlet boundary condition first eigenvalue of the operator $-\Delta$, and

$$2 < p \leq \begin{cases} \frac{2n}{n-2}, & \text{for } \omega > 0, \\ \frac{2n-2}{n-2}, & \text{for } \omega = 0, \end{cases} \quad \text{if } n \geq 3, \quad 2 < p < \infty, \quad \text{if } n = 1, 2.$$

They have shown the global existence of solutions with initial data and uniformly bounded in the natural phase space $H_0^1(\Omega)$ for system (5). In addition, they also obtained blow-up results in correspondence with initial data (u_0, u_1) having arbitrarily large initial energy, for details please see [16].

By analyzing the aforementioned articles and reviewing some current literature, for example [15,17–19], we find that the form of the nonlinear source term $f(u)$ is relatively simple and that few scholars carried out their study on complex forms. On consideration of the above content, once we encounter the initial boundary value problem of a wave equation with summation form and variable coefficient nonlinear source terms, what happens to the behavior of the solution of this system?

Therefore, the main purpose of this article is to give some sufficient conditions of global existence and finite time blow-up of the solutions for system (1). However, we have to face the following difficulties:

(i) How to handle the aforementioned nonlinear source terms, whether the existence of local solutions of system (1) can be obtained?

(ii) Under this type of nonlinear source terms $a(y)|u|^{p-2}u - b(y)|u|^{q-2}u$, how to obtain the invariant set of the solution of system (1)?

(iii) How to apply potential well theory, concave function and energy estimation method, respectively, to obtain a sufficient condition which is the existence of global solution and explosion of local solution under low initial energy $E(0) < d$ and critical initial energy $E(0) = d$.

In order to overcome the aforementioned difficulties, inspired by [14,18], a potential well method was employed in our article. By this method, some new results on the global well-posedness of solutions of system (1) were derived. In addition, we also constructed some sufficient conditions about finite time blow-up of the solutions for system (1).

This article is organized as follows. In Section 2, some preliminary results are given. In Section 3, we apply the Galerkin method to prove the existence of the local solution. In Sections 4 and 5, we prove global existence and finite time blow-up of the solution at low initial energy $E(0) < d$ and critical initial energy $E(0) = d$. In Section 6, we give an application that shows a lower estimate of the solution's blow-up time for system (1).

2 Notations and set up

In order to simplify the notation, we introduce the following abbreviations:

$$L^p(\Omega) = L^p,\ \|\cdot\| = \|\cdot\|_{L^2(\Omega)},\ \|\cdot\|_p = \|\cdot\|_{L^p(\Omega)},\ (u, v) = \int_\Omega uv\mathrm{d}x.$$

Moreover, for a weight function $\psi \geq 0$, L^2_ψ denotes the space of measurable functions φ so that $\sqrt[p]{\psi}\varphi \in L^p$ with norm

$$\|\varphi\|_{L^p_\psi} = \left(\int_\Omega \psi|\varphi|^p\,\mathrm{d}x\right)^{\frac{1}{p}}.$$

In this article, first, we introduce Nehari functional and potential functional, respectively, as follows:

$$I(u) = \|u\|^2 + \|\nabla u\|^2 - \|u\|^p_{L^p_{a(y)}} + \|u\|^q_{L^q_{b(y)}},$$

$$J(u) = \frac{1}{2}\|u\|^2 + \frac{1}{2}\|\nabla u\|^2 - \frac{1}{p}\|u\|^p_{L^p_{a(y)}} + \frac{1}{q}\|u\|^q_{L^q_{b(y)}}.$$

Next, we introduce a potential well depth, which has the following form:

$$d = \inf_{u\in H^1_0(\Omega)\backslash\{0\}} \max_{\lambda\geq 0} J(\lambda u). \tag{6}$$

We define Nehari manifold

$$N = \{u \in H^1_0(\Omega)\backslash\{0\} : I(u) = 0\}.$$

It is easy to know that the manifold N can be divided into two unbounded sets:

$$N_+ = \{u \in H^1_0(\Omega) : I(u) > 0\} \cup \{0\},$$

$$N_- = \{u \in H^1_0(\Omega) : I(u) < 0\}.$$

Moreover, we also define the sublevels of J

$$J^a = \{u \in H^1_0(\Omega) : J^a \leq a\} \quad (a \in \mathbb{R}).$$

We introduce the stable set W_1 and the unstable set W_2 defined by

$$W_1 = J^d \cap N_+ \quad \text{and} \quad W_2 = J^d \cap N_-.$$

In addition, by Theorem 4.2 of [20], it can also be characterized as

$$d = \inf J(u), \quad u \in N. \tag{7}$$

We consider the energy functional about system (1)

$$E(t) = \frac{1}{2}\|u_t\|^2 + J(u).$$

Finally, C is a generic constant that can change from one line to another.

3 Local existence of solutions

In this section, we focus on the existence of local solution and uniqueness for system (1). Through solution of system (1) over $[0, T]$, we mean a function

$$u(x,t) \in C^0(H_0^1(\Omega),[0,T]) \cap C^1(L^2(\Omega),[0,T]) \cap C^2(H_0^{-1}(\Omega),[0,T]),$$

such that $u(x,0) = u_0$, $u_t(x,0) = u_1$ and

$$\langle u_{tt}, \eta\rangle + \int_\Omega \nabla u \nabla \eta + \int_\Omega u\eta = \int_\Omega (a(y)|u|^{p-2}u - b(y)|u|^{q-2}u)\eta \tag{8}$$

about all $\eta \in H_0^1(\Omega)$ and a.e. $t \in [0, T]$.

First, we give the result of the local existence and uniqueness for the solutions of system (1).

Theorem 3.1. *Assume $u_0 \in H_0^1(\Omega)$, $u_1 \in L^2(\Omega)$ hold. Then there exists $T > 0$ and a unique local solution of system* (1) *over* $[0, T]$. *Furthermore, if*

$$T_{\max} = \sup\{T > 0 : u = u(x,t) \quad \textit{exists } [0,T]\} < \infty,$$

then

$$\lim_{t\to T_{\max}} \|u\|_{2p} = \infty.$$

Definition 3.1. If $T_{\max} < \infty$, we say that the solution of system (1) is explosive and $T_{\max}$ is called the blow-up time. If $T_{\max} = \infty$, we say the solution is global.

Now, we introduce a space $\mathcal{H} = C([0,T], H_0^1(\Omega)\bigcap C^1[0,T], L^2(\Omega))$ endow the norm with

$$\|u\|_{\mathcal{H}} = \max_{t\in[0,T]} (\|u\|^2 + \|\nabla u\|^2 + \|u_t\|_2^2).$$

Lemma 3.1. *For every $T > 0$, every $u \in \mathcal{H}$, and every initial data $u_0 \in H_0^1(\Omega)$, $u_1 \in L^2(\Omega)$, there exists a unique*

$$\vartheta \in \mathcal{H} \cap C^2([0,T]; H_0^{-1}(\Omega)) \quad \textit{such that } \vartheta_t \in L^2([0,T], H_0^1(\Omega)), \tag{9}$$

which satisfies the following equation:

$$\begin{cases} \vartheta_{tt} - \Delta\vartheta + \vartheta = a(y)|u|^{p-2}u - b(y)|u|^{q-2}u, \\ \vartheta(x,0) = \vartheta_0(x), \\ \vartheta_t(x,0) = \vartheta_1(x), \\ \vartheta(x,t)|_{\partial\Omega} = 0. \end{cases} \tag{10}$$

Proof. We consider using the Galerkin method and combining prior estimation to prove the local existence of the weak solution of system (1). For every $m \geq 1$, let $\{\psi_j(x)\}_{j=1}^m$ be the standard orthogonal basis of space $\mathcal{H}$.

Suppose that space W_m is spanned by

$$W_m = \{\psi_1, \psi_2, \ldots, \psi_m\}.$$

Taking $u_{0m}, u_{1m} \in W_m$, s.t, $m \to +\infty$,

$$u_{0m} = \sum_{i=1}^m a_i^m \psi_j(x) \to u_0 \quad \text{strong convergence in } \mathcal{H}, \tag{11}$$

$$u_{1m} = \sum_{i=1}^m b_i^m \psi_j(x) \to u_1 \quad \text{strong convergence in } \mathcal{H}. \tag{12}$$

Construct approximation solution of system (1)

$$u_m(x,t)=\sum_{i=1}^{m}c_i^m(t)\psi_i(x),\quad m=1,2,\dots,\tag{13}$$

satisfying

$$\left(\frac{\partial^2 u_m}{\partial t^2},\psi_k\right)-(\Delta u_m,\psi_k)+(u_m,\psi_k)=(a(y)|u_m|^{p-2}u_m-b(y)|u_m|^{q-2}u_m,\psi_k),\quad k=1,2,\dots,m.\tag{14}$$

Note

$$\left(\frac{\partial^2 u_m}{\partial t^2},\psi_k\right)=\sum_{i=1}^{m}\frac{d^2}{dt^2}c_i^m(t)(\psi_i,\psi_k)=\frac{d^2}{dt^2}c_k^m(t),$$
$$(\Delta u_m,\psi_k)=\sum_{i=1}^{m}(\Delta\psi_i,\psi_k)=-\frac{1}{\lambda_k}c_k^m(t),$$
$$(u_m,\psi_k)=\sum_{i=1}^{m}(c_i^m(t)\psi_i,\psi_k)=c_k^m(t).$$

Let

$$\begin{aligned}f_k(t,x)&=(a(y)|u_m|^{p-2}u_m-b(y)|u_m|^{q-2}u_m,\psi_k)\\&=(a(y)|\sum_{i=1}^{m}c_i^m(t)\psi_i(x)|^{p-2}-b(y)|\sum_{i=1}^{m}c_i^m(t)\psi_i(x)|^{q-2})c_k^m(t).\end{aligned}$$

So that, we can obtain that

$$\frac{d^2c_k^m(t)}{dt^2}+\left(1+\frac{1}{\lambda_k}\right)c_k^m(t)=f_k(t,x),\quad k=1,2,\dots,m,\tag{15}$$

$$c_k^m(0)=a_k^m,\quad k=1,2,\dots,m,\tag{16}$$

$$(c_k^m(0))'=b_k^m,\quad k=1,2,\dots,m.\tag{17}$$

Based on the theory of linear ordinary differential equation (15), there exists a unique global solution $c_k^m(t)\in[0,T]$. Then we obtain a unique $u_m(x,t)\in C([0,T],H_0^1(\Omega))$ defined by (13) that satisfies (14). Multiplying (14)$\frac{d}{dt}c_k^m(t)$ and summing, we have

$$\left(\frac{\partial^2 u_m}{\partial t^2},\frac{\partial u_m}{\partial t}\right)+\left(u_m,\frac{\partial u_m}{\partial t}\right)=\left(\Delta u_m,\frac{\partial u_m}{\partial t}\right)+\left(a(y)|u_m|^{p-2}u_m-b(y)|u_m|^{q-2}u_m,\frac{\partial u_m}{\partial t}\right).\tag{18}$$

Next, (18) from $(0,t)$ integrating once, for every $m\ge 1$, we obtain

$$\|u_{mt}\|^2+\|\nabla u_m\|^2+\|u_m\|^2=\|u_{1m}\|^2+\|\nabla u_{0m}\|^2+\|u_{0m}\|^2+\frac{2}{p}\int_0^t\|u_m\|_{L^p_{a(y)}}^p d\tau-\frac{2}{q}\int_0^t\|u_m\|_{L^q_{b(y)}}^q d\tau.\tag{19}$$

Now, we estimate the last two terms in the right-hand side of (19).

$$\begin{aligned}&\frac{2}{p}\int_0^t\|u_m\|_{L^p_{a(y)}}^p d\tau-\frac{2}{q}\int_0^t\|u_m\|_{L^q_{b(y)}}^q d\tau\\&\le\frac{2}{p}\int_0^T\|u_m\|_{L^p_{a(y)}}^p d\tau+\frac{2}{q}\int_0^T\|u_m\|_{L^q_{b(y)}}^q d\tau\\&\le\frac{2}{p}\left(\int_0^T\|a(y)\|^2 d\tau+\int_0^T\|u_m\|_{L^{2p}}^{2p}d\tau\right)+\frac{2}{q}\left(\int_0^T\|b(y)\|^2 d\tau+\int_0^T\|u_m\|_{L^{2q}}^{2q}d\tau\right)\\&\le\frac{2}{p}M_1T+\frac{2}{p}M_2T+\frac{2}{q}M_3T+\frac{2}{q}M_4T\\&\le CT,\end{aligned}\tag{20}$$

where $\|a(y)\|^2 \le M_1$, $\|b(y)\|^2 \le M_2$, $\|u_m\|_{L^{2p}}^{2p} \le M_3$, $\|u_m\|_{L^{2q}}^{2q} \le M_4$, $C = \max\left\{\frac{2M_1+2M_2}{p}, \frac{2M_3+2M_4}{q}\right\}$.

Recalling that u_{0m} and u_{1m} converge, from (19) and (20) we obtain

$$\|u_m\|_{\mathcal{H}}^2 + \int_0^t \|u_m\|_{L^p}^p \mathrm{d}x \le C_T$$

for all $m \ge 1$. In addition, C_T is independent of m. By this uniform estimate and Grönwalląfs inequality, we obtain

$$\begin{aligned} \|u_m\|_{(L^\infty([0,T]),H_0^1(\Omega))} &\le C, \\ \|u_{mt}\|_{L^\infty([0,T]),H_0^1(\Omega)\cap L^2([0,T],H_0^1(\Omega))} &\le C, \\ \|u_{mtt}\|_{L^2([0,T]),H^{-1}(\Omega)} &\le C. \end{aligned}$$

Hence, up to a subsequence, we can derive a weak solution ϑ of (10) with the above regularity by taking the limit in (13). Due to $\vartheta \in H^1([0,T], H_0^1(\Omega))$, we have $\vartheta \in C([0,T], H_0^1(\Omega))$. Furthermore, since $\vartheta_t \in L^\infty([0,T], L^2(\Omega)) \cap L^2([0,T], H_0^1(\Omega))$ and $\vartheta_{tt} \in L^2([0,T], H^{-1}(\Omega))$, we obtain $\vartheta_t \in L^2([0,T], L^2(\Omega))$. Finally, from (10), we can easily obtain $\vartheta_{tt} \in C^0([0,T], H^{-1}(\Omega))$. Therefore, there exists a ϑ, which not only satisfies (10), but also (9). The existence of ϑ is obtained.

Next, the proof by contradiction is used to prove uniqueness of this proposition. If ϑ and v were two solutions of (10), which satisfy the same initial data, by subtracting the equations and testing with $\vartheta_t - v_t$, instead of (19), we obtain that

$$\|\vartheta_t - v_t\|^2 + \|\nabla(\vartheta - v)\|^2 + \|\vartheta - v\|^2 = 0.$$

Obviously, it immediately yields $\vartheta = v$. We finish the proof of this lemma.

Let $u_0 \in H_0^1(\Omega)$, $u_1 \in L^2(\Omega)$ and $R^2 = 2(\|u_1\|^2 + \|u_0\|^2 + \|\nabla u_0\|^2)$, for any $T > 0$, we consider

$$N_T = \{u \in \mathcal{H} | u(x,0) = u_0, u_t(x,0) = u_1, \|u\|_{\mathcal{H}} \le R\}.$$

Now, through Lemma 3.1, we define $\Gamma(u) = \vartheta$ for any $u \in N_T$, where ϑ is the unique solution for (10). Next, we shall show that Γ is a contractive operator which satisfies $\Gamma(u) \subseteq \mathcal{H}$ for a suitable T. Obviously, given $u \in N_T$, the corresponding solution $\vartheta = \Gamma(u)$ satisfies the following energy identity for all $t \in (0, T]$,

$$\|\vartheta_t\|^2 + \|\nabla\vartheta\|^2 + \|\vartheta\|^2 = \|\vartheta_1\|^2 + \|\nabla\vartheta_0\|^2 + \|\vartheta_0\|^2 + \frac{2}{p}\int_0^t \|\vartheta\|_{L_{a(y)}^p}^p \mathrm{d}\tau - \frac{2}{q}\int_0^t \|\vartheta\|_{L_{b(y)}^q}^q \mathrm{d}\tau. \tag{21}$$

For the last two terms, we apply the same method to estimate (although slightly differently) as for (20) and we obtain

$$\frac{2}{p}\int_0^t \|\vartheta\|_{L_{a(y)}^p}^p \mathrm{d}\tau - \frac{2}{q}\int_0^t \|\vartheta\|_{L_{b(y)}^q}^q \mathrm{d}\tau \le CT \tag{22}$$

for all $t \in (0, T]$. Through (21), (22) and taking the maximum over $[0, T]$, we can obtain

$$\|\vartheta\|_{\mathcal{H}}^2 \le \frac{1}{2}R^2 + cT + TR^{2(p-1)}.$$

Choosing T sufficiently small, we obtain $\|\vartheta\|_{\mathcal{H}} \le R$, which shows that $\Gamma(N_T) \subseteq N_T$. Next, we take $\eta_1, \eta_2 \in N_T$. Let $\vartheta_1 = \Gamma(\eta_1)$, $\vartheta_2 = \Gamma(\eta_2)$, and substitute them into (10), respectively, setting $\vartheta = \vartheta_1 - \vartheta_2$, we obtain for all $\eta \in H_0^1(\Omega)$ and a.e. $t \in [0, T]$

$$\begin{aligned} \langle \vartheta_{tt}, \eta\rangle + \int_\Omega \nabla\vartheta\nabla\eta + \int_\Omega \vartheta\eta &= \int_\Omega (a(y)|\vartheta_1|^{p-2}\vartheta_1 - b(y)|\vartheta_1|^{q-2}\vartheta_1)\eta - (a(y)|\vartheta_2|^{p-2}\vartheta_2 - b(y)|\vartheta_2|^{q-2}\vartheta_2)\eta \\ &= \int_\Omega \xi(t)(\vartheta_1 - \vartheta_2)\eta. \end{aligned} \tag{23}$$

Here, $\xi = \xi(t) \geq 0$ is given by the Lagrange theorem so that $\xi(t) \leq (p-1)(a(y)-b(y))(\vartheta_1+\vartheta_2)^{p+q-2}$. By taking $\eta = \vartheta_t$ in (23) and (10), we obtain

$$\|\vartheta_t\|^2 + \|\nabla\vartheta\|^2 + \|\vartheta\|^2 \leq \|\vartheta_1\|^2 + \|\nabla\vartheta_0\|^2 + \|\vartheta_0\|^2 + 2\int_0^t \xi(t)(\vartheta_1 - \vartheta_2)\mathrm{d}\tau. \tag{24}$$

Be similar to the discussion above, we obtain

$$\|\Gamma(\vartheta_1) - \Gamma(\vartheta_2)\|_{\mathcal{H}} \leq cR^{2p+2q-4}T\|\vartheta_1 - \vartheta_2\|^2_{\mathcal{H}} \leq \delta\|\vartheta_1 - \vartheta_2\|^2_{\mathcal{H}}$$

for some $0 < \delta < 1$ and assume T is sufficiently small. Therefore, by the contraction mapping principle, there exists a unique (weak) solution u for system (1) defined on $[0, T]$. The main statement of Theorem 3.1 is proved. Next, we concern the remain assertion. By the construction and analysis above, we know that the local existence time of u merely depends (through R) on the norms of the initial data, so that, once $\|u\|_{\mathcal{H}}$ continues to be bounded, the solution may be continued, also see [19], for a similar argument. Therefore, if $T_{\max} < \infty$, we obtain

$$\lim_{t\to T_{\max}} (\|\nabla u\|^2 + \|u_t\|^2) = \lim_{t\to T_{\max}} \|u\|^2_{\mathcal{H}} = \infty. \tag{25}$$

Note that

$$E(t) = \frac{1}{2}\|u_t\|^2 + \frac{1}{2}\|u\|^2 + \frac{1}{2}\|\nabla u\|^2 - \frac{1}{p}\|u\|^p_{L^p_{a(y)}} + \frac{1}{q}\|u\|^q_{L^q_{b(y)}}.$$

By multiplying the first equality of (1) with u_t and integrating with respect to t, we have

$$E(t) + \int_t^r \|u\|^2\mathrm{d}\tau = E(r) \tag{26}$$

for all $r \in [0, T_{\max}]$. In this case, $E(t)$ is nonincreasing, so that

$$\begin{aligned}\frac{1}{2}\|u_t\|^2 + \frac{1}{2}\|u\|^2 + \frac{1}{2}\|\nabla u\|^2 &\leq \frac{1}{p}\|u\|^p_{L^p_{a(y)}} - \frac{1}{q}\|u\|^q_{L^q_{b(y)}} + E(0)\\ &\leq \frac{1}{p}\|u\|^p_{L^p_{a(y)}} + E(0)\\ &\leq \|a(y)\|^2 + \|u\|^{2p}_{2p} + E(0)\\ &\leq M_1 + \|u\|^{2p}_{2p} + E(0),\end{aligned} \tag{27}$$

for all $t \in [0, T_{\max})$. Together with (25), we must obtain

$$\lim_{t\to T_{\max}} \|u\|_{2p} = \infty.$$

The proof of Theorem 3.1 is now completed. □

4 Global existence and finite time blow up when $E(0) < d$

Now, let us turn to the global existence of solutions starting with suitable initial data and low initial energy, that depends on Theorem 3.1.

Theorem 4.1. *If $u_0 \in H^1_0(\Omega)$, $u_1 \in L^2(\Omega)$, $E(0) < d$ hold, u will be the unique local solution. Moreover, suppose that $u(t) \in W_1$, for $t \in [0, T_{\max}]$. Then $T_{\max} = \infty$. Namely, system (1) admits a global weak solution $u(x, t)$.*

Proof. Through (26) we infer that the energy map $E(t)$ is decreasing. From the above condition, we have

$$u(t) \in W_1, E(t) < d \quad \text{for every } t \in (0, T_{\max}). \tag{28}$$

In fact, if the above situation is not true, there exists $t_* > 0$ such that $u(t_*) \in W_2$. According to the variational characterization (7) of d, we have

$$d \leq J(t^*) \leq E(t^*) < d. \tag{29}$$

Obviously, it is contradictory to (28). Therefore, $u(t) \in W_1$ for every $t \in [0, T_{\max}]$. As a further consequence of (28), a simple computation entails

$$J(u(t)) \geq \frac{pq - 2q + 2p}{2pq}\|\nabla u\|^2 \quad \text{for every } t \in (0, T_{\max}).$$

Through (26), we obtain

$$\frac{1}{2}\|u_t\|^2 + J(u) + \int_0^r \|u\|^2 \mathrm{d}\tau = E(0) < d. \tag{30}$$

This implies $\|u\|_{H^1} \leq C$ and $\|u_t\| \leq C$. Hence $T = \infty$.

Note that not all local solutions of system (1) are global in time. Particularly, in low initial energy $E(0)$ and $u_0 \in W_2$, the local solutions usually blow up. In the next theorem, we applied concavity method which was introduced by Levine in [21,22] to show the finite time blow up of some solutions of system (1) under $E(0) < d$. □

Theorem 4.2. *Assume that $u_0 \in H_0^1(\Omega)$, $u_1 \in L^2(\Omega)$ and* (2) *hold, $u(x, t)$ is the unique local solution to system* (1), *then the solution is blow up in finite time if and only if there exists $t^* \in [0, T_{\max}]$ such that*

$$u(t^*) \in W_2 \quad \text{and} \quad E(t^*) < d.$$

In addition, we obtain a lower estimation of the blow-up time t^* of the following solution:

$$t^* \geq \frac{p\mu H^{\frac{2-p}{p}}(0)}{(p-2)8p^2 C^2 |\Omega|^{\frac{1}{p}+\frac{1}{2N}+1}}.$$

Proof. We assume that there exists $t^* \geq 0$ such that $u(t^*) \in W_2$, $E(t^*) < d$. Without loss of generality, we assume that $t^* = 0$, and by (26), we can know that $E(t) < d$ for all $t > 0$, so that $u \notin N$. This shows that $u(t) \in W_2$ for all $t \in [0, T_{\max}]$.

Assume by contradiction that the solution u is global. Then, for any $T > 0$, we consider $G : [0, T] \to R_+$ defined by

$$G(t) = \|u(t)\|^2. \tag{31}$$

Since $G(t)$ is continuous and $G(t) \geq 0$ for all $[0, T]$, there exists $\theta > 0$ such that

$$G(t) \geq \theta \quad \text{for all } t \in [0, T]. \tag{32}$$

Furthermore,

$$G'(t) = 2\int uu_t \mathrm{d}x$$

and

$$G''(t) = 2\|u_t\|^2 + 2(u_{tt}, u_t). \tag{33}$$

Note that $u = u(x, t)$ satisfies the first equality of system (1)

$$u_{tt} - \Delta u + u = a(y)|u|^{p-2}u - b(y)|u|^{q-2}u.$$

Next, multiplying the above equation by u and integrating with respect to x over Ω, we obtain

$$(u_{tt}, u) - (\Delta u, u) + (u, u) = (a(y)|u|^{p-2}u - b(y)|u|^{q-2}u, u).$$

Therefore,

$$(u_{tt}, u) = -\|\nabla u\|^2 - \|u\|^2 + \|u\|^p_{L^p(a(y))} - \|u\|^q_{L^q(b(y))} = -I(u).$$

$G''(t)$ can be expressed as

$$G''(t) = 2(\|u_t\|^2 - I(u)) = 2\Big(\|u_t\|^2 - \|u\|^2 - \|\nabla u\|^2 + \|u\|^p_{L^p(a(y))} - \|u\|^q_{L^q(b(y))}\Big). \tag{34}$$

Next, we estimate $G''(t)$. From (27), we obtain

$$\frac{1}{2}\|u_t\|^2 + \frac{1}{2}\|u\|^2 + \frac{1}{2}\|\nabla u\|^2 - \frac{1}{2}\|u\|^p_{L^p(a(y))} + \frac{1}{2}\|u\|^q_{L^q(b(y))} \le \left(\frac{1}{p} - \frac{1}{2}\right)\|u\|^p_{L^p(a(y))} - \left(\frac{1}{q} - \frac{1}{2}\right)\|u\|^q_{L^q(b(y))} + E(0).$$

Then

$$\frac{1}{2}\|u_t\|^2 + \frac{1}{2}I(u) \le \left(\frac{1}{p} - \frac{1}{2}\right)\|u\|^p_{L^p(a(y))} - \left(\frac{1}{q} - \frac{1}{2}\right)\|u\|^q_{L^q(b(y))} + E(0)$$
$$I(u) \le 2E(0) + C\Big(\|u\|^p_{L^p(a(y))} - \|u\|^q_{L^q(b(y))}\Big) - \|u_t\|^2.$$

Since $E(t) < E(0) < d$, we have

$$\begin{aligned}
&\frac{1}{2}\|u_t\|^2 - \frac{1}{p}\|u\|^p_{L^p_{a(y)}} + \frac{1}{q}\|u\|^q_{L^q_{b(y)}} < d,\\
&\frac{C}{p}\Big(\|u\|^p_{L^p_{a(y)}} - \|u\|^q_{L^p_{b(y)}}\Big) - \frac{2E(0)C}{p} \ge -2d - \frac{2E(0)C}{p} + \|u_t\|^2,\\
&2\Big(\|u\|^p_{L^p_{a(y)}} - \|u\|^q_{L^p_{b(y)}}\Big) - 4E(0) \ge p\|u_t\|^2,\\
&-2I(u) \ge (2 + p)\|u_t\|^2.
\end{aligned}$$

Therefore,

$$G''(t) \ge (4 + p)\|u_t\|^2 \ge 4\|u_t\|^2.$$

Through the Schwarz inequality, we have

$$(G'(t))^2 \le 4\|u\|^2\|u_t\|^2 = 4G(t)\|u_t\|^2, \tag{35}$$

so that

$$G(t)G''(t) - (G'(t))^2 \ge 0. \tag{36}$$

Moreover, we introduce $\ln G(t)$, by computing, we have

$$(\ln G(t))' = \frac{G'(t)}{G(t)}, (\ln G(t))'' = \left(\frac{G'(t)}{G(t)}\right)' = \frac{G''(t)G(t) - G'^2(t)}{G^2(t)}. \tag{37}$$

Integrating the second and the first expression of (37), we can obtain

$$(\ln G(t))' = (\ln G(t_0))' - \int_{t_0}^{t} \frac{G''(\tau)G(\tau) - G'^2(\tau)}{G^2(\tau)}\mathrm{d}\tau,$$

$$\ln G(t) - \ln G(t_0) = \int_{t_0}^{t} (\ln G(\tau))'\,\mathrm{d}\tau \ge \frac{G'(t_0)}{G(t_0)}(t - t_0).$$

We can deduce

$$G(t) \geq G(t_0)e^{\frac{G'(t_0)}{G(t_0)}(t-t_0)}.$$

Then

$$\lim_{t\to\infty} G(t) = \infty,$$

which contradicts $T = \infty$. We completed the main proof of this theorem. □

Next, we give a lower estimation of the blow-up time t^*. First, we introduce the definition

$$H(t) = \int_\Omega a(y)|u|^p \,\mathrm{d}x - \mu \int_0^t \int_\Omega |\nabla u_\tau|^2 \,\mathrm{d}x\mathrm{d}\tau,$$

where μ is a positive constant. Moreover, $H(0) = \int_\Omega a(y)|u|^p \,\mathrm{d}x$.

Calculating $H'(t)$, we can obtain

$$H'(t) = p\int_\Omega a(y)|u|^{p-2}uu_t \mathrm{d}x - \mu\int_\Omega |\nabla u_\tau|^2 \,\mathrm{d}x. \tag{38}$$

Applying the Hölder inequality and the Sobolev embedding inequality, we have

$$\begin{aligned}
p\int_\Omega a(y)|u|^{p-2}uu_t \mathrm{d}x &\leq pM_1\int_\Omega |u|^{p-1}|u_t|\mathrm{d}x \\
&\leq pM_1\left(\int_\Omega |u|^{\frac{2(p-1)N}{N+1}}\,\mathrm{d}x\right)^{\frac{N+1}{2N}}\left(\int_\Omega |u_t|^{\frac{2N}{N-1}}\,\mathrm{d}x\right)^{\frac{N-1}{2N}} \\
&\leq pM_1\left(\int_\Omega |u|^{\frac{2(p-1)N}{N+1}}\,\mathrm{d}x\right)^{\frac{N+1}{2N}}\left(\int_\Omega |u_t|^2\,\mathrm{d}x\right)^{\frac{1}{2}} \\
&\leq \frac{8p^2C^2}{\mu}\left(\int_\Omega |u|^{\frac{2(p-1)N}{N+1}}\,\mathrm{d}x\right)^{\frac{N+1}{2N}} + \mu\int_\Omega |\nabla u_t|^2\,\mathrm{d}x.
\end{aligned}$$

Moreover,

$$\left(\int_\Omega |u|^{\frac{2(p-1)N}{N+1}}\,\mathrm{d}x\right)^{\frac{N+1}{2N}} \leq [\int_\Omega |u|^p\,\mathrm{d}x]^{\frac{2p-2}{p}}|\Omega|^{\frac{2}{p}+\frac{1}{2N}-1}. \tag{39}$$

Combining (38) and (39), we have

$$H'(t) \leq \frac{8p^2C^2|\Omega|^{\frac{2}{p}+\frac{1}{2N}-1}}{\mu}H^{\frac{2p-2}{p}}(t). \tag{40}$$

Integrating (40) with respect to t form 0 to t^*, we have

$$\int_0^{t^*} H'(t)H(t)^{\frac{2}{p}-2}(t)\mathrm{d}t \leq \frac{8p^2C^2|\Omega|^{\frac{2}{p}+\frac{1}{2N}-1}}{\mu}t^*.$$

Due to $\lim_{t\to t^*}\|u\|_{2p} = \infty$, $\lim_{t\to t^*}H(t) = \infty$, we have that

$$t^* \geq \frac{p\mu H^{\frac{2-p}{p}}(0)}{(p-2)8p^2C^2|\Omega|^{\frac{1}{p}+\frac{1}{2N}+1}}.$$

5 Global existence and finite time blow up when $E(0) = d$

In this section, we will use potential well method to prove global existence and finite time blow-up of the solution to system (1) at critical initial energy level $E(0) = d$.

Theorem 5.1. *Let $u_0 \in H_0^1(\Omega)$, $u_1 \in L^2(\Omega)$. If $E(0) = d$, $u_0 \in W_1$, then system* (1) *admits a global weak solution $u(x, t)$.*

Proof. First, let $\mu_m = 1 - \frac{1}{m}$, $u_{0m}(x) = \mu_m u_0(x)$, $u_{1m}(x) = \mu_m u_1(x)$, $m = 2, 3, \ldots$. Let us consider the initial conditions

$$u(x, 0) = u_{0m}(x), u_t(x, 0) = u_1(x) \tag{41}$$

and the corresponding system (1). Due to $u \in W_1$, we have $I(u_0) > 0$ or $u_0 = 0$. Next, we prove this theorem considering two cases (i) and (ii).

Case i: If $I(u_0) > 0$, it implies $I(u_{0m}) > 0$. Furthermore,

$$\begin{aligned} J(u_{0m}) &\geq \frac{1}{2}\|u_{0m}\|^2 + \frac{1}{2}\|\nabla u_{0m}\|^2 - \frac{1}{p}\left(\|u_{0m}\|_{L^p_{a(y)}}^p - \|u_{0m}\|_{L^q_{b(y)}}^q\right) \\ &= \frac{p-2}{2p}\|u_{0m}\|_{H^1}^2 + \frac{1}{p}I(u_{0m}) \\ &> 0 \end{aligned}$$

and

$$0 < E_m(0) \equiv \frac{1}{2}\|u_{0m}\|^2 + J(u_{0m}) < \frac{1}{2}\|u_1\|^2 + J(u_0) < d.$$

By Theorem 4.1, it follows systems (1) and (41) admit a global weak solution $u_m(x, t) \in L^\infty((0, \infty); H_0^1(\Omega))$, $u_{mt}(x, t) \in L^\infty((0, \infty); L^2(\Omega))$ and $u_m(x, t) \in W_1$ for every m. Similarly as the proof of Theorem 4.1, we can derive the conclusion.

Case ii: If $u_0 = 0$, it implies $J(u_0) = 0$ and $\frac{1}{2}\|u_1\|^2 = E(0) = d$. Moreover,

$$0 < E_m(0) = \frac{1}{2}\|u_{1m}(x)\|^2 + J(u_{0m}(x)) = \frac{1}{2}\|\mu_m u_1(x)\|^2 < d.$$

Through Theorem 4.1, for every m, systems (1) and (41) admit a global weak solution $u_m(x, t) \in L^\infty((0, \infty); H_0^1(\Omega))$, $u_{mt}(x, t) \in L^\infty((0, \infty); L^2(\Omega))$ and $u_m(x, t) \in W_1$. The remainder proof is similar to part (i) of this theorem. □

Next, to prove finite time blow up of the solution of system (1) under $E(0) = d$, we first present the following lemma.

Lemma 5.1. *Let $u_0 \in H_0^1(\Omega)$, $u_1 \in L^2(\Omega)$, $E(0) = d$ and $u_0 \in N_-$, then all the weak solutions of system* (1) *belong to N_-.*

Proof. Let $u(x, t)$ be any weak solution under critical initial energy $E(0) = d$ of system (1), which satisfies $u_0 \in N_-$. T is the maximum existence time of the solution $u(x, t)$.

Next, we apply reduction to absurdity to prove $u(x, t) \in N_-$, $0 < t < T$. Assume this was not the case, then there exists a time $t_0 \in (0, T)$, that is, $I(t_0) = 0$. Moreover, for any $t \in [0, t_0)$, $I(t) < 0$. Based on the value of potential well depth, we can obtain $J(t_0) \geq d$. Combine that with the energy (26) and (30), we have

$$\frac{1}{2}\|u_t\|^2 + J(u) + \int_0^t \|u\|^2 \mathrm{d}\tau = E(t) + \int_0^t \|u\|^2 \mathrm{d}\tau = E(0) = d.$$

Therefore, we can derive

$$\|u_t(t_0)\|^2 + \int_0^t \|u\|^2 \mathrm{d}\tau = 0.$$

From the aforementioned equation, for $x \in \Omega$, $0 \le t \le t_0$, we can obtain $\frac{\mathrm{d}u}{\mathrm{d}t} = 0$. Obviously, $u(x,t) = u_0(x)$, so that, $I(t_0) = I(u_0) > 0$. It is in contradiction to the previous assumption. Then the set N_- is invariant under the flow of system (1). □

Theorem 5.2. *Let $u_0 \in H_0^1(\Omega)$, $u_1 \in L^2(\Omega)$. If $E(0) = d$ and $u_0 \in N_-$, $(u_0, u_1) \ge 0$, then the local solution of system* (1) *blows up in finite time.*

Proof. Assume $u(x,t)$ is the solution for system (1) and satisfies $E(0) = d$, $I(u_0) < 0$. By the auxiliary function $G(t)$ as (31), then

$$G''(t) = 2\|u_t\|^2 + 2(u_{tt}, u_t) = 2(\|u_t\|^2 - I(u)).$$

From definition of $I(u)$, $J(u)$ and $E(t)$, we arrive at

$$\frac{1}{2}\|u_t\|^2 + \frac{p-1}{2p}\|u\|_{H^1}^2 + \frac{1}{p}I(u) \le \frac{1}{2}\|u_t\|^2 + J(u) = E(t) \le E(0) = d,$$

so that we have

$$\begin{aligned} G''(t) &> (p+2)\|u_t\|^2 + (p-1)\|u\|_{H^1}^2 - 2pd \\ &> (p+2)\|u_t\|^2 + (p-1)\|u\|^2 - 2pd \\ &= (p+2)\|u_t\|^2 + (p-1)G(t) - 2pd. \end{aligned}$$

According to Lemma 5.1, we know that $I(u) < 0$, then

$$G''(t) = 2(\|u_t\|^2 - I(u)) > 0, \quad 0 < t < \infty.$$

Moreover, thinking back to $G'(0) = 2(u_0, u_1) \ge 0$. It shows that $M'(t)$ is increasing for $t \in (0, \infty)$. We can obtain that $G'(t) \ge G'(t_0) \ge 0$ for $t > t_0$ and

$$G(t) \ge G'(t_0)(t - t_0) + G(t_0)(t - t) \ge G'(t_0)(t - t_0).$$

Hence, we can derive $(p-1)G(t) \ge 2pd$, then $G''(t) \ge (p+2)\|u_t\|^2$. Combining with (35) and applying the same as method of proof of Theorem 4.2, we can obtain the conclusion. □

6 Application

Now, we give the following example to show that lower estimate of the solution's blow-up time of system (1) relies on the conclusions of Theorem 4.2.

Example. Let $u \in C^2(\Omega \times (0,T))$ be a solution of the following system:

$$\begin{cases} u_{tt} - \Delta u + u = 2\sin y|u|^2 u - \cos y|u|u, \\ u(x,0) = |x|, \\ u_t(x,0) = 2 + |x|, \\ u(x,t)|_{\partial\Omega} = 0, \end{cases}$$

where $y \in (0, \pi)$, $x = (x_1, x_2, x_3, x_4) \in \Omega$, $|x| = \sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2}$, and $|\Omega| = \{x||x|=1\}$, $p = 4, q = 3, N = 4$, $\mu = 2$, $C^2 = 10$. Furthermore, we noted that

$$H(0) = \int_{\Omega} \sin y|x|^4 \, dx \leq 1,$$

so that, according to Theorem 4.2, we obtain a lower estimate of blows up time t^* as follows:

$$t^* \geq \frac{p\mu H^{\frac{2-p}{p}}(0)}{(p-2)8p^2C^2|\Omega|^{\frac{1}{p}+\frac{1}{2N}+1}} \geq 0.0125.$$

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References

[1] T. D. Lee, *Particle Physics and Introduction to Field Theory*, Harwood Academic Publishers, New York, 1981.
[2] P. J. Drazin and R. S. Johnson, *Solitons: An Introduction*, Cambridge University Press, Cambridge, 1989.
[3] T. Cazenave, *Uniform estimates for solutions of nonlinear Klein-Gordon equations*, J. Funct. Anal. **60** (1985), no. 1, 36–55, DOI: https://doi.org/10.1016/0022-1236(85)90057-6.
[4] F. E. Browder, *On nonlinear wave equations*, Math. Z. **80** (1962), 249–264, DOI: https://doi.org/10.1007/BF01162382.
[5] J. C. H. Simon, *A wave operator for a non-linear Klein-Gordon equation*, Lett. Math. Phys. **7** (1983), no. 5, 387–398, DOI: https://doi.org/10.1007/BF00398760.
[6] M. Nakao, *Remarks on the energy decay for nonlinear wave equations with nonlinear localized dissipative terms*, Nonlinear. Anal. **73** (2010), no. 7, 2158–2169, DOI: https://doi.org/10.1016/j.na.2010.05.042.
[7] Y. B. Yang and R. Z. Xu, *Finite time blowup for nonlinear Klein-Gordon equation with arbitrarily positive initial energy*, Appl. Math. Lett. **77** (2018), 21–26, DOI: https://doi.org/10.1016/j.aml.2017.09.014.
[8] M. O. Korpusov, A. N. Levashov, and D. V. Lukyanenko, *Analytical-numerical study of finite-time blow-up of the Solution to the initial-boundary value problem for the nonlinear Klein-Gordon equation*, Comput. Math. Math. Phys. **60** (2020), no. 1, 1452–1460, DOI: https://doi.org/10.1134/S0965542520090109.
[9] K. T. Li and Q. D. Zhang, *Existence and nonexistence of global solutions for the equation of dissociation of crystals*, J. Differential Equations **146** (1998), no. 1, 5–21, DOI: https://doi.org/10.1006/jdeq.1998.3409.
[10] Z. H. Gan, B. L. Guo, and J. Zhang, *Sharp threshold of global existence for the Klein-Gordon equations with critical nonlinearity*, Acta Math. Appl. Sin. Engl. Ser. **25** (2009), no. 2, 273–282, DOI: https://doi.org/10.1007/s10255-007-7085-7.
[11] J. Ginibre and G. Velo, *The global Cauchy problem for the nonlinear Klein-Gordon equation-II*, Ann. Inst. H. Poincare C Anal. Non Lineaire **6** (1989), no. 1, 15–35, DOI: https://doi.org/10.1016/S0294-1449(16)30329-8.
[12] J. Lu and Q. Miao, *Sharp threshold of global existence and blow-up of the combined nonlinear Klein-Gordon equation*, J. Math. Anal. Appl. **474** (2019), no. 2, 814–832, DOI: https://doi.org/10.1016/j.jmaa.2019.01.058.
[13] Y. J. Wang, *A sufficient condition for finite time blow up of the nonlinear Klein-Gordon equations with arbitrarily positive initial energy*, Proc. Amer. Math. Soc. **136** (2008), no. 10, 3477–3482, DOI: https://doi.org/10.1090/S0002-9939-08-09514-2.
[14] R. Z. Xu, *Global existence, blow up and asymptotic behavior of solutions for nonlinear Klein-Gordon equation with dissipative term*, Math. Methods Appl. Sci. **33** (2010), no. 7, 831–844, DOI: https://doi.org/10.1002/mma.1196.
[15] L. E. Payne and D. H. Sattinger, *Sadle points and instability of nonlinear hyperbolic equations*, Israel J. Math. **23** (1975), no. 3, 273–303, DOI: https://doi.org/10.1007/BF02761595.
[16] F. Gazzola and M. Squassina, *Global solutions and finite time blow up for damped semilinear wave equations*, Ann. Inst. H. Poincare C Anal. Non Lineaire **23** (2006), no. 2, 185–207, DOI: https://doi.org/10.1016/j.anihpc.2005.02.007.
[17] T. L. Chen and Y. X. Chen, *On family of potential wells and applications to semilinear hyperbolic equations and parabolic equations*, Nonlinear. Anal. **198** (2020), 1–19, DOI: https://doi.org/10.1016/j.na.2020.111898.

[18] Y. C. Liu and J. S. Zhao, *On potential wells and applications to semilinear hyperbolic equations and parabolic equation*, Nonlinear. Anal. **64** (2006), 2665–2687, DOI: https://doi.org/10.1016/j.na.2005.09.011.
[19] K. Ono, *On global existence, asymptotic stability and blowing up of solutions for some degenerate non-linear wave equations of Kirchhoff type with a strong dissipation*, Math. Methods Appl. Sci. **20** (1997), no. 2, 151–177.
[20] M. Willem, *Minimax theorems*, Progress in Nonlinear Differential Equations and Their Applications, vol. 24, Birkhäuser Boston, Boston, 1996.
[21] H. A. Levine, *Instability and nonexistence of global solutions to nonlinear wave equations of the form* $Pu_{tt} = -Au + F(u)$, Trans. Amer. Math. Soc. **192** (1974), 1–21, DOI: https://doi.org/10.1090/S0002-9947-1974-0344697-2.
[22] H. A. Levine, *Some additional remarks on the nonexistence of global solutions to nonlinear wave equations*, SIAM J. Math. Anal. **5** (1974), no. 1, 138–146, DOI: https://doi.org/10.1137/0505015.