Research Article

Bo Kong and Xiying Zheng*

Non-binary quantum codes from constacyclic codes over $\mathbb{F}_q[u_1, u_2, \ldots, u_k]/\langle u_i^3 = u_i, u_iu_j = u_ju_i\rangle$

https://doi.org/10.1515/math-2022-0459
received August 3, 2021; accepted June 24, 2022

Abstract: Let $q = p^m$, p be an odd prime, and $R_k = \mathbb{F}_q[u_1, u_2, \ldots, u_k]/\langle u_i^3 = u_i, u_iu_j = u_ju_i\rangle$, where $k \geq 1$ and $1 \leq i, j \leq k$. In this article, we define a Gray map from R_k^n to $\mathbb{F}_q^{3^k n}$. We study constacyclic codes over R_k and construct non-binary quantum codes over $\mathbb{F}_q$.

Keywords: constacyclic codes, quantum codes, Gray map, dual-containing codes

MSC 2020: 94B05, 94B15

1 Introduction

Recently, constructing quantum error-correcting codes has important significance in theory and practice. Calderbank et al. [1] gave a way to construct quantum error correcting codes from classical error correcting codes. Constacyclic codes that have good error-correcting properties are an important class of linear codes. Constacyclic codes also have rich algebraic structures that can be encoded with shift registers. Due to their rich algebraic structure, constacyclic codes over finite fields have been studied by many authors [2–4], and many good quantum codes have been constructed by using classical cyclic and constacyclic codes over finite fields [5–8].

In recent years, there are a lot of works about constacyclic codes over finite rings of the form $\mathbb{F}_{p^m} + u\mathbb{F}_{p^m} + \cdots + u^{e-1}\mathbb{F}_{p^m}$ by many authors [9–12], where $u^e = 0$. The class of finite commutative rings of the form $R + uR$ has been studied by many authors [13,14], where $u^2 = 1$. The class of finite commutative rings of the form $\mathbb{F}_{p^m}[u_1, u_2, \ldots, u_k]/\langle u_i^2 = u_i, u_iu_j = u_ju_i\rangle$ has been studied by many authors [15–18], where $u_i^2 = u_i$. Due to their rich algebraic structure, many good quantum codes have been constructed by using classical cyclic and negacyclic codes, and there are a few quantum codes constructed by using constacyclic codes over finite rings. Dertli and Cengellenmis [19] constructed quantum codes from constacyclic codes over ring $\mathbb{F}_p + u\mathbb{F}_p + v\mathbb{F}_p$ with $u^2 = u$, $v^2 = v$, and $uv = vu = 0$. Wang et al. [20] constructed non-binary quantum codes from $(1 - 2v)$-constacyclic codes over $\mathbb{F}_{q^2} + v\mathbb{F}_{q^2}$ with $v^2 = v$. Gowdhaman et al. [21] constructed quantum codes from λ-constacyclic codes over the ring $\frac{\mathbb{F}_p[u,v]}{\langle v^3 - v, u^3 - u, uv - vu\rangle}$. Li et al. [22] constructed quantum error correcting codes by Hermitian construction and obtained some good quantum codes. In [23], quantum codes from cyclic codes over $\mathbb{F}_2 + u\mathbb{F}_2 + v\mathbb{F}_2 + uv\mathbb{F}_2$ for arbitrary length n were constructed. In [24], the structure of cyclic codes over the ring $\mathbb{F}_q + v_1\mathbb{F}_q + \cdots + v_r\mathbb{F}_q$ was studied, and quantum codes from cyclic codes were constructed. Furthermore, some new non-binary quantum codes were obtained. In [25],

* **Corresponding author: Xiying Zheng,** Faculty of Engineering, Huanghe Science and Technology University, Zhengzhou 450063, China, e-mail: zxyccnu@163.com
Bo Kong: School of Statistics and Mathematics, Henan Finance University, Zhengzhou 450046, China, e-mail: kongbo666@163.com

quantum codes over $\mathbb{F}_p$ were constructed by using the cyclic codes of length n over $\mathbb{F}_p[u, v, w]/\langle u^2 - 1, v^2 - 1, w^2 - 1, uv - vu, vw - wv, wu - uw\rangle$. In [26], some non-binary quantum codes were obtained from $(1 - 2v)$-constacyclic codes over the finite non-chain ring $\mathbb{F}_q + u\mathbb{F}_q + v\mathbb{F}_q + uv\mathbb{F}_q$. Grassl et al. [27] presented families of non-binary quantum codes which were optimal in the sense that the minimum distance was maximal.

The purpose of this article is to continue this line of research. First, we determine the algebraic structures of all λ-constacyclic codes of $R_k = \mathbb{F}_q[u_1, u_2, \ldots, u_k]/\langle u_i^3 = u_i, u_iu_j = u_ju_i\rangle$. Second, we construct quantum codes from constacyclic codes over R_k.

The rest of this article is arranged as follows: In Section 2, we give some results of R_k and the definition of the Gray map from R_k^n to $\mathbb{F}_q^{3^kn}$. In Section 3, we discuss the algebraic structure of constacyclic codes over R_k. In Section 4, we give the parameters of quantum error correcting codes from constacyclic codes over R_k.

2 Preliminaries

Let $\mathbb{F}_q$ be a finite field with q elements, where p is an odd prime and $q = p^m$, and let

$$R_k = \mathbb{F}_q[u_1, u_2, \ldots, u_k]/\langle u_i^3 = u_i, u_iu_j = u_ju_i\rangle.$$

Clearly, R_k is a Frobenius ring but not local, and R_k has cardinality $q^{(3^k)}$.

Lemma 2.1. *Let $I = \langle w_1, w_2, \ldots, w_k\rangle$, where $w_i \in \left\{1 - u_i^2, \frac{u_i^2 + u_i}{2}, \frac{u_i^2 - u_i}{2}\right\}$, then I is an ideal of R_k, and the cardinality of I is $q^{(3^k-1)}$, and the number of such ideals is 3^k.*

Proof. The elements of I are of the form

$$\left\{\sum a_{i_1i_2\cdots i_k}w_1^{i_1}w_2^{i_2}\cdots w_k^{i_k} | a_{i_1i_2\cdots i_k} \in \mathbb{F}_q, i_s \in \{0, 1, 2\},\ s = 1, 2, \ldots, k\right\}.$$

It is easy to see that there are $3^k - 1$ choices of $w_1^{i_1}w_2^{i_2}\cdots w_k^{i_k}$. So, I has cardinality $q^{(3^k-1)}$. It is easy to see that the other ideals are isomorphic to I, for each w_i has three choices, so the number of such ideals is 3^k. □

Let $\varpi_i = \langle w_{i1}, w_{i2}, \ldots, w_{ik}\rangle$ be an ideal in Lemma 2.1, where $w_{ij} \in \left\{1 - u_j^2, \frac{u_j^2 + u_j}{2}, \frac{u_j^2 - u_j}{2}\right\}$, $1 \le i \le 3^k$, $1 \le j \le k$.

Let $\varsigma_i = w_{i1}w_{i2}\cdots w_{ik}$, where $w_{ij} \in \left\{1 - u_j^2, \frac{u_j^2 + u_j}{2}, \frac{u_j^2 - u_j}{2}\right\}$, $i = 1, 2, \ldots, 3^k$, $j = 1, 2, \ldots, k$. We can have that

$$(1 - u_j^2)^2 = 1 - u_j^2, \qquad \left(\frac{u_j^2 + u_j}{2}\right)^2 = \frac{u_j^2 + u_j}{2}, \quad \left(\frac{u_j^2 - u_j}{2}\right)^2 = \frac{u_j^2 - u_j}{2},$$

$$(1 - u_j^2)\left(\frac{u_j^2 + u_j}{2}\right) = 0, \quad (1 - u_j^2)\left(\frac{u_j^2 - u_j}{2}\right) = 0, \quad \left(\frac{u_j^2 - u_j}{2}\right)\left(\frac{u_j^2 + u_j}{2}\right) = 0,$$

$\varsigma_i\varsigma_j = 0$, when $i \ne j$, and $\varsigma_i^2 = \varsigma_i$, when $i = 1, 2, \ldots, 3^k$.

By the induction method over R_k, we can obtain that $1 = \varsigma_1 + \varsigma_2 + \cdots + \varsigma_{3^k}$ and $R_k = \varsigma_1R_k \oplus \varsigma_2R_k \oplus \cdots \oplus \varsigma_{3^k}R_k$. $\forall r \in R_k$, then r can be expressed uniquely as the form $r = r_1\varsigma_1 + r_2\varsigma_2 + \cdots + r_{3^k}\varsigma_{3^k}$, where $r_i \in \mathbb{F}_q$, $i = 1, 2, \ldots, 3^k$.

By the same method of Theorem 2.3 in [18], we have the following theorem.

Theorem 2.1. *$\varpi_1, \varpi_2, \ldots, \varpi_{3^k}$ are maximal ideals of R_k, and $R_k \cong \mathbb{F}_q^{3^k}$.*

By the aforementioned theorem, it can be easily seen that R_k is a principal ideal ring, not a chain ring.

For $a = a_1\varsigma_1 + a_2\varsigma_2 + \cdots + a_{3^k}\varsigma_{3^k} \in R_k$, we can define $\phi_k : R_k \to \mathbb{F}_q^{3^k}$ by $a \mapsto (a_1, a_2, \ldots, a_{3^k})$, and we expand ϕ_k as

$$\phi_k : R_k^n \to \mathbb{F}_q^{3^k n},$$

$$(a_0, a_1, \ldots, a_{n-1}) \mapsto \left(a_{1,0}, \ldots, a_{1,n-1}, a_{2,0}, \ldots, a_{2,n-1}, \ldots, a_{3^k,0}, \ldots, a_{3^k,n-1}\right),$$

where $a_i = a_{1,i}\varsigma_1 + a_{2,i}\varsigma_2 + \cdots + a_{3^k,i}\varsigma_{3^k} \in R_k$, $i = 1, 2, \ldots, 3^k$.

Let R_k^n be the R_k-submodule, if C is an R_k-submodule of R_k^n, then C is a linear code of length n over R_k. Every codeword $c = (c_0, c_1, \ldots, c_{n-1}) \in C$ can be represented as

$$c = (c_0, c_1, \ldots, c_{n-1}) \ \leftrightarrow \ c(x) = \sum_{i=0}^{n-1} c_i x^i \in R_k[x].$$

If C is invariant under constacyclic shift operator $\sigma_\lambda : R_k^n \to R_k^n$ by

$$\sigma_\lambda(c_0, c_1, \ldots, c_{n-1}) = (\lambda c_{n-1}, c_0, \ldots, c_{n-2}),$$

then C is called a λ-constacyclic code of length n over R_k.

Let $x = (x_0, x_1, \ldots, x_{n-1})$ and $y = (y_0, y_1, \ldots, y_{n-1}) \in R_k^n$. The Euclidean inner product of x and y is defined by $x \cdot y = \sum_{i=0}^{n-1} x_i y_i$. If $x \cdot y = 0$, then x and y are orthogonal.

If C is a linear code, the Euclidean dual code $C^\perp = \{x | \forall y \in C, x \cdot y = 0\}$ is a linear code too. A code C is Euclidean self-orthogonal if $C \subseteq C^\perp$, and Euclidean self-dual if $C = C^\perp$.

Let $w_H(\phi_k(r))$ denote the Hamming weight of the image of r under ϕ_k, $\forall r \in R_k$, the Lee weight of r is defined as $w_L(r) = w_H(\phi_k(r))$.

$\forall r = (x_1, x_2, \ldots, x_n) \in R_k^n$, the Lee weight of r is defined as $w_L(r) = \sum_{i=1}^n w_L(x_i)$, the Lee distance of codewords x, y over R_k^n is defined as $d_L(x, y) = w_L(x - y)$, and the Lee distance of C is defined as

$$d_L(C) = \min\{d_L(x - y), x, y \in C, x \neq y\}.$$

By the definition above, it is easy to see that ϕ_k is both a distance preserving map and a linear map from R_k^n to $\mathbb{F}_q^{3^k n}$.

3 Constacyclic codes over R_k

$\forall x = \sum_{j=1}^{3^k} x_j\varsigma_j, y = \sum_{j=1}^{3^k} y_j\varsigma_j \in R_k^n$, where $x_j = (x_{0j}, x_{1j}, \ldots, x_{n-1,j}) \in \mathbb{F}_q^n$, $y_j = (y_{0j}, y_{1j}, \ldots, y_{n-1,j}) \in \mathbb{F}_q^n$, we can have that

$$x \cdot y = \sum_{j=1}^{3^k} (x_j \cdot y_j)\varsigma_j.$$

Let C be a linear code of length n over R_k. Let

$$C_j = \left\{ x_j \in \mathbb{F}_q^n | \sum_{i=1}^{3^k} x_i\varsigma_i \in C, x_i \in \mathbb{F}_q^n, \right\}, \quad j = 1, 2, \ldots, 3^k.$$

Clearly, $C_1, C_2, \ldots, C_{3^k}$ are linear codes of length n over $\mathbb{F}_q$, and $C = \oplus_{j=1}^{3^k} \varsigma_j C_j$, $|C| = \prod_{j=1}^{3^k} |C_j|$.

Lemma 3.1. *Let $(\lambda_1\varsigma_1 + \lambda_2\varsigma_2 + \cdots + \lambda_{3^k}\varsigma_{3^k})$ be an element over R_k. Then $(\lambda_1\varsigma_1 + \lambda_2\varsigma_2 + \cdots + \lambda_{3^k}\varsigma_{3^k})$ is a unit over R_k if and only if $\lambda_1, \lambda_2, \ldots, \lambda_{3^k}$ are units over $\mathbb{F}_q$.*

Proof. Let $(\lambda_1\varsigma_1 + \lambda_2\varsigma_2 + \cdots + \lambda_{3^k}\varsigma_{3^k})$ be a unit of R_k, then there exist $\beta_1, \beta_2, \ldots, \beta_{3^k} \in \mathbb{F}_q$ such that

$$\left(\lambda_1\varsigma_1 + \lambda_2\varsigma_2 + \cdots + \lambda_{3^k}\varsigma_{3^k}\right)\left(\beta_1\varsigma_1 + \beta_2\varsigma_2 + \cdots + \beta_{3^k}\varsigma_{3^k}\right) = 1.$$

Then $(\lambda_1\beta_1\varsigma_1 + \lambda_2\beta_2\varsigma_2 + \cdots + \lambda_{3^k}\beta_{3^k}\varsigma_{3^k}) = 1$, which implies $(\lambda_1\beta_1\varsigma_1 + \lambda_2\beta_2\varsigma_2 + \cdots + \lambda_{3^k}\beta_{3^k}\varsigma_{3^k})\varsigma_i = \lambda_i\beta_i\varsigma_i = \varsigma_i$, so $\lambda_i\beta_i = 1$, where $i = 1, 2, \ldots, 3^k$. So $\lambda_1, \lambda_2, \ldots, \lambda_{3^k}$ are units over $\mathbb{F}_q$.

Conversely, if $\lambda_1, \lambda_2, \ldots, \lambda_{3^k}$ are units over $\mathbb{F}_q$, then there exist $\beta_1, \beta_2, \ldots, \beta_{3^k} \in \mathbb{F}_q$ such that $\lambda_i\beta_i = 1$, where $i = 1, 2, \ldots, 3^k$. We can obtain that $(\lambda_1\varsigma_1 + \lambda_2\varsigma_2 + \cdots + \lambda_{3^k}\varsigma_{3^k})(\beta_1\varsigma_1 + \beta_2\varsigma_2 + \cdots + \beta_{3^k}\varsigma_{3^k}) = \lambda_1\beta_1\varsigma_1 + \lambda_2\beta_2\varsigma_2 + \cdots + \lambda_{3^k}\beta_{3^k}\varsigma_{3^k} = \varsigma_1 + \varsigma_2 + \cdots + \varsigma_{3^k} = 1$. So $(\lambda_1\varsigma_1 + \lambda_2\varsigma_2 + \cdots + \lambda_{3^k}\varsigma_{3^k})$ is a unit over R_k. □

Theorem 3.1. *Let $C = \oplus_{j=1}^{3^k}\varsigma_j C_j$ be a linear code of length n over R_k, then C is a $(\lambda_1\varsigma_1 + \lambda_2\varsigma_2 + \cdots + \lambda_{3^k}\varsigma_{3^k})$-constacyclic code over R_k if and only if C_i is λ_i-constacyclic codes over $\mathbb{F}_q$, where $i = 1, 2, \ldots, 3^k$, and $\lambda_1\varsigma_1 + \lambda_2\varsigma_2 + \cdots + \lambda_{3^k}\varsigma_{3^k}$ is a unit of R_k.*

Proof. This can be proved by the same method of Theorem 4.1 in [18]. □

Theorem 3.2. *Let $C = \oplus_{j=1}^{3^k}\varsigma_j C_j$ be a $(\lambda_1\varsigma_1 + \lambda_2\varsigma_2 + \cdots + \lambda_{3^k}\varsigma_{3^k})$-constacyclic code of length n over R_k, then $C = \langle \varsigma_1 g_1(x) + \varsigma_2 g_2(x) + \cdots + \varsigma_{3^k} g_{3^k}(x)\rangle$, where g_i is the generator polynomial of C_i, $i = 1, 2, \ldots, 3^k$.*

Proof. This can be proved by the same method of Theorem 4.2 in [18]. □

Theorem 3.3. *Let $C = \langle \varsigma_1 g_1(x) + \varsigma_2 g_2(x) + \cdots + \varsigma_{3^k} g_{3^k}(x)\rangle$ be a $(\lambda_1\varsigma_1 + \lambda_2\varsigma_2 + \cdots + \lambda_{3^k}\varsigma_{3^k})$-constacyclic code of length n over R_k. Then $C^\perp = \langle \varsigma_1 f_1^*(x) + \varsigma_2 f_2^*(x) + \cdots + \varsigma_{3^k} f_{3^k}^*(x)\rangle$ is a $(\lambda_1^{-1}\varsigma_1 + \lambda_2^{-1}\varsigma_2 + \cdots + \lambda_{3^k}^{-1}\varsigma_{3^k})$-constacyclic code of length n over R_k, $|C^\perp| = q^{\left(\sum_{i=1}^{3^k}\deg(g_i)\right)}$, where $f_i^*(x)$ is the reciprocal polynomial of $f_i(x)$, i.e., $f_i(x) = (x^n - \lambda_i)/g_i(x)$, $f_i^*(x) = x^{\deg(f_i)}f_i(x^{-1})$, for $i = 1, 2, \ldots, 3^k$.*

Proof. Let $C = \langle \varsigma_1 g_1(x) + \varsigma_2 g_2(x) + \cdots + \varsigma_{3^k} g_{3^k}(x)\rangle$ be a λ-constacyclic code of length n over R_k, where $\lambda = \lambda_1\varsigma_1 + \lambda_2\varsigma_2 + \cdots + \lambda_{3^k}\varsigma_{3^k}$.

$\forall x = (x_0, x_1, \ldots, x_{n-1}) \in C^\perp$, $\forall y = (y_0, y_1, \ldots, y_{n-1}) \in C$, then $\sigma_\lambda^{n-1}(y) = (\lambda y_1, \lambda y_2, \ldots, \lambda y_{n-1}, y_0) \in C$, we can obtain that

$$\begin{aligned} 0 &= x \cdot \sigma_\lambda^{n-1}(y) \\ &= \lambda x_0 y_1 + \lambda x_1 y_2 + \cdots + \lambda x_{n-2} y_{n-1} + x_{n-1} y_0 \\ &= \lambda(x_0 y_1 + x_1 y_2 + \cdots + x_{n-2} y_{n-1} + \lambda^{-1} x_{n-1} y_0) \\ &= \lambda \sigma_{\lambda^{-1}}(x) \cdot y. \end{aligned}$$

We have $\sigma_{\lambda^{-1}}(x) \in C^\perp$, so $C^\perp$ is a λ^{-1}-constacyclic code.

By Lemma 3.1, $\lambda^{-1} = \lambda_1^{-1}\varsigma_1 + \lambda_2^{-1}\varsigma_2 + \cdots + \lambda_{3^k}^{-1}\varsigma_{3^k}$, so $C^\perp$ is a $(\lambda_1^{-1}\varsigma_1 + \lambda_2^{-1}\varsigma_2 + \cdots + \lambda_{3^k}^{-1}\varsigma_{3^k})$-constacyclic code of length n over R_k.

Let $D_i = \langle f_i^*(x)\rangle$, where $f_i^*(x)$ is the reciprocal polynomial of $f_i(x)$, i.e., $f_i(x) = (x^n - \lambda_i)/g_i(x)$, $f_i^*(x) = x^{\deg(f_i)}f_i(x^{-1})$, g_i is the generator polynomial of C_i, $i = 1, 2, \ldots, 3^k$.

Let $D = \oplus_{j=1}^{3^k}\varsigma_j D_j$, then $|D| = \prod_{j=1}^{3^k}|D_j| = q^{(\sum_{i=1}^{3^k}\deg(g_i))}$, and it follows that D has the form

$$D = \langle \varsigma_1 f_1^*(x), \varsigma_2 f_2^*(x), \ldots, \varsigma_{3^k} f_{3^k}^*(x)\rangle.$$

Let $\tilde{D} = \langle \varsigma_1 f_1^*(x) + \varsigma_2 f_2^*(x) + \cdots + \varsigma_{3^k} f_{3^k}^*(x)\rangle$. We can have that $\tilde{D} \subseteq D$.

Note that

$$\varsigma_i\left[\varsigma_1 f_1^*(x) + \varsigma_2 f_2^*(x) + \cdots + \varsigma_{3^k} f_{3^k}^*(x)\right] = \varsigma_i f_i^*(x),$$

where $i = 1, 2, \ldots, 3^k$.

We can obtain that $D \subseteq \tilde{D}$. So, $\tilde{D} = D$, and $D = \langle \varsigma_1 f_1^*(x) + \varsigma_2 f_2^*(x) + \cdots + \varsigma_{3^k} f_{3^k}^*(x) \rangle$.

Note that

$$\begin{aligned}
&\left(\varsigma_1 f_1^*(x) + \varsigma_2 f_2^*(x) + \cdots + \varsigma_{3^k} f_{3^k}^*(x)\right)\left(\varsigma_1 g_1^*(x) + \varsigma_2 g_2^*(x) + \cdots + \varsigma_{3^k} g_{3^k}^*(x)\right) \\
&= \varsigma_1 f_1^*(x) g_1^*(x) + \varsigma_2 f_2^*(x) g_2^*(x) + \cdots + \varsigma_{3^k} f_{3^k}^*(x) g_{3^k}^*(x) \\
&= \varsigma_1(1 - x^n \lambda_1) + \varsigma_2(1 - x^n \lambda_2) + \cdots + \varsigma_{3^k}\left(1 - x^n \lambda_{3^k}\right) \\
&= 1 - x^n\left(\lambda_1 \varsigma_1 + \lambda_2 \varsigma_2 + \cdots + \lambda_{3^k} \varsigma_{3^k}\right) \\
&= -\left(x^n - \left(\lambda_1^{-1} \varsigma_1 + \lambda_2^{-1} \varsigma_2 + \cdots + \lambda_{3^k}^{-1} \varsigma_{3^k}\right)\right)\left(\lambda_1 \varsigma_1 + \lambda_2 \varsigma_2 + \cdots + \lambda_{3^k} \varsigma_{3^k}\right).
\end{aligned}$$

So, $(\varsigma_1 f_1^*(x) + \varsigma_2 f_2^*(x) + \cdots + \varsigma_{3^k} f_{3^k}^*(x)) | (x^n - (\lambda_1^{-1} \varsigma_1 + \lambda_2^{-1} \varsigma_2 + \cdots + \lambda_{3^k}^{-1} \varsigma_{3^k}))$, we can obtain that $D \subseteq C^\perp$.

For R_k is a Frobenius ring, $|C||C^\perp| = |R_k|^n$, hence $|C^\perp| = \frac{|R_k|^n}{|C|} = q^{(\sum_{i=1}^{3^k} \deg(g_i))} = |D|$.

So, $C^\perp = D = \langle \varsigma_1 f_1^*(x) + \varsigma_2 f_2^*(x) + \cdots + \varsigma_{3^k} f_{3^k}^*(x) \rangle$. □

4 Quantum codes from constacyclic codes over R_k

Theorem 4.1. *Let $C = \oplus_{j=1}^{3^k} \varsigma_j C_j$ be a linear code of length n over R_k, then $C^\perp = \sum_{j=1}^{3^k} \varsigma_j C_j^\perp$, where $C_j^\perp$ is a Euclidean dual code of C_j, where $j = 1, 2, \ldots, 3^k$.*

Proof. Let $\tilde{C} = \oplus_{j=1}^{3^k} \varsigma_j C_j^\perp$. $\forall x = \sum_{j=1}^{3^k} \varsigma_j x_j \in C$, $\tilde{x} = \sum_{j=1}^{3^k} \varsigma_j \tilde{x}_j \in \tilde{C}$, $x \cdot \tilde{x} = \sum_{j=1}^{3^k} (x_j \tilde{x}_j) \varsigma_j$, where $x_j \in C_j$, and $\tilde{x}_j \in C_j^\perp$. We can have $x \cdot \tilde{x} = 0$, so $\tilde{C} \subseteq C^\perp$.

For R_k is a Frobenius ring, $|C||C^\perp| = |R_k|^n$. Hence,

$$|\tilde{C}| = \prod_{j=1}^{3^k} |C_j^\perp| = \prod_{j=1}^{3^k} \frac{p^n}{|C_j|} = \frac{|R_k|^n}{|C|} = |C^\perp|.$$

It follows that

$$C^\perp = \tilde{C}.$$ □

Theorem 4.2. *Let $C = \oplus_{j=1}^{3^k} \varsigma_j C_j$ be a linear code of length n over R_k, then C is a Euclidean self-orthogonal code over R_k if and only if C_j is a Euclidean self-orthogonal code over $\mathbb{F}_q$, where $j = 1, 2, \ldots, 3^k$.*

Proof. By Theorem 4.1, $C \subseteq C^\perp$ if and only if $C_j \subseteq C_j^\perp$, so if C is a Euclidean self-orthogonal code over R_k, then C_j is a Euclidean self-orthogonal code over $\mathbb{F}_q$, where $j = 1, 2, \ldots, 3^k$. □

Let C be a linear code of length n over R_k, for any $c = c_1 \varsigma_1 + c_2 \varsigma_2 + \cdots + c_{3^k} \varsigma_{3^k} \in C$, and $\phi_k(c) = (c_1, c_2, \ldots, c_{3^k}) \in \mathbb{F}_q^{3^k n}$.

Theorem 4.3. *Let $C = \oplus_{j=1}^{3^k} \varsigma_j C_j$ be a linear code of length n over R_k with $|C| = q^l$ and the minimum Lee distance $d_L(C) = d$. Then $\phi_k(C)$ is a linear code with parameter $[3^k n, l, d]$. If C is a Euclidean self-orthogonal code over R_k, then $\phi_k(C)$ is a Euclidean self-orthogonal code over $\mathbb{F}_q$.*

Proof. By the definition of Gray map ϕ_k, we can know that $\phi_k(C)$ is a linear code with parameter $[3^k n, l, d]$.

Let C be a Euclidean self-orthogonal code, $x = \sum_{j=1}^{3^k} \varsigma_j x_j \in C$, and $y = \sum_{j=1}^{3^k} \varsigma_j y_j \in C$, where $x_j, y_j \in \mathbb{F}_q^n$, then $x \cdot y = \sum_{j=1}^{3^k} \varsigma_j x_j y_j = 0$, which implies that $x_j y_j = 0$, so

$$\phi_k(x)\cdot\phi_k(y)=\sum_{j=1}^{3^k}x_jy_j=0.$$

So $\phi_k(C)$ is a Euclidean self-orthogonal code over $\mathbb{F}_q$. □

Lemma 4.1. [26] *Let C be a constacyclic code with generator polynomial $g(x)$ over $\mathbb{F}_q$. Then, C contains its dual code if and only if $x^n-\lambda\equiv 0(\mathrm{mod}\ g(x)g^*(x))$, where $g^*(x)$ is the reciprocal polynomial of $g(x)$, $\lambda=\pm1$.*

Theorem 4.4. [27] *Let C_1 and C_2 be $[n,k_1,d_1]q$ and $[n,k_2,d_2]q$ linear codes over $\mathbb{F}_q$, with $C_2^{\perp}\subseteq C_1^{\perp}$. Furthermore, let $d=\min(d_1,d_2)$. Then, there exists a quantum error-correcting code C with parameters $C=[n,k_1+k_2-n,\geq d]_q$. In particular, if $C_1^{\perp}\subseteq C_1$, then there exists a quantum error-correcting code C with parameters $C=[n,2k_1-n,\geq d_1]$.*

Theorem 4.5. *Let $C=\oplus_{j=1}^{3^k}\varsigma_jC_j$ be a $(\lambda_1\varsigma_1+\lambda_2\varsigma_2+\cdots+\lambda_{3^k}\varsigma_{3^k})$-constacyclic code over R_k, where $g_i(x)$ is the generator of polynomial of C_i. Then $C^{\perp}\subseteq C$ if and only if $C_i^{\perp}\subseteq C_i$ for $i=1,2,\ldots,3^k$.*

Proof. If $C_i^{\perp}\subseteq C_i$ for $i=1,2,\ldots,3^k$, then $\varsigma_iC_i^{\perp}\subseteq\varsigma_iC_i$, $\varsigma_1C_1^{\perp}\oplus\varsigma_2C_2^{\perp}\oplus\cdots\oplus\varsigma_{3^k}C_{3^k}^{\perp}\subseteq\varsigma_1C_1\oplus\varsigma_2C_2\oplus\cdots\oplus\varsigma_{3^k}C_{3^k}$. So $C^{\perp}\subseteq C$. Conversely, if $C^{\perp}\subseteq C$, then, $\varsigma_1C_1^{\perp}\oplus\varsigma_2C_2^{\perp}\oplus\cdots\oplus\varsigma_{3^k}C_{3^k}^{\perp}\subseteq\varsigma_1C_1\oplus\varsigma_2C_2\oplus\cdots\oplus\varsigma_{3^k}C_{3^k}$, so, $C_i^{\perp}\subseteq C_i$ for $i=1,2,\ldots,3^k$. □

According to Lemma 4.1 and Theorem 4.5, we obtain the following corollary directly.

Corollary 4.1. *Let $C=\oplus_{j=1}^{3^k}\varsigma_jC_j$ be a $(\lambda_1\varsigma_1+\lambda_2\varsigma_2+\cdots+\lambda_{3^k}\varsigma_{3^k})$-constacyclic code over R_k. Then $C^{\perp}\subseteq C$ if and only if C_i is λ_i-constacyclic codes over $\mathbb{F}_q$, where $\lambda_i=\pm1$, $i=1,2,\ldots,3^k$.*

Using Theorems 4.3–4.5, we can construct quantum codes.

Theorem 4.6. *Let $C=\oplus_{j=1}^{3^k}\varsigma_jC_j$ be a $(\lambda_1\varsigma_1+\lambda_2\varsigma_2+\cdots+\lambda_{3^k}\varsigma_{3^k})$-constacyclic code over R_k. If C_i is λ_i-constacyclic code over $\mathbb{F}_q$, where $\lambda_i=\pm1$, $i=1,2,\ldots,3^k$, then $C^{\perp}\subseteq C$ and there exists a quantum error-correcting code with parameters $[3^kn,2l-3^kn,\geq d_L]_q$, where d_L is the minimum Lee weight of code C and l is the dimension of the linear code $\phi_k(C)$.*

Example 4.1. Let $n=30$ and $R_2=\mathbb{F}_5[u_1,u_2]/\langle u_i^3=u_i,u_iu_j=u_ju_i\rangle$, $\varsigma_1=(1-u_1^2)(1-u_2^2)$, $\varsigma_2=\frac{(1-u_1^2)(u_2^2+u_2)}{2}$, $\varsigma_3=\frac{(1-u_1^2)(u_2^2-u_2)}{2}$, $\varsigma_4=\frac{(1-u_2^2)(u_1^2+u_1)}{2}$, $\varsigma_5=\frac{(u_1^2+u_1)(u_2^2+u_2)}{4}$, $\varsigma_6=\frac{(u_1^2+u_1)(u_2^2-u_2)}{4}$, $\varsigma_7=\frac{(1-u_2^2)(u_1^2-u_1)}{2}$, $\varsigma_8=\frac{(u_1^2-u_1)(u_2^2+u_2)}{4}$, $\varsigma_9=\frac{(u_1^2-u_1)(u_2^2-u_2)}{4}$.

$x^{30}+1=(x+2)^5(x+3)^5(x^2+2x+4)^5(x^2+3x+4)^5$,
$x^{30}-1=(x+1)^5(x+4)^5(x^2+x+1)^5(x^2+4x+1)^5$ in $\mathbb{F}_5(x)$.

Let C be a $(1-2u_1^2u_2^2)$-constacyclic code of length 30 over $R_2=\mathbb{F}_5[u_1,u_2]/\langle u_i^3=u_i,u_iu_j=u_ju_i\rangle$. Let $g(x)=\varsigma_1g_1+\varsigma_2g_2+\varsigma_3g_3+\varsigma_4g_4+\varsigma_5g_5+\varsigma_6g_6+\varsigma_7g_7+\varsigma_8g_8+\varsigma_9g_9$ be the generator polynomial of C, where $g_1=g_2=g_3=x+1$, $g_4=g_7=x+4$, $g_5=g_6=x+2$, and $g_8=g_9=x+3$. By Theorem 4.5, we have $C^{\perp}\subseteq C$, $\phi_2(C)$ is a linear code over $\mathbb{F}_5$ with parameters [270, 261, 2]. By Theorem 4.6, we obtain a quantum error-correcting code with parameters $[270,252,\geq2]_5$.

Example 4.2. Let $n=15$ and $R_2=\mathbb{F}_7[u_1,u_2]/\langle u_i^3=u_i,u_iu_j=u_ju_i\rangle$, $\varsigma_1=(1-u_1^2)(1-u_2^2)$, $\varsigma_2=\frac{(1-u_1^2)(u_2^2+u_2)}{2}$, $\varsigma_3=\frac{(1-u_1^2)(u_2^2-u_2)}{2}$, $\varsigma_4=\frac{(1-u_2^2)(u_1^2+u_1)}{2}$, $\varsigma_5=\frac{(u_1^2+u_1)(u_2^2+u_2)}{4}$, $\varsigma_6=\frac{(u_1^2+u_1)(u_2^2-u_2)}{4}$, $\varsigma_7=\frac{(1-u_2^2)(u_1^2-u_1)}{2}$, $\varsigma_8=\frac{(u_1^2-u_1)(u_2^2+u_2)}{4}$, $\varsigma_9=\frac{(u_1^2-u_1)(u_2^2-u_2)}{4}$.

$x^{15}-1=(x+3)(x+5)(x+6)(x^4+x^3+x^2+x+1)(x^4+2x^3+4x^2+x+2)(x^4+4x^3+2x^2+x+4)$,
$x^{15}+1=(x+1)(x+2)(x+4)(x^4+3x^3+2x^2+6x+4)(x^4+5x^3+4x^2+6x+2)(x^4+6x^3+x^2+6x+1)$.

Let C be a $(1-2u_1^2u_2^2)$-constacyclic code of length 15 over $R_2=\mathbb{F}_7[u_1,u_2]/\langle u_i^3=u_i, u_iu_j=u_ju_i\rangle$. Let $g_1=x^4+x^3+x^2+x+1$, $g_2=g_3=x^4+2x^3+4x^2+x+2$, $g_4=g_7=x^4+4x^3+2x^2+x+4$, $g_5=x^4+3x^3+2x^2+6x+4$, $g_6=x^4+5x^3+4x^2+6x+2$, and $g_8=g_9=x^4+6x^3+x^2+6x+1$. By Theorem 4.5, we have $C^\perp\subseteq C$, $\phi_2(C)$ is a linear code over $\mathbb{F}_7$ with parameters $[135, 99, 4]$. By Theorem 4.6, we obtain a quantum error-correcting code with parameters $[135, 63, \geq 4]_7$.

Example 4.3. Let $n=21$ and $R_1=\mathbb{F}_7[u_1]/\langle u_1^3=u_1\rangle$, $\varsigma_1=(1-u_1^2)$, $\varsigma_2=\frac{(u_1^2-u_1)}{2}$, $\varsigma_3=\frac{(u_1^2+u_1)}{2}$.

$x^{21}-1=(x+3)^7(x+5)^7(x+6)^7$, $x^{21}+1=(x+1)^7(x+2)^7(x+4)^7$.

Let C be a $(1-2u_1^2)$-constacyclic code of length 15 over R_1. $g_1=x+3$, $g_2=x+2$, $g_3=x+4$. By Theorem 4.5, we have $C^\perp\subseteq C$, $\phi_1(C)$ is a linear code over $\mathbb{F}_7$ with parameters $[63, 60, 2]$. By Theorem 4.6, we obtain a quantum error-correcting code with parameters $[63, 57, \geq 2]_7$.

Example 4.4. Let $n=20$ and $R_2=\mathbb{F}_5[u_1,u_2]/\langle u_i^3=u_i, u_iu_j=u_ju_i\rangle$, $\varsigma_1=(1-u_1^2)(1-u_2^2)$, $\varsigma_2=\frac{(1-u_1^2)(u_2^2+u_2)}{2}$, $\varsigma_3=\frac{(1-u_1^2)(u_2^2-u_2)}{2}$, $\varsigma_4=\frac{(1-u_2^2)(u_1^2+u_1)}{2}$, $\varsigma_5=\frac{(u_1^2+u_1)(u_2^2+u_2)}{4}$, $\varsigma_6=\frac{(u_1^2+u_1)(u_2^2-u_2)}{4}$, $\varsigma_7=\frac{(1-u_2^2)(u_1^2-u_1)}{2}$, $\varsigma_8=\frac{(u_1^2-u_1)(u_2^2+u_2)}{4}$, and $\varsigma_9=\frac{(u_1^2-u_1)(u_2^2-u_2)}{4}$.

$x^{20}-1=(x-1)^4(x-2)^5(x-3)^5(x-1)^5$, $x^{20}+1=(x^2-3)^5(x^2-2)^5$.

Let C be a $(1-2u_1^2-2u_2^2-2u_1^2u_2^2)$-constacyclic code of length 20 over R_2. $g_1=(x-3)^2$, $g_2=g_3=\cdots=g_9=x^2-3$. By Theorem 4.5, we have $C^\perp\subseteq C$, $\phi_2(C)$ is a linear code over $\mathbb{F}_5$ with parameters $[180, 162, 3]$. By Theorem 4.6, we obtain a quantum error-correcting code with parameters $[180, 144, \geq 3]_5$.

5 Conclusion

In this article, by studying the structure of constacyclic codes over $R_k=\mathbb{F}_q[u_1,u_2,\ldots,u_k]/\langle u_i^3=u_i, u_iu_j=u_ju_i\rangle$, we construct some non-binary quantum codes from constacyclic codes over the finite non-chain ring R_k.

Acknowledgements: The authors would like to thank the referees and the editor for their careful reading of the article and for valuable comments and suggestions, which improved the presentation of this manuscript.

Funding information: This work was supported by the Key Technologies Research and Development Program of Henan Province (No. 212102210573), the Key Scientific Research Project Plan of Henan Province colleges and universities (No. 20A110015), and Zhengzhou Special Fund for Basic Research and applied basic research (No. ZZSZX202111).

Author contributions: The authors applied the SDC approach for the sequence of authors.

Conflict of interest: The authors declare no conflict of interest.

References

[1] A. R. Calderbank, E. M. Rains, P. M. Shor, and N. J. A. Sloane, *Quantum error correction via codes over GF(4)*, IEEE Trans. Inform. Theory **44** (1998), 1369–1387, DOI: https://doi.org/10.1109/ISIT.1997.613213.

[2] B. Chen, H. Q. Dinh, and H. Liu, *Repeated-root constacyclic codes of length $2l^mp^n$*, Finite Fields Appl. **33** (2015), 137–159, DOI: https://doi.org/10.1016/j.ffa.2014.11.006.

[3] B. Chen, Y. Fan, L. Lin, and H. Liu, *Constacyclic codes over finite fields*, Finite Fields Appl. **18** (2012), 1217–1231, DOI: https://doi.org/10.1016/j.ffa.2012.10.001.

[4] H. Q. Dinh, *Repeated-root constacyclic codes of length* $2p^s$, Finite Fields Appl. **18** (2012), 133–143, DOI: https://doi.org/10.1016/j.ffa.2011.07.003.

[5] L. Wang and S. Zhu, *New quantum MDS codes derived from constacyclic codes*, Quantum Inf. Process. **14** (2015), 881–889, DOI: https://doi.org/10.1007/s11128-014-0903-y.

[6] X. Hu, G. Zhang, and B. Chen, *Constructions of new nonbinary quantum codes*, Int. J. Theor. Phys. **54** (2015), 92–99, DOI: https://doi.org/10.1007/s10773-014-2204-8.

[7] X. Kai, S. Zhu, and P. Li, *Constacyclic codes and some new quantum MDS codes*, IEEE Trans. Inform. Theory **60** (2014), 2080–2085, DOI: https://doi.org/10.1109/TIT.2014.2308180.

[8] R. Li, and Z. Xu, *Construction of* $[[n, n-4, 3]]_q$ *quantum codes for odd prime* q, Phys. Rev. A. **82** (2010), 052316, DOI: https://doi.org/10.1103/PhysRevA.82.052316.

[9] B. Chen, H. Q. Dinh, H. Liu, and L. Wang, *Constacyclic codes of length* $2p^s$ *over* $\mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$, Finite Fields Appl. **37** (2016), 108–130, DOI: https://doi.org/10.1016/j.ffa.2015.09.006.

[10] R. Sobhani, *Complete classification of* $(\delta + \alpha u^2)$ *-constacyclic codes of length* p^k *over* $\mathbb{F}_{p^m} + u\mathbb{F}_{p^m} + u^2\mathbb{F}_{p^m}$, Finite Fields Appl. **34** (2015), 123–138, DOI: https://doi.org/10.1016/j.ffa.2015.01.008.

[11] X. Liu and X. Xu, *Cyclic and negacyclic codes of length* $2p^s$ *over* $\mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$, Acta Math. Sci. **34** (2014), 829–839.

[12] X. Kai, S. Zhu, and P. Li, $(1 + \lambda u)$*-constacyclic codes over* $\mathbb{F}_p[u]\langle u^m\rangle$, J. Franklin Inst. **347** (2010), 751–762, DOI: https://doi.org/10.1016/j.jfranklin.2010.02.003.

[13] Y. Cengellenmis, *On the cyclic codes over* $\mathbb{F}_3 + v\mathbb{F}_3$, Int. J. Algebra **6** (2010), 253–259, http://www.m-hikari.com/ija/ija-2010/ija-5-8-2010/index.html.

[14] M. Özen, F. Z. Uzekmek, N. Aydin, and N. T. Özzaim, *Cyclic and some constacyclic codes over the ring* $\mathbb{Z}_4[u]\langle u^2 - 1\rangle$, Finite Fields Appl. **38** (2016), 27–39, DOI: https://doi.org/10.1016/j.ffa.2015.12.003.

[15] Y. Cengellenmis, A. Dertli, and S. T. Dougherty, *Codes over an infinite family of rings with a Gray map*, Des. Codes Cryptogr. **72** (2014), 559–580, DOI: https://doi.org/10.1007/s10623-012-9787-y.

[16] P. Li, X. Guo, S. Zhu, and X. Kai, *Some results on linear codes over the ring* $\mathbb{Z}_4 + u\mathbb{Z}_4 + v\mathbb{Z}_4 + uv\mathbb{Z}_4$, J. Appl. Math. Comput. **54** (2017), 307–324, DOI: https://doi.org/10.1007/s12190-016-1011-1.

[17] X. Zheng and B. Kong, *Cyclic codes and* $\lambda_1 + \lambda_2 u + \lambda_3 v + \lambda_4 uv$ *-constacyclic codes over* $\mathbb{F}_p + u\mathbb{F}_p + v\mathbb{F}_p + uv\mathbb{F}_p$, Appl. Math. Comput. **306** (2017), 86–91, DOI: https://doi.org/10.1016/j.amc.2017.02.017.

[18] X. Zheng and B. Kong, *Constacyclic codes over* $\mathbb{F}_{p^m}[u_1, u_2, \ldots, u_k]\langle u_i^2 = u_i, u_iu_j = u_ju_i\rangle$, Open Math. **16** (2018), 490–497, DOI: https://doi.org/10.1515/math-2018-0045.

[19] A. Dertli and Y. Cengellenmis, *Quantum codes obtained from some constacyclic codes over a family of finite rings* $\mathbb{F}_P + u\mathbb{F}_p + v\mathbb{F}_p$, Math. Comput. Sci. **14** (2020), 437–441, DOI: https://doi.org/10.1007/s11786-019-00426-3.

[20] Y. Wang, X. Kai, Z. Sun, and S. Zhu, *Quantum codes from Hermitian dual-containing constacyclic codes over* $\mathbb{F}_{q^2} + v\mathbb{F}_{q^2}$, Quantum Inf. Process. **20** (2021), 122, DOI: https://doi.org/10.1007/s11128-021-03052-w.

[21] K. Gowdhaman, C. Mohan, D. Chinnapillai, and J. Gao, *Construction of quantum codes from* λ *-constacyclic codes over the ring* $\frac{\mathbb{F}_p[u,v]}{<v^3-v, u^3-u, uv-vu>}$, J. Appl. Math. Comput. **65** (2021), 611–622.

[22] J. Li, J. Gao, F. Fu, and F. Ma, $\mathbb{F}_q R$*-linear skew constacyclic codes and their application of constructing quantum codes*, Quantum Inf. Process **19** (2020), 193, DOI: https://doi.org/10.1007/s11128-020-02700-x.

[23] A. Dertli, Y. Cengellenmis, and S. Eren, *On quantum codes obtained from cyclic codes over* A_2, Int. J. Quantum Inf. **13** (2015), 1550031, DOI: https://doi.org/10.1142/S0219749915500318.

[24] Y. Gao, J. Gao, and F. Fu, *Quantum codes from cyclic codes over the ring* $\mathbb{F}_q + v_1\mathbb{F}_q + \cdots + v_r\mathbb{F}_q$, Appl. Algebra Engrg. Comm. Comput. **30** (2019), 161–174, DOI: https://doi.org/10.1007/s00200-018-0366-y.

[25] H. Islam and O. Prakash, *Quantum codes from the cyclic codes over* $\mathbb{F}_p[u, v, w]\langle u^2 - 1, v^2 - 1, w^2 - 1, uv - vu, vw - wv, wu - uw\rangle$, J. Appl. Math. Comput. **60** (2019), 625–635, DOI: https://doi.org/10.1007/s12190-018-01230-1.

[26] J. Li, J. Gao, and Y. Wang, *Quantum codes from* $(1 - 2v)$ *-constacyclic codes over the ring* $\mathbb{F}_q + u\mathbb{F}_q + v\mathbb{F}_q + uv\mathbb{F}_q$, Discrete Math. Algorithms Appl. **10** (2018), 1850046, DOI: https://doi.org/10.1142/S1793830918500465.

[27] M. Grassl, T. Beth, and M. Rötteler, *On optimal quantum codes*, Int. J. Quantum Inf. **02** (2004), 1550031, DOI: https://doi.org/10.1142/S0219749904000079.