Research Article

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Efficient fixed-point iteration for generalized nonexpansive mappings and its stability in Banach spaces

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Abstract: The aim of this article is to design a new iteration process for solving certain fixed-point problems. In particular, we prove weak and strong convergence theorems for generalized nonexpansive mappings in the framework of uniformly convex Banach spaces. In addition, we discuss the stability of the solution under mild conditions. Further, we provide some numerical examples to indicate that the proposed method works properly.

Keywords: generalized nonexpansive mapping, uniformly convex Banach space, iteration process, fixed point problem

MSC 2020: 47H09, 47H10

1 Introduction and preliminaries

In the last few decades, metric fixed-point theory is one of the hot topics for researchers in mathematics and applied sciences due to its wide application potential in nonlinear systems. The power of the metric fixed-point theory is to combine functional analysis, topology, and geometry, in a unique way. Accordingly, the problems in qualitative science (engineering, biology, chemistry, economics, technology, game theory, computer science, etc.) can be transformed and solved in the context of metric fixed-point theory. The pioneering work of the theory was announced by Banach in 1922, which guarantees both the existence and uniqueness of the fixed point. Indeed, it also shows a way to obtain the desired fixed point. Notice that finding a fixed point is equivalent to saying that the transferred real-world problem has a unique solution.

On the basis of this motivation, in the last few decades, several researchers have been investigating the existence (and if possible, the uniqueness) of a fixed point of distinct operators in the setting of various spaces. We emphasize that the existence of a fixed point and finding the existence fixed point are two different tasks. It is clear that the second task is more difficult one. For this reason, for finding a fixed point, several distinct iteration processes were defined and studied. Among all, we count the most interesting and useful iteration as follows: Mann iteration process [1], Ishikawa iteration process [2], K-iteration process [3],

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M^*-iteration process [4], K^*-iteration process [5], M-iteration process [6], J-iteration process [7], D-iteration process [8], and its error in [9]; see also, Agarwal et al. [10], Noor [11], Abbas and Nazir [12], and Ullah et al. [13].

Motivated by the aforementioned facts, in this article, we introduce a new iteration process, namely, D-plus iteration process. In addition, we prove its stability under suitable conditions. We present a comparison of the proposed iteration process with S-iteration process [10] and Picard-S iteration process [3]. We conclude that our method can outperform them in terms of number of iterations. Finally, we prove weak and strong convergence theorems for Suzuki generalized nonexpansive mappings in the setting of uniformly convex Banach spaces.

We next recall some useful definitions and basic concepts for this article.

Let M be a nonempty subset of a Banach space X and $F : M \to M$. We denote by Fix(F) the fixed-point set of F, that is, $\text{Fix}(F) = \{x \in M : Fx = x\}$. A mapping $F : M \to M$ is said to be a contraction if there exists $k \in (0, 1)$ such that for all $r, s \in M$, $\|Fr - Fs\| \le k\|r - s\|$. If $k = 1$, then F is called nonexpansive and quasi nonexpansive if for all $r \in M$ and $p \in \text{Fix}(F)$, $\|Fr - p\| \le \|r - p\|$. A mapping F is said to be generalized nonexpansive if for all $r, s \in X$,

$$1/2\|r - Fr\| \le \|r - s\| \Rightarrow \|Fr - Fs\| \le \|r - s\|.$$

Definition 1. (See, e.g., [14]) A Banach space X is called uniformly convex if for each $\varepsilon \in (0, 2]$ there exists $\delta > 0$ such that for $r, s \in X$ with $\|r\| \le 1$ and $\|s\| \le 1$, $\|r - s\| > \varepsilon$ implies $\|\frac{r+s}{2}\| \le \delta$.

Definition 2. (See, e.g., [14]) A mapping $F : M \to M$ is said to satisfy condition (C) if for all $\xi, \eta \in M$, we have

$$1/2\|\xi - F\xi\| \le \|\xi - \eta\| \Rightarrow \|F\xi - F\eta\| \le \|\xi - \eta\|.$$

Indeed, this notion of Suzuki [14] was improved in [15].

Definition 3. [16] A Banach space X is said to satisfy Opial's property [2] if for each sequence $\{\xi_n\}$ in X converging weakly to $\xi \in X$, we have

$$\limsup_{n\to\infty}\|\xi_n - \xi\| < \limsup_{n\to\infty}\|\xi_n - \eta\|$$

for all $\eta \in X$ such that $\eta \neq \xi$.

Lemma 1. (See, e.g., [[1], Proposition 3]). *Let M be a nonempty subset of a Banach space X and $F : M \to M$. Suppose that X satisfies Opial's property. Assume that F is a Suzuki generalized nonexpansive mapping. If $\{\xi_n\}$ converges weakly to t and* $\lim_{n\to\infty}\|F\xi_n - \xi_n\| = 0$, *then $F(t) = t$, that is, $I - F$ is demiclosed at zero.*

Lemma 2. ([1], Theorem 5). *Let M be a weakly compact convex subset of a uniformly convex Banach space X. Let $F : M \to M$. Assume that F is a Suzuki generalized nonexpansive mapping. Then F has a fixed point.*

Definition 4. [17] Let $\{r_n\}_{n=0}^{\infty}$ and $\{s_n\}_{n=0}^{\infty}$ be two sequences that converge to the same fixed point p and $\|r_n - p\| \le a_n$, and $\|s_n - p\| \le b_n$ for all $n \ge 0$. If the sequence $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ converge to a and b, respectively, and $\lim_{n\to\infty}\frac{\|a_n - a\|}{\|b_n - b\|} = 0$, then we say that $\{r_n\}_{n=0}^{\infty}$ converges to p faster than $\{s_n\}_{n=0}^{\infty}$.

Definition 5. [18] Let $\{u_n\}_{n=0}^{\infty}$ be a sequence in M. Then an iteration procedure $r_{n+1} = f(F, r_n)$ converging to a fixed point p is said to be F-stable or stable with respect to F, if for $\varepsilon_n = \|t_n + 1 - f(F : u_n)\|$, $n \in N$, we have $\lim_{n\to\infty}\varepsilon_n = 0$ if and only if $\lim_{n\to\infty}u_n = p$.

Lemma 3. [19] *Let $\{r_n\}_{n=0}^{\infty}$ and $\{t_n\}_{n=0}^{\infty}$ be nonnegative real sequences satisfying the relation $r_{n+1} \le (1 - t_n)r_n + t_n$, where $t_n \in (0, 1)$ for all $n \in N$, $\Sigma_{n=0}^{\infty}t_n = \infty$ and $\frac{r_n}{t_n} \to 0$ as $n \to \infty$. Then* $\lim_{n\to\infty}r_n = 0$.

Lemma 4. [20] *Suppose that X is a uniformly convex Banach space and let $\{u_n\}$ be real sequence such that $0 < p \le u_n \le q < 1$ for all $n \ge 1$. Let $\{r_n\}$ and $\{s_n\}$ be sequences in X such that $\limsup_{n\to\infty}\|r_n\| \le r$, $\limsup_{n\to\infty}\|s_n\| \le r$, and $\limsup_{n\to\infty}\|u_n r_n + (1-u_n)s_n)\| = r$ hold for some $r \ge 0$. Then $\lim_{n\to\infty}\|r_n - s_n\| = 0$.*

Proposition 1. (See, e.g., [14]) *Let M be a nonempty subset of a Banach space X and $F : M \to M$ be a mapping. Then*

(*i*) *If F is nonexpansive, then F is a Suzuki generalized nonexpansive mapping.*

(*ii*) *If F is a Suzuki generalized nonexpansive mapping and has a fixed point, then F is a quasi nonexpansive mapping.*

Also, the author in [14] proved the following lemma (see Lemma 7 in [14]).

Lemma 5. [14] *Let M be a nonempty subset of a Banach space X and $F : M \to M$ be a Suzuki generalized nonexpansive mapping. Then, for all $r, s \in X$, we have*

$$\|Fr - Fs\| \le 3\|Fr - r\| + \|r - s\|.$$

Let M be a nonempty closed convex subset of a Banach space X, and let $\{r_n\}$ be a bounded sequence in X. For $s \in X$, we set

$$r(s, \{r_n\}) = \limsup_{n\to\infty}\|r_n - s\|.$$

The asymptotic radius of $\{r_n\}$ relative to M is given by

$$r(M, \{r_n\}) = \inf\{r(s, \{r_n\}) : s \in M\},$$

and the asymptotic center of $\{r_n\}$ relative to M is the set

$$A(M, \{r_n\}) = \{s \in M : r(s, \{r_n\})\} = r(M, \{r_n\}).$$

It is known that, in a uniformly convex Banach space, $A(M, \{\xi_n\})$ consists of exactly one point.

Next we discuss the existing iterative process.

Throughout this section, we suppose that $\{\theta_n\}_{n=0}^{\infty}$, $\{\eta_n\}_{n=0}^{\infty}$ and $\{\vartheta_n\}_{n=0}^{\infty}$ are real sequences in $[0, 1]$ and C is a nonempty subset of Banach space X.

In 2016, the authors in [21] introduced a new iteration process as follows:

$$\begin{cases} r_0 \in C, \\ t_n = (1-\vartheta_n)r_n + \vartheta_n F r_n, \\ s_n = F((1-\theta_n)Fr_n + \theta_n F t_n), \\ r_{n+1} = F s_n. \end{cases} \tag{1}$$

Subsequently, the authors in [22] introduced a new iteration process as follows:

$$\begin{cases} r_0 \in C, \\ t_n = (1-\vartheta_n)r_n + \vartheta_n F r_n, \\ s_n = (1-\theta_n)t_n + \theta_n F t_n, \\ r_{n+1} = (1-\eta_n)F t_n + \eta_n F s_n. \end{cases} \tag{2}$$

In 2017, the authors in [4] introduced the following iteration process known as M^*-iteration process:

$$\begin{cases} r_0 \in C, \\ t_n = (1-\vartheta_n)r_n + \vartheta_n F r_n, \\ s_n = F((1-\theta_n)r_n + \theta_n F t_n), \\ r_{n+1} = F s_n. \end{cases} \tag{3}$$

Recently, in 2018, the authors in [3] introduced the following iteration process called K-iteration process and proved weak and strong convergence theorems for fixed points of Suzuki generalized nonexpansive mappings in the setting of uniformly convex Banach spaces.

$$\begin{cases} r_0 \in C, \\ t_n = (1-\vartheta_n)r_n + \vartheta_n F r_n, \\ s_n = F((1-\theta_n)Fr_n + \theta_n F t_n), \\ r_{n+1} = F s_n. \end{cases} \tag{4}$$

They have demonstrated that the K-iteration process converges faster than the S-iteration process, Picard-S iteration process, M-iteration process, and M^*-iteration process. In 2018, the author in [5] introduced the K^* iteration process and showed that K^* iteration process converges faster than Picard-S iteration process and S-iteration process.

$$\begin{cases} r_0 \in C, \\ t_n = (1-\vartheta_n)r_n + \vartheta_n F r_n, \\ s_n = F((1-\theta_n)t_n + \theta_n F t_n), \\ r_{n+1} = F s_n. \end{cases} \tag{5}$$

In the same year, the authors in [6] introduced M-iteration process as follows:

$$\begin{cases} r_0 \in C, \\ t_n = (1-\vartheta_n)r_n + \vartheta_n F r_n, \\ s_n = F t_n, \\ r_{n+1} = F s_n. \end{cases} \tag{6}$$

Recently, in 2019, the authors in [7] introduced the new iteration process called J-iteration process as follows:

$$\begin{cases} r_0 \in C, \\ t_n = F((1-\vartheta_n)r_n + \vartheta_n F r_n), \\ s_n = F((1-\theta_n)t_n + \theta_n F t_n), \\ r_{n+1} = F s_n. \end{cases} \tag{7}$$

By numerical examples, it was demonstrated that J-iteration process converges faster than some known iteration processes. They also discussed the stability of the proposed iteration and proved fixed-point results in the context of the uniformly convex Banach spaces for Suzuki generalized nonexpansive mappings. In 2021, the authors in [8] introduced a new iteration process, namely, D-iteration process as follows:

$$\begin{cases} \xi_0 \in C \\ \omega_n = F((1-\vartheta_n)\xi_n + \vartheta_n F \xi_n) \\ \eta_n = F((1-\theta_n)F\xi_n + \theta_n F \omega_n) \\ \xi_{n+1} = F\eta_n. \end{cases} \tag{8}$$

They proved that their iteration process (9) has a better convergence rate than (1), (4), (5), and (7). Furthermore, in [9], they proved that their D-iteration process is stable. Also they have proved the data dependency result and the error estimation for D-iteration process.

2 Main results

In this section, we present a new iteration process and analytically prove that it converges strongly to unique fixed point as well as stable and also prove that has better convergence rate than the existing iteration process.

First, we introduce a new iteration process called D-plus iteration process. It is defined as follows:

$$\begin{cases} r_0 \in C, \\ t_n = F((1-\vartheta_n)r_n + \vartheta_n F r_n), \\ s_n = F((1-\theta_n)t_n + \theta_n F t_n), \\ r_{n+1} = F((1-\eta_n)Ft_n + \eta_n F s_n). \end{cases} \tag{9}$$

We prove that D-plus iteration process converges faster than some existing iteration processes. We give some numerical experiments to show that the proposed iteration process has a better convergence rate than S-iteration process and Picard-S iteration process. It is shown that our iteration process is free from the selection of initial value. Its stability is also established under mild conditions.

In this section, we also generalized the strong convergence theorem "Theorem 2.1" for our iteration process, which shows that our iteration process strongly converge to unique fixed point. We also generalized some comparison result to represent that our iteration process is the fast convergent one.

Theorem 2.1. *Let C be a nonempty closed convex subset of a Banach space X and $F : C \to C$ be a contraction mapping. Let $\{r_n\}_{n=0}^{\infty}$ be a sequence generated by D-plus iteration process with real sequences $\{\theta_n\}_{n=0}^{\infty}$ and $\{\vartheta_n\}_{n=0}^{\infty}$ in $[0\ 1]$ satisfying $\Sigma_{n=0}^{\infty}\vartheta_n = \infty$ or $\Sigma_{n=0}^{\infty}\theta_n = \infty$. Then $\{r_n\}_{n=0}^{\infty}$ converges strongly to a unique fixed point of F.*

Proof. Since F is a contraction in a Banach space, and F has a unique fixed point in C. Let us suppose that p is a fixed point of F. So we obtain

$$\begin{aligned} \|t_n - p\| &= \|F((1-\vartheta_n)r_n + \vartheta_n F r_n) - Fp\| \\ &\le k\|(1-\vartheta_n)r_n + \vartheta_n F r_n - p\| \\ &\le k\|(1-\vartheta_n)(r_n - p) + \vartheta_n(Fr_n - p)\| \\ &\le k(1-\vartheta_n)\|(r_n - p)\| + \vartheta_n\|Fr_n - p\| \\ &\le k\{(1-\vartheta_n)\|r_n - p\| + k\vartheta_n\|r_n - p\|\} \\ &= k\{1 - \vartheta_n(1-k)\}\|r_n - p\|. \end{aligned}$$

Also we have

$$\begin{aligned} \|s_n - p\| &= \|F((1-\theta_n)t_n + \theta_n F t_n) - Fp\| \\ &\le k\|(1-\theta_n)t_n + \theta_n F t_n - p\| \\ &\le k\|(1-\theta_n)(t_n - p) + \theta_n(Ft_n - p)\| \\ &\le k(1-\theta_n)\|t_n - p\| + \theta_n\|Ft_n - p\| \\ &\le k\{(1-\theta_n)\|t_n - p\| + k\theta_n\|t_n - p\|\} = k\|t_n - p\| \\ &\le k^2\{1 - \vartheta_n(1-k)\}\|r_n - p\|. \end{aligned}$$

It follows that

$$\begin{aligned} \|r_{n+1} - p\| &= \|F((1-\eta_n)Ft_n + \eta_n F s_n) - Fp\| \\ &\le k[(1-\eta_n)\|Ft_n - p\| + \eta_n\|Fs_n - p\|] \\ &\le k[(1-\eta_n)k\|t_n - p\| + \eta_n k\|s_n - p\|] \\ &\le k^2[(1-\eta_n)\|t_n - p\| + \eta_n\|s_n - p\|] \\ &\le k^2[(1-\eta_n)\|t_n - p\| + k\eta_n\|t_n - p\|] \\ &= k^2\{1 - \eta_n(1-k)\}\|t_n - p\| \\ &\le k^2\{1 - \eta_n(1-k)\}\{k\{1 - \vartheta_n(1-k)\}\|r_n - p\|\} \\ &= k^3\{1 - \eta_n(1-k)\}\{1 - \vartheta_n(1-k)\}\|r_n - p\|. \end{aligned}$$

By repeating the aforementioned process, we obtain

$$
\begin{aligned}
\|r_n - p\| &\le k^3\{1 - \eta_{n-1}(1-k)\}\{1 - \vartheta_{n-1}(1-k)\}\|r_{n-1} - p\| \\
\|r_{n-1} - p\| &\le k^3\{1 - \eta_{n-2}(1-k)\}\{1 - \vartheta_{n-2}(1-k)\}\|r_{n-2} - p\| \\
\|r_{n-2} - p\| &\le k^3\{1 - \eta_{n-3}(1-k)\}\{1 - \vartheta_{n-3}(1-k)\}\|r_{n-3} - p\| \\
&\vdots \\
\|r_1 - p\| &\le k^3\{1 - \eta_0(1-k)\}\{1 - \vartheta_0(1-k)\}\|r_0 - p\|.
\end{aligned}
$$

Therefore, we obtain

$$\|r_{n+1} - p\| \le k^{3(n+1)}\|r_0 - p\|\prod_{i=0}^{n}\{1 - \eta_i(1-k)\}\{1 - \vartheta_i(1-k)\}.$$

Since $(1-k) > 0$ and $\vartheta_n \le 1$ for all $n \in N$. Therefore, we obtain $1 - \vartheta_n(1-k) < 1$ and $1 - \eta_n(1-k) < 1$ for all $n \in N$. We know that $1 - r \le e^{-r}$ for all $r \in [0, 1]$. So we have

$$\|r_{n+1} - p\| \le k^{3(n+1)}\|r_0 - p\|e^{-(1-k)\sum_{i=0}^{n}\vartheta_i\sum_{i=0}^{n}\vartheta_i}.$$

Thus, taking the limits $n \to \infty$ both sides, we obtain $\lim_{n\to\infty}\|r_n - p\| = 0$. □

Remark 1. From Theorem 2.1, by replacing the condition $\Sigma_{n=0}^{\infty}\vartheta_n = \infty$ by $\Sigma_{n=0}^{\infty}\theta_n = \infty$ and putting $\Sigma_{n=0}^{\infty}\eta_n = 0$, then $\|t_n - p\| \le k\|r_n - p\|$ and we obtain $\|s_n - p\| \le k^2\{1 - \theta_n(1-k)\}\|r_n - p\|$. Thus

$$\|r_{n+1} - p\| \le k^{3(n+1)}\|r_0 - p\|\prod_{i=0}^{n}\{1 - \theta_i(1-k)\}.$$

Therefore, we obtain the desired result.

Theorem 2.2. *Let M be a nonempty closed convex subset of a Banach space X and $F : M \to M$ be a contraction with a fixed point p. For a given $r_0 = u_0$, let $\{r_n\}_{n=0}^{\infty}$ and $\{u_n\}_{n=0}^{\infty}$ be a sequence generated by D-plus iteration process and K^*-iteration process as in* [5], *respectively, with real sequences $\{\theta_n\}_{n=0}^{\infty}$, $\{\vartheta_n\}_{n=0}^{\infty}$ and $\{\eta_n\}_{n=0}^{\infty}$ in $[0, 1]$ satisfying $\vartheta \le \vartheta_n < 1$ for some $\theta, \vartheta > 0$ and for all $n \in N$. Then $\{r_n\}_{n=0}^{\infty}$ converges to p faster than $\{u_n\}_{n=0}^{\infty}$.*

Proof. From inequality (10) of Theorem 3.2 in [5], we have

$$\|u_{n+1} - p\| \le k^{2(n+1)}\|u_0 - p\|\prod_{i=0}^{n}\{1 - \theta_i(1-k)\}.$$

Since $\theta \le \theta_n$ and for all $n \in N$, we obtain

$$\|u_{n+1} - p\| \le k^{2(n+1)}\|u_0 - p\|\{1 - \theta_i(1-k)\}^{n+1}.$$

Also, from Remark 1, we obtain

$$\|r_{n+1} - p\| \le k^{3(n+1)}\|r_0 - p\|\prod_{i=0}^{n}\{1 - \theta_i(1-k)\}.$$

Moreover, $\theta \le \theta_n$ for all $n \in N$ gives

$$\|r_{n+1} - p\| \le k^{3(n+1)}\|r_0 - p\|\{1 - \theta_i(1-k)\}^{n+1}.$$

So we have

$$a_n = k^{2(n+1)}\|u_0 - p\|\{1 - \theta_i(1-k)\}^{n+1}$$

and

$$b_n = k^{3(n+1)}\|r_0 - p\|\{1 - \theta_i(1-k)\}^{n+1}.$$

Then

$$\frac{b_n}{a_n} = \frac{k^{3(n+1)}\|r_0 - p\|\{1 - \theta_i(1-k)\}^{n+1}}{k^{2(n+1)}\|u_0 - p\|\{1 - \theta_i(1-k)\}} = k^{n+1}.$$

Thus, we obtain $\lim_{n\to\infty}\frac{b_n}{a_n} = 0$. Hence, the result follows. □

Now we prove that D-plus converges faster than K-iteration process [3].

Theorem 2.3. *Let M be a nonempty closed convex subset of a Banach space X and $F : M \to M$ be a contraction with a fixed point p. For a given $r_0 = u_0$, let $\{r_n\}_{n=0}^{\infty}$ and $\{u_n\}_{n=0}^{\infty}$ be a sequence generated by D-plus iteration process and K-iteration process* [3], *respectively, with real sequences $\{\theta_n\}_{n=0}^{\infty}$, $\{\vartheta_n\}_{n=0}^{\infty}$ and $\{\eta_n\}_{n=0}^{\infty}$ in* $[0, 1]$ *satisfying $\vartheta \le \vartheta_n < 1$ for some $\theta, \vartheta > 0$ and for all $n \in N$. Then $\{r_n\}_{n=0}^{\infty}$ converges to p faster than $\{u_n\}_{n=0}^{\infty}$.*

Proof. From Theorem 2.1, we have

$$\|r_{n+1} - p\| \le k^{3(n+1)}\|r_0 - p\|\{1 - \theta_i(1-k)\}^{n+1}.$$

Since $\theta \le \theta_n$ and for all $n \in N$, we obtain

$$\|r_{n+1} - p\| \le k^{3(n+1)}\|r_0 - p\|\{1 - \theta_i(1-k)\}^{n+1}.$$

Let $a_n = k^{3(n+1)}\|r_0 - p\|\{1 - \theta_i(1-k)\}^{n+1}$.

Now, from Theorem 3.2 in [3], we have

$$\|u_{n+1} - p\| \le k^{3(n+1)}\|u_0 - p\|\prod_{i=0}^{n}\{1 - \vartheta\theta_i(1-k)\}.$$

Since $\vartheta \le \vartheta_n$ and for all $n \in N$, we obtain

$$\|u_{n+1} - p\| \le k^{3(n+1)}\|u_0 - p\|\{1 - \vartheta\theta_i(1-k)\}^{n+1}.$$

Here, we define

$$b_n = k^{3(n+1)}\|u_0 - p\|\{1 - \vartheta\theta_i(1-k)\}^{n+1}.$$

Then

$$\frac{a_n}{b_n} = \frac{k^{3(n+1)}\|r_0 - p\|\{1 - \theta_i(1-k)\}^{n+1}}{k^{3(n+1)}\|(u_0 - p)\|\{1 - \vartheta\theta_i(1-k)\}^{n+1}} = \frac{\|r_0 - p\|\{1 - \theta_i(1-k)\}^{n+1}}{\|u_0 - p\|\{1 - \vartheta\theta_i(1-k)\}^{n+1}}.$$

Thus, taking limit as $n \to \infty$, we have $\lim_{n\to\infty}\frac{a_n}{b_n} = 0$. Hence, the result follows. □

Now we prove that D-plus converges faster than M-iteration process [6].

Theorem 2.4. *Let M be a nonempty closed convex subset of a Banach space X and $F : M \to M$ be a contraction with a fixed point p. For a given $r_0 = u_0$, let $\{r_n\}_{n=0}^{\infty}$ and $\{u_n\}_{n=0}^{\infty}$ be sequences generated by D-plus iteration process and M-iteration process* [6], *respectively, with real sequences $\{\theta_n\}_{n=0}^{\infty}$, $\{\vartheta_n\}_{n=0}^{\infty}$ and $\{\eta_n\}_{n=0}^{\infty}$ in* $[0, 1]$ *satisfying $\vartheta \le \vartheta_n < 1$ for some $\theta, \vartheta > 0$ and for all $n \in N$. Then $\{r_n\}_{n=0}^{\infty}$ converges to p faster than $\{u_n\}_{n=0}^{\infty}$.*

Proof. From Theorem 2.1, we have

$$\|r_{n+1} - p\| \le k^{3(n+1)}\|r_0 - p\|\{1 - \theta_i(1-k)\}^{n+1}.$$

Since $\theta \le \theta_n$ and for all $n \in N$, we obtain

$$\|r_{n+1} - p\| \le k^{3(n+1)}\|r_0 - p\|\{1 - \theta_i(1-k)\}^{n+1}.$$

We note that M-iteration is defined by

$$\begin{cases} u_0 \in C, \\ w_n = (1-\vartheta_n)u_n + \vartheta_n F u_n, \\ v_n = F w_n, \\ u_{n+1} = F v_n. \end{cases} \tag{10}$$

Then we have

$$\begin{aligned} \|w_n - p\| &= \|((1-\vartheta_n)u_n + \vartheta_n F u_n) - Fp\| \\ &\le \|(1-\vartheta_n)u_n + \vartheta_n F u_n - p\| \\ &\le \|(1-\vartheta_n)(u_n - p) + \vartheta_n(F u_n - p)\| \\ &\le (1-\vartheta_n)\|u_n - p\| + \vartheta_n\|F u_n - p\| \\ &\le (1-\vartheta_n)\|u_n - p\| + k\vartheta_n\|u_n - p\| \\ &= k\{1 - \vartheta_n(1-k)\}\|u_n - p\|. \end{aligned}$$

Now,

$$\|v_n - p\| \le \|F w_n - p\| \le k\|w_n - p\| \le k\{1 - \vartheta_n(1-k)\}\|u_n - p\|.$$

Therefore, we obtain

$$\|u_{n+1} - p\| \le \|F v_n - p\| \le k\|v_n - p\| \le k^2\{1 - \vartheta_n(1-k)\}\|u_n - p\|.$$

By repeating the aforementioned process, we obtain

$$\begin{aligned} \|u_n - p\| &\le k^2\{1 - \vartheta_{n-1}(1-k)\}\|u_{n-1} - p\| \\ \|u_{n-1} - p\| &\le k^2\{1 - \vartheta_{n-2}(1-k)\}\|u_{n-2} - p\| \\ \|u_{n-2} - p\| &\le k^2\{1 - \vartheta_{n-3}(1-k)\}\|u_{n-3} - p\| \\ &\vdots \\ \|u_1 - p\| &\le k^2\{1 - \vartheta_0(1-k)\}\|u_0 - p\|. \end{aligned}$$

Therefore, we obtain $\|u_{n+1} - p\| \le k^{2(n+1)}\|u_0 - p\|\prod_{i=0}^{n}\{1 - \vartheta_i(1-k)\}$.

Now, since $\vartheta \le \vartheta_n$ and for all $n \in N$, we obtain

$$\|u_{n+1} - p\| \le k^{2(n+1)}\|u_0 - p\|\{1 - \vartheta\theta_i(1-k)\}^{n+1}.$$

Let $b_n = k^{2(n+1)}\|u_0 - p\|\{1 - \vartheta\theta_i(1-k)\}^{n+1}$. Then, we have

$$\frac{a_n}{b_n} = \frac{\|r_{n+1} - p\| \le k^{3(n+1)}\|r_0 - p\|\{1 - \theta_i(1-k)\}^{n+1}}{k^{2(n+1)}\|u_0 - p\|\{1 - \vartheta\theta_i(1-k)\}^{n+1}}.$$

Thus, taking limit as $n \to \infty$, we obtain $\lim_{n\to\infty}\frac{a_n}{b_n} = 0$. Hence, the result follows. □

Next, we prove that D-plus converges faster than that of the M^*-iteration process [4]. Here, we consider the rate of convergence of M^*-iteration process under contraction and compare it with the D-plus iteration process.

Theorem 2.5. *Let M be a nonempty closed convex subset of a Banach space X and $F : M \to M$ be a contraction with a fixed point p. For a given $r_0 = u_0$, let $\{r_n\}_{n=0}^{\infty}$ and $\{u_n\}_{n=0}^{\infty}$ be sequences generated by D-plus iteration process and M^*-iteration process* [4], *respectively, with real sequences $\{\theta_n\}_{n=0}^{\infty}$, $\{\vartheta_n\}_{n=0}^{\infty}$ and $\{\eta_n\}_{n=0}^{\infty}$ in $[0, 1]$ satisfying $\vartheta \le \vartheta_n < 1$ for some $\theta, \vartheta > 0$ and for all $n \in N$. Then $\{r_n\}_{n=0}^{\infty}$ converges to p faster than $\{u_n\}_{n=0}^{\infty}$.*

Proof. From Theorem 2.1, we have

$$\|r_{n+1} - p\| \le k^{3(n+1)}\|r_0 - p\|\{1 - \theta_i(1-k)\}^{n+1}.$$

Since $\theta \leq \theta_n$ and for all $n \in N$, we obtain

$$\|r_{n+1} - p\| \leq k^{3(n+1)}\|r_0 - p\|\{1 - \theta_i(1 - k)\}^{n+1}.$$

We define

$$a_n = k^{3(n+1)}\|r_0 - p\|\{1 - \theta_i(1 - k)\}^{n+1}.$$

Note that M^*-iteration is defined by

$$\begin{cases} u_0 \in C, \\ w_n = (1 - \vartheta_n)u_n + \vartheta_n F u_n, \\ v_n = F((1 - \theta_n)u_n + \theta_n F w_n), \\ u_{n+1} = F v_n. \end{cases} \tag{11}$$

Then we have

$$\begin{aligned} \|w_n - p\| &= \|((1 - \vartheta_n)u_n + \vartheta_n F u_n) - Fp\| \\ &\leq \|(1 - \vartheta_n)u_n + \vartheta_n F u_n - p\| \\ &\leq \|(1 - \vartheta_n)(u_n - p) + \vartheta_n(F u_n - p)\| \\ &\leq (1 - \vartheta_n)\|u_n - p\| + \vartheta_n\|F u_n - p\| \\ &\leq (1 - \vartheta_n)\|u_n - p\| + k\vartheta_n\|u_n - p\| \\ &= \{1 - \vartheta_n(1 - k)\}\|u_n - p\|. \end{aligned}$$

Now

$$\begin{aligned} \|v_n - p\| &= \|F((1 - \theta_n)u_n + \theta_n F w_n) - Fp\| \\ &\leq k\|(1 - \theta_n)u_n + \theta_n F w_n - p\| \\ &\leq k\|(1 - \theta_n)(u_n - p) + \theta_n(F w_n - p)\| \\ &\leq k(1 - \theta_n)\|u_n - p\| + \theta_n\|F w_n - p\| \\ &\leq k\{(1 - \theta_n)\|u_n - p\| + k\theta_n\|w_n - p\|\} \\ &\leq k\{(1 - \theta_n)\|u_n - p\| + k\theta_n\{1 - \vartheta_n(1 - k)\}\|u_n - p\|\} \\ &= k\{(1 - \theta_n) + k\theta_n - k\theta_n\vartheta_n(1 - k)\}\|u_n - p\| \\ &= k\{1 - (1 - k)\theta_n - k\theta_n\vartheta_n(1 - k)\}\|u_n - p\| \\ &= k\{1 - \theta_n(1 - k)(1 - k\vartheta_n)\}\|u_n - p\|. \end{aligned}$$

It follows that

$$\|u_{n+1} - p\| \leq \|F v_n - p\| \leq k^2\{1 - \theta_n(1 - k)(1 - k\vartheta_n)\}\|u_n - p\|.$$

By repeating the aforementioned process, we obtain

$$\begin{aligned} \|u_n - p\| &\leq k^2\{1 - \theta_{n-1}(1 - k)(1 - k\vartheta_{n-1})\}\|u_{n-1} - p\| \\ \|u_{n-1} - p\| &\leq k^2\{1 - \theta_{n-2}(1 - k)(1 - k\vartheta_{n-2})\}\|u_{n-2} - p\| \\ \|u_{n-2} - p\| &\leq k^2\{1 - \theta_{n-3}(1 - k)(1 - k\vartheta_{n-3})\}\|u_{n-3} - p\| \\ &\vdots \\ \|u_1 - p\| &\leq k^2\{1 - \theta_0(1 - k)(1 - k\vartheta_0)\}\|u_0 - p\|. \end{aligned}$$

Therefore, we obtain

$$\|u_{n+1} - p\| \leq k^{2(n+1)}\|u_0 - p\|\prod_{i=0}^{n}\{1 - \theta_i(1 - k)(1 - k\vartheta_i)\}.$$

Now, since $\theta \leq \theta_n\vartheta \leq \vartheta_n$ and for all $n \in N$, we obtain

$$\|u_{n+1} - p\| \leq k^{2(n+1)}\|(u_0 - p)\|\{1 - \theta_i(1 - k)(1 - k\vartheta_i)\}^{n+1}.$$

Let $b_n = k^{2(n+1)}\|(u_0 - p)\|\{1 - \theta_i(1 - k)(1 - k\vartheta_i)\}^{n+1}$. Then

$$\frac{a_n}{b_n} = \frac{k^{3(n+1)}\|(r_0 - p)\|\{1 - \theta_i(1 - k)\}^{n+1}}{k^{2(n+1)}\|(u_0 - p)\|\{1 - \theta_i(1 - k)(1 - k\vartheta_i)\}^{n+1}} = \frac{k^{n+1}\{1 - \theta_i(1 - k)\}^{n+1}}{\{1 - \theta_i(1 - k)(1 - k\vartheta_i)\}^{n+1}}.$$

Thus, taking limit as $n \to \infty$, we obtain $\lim_{n\to\infty}\frac{a_n}{b_n} = 0$. Hence, the result follows. □

Next we prove that our new iteration D-plus is stable.

Theorem 2.6. *Let M be a nonempty closed convex subset of a Banach space X and $F : M \to M$ be a contraction. Let $\{r_n\}_{n=0}^{\infty}$ be a sequence generated by D-plus iteration process, with real sequences $\{\theta_n\}_{n=0}^{\infty}$, $\{\vartheta_n\}_{n=0}^{\infty}$ and $\{\eta_n\}_{n=0}^{\infty}$ in $[0, 1]$ satisfying $\sum_{n=0}^{\infty}\vartheta_n = \infty$. Then D-plus iterative process is stable.*

Proof. Let $\{r_n\}_{n=0}^{\infty}$ be a sequence in C. Suppose that the sequence generated by D-plus iteration process is defined by $r_{n+1} = f(F; r_n)$ converging to unique fixed point p (follows from Theorem 2.1). Set

$$\varepsilon_n = \|r_{n+1} - f(F; r_n)\|.$$

We will prove that $\lim_{n\to\infty}\varepsilon_n = 0$ if and only if $\lim_{n\to\infty}r_n = p$.

Let $\lim_{n\to\infty}\varepsilon_n = 0$. Then, we have

$$\|r_{n+1} - p\| \le \|r_{n+1} - f(F, r_n)\| + \|f(F, r_n) - p\| = \varepsilon_n + \|r_{n+1} - p\|.$$

From Theorem 2.1, we obtain

$$\|r_{n+1} - p\| \le k^3\{1 - \eta_n(1 - k)\}\{1 - \vartheta_n(1 - k)\}\|r_n - p\|.$$

Since $0 < k < 1$, $0 \le \eta_n \le 1$, and $0 \le \vartheta_n \le 1$ for all $n \in N$ and $\lim_{n\to\infty}\varepsilon_n = 0$, and by using Lemma 3, we obtain $\lim_{n\to\infty}\|r_n - p\| = 0$. Hence, $\lim_{n\to\infty}r_n = p$.

Conversely, let $\lim_{n\to\infty}r_n = p$. Then we have

$$\begin{aligned}\varepsilon_n = \|r_{n+1} - f(F, r_n)\| &\le \|r_{n+1} - p\| + \|f(F, r_n) - p\|,\\ &\le \|r_{n+1} - p\| + k^3\{1 - \eta_n(1 - k)\}\{1 - \vartheta_n(1 - k)\}\|x_n - p\|.\end{aligned}$$

Therefore, we have $\lim_{n\to\infty}\varepsilon_n = 0$. Hence, D-plus iteration process is stable. □

Remark 2. As after reading literature, there raise a question, is it possible to develop an iteration process that has better convergence rate? The main objective of this article is to present an iterative process that has better convergence rate and stable. To fulfil this aim, we attain the aforementioned mention result (Theorems 2.1–2.6).

Theorem 2.1 is the main result, which shows that our iterative process strongly converges to unique fixed point. Theorems 2.2–2.5 show the analytic comparison of our iteration process with existing iterative process. The last one result represents that our iteration process is stable.

In next section, we present weak and strong convergence result in the setting of uniformly convex Banach spaces.

3 Convergence analysis

Lemma 6. *Let M be a nonempty closed convex subset of a Banach space X, and let $F : M \to M$ be a mapping satisfying condition (C) with $\mathrm{Fix}(F) \ne \varnothing$. For arbitrary chosen $r_0 \in M$, let the sequence $\{r_n\}$ be generated by D-plus iteration process. Then $\lim_{n\to\infty}\|r_n - p\|$ exists for any $p \in \mathrm{Fix}(F)$.*

Proof. Let $p \in \mathrm{Fix}(F)$ and $t \in C$. Since F satisfies condition (C), it follows that $1/2\|p - Ft\| = 0 \le \|p - t\|$ implies $\|Fp - Ft\| \le \|p - t\|$. So by Proposition 1 (ii), we have

$$\begin{aligned}\|t_n - p\| &= \|(1-\vartheta_n)r_n + \vartheta_n Fr_n - p\| \\ &\le (1-\vartheta_n)\|r_n - p\| + \vartheta_n\|Fr_n - p\| \\ &\le (1-\vartheta_n)\|r_n - p\| + \vartheta_n\|r_n - p\| \\ &= \|r_n - p\|.\end{aligned}$$

It follows that

$$\begin{aligned}\|s_n - p\| &= \|F((1-\theta_n)t_n + \theta_n Ft_n) - p\| \\ &\le \|(1-\theta_n)Ft_n + \theta_n Ft_n - p\| \\ &\le (1-\theta_n)\|Ft_n - p\| + \theta_n\|Ft_n - p\| \\ &\le (1-\theta_n)\|t_n - p\| + \theta_n\|t_n - p\| \\ &= \|t_n - p\| \le \|r_n - p\|.\end{aligned}$$

Then

$$\begin{aligned}\|r_{n+1} - p\| &= \|F((1-\eta_n)Ft_n + \eta_n Fs_n) - Fp\| \\ &\le \|(1-\eta_n)(Ft_n - p) + \eta_n(Fs_n - p)\| \\ &\le [(1-\eta_n)\|Ft_n - p\| + \eta_n\|Fs_n - p\|] \\ &\le (1-\eta_n)\|t_n - p\| + \eta_n\|s_n - p\| \\ &\le (1-\eta_n)\|r_n - p\| + \eta_n\|r_n - p\| \\ &= \|r_n - p\|.\end{aligned}$$

This implies that $\{\|r_n - p\|\}$ is bounded and nonincreasing for all $p \in \mathrm{Fix}(F)$. Hence, $\lim_{n\to\infty}\|r_n - p\|$ exists as required. □

Theorem 3.1. *Let M be a nonempty closed convex subset of uniformly convex Banach space X, and let $F : M \to M$ be a mapping satisfying condition (C). For arbitrary chosen $r_0 \in M$, let the sequence $\{r_n\}$ be generated by D-plus iteration process for all $n \ge 1$ where $\{\theta_n\}$ and $\{\vartheta_n\}$ are real sequences in $[a, b]$ for some a, b with $0 < a \le b < 1$. Then* $\mathrm{Fix}(F) \ne \varnothing$ *if and only if $\{r_n\}$ is bounded and $\lim_{n\to\infty}\|Fr_n - r_n\| = 0$.*

Proof. Suppose $\mathrm{Fix}(F) \ne \varnothing$ and $p \in \mathrm{Fix}(F)$. Then, by Lemma 6, we have $\lim_{n\to\infty}\|r_n - p\|$ exists and $\{r_n\}$ is bounded. Let $\lim_{n\to\infty}\|r_n - p\| = r$. Then, by Lemma 6, we have

$$\limsup_{n\to\infty}\|t_n - p\| \le \limsup_{n\to\infty}\|r_n - p\| = r.$$

By Proposition 1 (ii), we obtain

$$\limsup_{n\to\infty}\|Tr_n - p\| \le \limsup_{n\to\infty}\|r_n - p\| = r.$$

On the other hand, we see that

$$\begin{aligned}\|r_{n+1} - p\| &= \|F((1-\eta_n)Ft_n + \eta_n Fs_n) - p\| \\ &= \|F((1-\eta_n)Ft_n + \eta_n Fs_n) - Fp\| \\ &\le \|(1-\eta_n)(Ft_n - p) + \eta_n(Fs_n - p)\| \\ &\le [(1-\eta_n)\|Ft_n - p\| + \eta_n\|Fs_n - p\|] \\ &\le (1-\eta_n)\|t_n - p\| + \eta_n\|s_n - p\| \\ &\le (1-\eta_n)\|r_n - p\| + \eta_n\|r_n - p\| \\ &= \|r_n - p\|.\end{aligned}$$

Therefore, $r \le \liminf_{n\to\infty}\|t_n - p\|$. Thus, we obtain

$$\begin{aligned}r &= \lim_{n\to\infty}\|t_n - p\| \\ &= \lim_{n\to\infty}\|(1-\vartheta_n)r_n + \vartheta_n Fr_n - p\| \\ &= \lim_{n\to\infty}\|\vartheta_n(Fr_n - p) + (1-\vartheta_n)(r_n - p)\|.\end{aligned}$$

Then, by using aforementioned inequalities and Lemma 4, we have

$$\lim_{n\to\infty} \|Fr_n - r_n\| = 0.$$

Conversely, suppose that $\{r_n\}$ is bounded and $\lim_{n\to\infty}\|Fr_n - r_n\| = 0$. Let $p \in A(M, \{r_n\})$. By Lemma 4, we have

$$r(Fp, \{r_n\}) = \limsup_{n\to\infty}\|r_n - Fp\| \le \limsup_{n\to\infty}(3\|Fr_n - r_n\| + \|r_n - p\|) \le \limsup_{n\to\infty}\|r_n - p\| = r(p, \{r_n\}).$$

This implies that $Fp \in A(C, \{r_n\})$. Since X is uniformly convex, $A(C, \{r_n\})$ is singleton, and hence, we have $Fp = p$. Hence, $\mathrm{Fix}(F) \neq \emptyset$.

Next, we prove strong and weak convergence results of sequences generated by D-plus iteration process for Suzuki generalized nonexpansive mappings in the setting of uniformly convex Banach spaces. □

Theorem 3.2. *Let M be a nonempty closed convex subset of a uniformly Banach space X, and let $F : M \to M$ be a mapping satisfying condition (C), where $\{\theta_n\}$ and $\{\vartheta_n\}$ are real sequences in $[a, b]$ for some a, b with $0 < a \le b < 1$. Such that $\mathrm{Fix}(F) \neq \emptyset$. Let X satisfy the Opial's property. For arbitrary chosen $r_0 \in M$, let the sequence $\{r_n\}$ be generated by D-plus iteration process for all $n \ge 1$. Then $\{r_n\}$ converges weakly to $p \in \mathrm{Fix}(F)$.*

Proof. Since $\mathrm{Fix}(F) \neq \emptyset$, so by Theorem 3.1, we have $\{r_n\}$ is bounded. So $\lim_{n\to\infty}\|Fr_n - r_n\| = 0$. Since X is uniformly convex hence reflexive, so by Eberlin's theorem, there exists a subsequence $\{r_{n_j}\}$ of $\{r_n\}$, which converges weakly to some $p_1 \in X$. Since M is closed and convex, by Mazur's theorem, $p_1 \in C$. By Lemma 2, we have $p_1 \in \mathrm{Fix}(F)$.

Now, we show that $\{r_n\}$ converges weakly to p_1. In fact, if this is not true, so there must exists a subsequence $\{r_{n_k}\}$ for $\{r_n\}$ such that $\{\xi_{n_k}\}$ converges weakly to $p_2 \in C$ and $p_2 \neq p_1$. By Lemma 2, $p_2 \in \mathrm{Fix}(F)$. Since $\lim_{n\to\infty}\|r_n - p\|$ exists for all $p \in \mathrm{Fix}(F)$. By Theorem 3.1 and Opial's property, we have

$$\begin{aligned}
\lim_{n\to\infty} \|r_n - p_1\| &= \lim_{j\to\infty} \|r_{n_j} - p_1\| \\
&< \lim_{j\to\infty} \|r_{n_j} - p_2\| \\
&= \lim_{n\to\infty} \|r_n - p_2\| \\
&= \lim_{k\to\infty} \|r_{n_k} - p_2\| \\
&< \lim_{k\to\infty} \|r_{n_k} - p_1\| \\
&= \lim_{n\to\infty} \|r_n - p_1\|,
\end{aligned}$$

which is contradiction. So $p_1 = p_2$. This implies that $\{r_n\}$ converges weakly to a fixed point of F. □

Theorem 3.3. *Let M be a nonempty closed convex subset of a uniformly Banach space X, and let $F : M \to M$ be a mapping satisfying condition (C) where $\{\theta_n\}$ and $\{\vartheta_n\}$ are real sequences in $[a, b]$ for some a, b with $0 < a \le b < 1$. Suppose that $\mathrm{Fix}(F) \neq \emptyset$. Let C be a nonempty convex subset of X. Then $\{r_n\}$ converges strongly to $p \in \mathrm{Fix}(F)$.*

Proof. By Lemma 2, we have $\mathrm{Fix}(F) \neq \emptyset$, and so, by Theorem 3.1, we obtain $\lim_{n\to\infty}\|Fr_n - r_n\| = 0$. Since M is compact, there exists a subsequence $\{r_{n_j}\}$ of $\{r_n\}$, which converges strongly to p for some $p \in C$. By Lemma 5, we have

$$\|r_{n_k} - Fp\| \le 3\|Fr_{n_k} - r_{n_k}\| + \|r_{n_k} - p\|, \quad \text{for all } n \ge 1.$$

Letting $k \to \infty$, we obtain $p \in \mathrm{Fix}(F)$. By Lemma 6, $\lim_{n\to\infty}\|r_n - p\|$ exists for every $p \in \mathrm{Fix}(F)$. So $\{r_n\}$ converges strongly to p. □

4 Numerical examples

In this section, we present a numerical example to support our analytic result of Section 2. First, we take a contraction map and calculate fixed point for it by using different iteration process. Graphically as well as with the help of table, we compare the calculation of our iteration process with the existing iteration process. Both "table and graphs" show the efficiency of our iteration process. As some of the iteration process of literature fails to converge at particular initial value. Their convergence depends on the selection of initial value. The objective of this article is to present the fastest convergent iterative method as well as its convergent independent from the selection of the initial value. In Example 2, we take different initial value for a contraction map in Example 1. Figures 1–4 show that either the initial value is above or below the fixed point, convergence of our iteration process does not effect.

Example 1. Let us define a function $F : R \to R$ by $F(r) = (r^2 - 8r + 40)^{\frac{1}{2}}$. Then clearly F is a contraction. Let $\theta_n = \frac{2n}{3n+1}$, $\vartheta_n = \frac{3n}{4n+5}$, and $\eta_n = \frac{4n}{5n+1}$. The initial value $r_0 = 40.5$ is given in Table 1. Figure 5 shows the convergence of iteration processes. The efficiency of D-plus iteration process is shown.

Table 1 presents that our iteration process is most efficient and fastest compared to the exiting iterative process of literature. We also represent efficiency of our iteration process graphically.

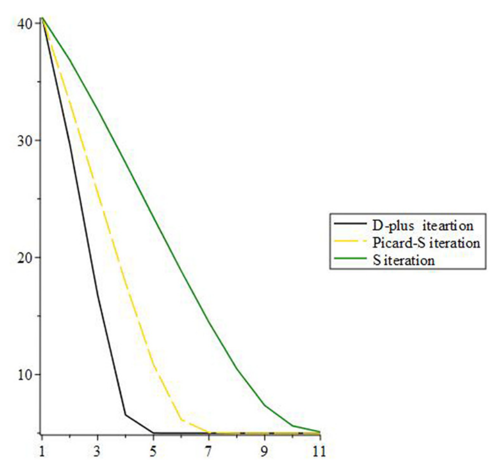

Figure 1: Convergence of D-plus iteration process when initial guess is 20.5.

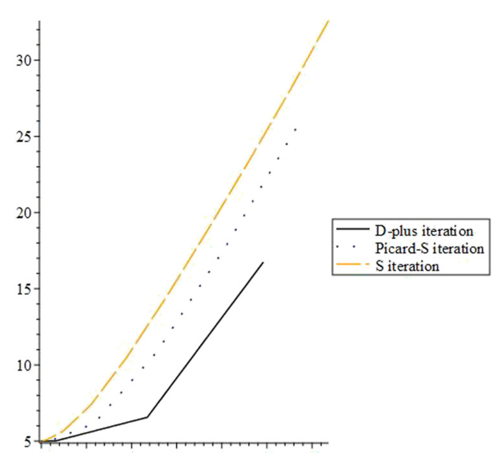

Figure 2: Convergence of D-plus iteration process when initial guess is 10.5.

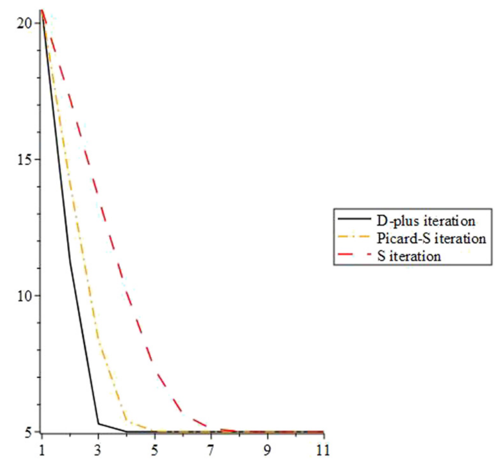

Figure 3: Convergence of D-plus iteration process when initial guess is 0.5.

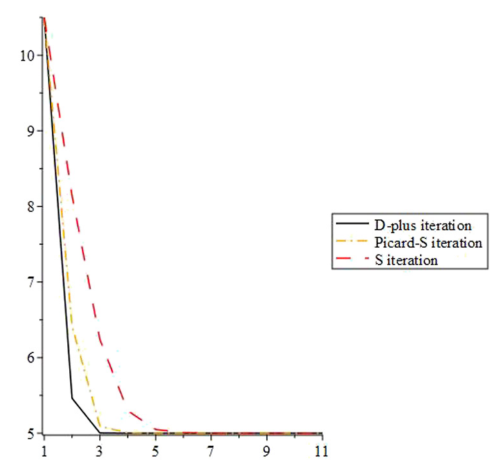

Figure 4: Convergence of D-plus iteration process when initial guess is −5.5.

Table 1: Convergence of D-plus, Picard-S, and S iteration processes in Example 1

	D-plus	Picard-S	S
r_0	40.5	40.5	40.5
r_1	29.599069	33.190836	36.827299
r_2	16.743028	25.484353	32.591647
r_3	6.5593682	17.820134	28.059416
r_4	5.0075393	10.863721	23.430599
r_5	5.0000166	6.1638489	18.843365
r_6	5	5.0537052	14.448288
r_7	5	5.0014824	10.481527
r_8	5	5.0000392	7.3683714
r_9	5	5.0000010	5.6315028
r_{10}	5	5.0000001	5.1036913

From Figures 5, 6, and Table 1, we can easily see that D-plus iteration process has a better convergence behavior than Picard-S and S-iteration processes.

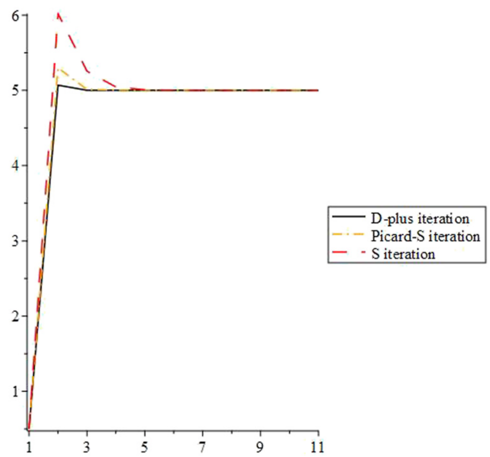

Figure 5: Convergence of D-plus, Picard-S and S-iteration processes in Example 1.

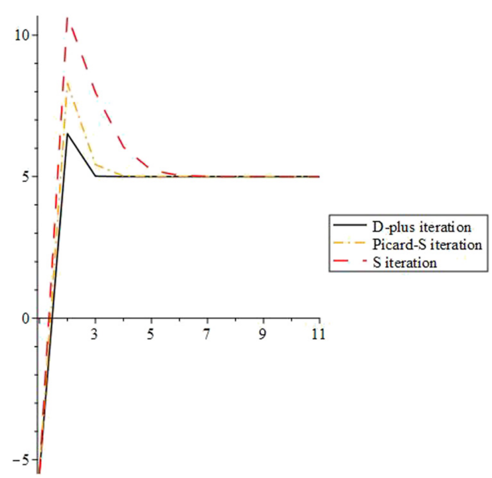

Figure 6: Convergence of D-plus, Picard-S and S-iteration processes in Example 1.

Hence, in Example 1, computationally as well as graphically, it is clear that D iteration process is more efficient then existing iterative process.

In the following example, we present graphical representation for different initial values of our iteration process.

Example 2. Let us define a function F as in Example 1. Let $r_0 = 20.5$, $r_0 = 10.5$, $r_0 = 0.5$, and $r_0 = -5.5$ be different initial values in D-plus iteration process in Figures 1–4, respectively.

On account of Figures 1–4, we can easily see that D-plus iteration process is independent from the selection of initial values.

5 Conclusion

In this article, we present a new instantly convergent iterative method to approximate fixed points of contractions. First, we have presented D-plus iteration process and then proved its convergence to a unique fixed point and its stability. Also, analytically and numerically showed that the proposed iteration process has a better convergence rate than some existing iteration processes defined in [3–8,15,21–23]. Furthermore, it was shown that the convergence of D-plus iteration process is independent from the choice of initial values.

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