Research Article

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Shooting method in the application of boundary value problems for differential equations with sign-changing weight function

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Abstract: In this paper, we use the shooting method to study the solvability of the boundary value problem of differential equations with sign-changing weight function:

$$\begin{cases} u''(t) + (\lambda a^{+}(t) - \mu a^{-}(t))g(u) = 0, \quad 0 < t < T, \\ u'(0) = 0, \quad u'(T) = 0, \end{cases}$$

where $a \in L[0, T]$ is sign-changing and the nonlinearity $g : [0, \infty) \to \mathbb{R}$ is continuous such that $g(0) = g(1) = g(2) = 0$, $g(s) > 0$ for $s \in (0, 1)$, $g(s) < 0$ for $s \in (1, 2)$.

Keywords: boundary value problem, sign-changing weight function, positive solutions, shooting method

MSC 2020: 34B15, 34B18

1 Introduction and main result

In this paper, we are interested in the multiplicity of positive solutions for the boundary value problem:

$$\begin{cases} u''(t) + a(t)g(u) = 0, \quad 0 < t < T, \\ u'(0) = 0, \quad u'(T) = 0, \end{cases} \tag{1.1}$$

where $a \in L[0, T]$ changes sign. Boundary value problem (1.1) describes many phenomena in applied mathematics. For example, the theory of nonlinear diffusion generated by nonlinear sources, biological models, and nuclear physics, where only positive solutions are meaningful, see [1–3].

Existence and multiplicity of positive solutions of (1.1) with a sign-changing weight function have been extensively studied, see [4,5]. In [6], the authors established multiplicity results of positive solutions with Dirichlet boundary conditions in relation to the nodal behavior of the weight $a(t)$. In [7], the authors further studied the influence of weight function to the problem (1.1) by defining the weight function as follows:

$$a(t) = a_{\lambda\mu}(t) := \lambda a^{+}(t) - \mu a^{-}(t),$$

where $a^{+}(t)$ and $a^{-}(t)$ denote the positive and the negative part of the function $a(t), \lambda > 0, \mu > 0$. They obtained the following multiplicity result:

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Theorem A. (Theorem 1.1, [7]) *Let* $g : [0,1] \to \mathbb{R}^+$ *be a locally Lipschitz continuous function satisfying*

$$g(0) = g(1) = 0, \lim_{s \to 0^+} \frac{g(s)}{s} = 0, \qquad (H_0)$$

and the weight term $a(t)$ *has two positive humps separated by a negative hump. Then, there exists* $\lambda_1 > 0$ *such that for each* $\lambda > \lambda_1$, *and there exists* $\mu_1(\lambda) > 0$ *such that for every* $\mu > \mu_1(\lambda)$, *problem*

$$\begin{cases} u''(t) + (\lambda a^+(t) - \mu a^-(t))g(u) = 0, & 0 < t < T, \\ u'(0) = 0, \quad u'(T) = 0 \end{cases} \qquad (1.2)$$

has least three positive solutions $u(t)$ *and* $0 < u(t) < 1$ *for all* $t \in [0, T]$.

A natural question that arises from the aforementioned quoted papers is whether the number of positive solutions to the problem (1.1) is related to the number of zeros of $g(s)$. For that reason, we would like to pursue further the investigation of the dynamical effects produced by the nonlinear term $g(s)$. Of course, this idea also has practical significance. For example, see [8,9], the classical application in population genetics

$$\begin{cases} u_t = d\Delta u + g(x)f(u), & \text{in } \Omega \times (0, \infty), \\ \partial_\nu u = 0, & \text{on } \partial\Omega \times (0, \infty), \end{cases} \qquad (1.3)$$

where $\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ is the Laplace operator, Ω is a bounded domain with smooth boundary $\partial\Omega$ in $\mathbb{R}^n$, ν denotes the unit outward normal to $\partial\Omega$, and ∂_ν is the normal derivative on $\partial\Omega$, g changes sign in Ω. We call this the "heterozygote superiority" case, when $f \in C^1[0,1]$ such that $f(0) = f(1) = 0, f'(0) > 0$, $f'(1) > 0$, and $f(u) > 0$ in $(0, \alpha)$, $f(u) < 0$ in $(\alpha, 1)$ for some $\alpha \in (0, 1)$. Under the condition that the spatial dimension $n = 1$, a steady-state solution of (1.3) satisfies

$$\begin{cases} du'' + g(x)f(u) = 0, & 0 < x < 1, \\ u'(0) = 0, \quad u'(1) = 0. \end{cases} \qquad (1.4)$$

The aim of the present paper is to show how the three solutions theorem in [7] generalizes in case we increase the number of zeros of $g(s)$. We follow closely the arguments of [7], actually, we are able to deal with more general nonlinearities $g(s)$. To keep the situation simple enough, we consider $g(s)$ has three zeros. Namely, we study the indefinite weight boundary value problem (1.2) under the assumptions:

(H_1) $g : [0, \infty) \to \mathbb{R}$ is locally Lipschitz continuous with $g(0) = g(1) = g(2) = 0$, $\lim_{s \to 2} \frac{g(s)}{2-s} = 0$; $g(s) > 0$ for $s \in (0,1)$, $g(s) < 0$ for $s \in (1,2)$;

(H_2) $a \in L[0,T]$, there exist σ, τ with $0 < \sigma < \tau < T$ such that

$$\begin{cases} a^+(t) > 0, a^-(t) \equiv 0, & t \in [0, \sigma], \\ a^+(t) \equiv 0, a^-(t) > 0, & t \in [\sigma, \tau], \\ a^+(t) > 0, a^-(t) \equiv 0, & t \in [\tau, T]. \end{cases}$$

Let (H_0), (H_1), and (H_2) hold, we can get six solutions $u(t)$ of problem (1.2), of which three solutions $0 < u(t) < 1$ for all $t \in [0, T]$ have been found in paper [7], and the purpose of this paper is to find the other three solutions $u(t)$ of problem (1.2), which satisfy $1 < u(t) < 2$ for all $t \in [0, T]$. The main result of the paper is the following.

Theorem 1.1. *Let* (H_1) *and* (H_2) *hold. Then, there exists* $\lambda_2 > 0$ *such that for each* $\lambda > \lambda_2$, *there exists* $\mu_2(\lambda) > 0$ *such that for every* $\mu > \mu_2(\lambda)$, *problem* (1.2) *has three positive solutions* $u(t)$ *and* $1 < u(t) < 2$ *for all* $t \in [0, T]$.

Remark 1.1. Note that when $g(s)$ only has two zeros $s = 0$ and $s = 1$, then, condition (H_1) will degenerate into condition (H_0), and the corresponding Theorem 1.1 will degenerate into Theorem A in [7]. Therefore, the results of this paper can be regarded as a direct generalization of [7].

2 Proof of main theorem

To prove our main theorem, we need some preliminary results.

In this section, we will find three positive solutions $u(t)$ of problem (1.2) and $1 < u(t) < 2$ for all $t \in [0, T]$. Therefore, we can further rewrite problem (1.2) as follows:

$$\begin{cases} u''(t) + (\lambda a^+(t) - \mu a^-(t))g^*(u) = 0, \quad 0 < t < T, \\ 1 < u(t) < 2, \\ u'(0) = 0, \quad u'(T) = 0, \end{cases} \tag{2.1}$$

where $g^*(u)$ is defined as follows:

$$g^*(u) = \begin{cases} 0, & u \leq 1, \\ g(u), & 1 < u < 2, \\ 0, & u \geq 2. \end{cases}$$

First, studying problem (2.1) in the interval $[0, \sigma]$ and the equation can be simplified to

$$u''(t) = -\lambda a^+(t)g^*(u). \tag{2.2}$$

Lemma 2.1. *Let $\lambda > 0$, $m_1 \in (1, 2)$, and $t_1 \in (0, \sigma)$. Then, for every $\omega \geq \frac{2 - m_1}{\sigma - t_1}$, solution $u(t)$ of (2.2) with $u(t_1) \geq m_1$, $u'(t_1) \geq \omega$ satisfies $u(\sigma) \geq 2$, $u'(\sigma) \geq \omega$.*

Proof. Let $u(t)$ be a solution of (2.2) with $u(t_1) \geq m_1$, $u'(t_1) \geq \omega$. Since

$$u''(t) = -\lambda a^+(t)g^*(u) = \lambda a^+(t)|g^*(u)| \geq 0,$$

by the monotonicity of $u'(t)$ on $[0, \sigma]$, we obtain

$$u'(t) \geq u'(t_1) \geq \omega, \quad t_1 < t < \sigma. \tag{2.3}$$

By integrating (2.3) on $[t_1, \sigma] \subseteq [0, \sigma]$, we immediately obtain

$$u(\sigma) \geq u(t_1) + \omega(\sigma - t_1) \geq 2.$$ □

Lemma 2.2. *Let $\lambda > 0$, $t_1 \in (0, \sigma)$, $m_0, m_1 \in (1, 2)$ such that $1 < m_0 < m_1 < 2$. Given*

$$\lambda^*(m_0, m_1, t_1) = \frac{m_1 - m_0}{\min_{m_0 \leq u \leq m_1}|g^*(u)| \int_0^{t_1}\left(\int_0^s a^+(h)\mathrm{d}h\right)\mathrm{d}s}$$

and $\omega < \frac{m_1 - m_0}{t_1}$. Then, for every $\lambda > \lambda^*(m_0, m_1, t_1)$, solution $u(t)$ of (2.2) with initial conditions $u(0) = m_0$, $u'(0) = 0$ satisfies $u(t_1) > m_1$ and $u'(t_1) > \omega$.

Proof. By integrating (2.2) on $[0, t] \subseteq [0, \sigma]$, we have

$$u'(t) = \int_0^t (\lambda a^+(s)|g^*(u)|)\mathrm{d}s \geq 0,$$

and therefore, $u(t)$ monotonically increasing on $(0, \sigma)$.

We suppose $u(t_1) \leq m_1$ holds. Then

$$1 < m_0 < u(t) < m_1 < 2, \quad 0 < t < t_1.$$

Furthermore, we have

$$u(t_1) \geq m_0 + \lambda \min_{m_0 \leq u \leq m_1} |g^*(u)| \int_0^{t_1}\left(\int_0^s a^+(h)\mathrm{d}h\right)\mathrm{d}s,$$

when $\lambda > \lambda^*$, we obtain

$$u(t_1) > m_1,$$

which implies a contradiction.

Similarly, we suppose $u'(t_1) \le \omega$ holds. By the monotonicity of $u'(t)$ in $[0, \sigma]$, we have

$$u'(t) \le \omega, \quad 0 < t < t_1.$$

Integrate on $[0, t_1] \subseteq [0, \sigma]$, we obtain

$$u(t_1) \le m_0 + \omega t_1 < m_1,$$

which implies a contradiction, and Lemma 2.2 is proved. □

Lemma 2.3. *Let $\lambda > 0$ and $m_1 \in (1, 2)$. Then, for any $\varepsilon > 0$, there exists $\delta_\varepsilon > 0 (\delta_\varepsilon < 2 - m_1)$ such that the following holds: for any $m \in (2 - \delta_\varepsilon, 2)$, solution $u(t)$ of (2.2) with initial conditions $u(0) = m$, $u'(0) = 0$ satisfies $u(t) < 2$ and $u'(t) > 0$ for all $t \in [0, \sigma]$.*

Proof. Let λ and m_1 be fixed as in the statement and denote the supremum norm by $\|\cdot\|_\infty$. From (H_1), we have

$$\lim_{s \to 2^-} \frac{g^*(s)}{2 - s} = 0,$$

so, for all $\varepsilon > 0$, there exists $\delta_\varepsilon \in (0, 2 - m_1)$ such that

$$|g^*(s)| \le \varepsilon(2 - s), \quad s \in [2 - \delta_\varepsilon, 2].$$

For any $m \in (2 - \delta_\varepsilon, 2)$, we consider the solution $u(t)$ of (2.2) with $u(0) = m$ and $u'(0) = 0$.

We suppose that there exists $\sigma_1 \in (0, \sigma)$ such that $u(t) < 2$ for all $t \in [0, \sigma_1)$ and $u(\sigma_1) = 2$. Without less of generality, we choose $\varepsilon < \frac{2 - m}{\lambda \| a^+(t) \|_\infty \int_0^{\sigma_1} \int_0^t (2 - u(s)) \mathrm{d}s \mathrm{d}t}$.

By integrating of (2.2) on $[0, t] \subseteq [0, \sigma_1)$, we have

$$u'(t) = \lambda \int_0^t a^+(s)|g^*(u)| \mathrm{d}s \le \lambda\varepsilon \|a^+(t)\|_\infty \int_0^t (2 - u(s)) \mathrm{d}s.$$

Furthermore, we obtain

$$2 = u(\sigma_1) \le m + \lambda\varepsilon \|a^+(t)\|_\infty \int_0^{\sigma_1} \int_0^t (2 - u(s)) \mathrm{d}s \mathrm{d}t < 2,$$

which implies a contradiction. □

Second, we consider problem (2.1) in the interval $[\sigma, \tau]$, where the equation can be simplified to

$$u''(t) = \mu a^-(t) g^*(u). \tag{2.4}$$

Lemma 2.4. *Let $\lambda > 0$, $\mu > 0$ for any $v > 0$. If $u(t)$ is the solution of the initial problem*

$$\begin{cases} u''(t) + (\lambda a^+(t) - \mu a^-(t)) g^*(u) = 0, & \sigma < t < T, \\ u(\sigma) = 2, \quad u'(\sigma) = v, \end{cases} \tag{2.5}$$

then $u(t) > 2$, $u'(t) > 0$ for all $t \in (\sigma, T)$.

Proof. Suppose that $[\sigma, t^*] \subseteq [\sigma, T]$ is the maximal interval such that $u'(t) \ge 0$ for all $t \in [\sigma, t^*]$ and $t^* < T$. We immediately obtain $u(t) > 2$ for all $t \in [\sigma, t^*]$, by integrating of (2.1) on $[\sigma, t^*]$, we have

$$u'(\sigma) = u'(t^*),$$

which implies a contradiction. □

Lemma 2.5. *Let* $\mu > 0$, $m_2 \in (1, 2)$ *and* $t_2 \in (\sigma, \tau)$. *Then, for every* $\gamma \le \frac{1-m_2}{\tau - t_2}$, *any solution* $u(t)$ *of* (2.4) *with* $u(t_2) \le m_2$, $u'(t_2) \le \gamma$ *satisfies* $u(\tau) \le 1$ *and* $u'(\tau) \le \gamma$.

Proof. Let $u(t)$ be a solution of (2.4) with $u(t_2) \le m_2$, $u'(t_2) \le \gamma$. Since

$$u''(t) = \mu a^-(t) g^*(u) \le 0,$$

we have

$$u'(t) \le u'(t_2) \le \gamma, \quad t \in [t_2, \tau]. \tag{2.6}$$

By integrating of (2.6) on $[t_2, \tau]$, we have

$$u(\tau) \le u(t_2) + \gamma(\tau - t_2) \le 1.$$

□

Lemma 2.6. *Let* m_2, m_3, *and* m^* *such that* $1 < m_2 < m_3 < m^* < 2$ *and* $\gamma_\sigma > 0$. *Given*

$$\mu^*(m_2, m_3, m^*, t_2, \gamma_\sigma) = \frac{m_2 - m_3 - \gamma_\sigma(t_2 - \sigma)}{\max_{m_2 \le u \le m^*} g^*(u) \int_\sigma^{t_2} \left(\int_\sigma^s a^-(h) \mathrm{d}h \right) \mathrm{d}s}, \quad \gamma > \frac{m_2 - m_3}{t_2 - \sigma},$$

and $t_2 \le \sigma + \frac{m^* - m_3}{\gamma_\sigma}$. *Then, for every* $\mu > \mu^*$, *any solution* $u(t)$ *of* (2.4) *with initial conditions* $u(\sigma) = m_3$, $u'(\sigma) = \gamma_\sigma$ *satisfies* $u(t_2) < m_2$ *and* $u'(t_2) < \gamma$.

Proof. Let $u(t)$ be a solution of (2.4) satisfies the initial conditions $u(\sigma) = m_3$ and $u'(\sigma) = \gamma_\sigma$.

We suppose $u(t_2) \ge m_2$ holds. Then, we immediately obtain $u(t) \ge m_2$ for all $t \in [\sigma, t_2]$. On the other hand,

$$u''(t) = \mu a^-(t) g^*(u) \le 0, \quad t \in [\sigma, \tau], \tag{2.7}$$

we have

$$u'(t) \le u'(\sigma), \quad t \in [\sigma, \tau]. \tag{2.8}$$

By integrating (2.8) on $[\sigma, t] \subseteq [\sigma, \tau]$, we obtain

$$u(t) \le \gamma_\sigma t - \gamma_\sigma \sigma + m_3, \quad t \in [\sigma, \tau],$$

in particular, $u(t) \le m^*$, $t \in [\sigma, t_2]$.

By integrating (2.7) twice on $[\sigma, t] \subseteq [\sigma, t_2]$, we have

$$\begin{aligned} u(t) &= u(\sigma) + u'(\sigma)(t - \sigma) + \mu \int_\sigma^t \left(\int_\sigma^s a^-(h) g^*(u) \mathrm{d}h \right) \mathrm{d}s \\ &\le m_3 + \gamma_\sigma(t - \sigma) + \mu \max_{m_2 \le u \le m^*} g^*(u) \int_\sigma^t \left(\int_\sigma^s a^-(h) \mathrm{d}h \right) \mathrm{d}s. \end{aligned}$$

When $\mu > \mu^*$, we have

$$u(t_2) < m_2,$$

which implies a contradiction.

And then, we suppose $u'(t_2) \ge \gamma$ holds. We immediately obtain $u'(t) \ge \gamma$, $t \in [\sigma, t_2]$, then

$$u(t_2) \ge m_3 + \gamma(t_2 - \sigma) \ge m_2,$$

contradiction and Lemma 2.6 is proved.

□

Finally, we studying problem (2.1) in the interval $[\tau, T]$. Similarly, the equation can also be simplified as (2.2), the situation is exactly symmetric to the described in Lemmas 2.1 and 2.2. We give the corresponding conclusions.

Lemma 2.7. *Let $\lambda > 0$, $m_5 \in (1,2)$, and $t_3 \in (\tau, T)$. Then, for every $\omega_1 \le \frac{m_5-2}{t_3-\tau}$, solution $u(t)$ of (2.2) with $u(t_3) \ge m_5$, $u'(t_3) \le \omega_1$ satisfies $u(\tau) \ge 2$ and $u'(\tau) \le \omega_1$.*

Lemma 2.8. *Let $\lambda > 0$, $t_3 \in (\tau, T)$, $m_4, m_5 \in (1,2)$ such that $1 < m_4 < m_5 < 2$. Given*

$$\lambda^{**}(m_4, m_5, t_3) = \frac{m_5 - m_4}{\min_{m_4 \le u \le m_5} |g^*(u)| \int_{t_3}^{T} \left(\int_s^T a^+(h)\mathrm{d}h \right) \mathrm{d}s}$$

and $\omega_1 > \frac{m_4-m_5}{T-t_3}$. Then, for every $\lambda > \lambda^{**}$, solution $u(t)$ of (2.2) with initial conditions $u(T) = m_4$, $u'(T) = 0$ satisfies $u(t_3) > m_5$ and $u'(t_3) < \omega_1$.

The proof process is completely similar to Lemmas 2.1 and 2.2, and it is omitted here.

Proof of Theorem 1.1. We show that problem (2.1) has at least three solutions through the following five steps.

Step 1. What needs to be explained is that $g(s)$ satisfies locally Lipschitz condition which ensure the uniqueness and the global existence of the solution $u(t, t_0, \alpha, \beta)$ for equation

$$u''(t) + (\lambda a^+(t) - \mu a^-(t))g^*(u) = 0, \quad 0 < t < T, \tag{2.9}$$

with the initial conditions $u(t_0) = \alpha$, $u'(t_0) = \beta$. In addition, the solution is continuously dependent on the initial value.

Step 2. In interval $[0, \sigma]$, let us fix $1 < m_0 < m_1 < 2$ and $0 < t_1 < \frac{\sigma(m_1-m_0)}{2-m_0}$. We immediately obtain $\frac{2-m_1}{\sigma-t_1} \le \omega < \frac{m_1-m_0}{t_1}$, so we can apply Lemmas 2.1 and 2.2 when $\lambda > \lambda^*(m_0, m_1, t_1)$, and for any μ, we have

$$u(\sigma, 0, m_0, 0) \ge 2, \quad u'(\sigma, 0, m_0, 0) \ge \omega.$$

We also have

$$u(\sigma, 0, 1, 0) = 1, \quad u'(\sigma, 0, 1, 0) = 0.$$

According to the intermediate value theorem, there exists an interval $[1, l_1] \subseteq [1, m_0]$ such that $u(\sigma, 0, l_1, 0) = 2$, $u'(\sigma, 0, l_1, 0) \ge 0$, and for all $\xi \in (1, l_1)$, $t \in [0, \sigma]$, we have $1 < u(t, 0, \xi, 0) < 2$, $u'(t, 0, \xi, 0) > 0$.

Furthermore, apply Lemma 2.3, there exists $m_6 \in (m_1, 2)$ such that

$$u(\sigma, 0, m_6, 0) < 2, \quad u'(\sigma, 0, 1, 0) > 0.$$

Similarly, there exists an interval $[l_2, 2]$ and $m_0 < l_2 < m_6$, such that $u(\sigma, 0, l_2, 0) = 2$, $u'(\sigma, 0, l_1, 0) > 0$, and for all $\xi \in (l_2, 2)$, $t \in [0, \sigma]$, we have $1 < u(t, 0, \xi, 0) < 2$, $u'(t, 0, \xi, 0) > 0$.

Step 3. In interval $[\tau, T]$. Analogously to Step 2, let us fix $1 < m_4 < m_5 < 2$ and $\frac{\tau(m_4-m_5)+T(m_5-2)}{m_4-2} < t_3 < T$. We obtain $\frac{m_4-m_5}{T-t_3} < \omega_1 \le \frac{m_5-2}{t_3-\tau}$, apply Lemmas 2.7 and 2.8 when $\lambda > \lambda^{**}(m_4, m_5, t_3)$, and for any μ, we have

$$u(\tau, T, m_4, 0) \ge 2, \quad u'(\tau, T, m_4, 0) \le \omega_1.$$

Thus, there exists an interval $[1, l_3] \subseteq [1, m_4]$ such that $u(\tau, T, l_3, 0) = 2$, $u'(\tau, T, l_3, 0) < 0$, and for all $\xi \in (1, l_3)$, $t \in [\tau, T]$, $1 < u(t, T, \xi, 0) < 2$.

Step 4. In interval $[\sigma, \tau]$, let

$$\lambda_2 = \max\{\lambda^*(m_0, m_1, t_1), \lambda^{**}(m_4, m_5, t_3)\}$$

and fix $\lambda > \lambda_2$. Take $p_1 \in (1, l_1)$ and $p_2 \in (l_2, 2)$, define

$$m_{3,i} = u(\sigma, 0, p_i, 0), \quad \gamma_{\sigma,i} = u'(\sigma, 0, p_i, 0), \quad i = 1, 2,$$

fix $m_i^*, m_{2,i}$, and $t_{2,i}$ such that $1 < m_{2,i} < m_{3,i} < m_i^* < 2$, and $t_{2,i} < \min\left\{\frac{\tau(m_{3,i}-m_{2,i})+\sigma(m_{2,i}-1)}{m_{3,i}-1}, \sigma + \frac{m_i^*-m_{3,i}}{\gamma_{\sigma,i}}\right\}$, then $\frac{m_{2,i}-m_{3,i}}{t_{2,i}-\sigma} < \gamma \le \frac{1-m_{2,i}}{\tau-t_{2,i}}$. Apply Lemmas 2.5 and 2.6 when $\mu > \mu^*(m_{2,i}, m_{3,i}, m_i^*, t_{2,i}, \gamma_{\sigma,i})$, we have

$$u(\tau, 0, p_i, 0) \le 1, \quad u'(\tau, 0, p_i, 0) \le 0, \quad i = 1, 2.$$

Meanwhile, apply Lemma 2.4, we have

$$u(\tau, 0, l_i, 0) > 2, \quad u'(\tau, 0, l_i, 0) > 0, \quad i = 1, 2.$$

According to the continuous dependence of the solutions upon the initial data and the Intermediate Value Theorem, for $\mu > \mu^*(\lambda)$, there exist three intervals

$$[q_1, r_1] \subseteq [p_1, l_1], \quad [q_2, r_2] \subseteq [l_2, p_2], \quad [q_3, r_3] \subseteq [p_2, 2],$$

such that

$$u(\tau, 0, r_i, 0) = 2, u(\tau, 0, q_i, 0) = 1, \quad i = 1, 2, 3,$$

and for all $\xi \in (q_i, r_i)$, $t \in [0, \tau]$, $1 < u(t, 0, \xi, 0) < 2$. Obviously, the three intervals do not intersect, and then we can find three connected region in $[0, T] \times (1, 2)$.

Step 5. In these connected regions, using the forward shooting method and the backward shooting method respectively, we can obtain at least three solutions to problem (2.1). At the same time, it is also the solution of problem (1.2). This completes the proof. □

Finally, we point out that, even if, for the sake of simplicity, we only consider the case that $g(s)$ has three zeros, it is reasonable to expect that some further multiplicity results can be proved also for non-linearity $g(s)$ with k zeros, yielding the existence of $3(k-1)$-positive solutions.

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