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Research Article

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Ground state solution for some new Kirchhoff-type equations with Hartree-type nonlinearities and critical or supercritical growth

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Abstract: In this article, we study two classes of Kirchhoff-type equations as follows:

$$\begin{cases} -\left(a + b\int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x\right)\Delta u + V(x)u = (I_\alpha * |u|^p)|u|^{p-2}u + f(u), \quad \text{in } \mathbb{R}^3, \\ u \in H^1(\mathbb{R}^3), \end{cases}$$

and

$$\begin{cases} -\left(a + b\int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x\right)\Delta u + V(x)u = (I_\alpha * |u|^p)|u|^{p-2}u + m|u|^{l-2}u, \quad \text{in } \mathbb{R}^3, \\ u \in H^1(\mathbb{R}^3), \end{cases}$$

where $a > 0, b \geq 0, \alpha \in (0, 3), (3 + \alpha)/3 < p < (3 + \alpha), l \geq 6, m > 0, V : \mathbb{R}^3 \to \mathbb{R}$ is a potential function and I_α is a Riesz potential whose order is $\alpha \in (0, 3)$. Under some assumptions on $V(x)$ and $f(u)$, we can prove that the equations have ground state solutions by variational methods.

Keywords: Kirchhoff equation, ground state solutions, Pohozaev identity, Nehari manifold

MSC 2020: 35J60, 35J35, 35A15

1 Introduction

In this article, we study the following two classes of Kirchhoff-type equations:

$$\begin{cases} -\left(a + b\int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x\right)\Delta u + V(x)u = (I_\alpha * |u|^p)|u|^{p-2}u + f(u), \quad \text{in } \mathbb{R}^3, \\ u \in H^1(\mathbb{R}^3), \end{cases} \tag{1}$$

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and

$$\begin{cases} -\left(a + b\int\limits_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x\right)\Delta u + V(x)u = (I_\alpha * |u|^p)|u|^{p-2}u + m|u|^{l-2}u, & \text{in } \mathbb{R}^3, \\ u \in H^1(\mathbb{R}^3), \end{cases} \tag{2}$$

where $a > 0$, $b \geq 0$, $\alpha \in (0, 3)$, $(3+\alpha)/3 < p < (3+\alpha)$, $l \geq 6$, $m > 0$, I_α is a Riesz potential whose order is $\alpha \in (0, 3)$. Here, $I_\alpha = \frac{\Gamma\left(\frac{3-\alpha}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right)\pi^{\frac{3}{2}}2^\alpha |x|^{3-\alpha}}$. Besides, $V(x) : \mathbb{R}^3 \to \mathbb{R}$ is a potential function satisfying:

(V1) $V \in C^1(\mathbb{R}^3)\bigcap L^\infty$, and there exists a constant $A \in (0, a)$ such that

$$|(\nabla V(x), x)| \leq \frac{A}{2|x|^2},$$

for all $x \in \mathbb{R}^3 \backslash \{0\}$,

(V2) there is a constant $V_\infty > 0$ such that for all $x \in \mathbb{R}^3$,

$$0 < V(x) \leq \liminf_{|y|\to+\infty} V(y) = V_\infty < +\infty,$$

(V3) $\inf_{x\in\mathbb{R}^3} V(x) := V_0 > 0$.

Furthermore, we suppose that the function $f \in C^1(\mathbb{R}, \mathbb{R})$ satisfies:

(f1) there exists a constant $C_0 > 0$ and $q \in (2, 6)$ such that $|f(t)| \leq C_0(1 + |t|^{q-1})$, $\quad \forall t \in \mathbb{R}$,

(f2) $f(t) = o(t)$ as $t \to 0$,

(f3) $\lim_{|t|\to+\infty} \frac{F(t)}{t^2} = \infty$, where $F(t) = \int_0^t f(s)\mathrm{d}s$,

(f4) $\frac{f(t)}{|t|}$ is increasing on $(-\infty, 0)\bigcup(0, +\infty)$.

In the past decades, many scholars have studied the existence of nontrivial solutions for the Kirchhoff-type problem:

$$\begin{cases} -\left(a + b\int\limits_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x\right)\Delta u + V(x)u = g(x, u), & \text{in } \mathbb{R}^3, \\ u \in H^1(\mathbb{R}^3), \end{cases} \tag{3}$$

where $a > 0$, $b \geq 0$, $V : \mathbb{R}^3 \to \mathbb{R}$ is a potential function and $g \in C(\mathbb{R}^3 \times \mathbb{R}, \mathbb{R})$. Problem (3) is a nonlocal problem because of the presence of the term $b\int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x$, which causes some mathematical difficulties, but at the same time makes the research of this problem particularly interesting. Besides, this problem has an interesting physical context. In fact, if we set $V(x) = 0$ and replace $\mathbb{R}^3$ by a bounded domain $\Omega \subset \mathbb{R}^3$ in (3), then we obtain the following Kirchhoff Dirichlet problem:

$$\begin{cases} -\left(a + b\int\limits_{\Omega} |\nabla u|^2 \mathrm{d}x\right)\Delta u = g(x, u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases}$$

It has relation to the stationary analogue of the equation:

$$\rho\frac{\partial^2 u}{\partial t^2} - \left(\frac{\rho_0}{h} + \frac{E}{2L}\int\limits_0^L \left|\frac{\partial u}{\partial x}\right| \mathrm{d}x\right)\frac{\partial^2 u}{\partial x^2} = 0,$$

which was proposed by G. Kirchhoff as an extension of classical D'Alemberts wave equations for free vibration of elastic strings. Kirchhoff's model considers the changes in length of the string, which were produced by transverse vibrations. Then J. L. Lions finished the previous work. He introduced a functional analysis approach. After that, more and more researchers have paid much attention to the problem (3). But most of their results need to assume:

1. V verifies (V): $\inf_{x\in\mathbb{R}^3} V(x) := V_0 > 0$ and for each $M > 0$.
$$\operatorname{meas}\{x \in \mathbb{R}^N : V(x) \le M\} < +\infty,$$
2. g satisfies classical Ambrosetti-Rabinowitz condition, i.e., $(A - R)$ condition: there exists $\mu > 2$ such that
$$0 < \mu G(x, s) \le sg(x, s)$$
for all $s > 0$.

In fact, (V) is sufficient to ensure that the embedding

$$\left\{u \in H^1(\mathbb{R}^3) : \int_{\mathbb{R}^3} V(x)|u|^2 \mathrm{d}x < +\infty\right\} \hookrightarrow\hookrightarrow L^p(\mathbb{R}^3), \quad 2 \le p < 6$$

is compact.

Unfortunately, there still are very few results of existence of ground state solution to (3) without (A-R) condition (see [1–3]).

In [3], Guo studied the following Kirchhoff-type problem:

$$\begin{cases} -\left(a + b\int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x\right)\Delta u + V(x)u = f(u), & \text{in } \mathbb{R}^3. \\ u \in H^1(\mathbb{R}^3). \end{cases} \tag{4}$$

He proved the existence of positive ground states to (4), and in his paper, he did not use (A-R) type condition. He defined a new manifold:

$$\mathcal{M} = \left\{u \in H^1(\mathbb{R}^3) : \frac{1}{2}\langle \Phi'(u), u\rangle + \mathcal{P}(u) = 0\right\},$$

which is named the Nehari-Pohozaev manifold. Here,

$$\Phi(u) = \frac{1}{2}\int_{\mathbb{R}^3} [a|\nabla u|^2 + V_\infty u^2]\mathrm{d}x + \frac{b}{4}\left(\int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x\right)^2 - \int_{\mathbb{R}^3} F(u)\mathrm{d}x$$

and

$$\mathcal{P}(u) = \frac{1}{2}\int_{\mathbb{R}^3} [a|\nabla u|^2 + 3V_\infty u^2]\mathrm{d}x + \frac{b}{2}\left(\int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x\right)^2 - 3\int_{\mathbb{R}^3} F(u)\mathrm{d}x$$

are the energy functional and Pohozaev identity for the "limit problem" of problem (4), respectively. He first applied the result, which was obtained for the related "limit problem" of (4) to obtain a minimizer for problem (4) on the Nehari-Pohozaev manifold.

We must point out that $f \in C^1$ and the fourth assumption about f are very important in Guo [3]. Actually, only under aforementioned assumptions, $\mathcal{M}$ is a C^1 manifold.

Most remarkably, as early as 2006, Ruiz [4] first proposed the prototype of this Nehari-Pohozaev manifold in his study of the Schrödinger-Poisson equation, which is a very great work.

On the other hand, when $a = 1$, $b = 0$, $f = 0$, equation (1) becomes

$$-\Delta u + V(x)u = (I_\alpha * |u|^p)|u|^{p-2}u. \tag{5}$$

We usually call it nonlinear Choquard-type equation. Its physical background can be found in [5], and the references therein. Besides, readers can see [6–13] for recent achievements.

Inspired by the aforementioned works, especially by [3,13–15], we now research problem (3) with Hartree-type nonlinearities $g(x,u) = (I_\alpha * |u|^p)|u|^{p-2}u + f(u)$, which may be regarded as a Kirchhoff-type perturbation to (5). As we all know, there are very few results to (3) with Hartree-type nonlinearities and critical or supercritical growth.

The main outcomes of our investigation are as follows.

Theorem 1.1. *If V satisfies* $(V1)$–$(V2)$, $f \in C^1(\mathbb{R},\mathbb{R})$ *verifies* $(f1)$–$(f4)$, *then problem* (1) *has a ground state solution.*

Theorem 1.2. *If V satisfies (V1)–(V3), then there exists some* $m_0 > 0$ *such that for* $m \in (0, m_0]$, *problem* (2) *has a ground state solution.*

For the convenience of expression, hereafter, we will use the following notations:
- $X := H^1(\mathbb{R}^3)$ is a space in which an equivalent norm is defined as follows:

$$\|u\| = \left[\int_{\mathbb{R}^3} (a|\nabla u|^2 + V(x)u^2)\mathrm{d}x\right]^{\frac{1}{2}},$$

- $L^s(\mathbb{R}^3)(1 \le s \le \infty)$ denotes the Lebesgue space in which the norm is defined as follows:

$$|u|_s = \left(\int_{\mathbb{R}^3} |u|^s \mathrm{d}x\right)^{1/s},$$

- For any $u \in H^1(\mathbb{R}^3)\backslash\{0\}$, u_t is denoted as follows:

$$u_t = \begin{cases} 0, & t = 0, \\ \sqrt{t}\,u\left(\frac{x}{t}\right), & t > 0. \end{cases}$$

- For any $x \in \mathbb{R}^3$ and $r > 0$, $B_r(x) := \{y \in \mathbb{R}^3 : |y - x| < r\}$.
- $C, C_1, C_2, \ldots$ denote positive constants, which are possibly different in different lines.

2 Preliminaries

Problem (1) has a variational structure, i.e., the critical points of the functional

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^3} [a|\nabla u|^2 + V(x)u^2]\mathrm{d}x + \frac{b}{4}\left(\int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x\right)^2 - \frac{1}{2p}\int_{\mathbb{R}^3} (I_\alpha * |u|^p)|u|^p \mathrm{d}x - \int_{\mathbb{R}^3} F(u)\mathrm{d}x \tag{6}$$

are weak solutions of problem (1).

Lemma 2.1. *Assume that* $(f1)$–$(f4)$ *hold, then we have*
1. *for all* $\varepsilon > 0$ *and* $q \in (2, 6)$, *there is a* $C_\varepsilon > 0$ *such that* $|f(t)| \le \varepsilon|t| + C_\varepsilon|t|^{q-1}$,
2. *for any* $s \ne 0$, $sf(s) > 2F(s)$ *and* $F(s) > 0$.

Proof. We could easily obtain the results by elementary calculation. □

Lemma 2.2. (Hardy-Littlewood-Sobolev inequality [16]). *Let* $0 < \alpha < N$, $p, q > 1$ *and* $1 \le r < s < \infty$ *be such that*

$$\frac{1}{p} + \frac{1}{q} = 1 + \frac{\alpha}{N}, \quad \frac{1}{r} - \frac{1}{s} = \frac{\alpha}{N}.$$

1. *For any* $f \in L^p(\mathbb{R}^N)$ *and* $g \in L^q(\mathbb{R}^N)$, *one has*

$$\left| \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{f(x)g(y)}{|x-y|^{N-\alpha}} \mathrm{d}x\mathrm{d}y \right| \le C(N, \alpha, p) \|f\|_{L^p(\mathbb{R}^N)} \|g\|_{L^q(\mathbb{R}^N)}.$$

2. *For any* $f \in L^r(\mathbb{R}^N)$, *one has*

$$\left\| \frac{1}{|\cdot|^{N-\alpha}} * f \right\|_{L^s(\mathbb{R}^N)} \le C(N, \alpha, r) \|f\|_{L^r(\mathbb{R}^N)}.$$

Lemma 2.3. (Brezis-Lieb lemma [17]) *Let* $s \in (1, \infty)$ *and* $\{w_n\}$ *be a bounded sequence in* $L^s(\mathbb{R}^N)$. *If* $w_n \to w$ *almost everywhere on* $\mathbb{R}^N$, *then for any* $q \in [1, s]$,

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} ||w_n|^q - |w_n - w|^q - |w|^q|^{\frac{s}{q}} \mathrm{d}x = 0 \tag{7}$$

and

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} ||w_n|^{q-1} w_n - |w_n - w|^{q-1}(w_n - w) - |w|^{q-1} w|^{\frac{s}{q}} \mathrm{d}x = 0. \tag{8}$$

Lemma 2.4. (Nonlocal Brezis lemma [6]) *Let* $\alpha \in (0, N)$, $N \ge 3$, $p \in \left[1, \frac{2N}{N+\alpha}\right)$ *and* $\{u_n\}$ *be a bounded sequence in* $L^{\frac{2Np}{N+\alpha}}(\mathbb{R}^N)$. *If* $u_n \to u$ *almost everywhere on* $\mathbb{R}^N$, *then*

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} (I_\alpha * |u_n|^p)|u_n|^p \mathrm{d}x - \int_{\mathbb{R}^N} (I_\alpha * |u_n - u|^p)|u_n - u|^p \mathrm{d}x = \int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p \mathrm{d}x.$$

Lemma 2.5. [13] *Let* $\alpha \in (0, N)$, $N \ge 3$, $p \in \left[1, \frac{2N}{N+\alpha}\right)$ *and* $\{u_n\}$ *be a bounded sequence in* $L^{\frac{2Np}{N+\alpha}}(\mathbb{R}^N)$. *If* $\{u_n\} \to u$ *almost everywhere on* $\mathbb{R}^N$, *then for any* $h \in L^{\frac{2Np}{N+\alpha}}(\mathbb{R}^N)$,

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} (I_\alpha * |u_n|^p)|u_n|^{p-2} u_n h \mathrm{d}x = \int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^{p-2} u h \mathrm{d}x.$$

Lemma 2.6. (Pohozaev identity [6,7,18,19]). *Suppose* $V(x)$ *satisfies* $(V1)$–$(V2)$ *and let* $u \in X$ *be a weak solution of Problem* (1), *then we have the following Pohozaev identity*:

$$\begin{aligned} 0 = P_V(u) = {} & \frac{a}{2} \int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x + \frac{3}{2} \int_{\mathbb{R}^3} V(x)|u|^2 \mathrm{d}x + \frac{1}{2} \left(\int_{\mathbb{R}^3} (\nabla V(x), x)|x|^2 \mathrm{d}x \right. \\ & + \frac{b}{2} \left(\int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x \right)^2 - \frac{3+\alpha}{2p} \int_{\mathbb{R}^3} (I_\alpha * |u|^p)|u|^p \mathrm{d}x - 3 \int_{\mathbb{R}^3} F(u) \mathrm{d}x. \end{aligned} \tag{9}$$

In particular, if $V \equiv V_\infty$, *we have*

$$\begin{aligned} 0 = P_\infty(u) = {} & \frac{a}{2} \int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x + \frac{3}{2} \int_{\mathbb{R}^3} V_\infty |u|^2 \mathrm{d}x + \frac{b}{2} \left(\int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x \right)^2 \\ & - \frac{3+\alpha}{2p} \int_{\mathbb{R}^3} (I_\alpha * |u|^p)|u|^p \mathrm{d}x - 3 \int_{\mathbb{R}^3} F(u) \mathrm{d}x. \end{aligned} \tag{10}$$

3 Ground state solution for the "limit problem" of equation (1)

In this section, we will investigate the following limit problem that is associated with problem (1):

$$\begin{cases} -\left(a + b \int\limits_{\mathbb{R}^3} |\nabla u|^2\right)\Delta u + V_\infty u = (I_\alpha * |u|^p)|u|^{p-2}u + f(u), \quad \text{in } \mathbb{R}^3, \\ u \in H^1(\mathbb{R}^3). \end{cases} \tag{11}$$

The asscoiated energy function is given by:

$$I_\infty(u) = \frac{1}{2}\int\limits_{\mathbb{R}^3} [a|\nabla u|^2 + V_\infty u^2]\mathrm{d}x + \frac{b}{4}\left(\int\limits_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x\right)^2 - \frac{1}{2p}\int\limits_{\mathbb{R}^3} (I_\alpha * |u|^p)|u|^p \mathrm{d}x - \int\limits_{\mathbb{R}^3} F(u)\mathrm{d}x. \tag{12}$$

We prove the following results.

Lemma 3.1. *Let $p \in \left(\frac{3+\alpha}{3}, 3 + \alpha\right)$, then I_∞ has no lower bounds.*

Proof. For $\forall u \in X\backslash\{0\}$ and $t > 0$, we have

$$\begin{aligned} I_\infty(u_t) &= I_\infty(\sqrt{t}u(t^{-1}x)) \\ &= \frac{at^2}{2}\int\limits_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x + \frac{t^4}{2}\int\limits_{\mathbb{R}^3} V_\infty u^2 \mathrm{d}x + \frac{bt^4}{4}\left(\int\limits_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x\right)^2 \\ &\quad - \frac{t^{p+3+\alpha}}{2p}\int\limits_{\mathbb{R}^3} (I_\alpha * |u|^p)|u|^p \mathrm{d}x - t^3\int\limits_{\mathbb{R}^3} F(\sqrt{t}u)\mathrm{d}x \to -\infty \end{aligned}$$

as $t \to \infty$, since $p + 3 + \alpha > 4$, and then we can obtain the conclusion. □

Next we define $\mathcal{M}_\infty = \{u \in X\backslash\{0\} : G_\infty(u) = 0\}$, where

$$\begin{aligned} G_\infty(u) &= \frac{1}{2}\langle I'_\infty(u), u\rangle + P_\infty(u) \\ &= a\int\limits_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x + 2\int\limits_{\mathbb{R}^3} V_\infty u^2 \mathrm{d}x + b\left(\int\limits_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x\right)^2 \\ &\quad - \frac{p+3+\alpha}{2p}\int\limits_{\mathbb{R}^3} (I_\alpha * |u|^p)|u|^p \mathrm{d}x - 3\int\limits_{\mathbb{R}^3} F(u)\mathrm{d}x - \frac{1}{2}\int\limits_{\mathbb{R}^3} f(u)u\mathrm{d}x = \left.\frac{\mathrm{d}I_\infty(u_t)}{\mathrm{d}t}\right|_{t=1}. \end{aligned} \tag{13}$$

Remark 3.2. For $t > 0$, we set

$$\begin{aligned} \gamma(t) = I_\infty(u_t) &= \frac{at^2}{2}\int\limits_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x + \frac{t^4}{2}\int\limits_{\mathbb{R}^3} V_\infty u^2 \mathrm{d}x + \frac{bt^4}{4}\left(\int\limits_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x\right)^2 \\ &\quad - \frac{1}{2p}t^{p+3+\alpha}\int\limits_{\mathbb{R}^3} (I_\alpha * |u|^p)|u|^p \mathrm{d}x - t^3\int\limits_{\mathbb{R}^3} F(\sqrt{t}u)\mathrm{d}x. \end{aligned} \tag{14}$$

Lemma 3.3. *Let c_1, c_2, and c_3 be constants, which are positive and $u \in X\backslash\{0\}$. Then, the function*

$$\eta(t) = c_1 t^2 + c_2 t^4 - c_3 t^{(p+3+\alpha)} - t^3\int\limits_{\mathbb{R}^3} F(\sqrt{t}u)\mathrm{d}x \quad \text{for } t \geq 0$$

has only one positive critical point, which corresponds to its maximal value.

Proof. One can obtain the result by elementary calculation. □

Lemma 3.4. *For any $u \in X\backslash\{0\}$, there exists only one $t_0 > 0$ such that $u_{t_0} \in \mathcal{M}_\infty$ and $I_\infty(u_{t_0}) = \max_{t>0} I_\infty(u_t)$.*

Proof. $I_\infty(u_t)$ has the form of the function $\eta(t)$ defined earlier. Since by Lemma 3.3, $\eta(t)$ has only one critical point $t_0 > 0$, which corresponds to its maximal value. Thus, $\eta(t_0) = \max_{t>0}\eta(t)$ and $\eta'(t_0) = 0$. It follows that $G_\infty(u_{t_0}) = t_0\eta'(t_0) = 0$. This implies $u_{t_0} \in \mathcal{M}_\infty$ and $I_\infty(u_{t_0}) = \max_{t>0} I_\infty(u_t)$. ☐

Lemma 3.5. *The functional I_∞ possesses the mountain-pass geometry, i.e.,*
(1) *there exists $\rho, \delta > 0$ such that $I_\infty \geq \delta$ for all $\|u\| = \rho$;*
(2) *there exists $e \in H^1(\mathbb{R}^3)$ such that $\|e\| > \rho$ and $I_\infty(e) < 0$.*

Proof. (1) By Lemmas 2.1(1) and 2.2, we have

$$I_\infty(u) \geq c_1\|u\|^2 - c_2\|u\|^{2p} - C_\varepsilon\|u\|^q.$$

Thus, there exists $\rho, \delta > 0$ such that $I_\infty \geq \delta$ for all $\|u\| = \rho > 0$ small enough.

(2) For any $u \in X\backslash\{0\}$, by the definition of $I_\infty(u_t)$, we see $I_\infty(u_t) < 0$ for $t > 0$ large. Note that

$$\|u_t\|^2 = at^2\int_{\mathbb{R}^3} |\nabla u|^2\,dx + t^4\int_{\mathbb{R}^3} V_\infty u^2\,dx.$$

Taking $e = u_{t_0}$, with $t_0 > 0$ large, then we have $\|e\| > \rho$ and $I_\infty(e) < 0$. ☐

Now we can define the mountain-pass level of I_∞:

$$c_\infty = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I_\infty(\gamma(t)) > 0,$$

where $\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = 0, I_\infty(\gamma(1)) < 0\}$.

Let

$$m_\infty = \inf_{u\in\mathcal{M}_\infty} I_\infty(u),$$

then for any $u \in \mathcal{M}_\infty$, we have

$$I_\infty(u) = I_\infty(u) - \frac{1}{4}G_\infty(u) \geq \frac{a}{4}\int_{\mathbb{R}^3} |\nabla u|^2\,dx \geq 0.$$

Thus, m_∞ is well defined. In addition, by the similar argument as Chapter 4 [19], we have the following property:

$$c_\infty = \inf_{u\in X\backslash\{0\}}\max_{t>0} I_\infty(u_t) = m_\infty = \inf_{u\in\mathcal{M}_\infty} I_\infty(u).$$

Lemma 3.6. *Assume that $(f1)$–$(f4)$ hold, then m_∞ is obtained.*

Proof. Let $\{u_n\} \subset \mathcal{M}_\infty$ be such that $I_\infty(u_n) \to m_\infty$. Since $G_\infty(u_n) = 0$, we have:

$$\begin{aligned}1 + m_\infty > I_\infty(u_n) &= I_\infty(u_n) - \frac{1}{4}G_\infty(u_n)\\ &= \frac{a}{4}\int_{\mathbb{R}^3} |\nabla u_n|^2\,dx + \frac{1}{8}\int_{\mathbb{R}^3} [f(u_n)u_n - 2F(u_n)]\,dx\\ &\quad + \frac{p-1+\alpha}{8p}\int_{\mathbb{R}^3} (I_\alpha * |u|^p)|u|^p\,dx\\ &> \frac{a}{4}\int_{\mathbb{R}^3} |\nabla u_n|^2\,dx\end{aligned}$$

for n large enough. Therefore, $\{|\nabla u_n|_2^2\}$ is bounded. So there exists $A \geq 0$ such that $\int_{\mathbb{R}^3} |\nabla u_n|^2 \mathrm{d}x \to A$. We claim that $\{|u_n|_2^2\}$ is also bounded. By Lemmas 2.1(1) and 2.2(1), the definition of $G_\infty(u_n)$ and Sobolev inequality, we have

$$\begin{aligned}
2\int_{\mathbb{R}^3} V_\infty u_n^2 \mathrm{d}x &= \frac{1}{2}\int_{\mathbb{R}^3} [6F(u_n)\mathrm{d}x + f(u_n)u_n]\mathrm{d}x + \frac{p+3+\alpha}{2p}\int_{\mathbb{R}^3} (I_\alpha * |u_n|^p)|u_n|^p \mathrm{d}x \\
&\quad - a\int_{\mathbb{R}^3} |\nabla u_n|^2 \mathrm{d}x - b\left(\int_{\mathbb{R}^3} |\nabla u_n|^2 \mathrm{d}x\right)^2 \\
&< \frac{1}{2}\int_{\mathbb{R}^3} [6F(u_n) + f(u_n)u_n]\mathrm{d}x + \frac{p+3+\alpha}{2p}\int_{\mathbb{R}^3} (I_\alpha * |u_n|^p)|u_n|^p \mathrm{d}x \\
&< \varepsilon|u_n|_2^2 + C_\varepsilon |u_n|_q^q + C|u_n|_{\frac{6p}{3+\alpha}}^{2p} \\
&< \varepsilon|\nabla u_n|_2^2 + C_\varepsilon |\nabla u_n|_2^q + C|\nabla u_n|_2^{2p},
\end{aligned}$$

which indicates $\{|u_n|_2^2\}$ is bounded. Then let $\delta = \lim_{n\to\infty} \sup_{y\in\mathbb{R}^3} \int_{B_1(y)} |u_n|^2 \mathrm{d}x$. Next we prove $\delta > 0$. If $\delta = 0$, then by Lions' concentration compactness principle [19], we have $u_n \to 0$ in $L^q(\mathbb{R}^3)$ for $q \in (2,6)$. From Lemma 2.2, we can obtain that

$$\int_{\mathbb{R}^3} (I_\alpha * |u_n|^p)|u_n|^p \mathrm{d}x \leq C|u_n|_{\frac{6p}{3+\alpha}}^{2p} \to 0,$$

since $2 < \frac{6p}{3+\alpha} < 6$. Together with $G_\infty(u_n) = 0$, we have $u_n \to 0$ in X. This conflicts with the fact that $c_\infty > 0$. So $\delta > 0$ and there exists $\{y_n\} \subset \mathbb{R}^3$ such that $\int_{B_1(y_n)} |u_n|^2 \mathrm{d}x \geq \frac{\delta}{2} > 0$. We set $v_n(x) = u_n(x + y_n)$, then

$$\|u_n\| = \|v_n\|, \qquad \int_{B_1(0)} |v_n|^2 \mathrm{d}x > \frac{\delta}{2}$$

and

$$I_\infty(v_n) \to m_\infty, \quad G_\infty(v_n) = 0.$$

Therefore, there exists $v \in X\backslash\{0\}$ such that

$$\begin{cases} v_n \rightharpoonup v & \text{in } X, \\ v_n \to v & \text{in } L^s_{loc}(\mathbb{R}^3), \quad \forall s \in [1,6) \\ v_n \to v & a.e. \ \text{ on } \mathbb{R}^3 \end{cases}$$

and $\int_{\mathbb{R}^3} |\nabla v_n|^2 \mathrm{d}x \to A$. Then, we set

$$I_{A,\infty}(u) = \frac{a+bA}{2}\int_{R^3} |\nabla u|^2 \mathrm{d}x + \frac{1}{2}\int_{R^3} V_\infty u^2 \mathrm{d}x - \frac{1}{2p}\int_{\mathbb{R}^3} (I_\alpha * |u|^p)|u|^p \mathrm{d}x - \int_{\mathbb{R}^3} F(u)\mathrm{d}x.$$

Since $I'_\infty(v_n) \to 0$ and $v_n \to v$ in X, we have $I'_{A,\infty}(v) = 0$. Besides v satisfied the following Pohozaev identity:

$$P_{A,\infty} = \frac{a+bA}{2}\int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x + \frac{3}{2}\int_{\mathbb{R}^3} V_\infty |u|^2 \mathrm{d}x - \frac{3+\alpha}{2p}\int_{\mathbb{R}^3} (I_\alpha * |u|^p)|u|^p \mathrm{d}x - 3\int_{\mathbb{R}^3} F(u)\mathrm{d}x = 0.$$

Suppose that $\int_{\mathbb{R}^3} |\nabla v|^2 \mathrm{d}x < A$, then we have

$$0 = \frac{\langle I'_{A,\infty}(v), v\rangle}{2} + P_{A,\infty}(v) > G_\infty(v).$$

Thus, by Lemma 3.4, there exists $t_0 \in (0,1)$ such that $v_{t_0} \in \mathcal{M}_\infty$. Hence, we have

$$
\begin{aligned}
c_\infty &\le I_\infty\big(v_{t_0}\big) \\
&= I_\infty\big(v_{t_0}\big) - \frac{1}{4}G_\infty\big(v_{t_0}\big) \\
&= \frac{at_0^2}{4}\int_{\mathbb{R}^3}|\nabla v|^2\,\mathrm{d}x + \frac{t_0^3}{8}\int_{\mathbb{R}^3}[f(\sqrt{t}v)\sqrt{t}v - 2F(\sqrt{t}v)]\mathrm{d}x + \frac{p+\alpha-1}{8p}t_0^{p+3+\alpha}\int_{\mathbb{R}^3}(I_\alpha * |v|^p)|v|^p\,\mathrm{d}x \\
&< \frac{a}{4}\int_{\mathbb{R}^3}|\nabla v|^2\,\mathrm{d}x + \frac{1}{8}\int_{\mathbb{R}^3}[f(v)v - 2F(v)]\mathrm{d}x + \frac{p+\alpha-1}{8p}\int_{\mathbb{R}^3}(I_\alpha * |v|^p)|v|^p\,\mathrm{d}x \\
&\le \lim_{n\to\infty}\frac{a}{4}\int_{\mathbb{R}^3}|\nabla v_n|^2\,\mathrm{d}x + \frac{1}{8}\int_{\mathbb{R}^3}[f(v_n)v_n - 2F(v_n)]\mathrm{d}x + \frac{p+\alpha-1}{8p}\int_{\mathbb{R}^3}(I_\alpha * |v_n|^p)|v_n|^p\,\mathrm{d}x \\
&= \lim_{n\to\infty}\left[I_\infty(v_n) - \frac{1}{4}G_\infty(v_n)\right] = m_\infty.
\end{aligned}
$$

This is a contradiction. Consequently, we obtain $\int_{\mathbb{R}^3}|\nabla v|^2\,\mathrm{d}x = A = \lim_{n\to\infty}\int_{\mathbb{R}^3}|\nabla v_n|^2\,\mathrm{d}x$. So $G_\infty(v) = 0$, $I_\infty(v) = m_\infty$. This completes the proof. □

Lemma 3.7. *If m_∞ is obtained at some $\tilde{u} \in \mathcal{M}_\infty$, then $\tilde{u}$ is a critical point of $I_\infty(u)$.*

Proof. It is obvious that $\tilde{u} \neq 0$ since $\tilde{u} \in \mathcal{M}_\infty$. Then we claim that for every fixed $v \in H^1(\mathbb{R}^3)$, there exists $\varepsilon > 0$ such that $\tilde{u} + sv \neq 0$ for all $s \in (-\varepsilon, \varepsilon)$. In fact, by contradiction, there exists a sequence $\{s_i\}_{i=1}^\infty$ such that $\lim_{i\to+\infty}s_i = 0$ and $\tilde{u} + s_i v = 0$ a.e. on $\mathbb{R}^3$. Letting $i \to +\infty$, we have $\tilde{u} = 0$ a.e. on $\mathbb{R}^3$, which is a contradiction with $\tilde{u} \neq 0$. Then by Lemma 3.3, there exists only one $t_0 > 0$ such that $(\tilde{u} + sv)_{t(s)} \in \mathcal{M}_\infty$. Now consider the function $\Phi(t, s) = G_\infty((\tilde{u} + sv)_t)$ defined for $(t, s) \in (0, +\infty) \times (-\varepsilon, \varepsilon)$. Since $\tilde{u} \in \mathcal{M}_\infty$, one has $\Phi(1, 0) = G_\infty(\tilde{u}) = 0$. Moreover, Φ is a C^1 function and

$$\left.\frac{\partial\Phi(t,s)}{\partial t}\right|_{(t,s)=(1,0)} < 0.$$

By the implicit function theorem, the function $t(s)$ is C^1 and being $t(0) = 1$, then one can know $t(s) \neq 0$ near 0. By letting $\gamma(s) = I_\infty((\tilde{u} + sv)_{t(s)})$, one has γ is differentiable for all small s and attains its minimum at $s = 0$. Therefore, we can deduce

$$
\begin{aligned}
0 &= \gamma'(0) \\
&= \left.\frac{\mathrm{d}I_\infty((\tilde{u} + sv)_{t(s)})}{\mathrm{d}s}\right|_{s=0} \\
&= \left.\frac{\partial I_\infty((\tilde{u} + sv)_t)}{\partial t}\right|_{(t,s)=(1,0)}\left.\frac{\mathrm{d}t}{\mathrm{d}s}\right|_{t=0} + \left.\frac{\partial I_\infty((\tilde{u} + sv)_t)}{\partial s}\right|_{(t,s)=(1,0)} \\
&= G_\infty(\tilde{u})t'(0) + \langle I_\infty'(\tilde{u}), v\rangle = \langle I_\infty'(\tilde{u}), v\rangle.
\end{aligned}
$$

Since $v \in X$ is arbitrary, we deduce that $I_\infty'(\tilde{u}) = 0$. □

Then by Lemmas 3.6 and 3.7, we can have the following result.

Theorem 3.8. *Under assumptions* $(f1)$–$(f4)$, *Problem* (11) *has a ground state solution.*

4 Ground state solution for problem (1)

Proposition 4.1. (See [20]) *Let $(X, \|\cdot\|)$ be a Banach space and $T \subset \mathbb{R}^+$ be an interval. $\Phi_\tau(u)$ is a family of C^1 functions on X of the following form*:

$$\Phi_\tau(u) = A(u) - \tau B(u), \forall \tau \in T,$$

with $B(u) \geq 0$, $\forall u \in X$ and either $A(u) \to +\infty$ or $B(u) \to +\infty$ as $\|u\| \to \infty$. Suppose that there are two points $v_1, v_2 \in X$ such that

$$c_\tau = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} \Phi_\tau(\gamma(t)) > \max\{\Phi_\tau(v_1), \Phi_\tau(v_2)\}, \quad \forall \tau \in T,$$

where $\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = v_1, \gamma(1) = v_2\}$, then for almost every $\tau \in T$, there is a bounded $(PS)_{c_\tau}$ sequence in X.

Set $T = [\delta, 1]$, where $\delta > 0$. We investigate a family of functionals on X with the following form:

$$I_{V,\tau}(u) = \frac{1}{2}\int_{\mathbb{R}^3} [a|\nabla u|^2 + V(x)u^2]\mathrm{d}x + \frac{b}{4}\left(\int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x\right)^2 - \tau\left[\frac{1}{2p}\int_{\mathbb{R}^3} (I_\alpha * |u|^p)|u|^p \mathrm{d}x + \int_{\mathbb{R}^3} F(u)\mathrm{d}x\right], \quad \forall \tau \in [\delta, 1].$$

Then we can set $I_{V,\tau}(u) = A(u) - \tau B(u)$. Here,

$$A(u) = \frac{1}{2}\int_{\mathbb{R}^3} [a|\nabla u|^2 + V(x)u^2]\mathrm{d}x + \frac{b}{4}\left(\int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x\right)^2 \to +\infty,$$

as $\|u\| \to +\infty$, and

$$B(u) = \frac{1}{2p}\int_{\mathbb{R}^3} (I_\alpha * |u|^p)|u|^p \mathrm{d}x + \int_{\mathbb{R}^3} F(u)\mathrm{d}x \geq 0.$$

Lemma 4.2. *Assume* (V2) *holds, then we have*
1. *there exists a $v \in X\backslash\{0\}$ such that $I_{V,\tau}(v) \leq 0$ for all $\tau \in [\delta, 1]$,*
2. *$c_\tau = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I_{V,\tau}(\gamma(t)) > \max\{I_{V,\tau}(0), I_{V,\tau}(v)\}$ for all $\tau \in [\delta, 1]$,*

where

$$\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = 0, \gamma(1) = v\}.$$

Proof. (1) Fix $u \in X\backslash\{0\}$, then for $\forall \tau \in [\delta, 1]$ and $t > 0$, we have

$$I_{V,\tau}(u_t) \leq I_{V_\infty,\delta}(u_t) = \frac{at^2}{2}\int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x + \frac{t^4}{2}\int_{\mathbb{R}^3} V_\infty u^2 \mathrm{d}x + \frac{bt^4}{4}\left(\int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x\right)^2 - \delta\left[\frac{t^{p+3+\alpha}}{2p}\int_{\mathbb{R}^3} (I_\alpha * |u|^p)|u|^p \mathrm{d}x + t^3\int_{\mathbb{R}^3} F(\sqrt{t}u)\mathrm{d}x\right] \to -\infty$$

as $t \to +\infty$. Just by taking $v = u_t$ with t large, one can have

$$I_{V,\tau}(u_t) \leq I_{V_\infty,\delta}(u_t) < 0.$$

(2) By Lemmas 2.1(1) and 2.2, we have

$$I_{V,\tau}(u) \geq c_1\|u\|^2 - c_2\|u\|^{2p} - C_\varepsilon\|u\|^q.$$

Since $p > 1$, $I_{V,\tau}(u)$ has a strictly local minima at 0, i.e., there exists $r > 0$ such that

$$b = \inf_{\|u\|=r} I_{V,\tau}(u) > 0 = I_{V,\tau}(0) \geq I_{V,\tau}(v),$$

and hence taking $u_t = v$, we obtain $c_\tau > \max\{I_{V,\tau}(0), I_{V,\tau}(u_t)\} = 0$. □

Lemma 4.3. (See [20]) *Assume the conditions of Proposition* 4.1 *hold, the map* $\tau \mapsto c_\tau$ *is nonincreasing and left continuous.*

By Theorem 3.8, we conclude that for $\forall\ \tau \in [\delta, 1]$, the "limit problem" of the following type

$$\begin{cases} -\left(a + b\int_{\mathbb{R}^3} |\nabla u|^2\right)\Delta u + V_\infty u = \tau[(I_\alpha * |u|^p)|u|^{p-2}u + f(u)], & \text{in } \mathbb{R}^3, \\ u \in H^1(\mathbb{R}^3) \end{cases} \tag{15}$$

has a ground state solution $u_\tau \in H^1(\mathbb{R}^3)$, i.e., for $\forall\ \tau \in [\delta, 1]$, there exists $u_\tau \in \mathcal{M}_\tau = \{u \in X\backslash\{0\} : G_{\infty,\tau}(u) = 0\}$ such that $I'_{V_\infty,\tau}(u_\tau) = 0$ and $I_{V_\infty,\tau}(u_\tau) = m_\tau = \inf_{u\in\mathcal{M}_\tau} I_{V_\infty,\tau}(u)$. Here,

$$\begin{aligned} I_{V_\infty,\tau}(u) = {} & \frac{1}{2}\int_{\mathbb{R}^3} [a|\nabla u|^2 + V_\infty u^2]\mathrm{d}x + \frac{b}{4}\left(\int_{\mathbb{R}^3} |\nabla u|^2\,\mathrm{d}x\right)^2 \\ & - \tau\left[\frac{1}{2p}\int_{\mathbb{R}^3} (I_\alpha * |u|^p)|u|^p\,\mathrm{d}x + \int_{\mathbb{R}^3} F(u)\mathrm{d}x\right], \end{aligned}$$

$$\begin{aligned} G_{\infty,\tau}(u) = {} & a\int_{\mathbb{R}^3} |\nabla u|^2\,\mathrm{d}x + 2\int_{\mathbb{R}^3} V_\infty u^2\mathrm{d}x + b\left(\int_{\mathbb{R}^3} |\nabla u|^2\,\mathrm{d}x\right)^2 \\ & - \frac{p+3+\alpha}{2p}\tau\int_{\mathbb{R}^3} (I_\alpha * |u|^p)|u|^p\,\mathrm{d}x - 3\tau\int_{\mathbb{R}^3} F(u)\mathrm{d}x - \frac{\tau}{2}\int_{\mathbb{R}^3} f(u)u\mathrm{d}x. \end{aligned}$$

Lemma 4.4. *Suppose that* $(V1)$–$(V2)$ *hold, and* $V(x) \neq V_\infty$, *then* $c_\tau < m_\tau$ *for* $\forall \tau \in [\delta, 1]$.

Proof. Let u_τ be the minimizer of m_τ. By Lemma 4.2, there exists $\tilde{t} \in (0, t_0)$ such that

$$\begin{aligned} c_\tau &= \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I_{V,\tau}(\gamma(t)) \le \max_{0<t<t_0} I_{V,\tau}\left(\sqrt{t}\,u_\tau\left(\frac{x}{t}\right)\right) = I_{V,\tau}\left(\sqrt{\tilde{t}}\,u_\tau\left(\frac{x}{\tilde{t}}\right)\right) \\ &< I_{V_\infty,\tau}\left(\sqrt{\tilde{t}}\,u_\tau\left(\frac{x}{\tilde{t}}\right)\right) \le \max_{t>0} I_{V_\infty,\tau}\left(\sqrt{t}\,u_\tau\left(\frac{x}{t}\right)\right) = I_{V_\infty,\tau}(u_\tau) = m_\tau. \end{aligned}$$ □

Next we provide the following global compactness lemma.

Lemma 4.5. (See [3,13]) *Suppose that* $(V1)$–$(V2)$ *and* $(f1)$–$(f4)$ *hold. For* $c > 0$ *and* $\tau \in [\delta, 1]$, *let* $\{u_n\} \subset X$ *be a bounded* $(PS)_c$ *sequence for* $I_{V,\tau}$. *Then there exists a* $u_0 \in X$ *and* $A \in \mathbb{R}$ *such that* $J'_{V,\tau}(u_0) = 0$. *Here,*

$$J_{V,\tau}(u) = \frac{a + bA^2}{2}\int_{\mathbb{R}^3} a|\nabla u|^2\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^3} V(x)u^2\mathrm{d}x - \tau\left[\frac{1}{2p}\int_{\mathbb{R}^3} (I_\alpha * |u|^p)|u|^p\,\mathrm{d}x + \int_{\mathbb{R}^3} F(u)\mathrm{d}x\right].$$

Moreover, either
1. $u_n \to u_0$ *strongly in* $H^1(\mathbb{R}^3)$, *or*
2. *there exists a finite(possibly empty) set* $u_1, u_2, \ldots, u_k \subset X$ *of nontrivial solutions of*

$$-(a + bA^2)\Delta u + V_\infty u = \tau[(I_\alpha * |u|^p)|u|^{p-2}u + f(u)],$$

and $y_n^i \subset \mathbb{R}^3$, $i = 1, 2, 3, \ldots, k(k \in N^+)$, *such that*

$$|y_n^i| \to \infty, \quad |y_n^i - y_n^j| \to \infty (i \neq i), \quad \text{as } n \to \infty;$$

$$c + \frac{bA^4}{4} = J_{V,\tau}(u_0) + \sum_{i=1}^{k} J_{V_\infty,\tau}(u_i);$$

$$\left\| u_n - u_0 - \sum_{i=1}^{k} (\cdot - y_n^i) u_i \right\| \to 0;$$

$$A^2 = |\nabla u_0|_2^2 + \sum_{i=1}^{k} |\nabla u_i|_2^2,$$

where

$$J_{V_\infty,\tau}(u) = \frac{a + bA^2}{2} \int_{\mathbb{R}^3} a|\nabla u|^2 \mathrm{d}x + \frac{1}{2} \int_{\mathbb{R}^3} V_\infty u^2 \mathrm{d}x - \tau \left[\frac{1}{2p} \int_{\mathbb{R}^3} (I_\alpha * |u|^p)|u|^p \mathrm{d}x + \int_{\mathbb{R}^3} F(u)\mathrm{d}x \right].$$

Lemma 4.6. (See [3,13]) *Suppose that* $(V1)$–$(V2)$ *and* $(f1)$–$(f4)$ *hold. For* $\tau \in [\delta, 1]$, *let* $\{u_n\} \subset X$ *be a bounded* $(PS)_{c_\tau}$ *sequence for* $I_{V,\tau}$. *Then there exists a nontrivial* $u_\tau \in X$ *such that* $u_n \to u_\tau$ *strongly in X.*

Now, we can prove the main theorem.

Proof of Theorem 1.1. In the view of Proposition 4.1 and Lemma 4.2, we see for a.e.$\tau \in [\delta, 1]$, there exists a bounded sequence $\{u_n\} \subset X$ such that $I_{V,\tau}(u_n) \to c_\tau$, $I'_{V,\tau}(u_n) \to 0$. By Lemma 4.6, $I_{V,\tau}$ has a nontrivial critical point $u_\tau \in X$ and $I_{V,\tau}(u_\tau) = c_\tau$ for a.e. $\tau \in [\delta, 1]$. Next, we choose an arbitrary sequence $\{\tau_n\} \subset [\delta, 1]$ with $\tau_n \to 1^-$, then we obtain a sequence $\{u_{\tau_n}\} \subset X$ such that $I'_{V,\tau_n}(u_{\tau_n}) = 0$ and $I_{V,\tau_n}(u_{\tau_n}) = c_{\tau_n}$. In the following, we show that $\{u_{\tau_n}\}$ is bounded in X. By $(V1)$ and Hardy inequality, using the similar argument in Lemma 3.5, we can derive that both $|\nabla u_{\tau_n}|_2$ and $|u_{\tau_n}|_2$ are bounded. Thus, $\{u_{\tau_n}\}$ is bounded in X.

On the other hand, since $\tau_n \to 1^-$, by Lemma 4.3, we have

$$\begin{aligned} \lim_{n\to\infty} I\big(u_{\tau_n}\big) &= \lim_{n\to\infty} I_{V,1}\big(u_{\tau_n}\big) \\ &= \lim_{n\to\infty} \left[I_{V,\tau_n}\big(u_{\tau_n}\big) + (\tau_n - 1)\left(\frac{1}{2p} \int_{\mathbb{R}^3} \big(I_\alpha * |u_{\tau_n}|^p\big)|u_{\tau_n}|^p \mathrm{d}x + \int_{\mathbb{R}^3} F\big(u_{\tau_n}\big)\mathrm{d}x \right) \right] \\ &= \lim_{n\to\infty} c_{\tau_n} = c_1 \end{aligned}$$

and

$$\begin{aligned} \lim_{n\to\infty} \langle I'\big(u_{\tau_n}\big), \varphi\rangle &= \lim_{n\to\infty} \langle I'_{V,1}\big(u_{\tau_n}\big), \varphi\rangle \\ &= \lim_{n\to\infty} \left[\langle I'_{V,\tau_n}\big(u_{\tau_n}\big), \varphi\rangle + (\tau_n - 1)\left(\int_{\mathbb{R}^3} \big(I_\alpha * |u_{\tau_n}|^p\big)|u_{\tau_n}|^{p-2} u_{\tau_n}\varphi \mathrm{d}x + \int_{\mathbb{R}^3} f\big(u_{\tau_n}\big)\varphi \mathrm{d}x \right) \right] \\ &= 0. \end{aligned}$$

That is $\{u_{\tau_n}\}$ is a bounded $(PS)_{c_1}$ sequence for I. Again by Lemma 4.6, there exists $u_0 \in X$ such that $I(u_0) = c_1$, $I'(u_0) = 0$, which means u_0 is a nontrivial solution of Problem (1).

Finally, we prove the existence of ground state solution. Set $m = \inf_S I(u)$, where $S = \{u \in X\backslash\{0\} : I'(u) = 0\}$. Now we show $0 < m < \infty$. Since $u_0 \in S$, we see $m \le c_1 < \infty$. For any $u \in S$, we have

$$\begin{aligned} 0 &= \langle I'(u), u\rangle \\ &= a \int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x + \int_{\mathbb{R}^3} V(x)u^2 \mathrm{d}x + b\left(\int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x \right)^2 - \int_{\mathbb{R}^3} (I_\alpha * |u|^p)|u|^p \mathrm{d}x - \int_{\mathbb{R}^3} f(u)u\mathrm{d}x \\ &\ge c_1\|u\|^2 - c_2\|u\|^{2p} - C_\varepsilon\|u\|^q. \end{aligned}$$

This indicates that $\|u\| \geq \delta$ for some $\delta > 0$. On the other hand, by the Pohozaev identity, i.e., $P_V(u) = 0$. Then by $(V1)$ and Hardy's inequality, we obtain:

$$\begin{aligned}
I(u) &= I(u) - \frac{1}{8}[\langle I'(u), u\rangle + 2P_V(u)] \\
&= \frac{a}{4}\int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x - \frac{1}{8}\int_{\mathbb{R}^3} (\nabla V(x), x)u^2 \mathrm{d}x \\
&\quad + \frac{p+\alpha-1}{8p}\int_{\mathbb{R}^3} (I_\alpha * |u|^p)|u|^p \mathrm{d}x + \frac{1}{8}\int_{\mathbb{R}^3} [uf(u) - 2F(u)]\mathrm{d}x \\
&\geq \frac{a}{4}\int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x - \frac{1}{8}\int_{\mathbb{R}^3} (\nabla V(x), x)u^2 \mathrm{d}x \\
&\geq \frac{a}{4}\int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x - \frac{A}{16}\int_{\mathbb{R}^3} \frac{u^2}{|x|^2} \mathrm{d}x \\
&\geq \frac{a}{4}\int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x - \frac{A}{4}\int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x \\
&= \frac{a-A}{4}\int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x.
\end{aligned}$$

This implies $m \geq 0$. In the following, let us rule out $m = 0$. If $m = 0$, then there exists minimizing sequence $\{u_n\} \subset S$ such that $I(u_n) \to 0$, which implies $\lim_{n\to\infty}\int_{\mathbb{R}^3}|\nabla u_n|^2 \mathrm{d}x = 0$. Since $\langle I'(u_n), u_n\rangle = 0$, we can infer $\lim_{n\to\infty}\int_{\mathbb{R}^3}|u_n|^2 dx = 0$. Therefore, $\lim_{n\to\infty}\|u_n\|^2 = 0$, which contradicts to $\|u_n\| > \delta$. This proves our claim.

Then let $\{u_n\} \subset S$ be a minimizing sequence such that $I'(u_n) = 0$ and $I(u_n) \to m$. By similar argument as mentioned earlier, one can conclude that $\{u_n\}$ is bounded. Again by Lemma 4.6, there exists a $u \in X$ such that $u_n \to u$ strongly in X. Thus, $I'(u) = 0, \quad I(u) = m$. This implies u is a ground state solution for Problem (1). Then we finish the proof. □

5 Ground state solution for problem (2)

As we all know, a weak solution of Problem (2) is a critical point of the following functional:

$$\begin{aligned}
J_m(u) = \frac{1}{2}\int_{\mathbb{R}^3} [a|\nabla u|^2 + V(x)u^2]\mathrm{d}x + \frac{b}{4}\left(\int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x\right)^2 \\
- \frac{1}{2p}\int_{\mathbb{R}^3} (I_\alpha * |u|^p)|u|^p \mathrm{d}x - \frac{m}{l}\int_{\mathbb{R}^3} |u|^l \mathrm{d}x.
\end{aligned}$$

But obviously we cannot apply variational methods directly because that the functional J_m is not well defined when $l > 6$. To solve this difficulty, we define the following function:

$$\varphi(t) = \begin{cases} |t|^{l-2}t & \text{if} \quad |t| \leq M, \\ M^{l-q}|t|^{q-2}t & \text{if} \quad |t| > M. \end{cases}$$

where $M > 0$. Then $\varphi \in C(\mathbb{R}, \mathbb{R})$ and $|\varphi(t)| \leq M^{l-q}|t|^{q-1}$ for all $t \in \mathbb{R}$. Thus, $m\varphi(t)(m > 0)$ verifies the conditions $(f1)$–$(f4)$. By Theorem 1.1, we know the equation:

$$-\left(a + b\int_{R^3} |\nabla u|^2\right)\Delta u + V(x)u = (I_\alpha * |u|^p)|u|^{p-2}u + m\varphi(t), \quad \text{in} \quad \mathbb{R}^3 \tag{16}$$

has a ground state solution u_m.

Let

$$\tilde{J}_m(u) = \frac{1}{2}\int_{\mathbb{R}^3}[a|\nabla u|^2 + V(x)u^2]\mathrm{d}x + \frac{b}{4}\left(\int_{\mathbb{R}^3}|\nabla u|^2\,\mathrm{d}x\right)^2 - \frac{1}{2p}\int_{\mathbb{R}^3}(I_\alpha * |u|^p)|u|^p\,\mathrm{d}x - m\int_{\mathbb{R}^3}\Phi(u)\mathrm{d}x,$$

where $\Phi(t) = \int_0^t \varphi(s)\mathrm{d}s$. As a consequence, $\tilde{J}_m(u_m) = c_m$, and $\tilde{J}'_m(u_m) = 0$. By characterization of minimax level, we can derive that

$$c_m = \inf_{\gamma\in\Gamma_m}\max_{t\in[0,1]}\tilde{J}_m(\gamma(t)),$$

where

$$\Gamma_m = \{\gamma \in C([0,1], X) : \gamma(0) = 0, \tilde{J}_m(\gamma(1)) < 0\}.$$

Next, set

$$J(u) = \frac{1}{2}\int_{\mathbb{R}^3}[a|\nabla u|^2 + V(x)u^2]\mathrm{d}x + \frac{b}{4}\left(\int_{\mathbb{R}^3}|\nabla u|^2\,\mathrm{d}x\right)^2 - \frac{1}{2p}\int_{\mathbb{R}^3}(I_\alpha * |u|^p)|u|^p\,\mathrm{d}x - \int_{\mathbb{R}^3}\Phi(u)\mathrm{d}x,$$

and

$$c = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]}J(\gamma(t)),$$

where

$$\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = 0, J(\gamma(1)) < 0\}.$$

Then obviously we have $\Gamma \subset \Gamma_m$ and $c_m \le c$.

Next we only need to prove the following lemmas.

Lemma 5.1. *The solution u_m satisfies $|\nabla u_m|_2^2 \le \frac{4c_m}{a-A}$, and there exists a constant $A^* > 0$, which is independent on m such that $|\nabla u_m|_2^2 \le A^*$*

Proof. By Pohozaev identity, $(V1)$ and Hardy inequality, we have

$$\begin{aligned}
c_m &= \tilde{J}_m(u_m) - \frac{1}{8}[\langle \tilde{J}'_m(u_m), u_m\rangle + 2P_V(u_m)]\\
&= \frac{a}{4}\int_{\mathbb{R}^3}|\nabla u_m|^2\mathrm{d}x - \frac{1}{8}\int_{\mathbb{R}^3}(\nabla V(x), x)u_m^2\mathrm{d}x\\
&\quad + \frac{p+\alpha-1}{8p}\int_{\mathbb{R}^3}(I_\alpha * |u_m|^p)|u_m|^p\,\mathrm{d}x + \frac{m}{8}\int_{\mathbb{R}^3}[u_m\varphi(u_m) - 2\Phi(u_m)]\mathrm{d}x\\
&\ge \frac{a}{4}\int_{\mathbb{R}^3}|\nabla u_m|^2\mathrm{d}x - \frac{1}{8}\int_{\mathbb{R}^3}(\nabla V(x), x)u_m^2\mathrm{d}x\\
&\ge \frac{a}{4}\int_{\mathbb{R}^3}|\nabla u_m|^2\mathrm{d}x - \frac{A}{16}\int_{\mathbb{R}^3}\frac{u_m^2}{|x|^2}\mathrm{d}x\\
&\ge \frac{a}{4}\int_{\mathbb{R}^3}|\nabla u_m|^2\mathrm{d}x - \frac{A}{4}\int_{\mathbb{R}^3}|\nabla u_m|^2\mathrm{d}x\\
&= \frac{a-A}{4}\int_{\mathbb{R}^3}|\nabla u_m|^2\mathrm{d}x,
\end{aligned}$$

which implies $|\nabla u_m|_2^2 \le \frac{4c_m}{a-A} \le \frac{4c}{a-A} := A^* > 0$. This completes the proof. □

By some parts of the ideas of the proof, which comes from [14,15], we can obtain the following lemma.

Lemma 5.2. *There exist two constants $B, D > 0$, which is independent on m such that $\|u_m\|_{L^\infty} \le B(1+m)^D$.*

Proof. Set $I > 2$, $r > 0$ and $\tilde{u}_m^I := b(u_m)$, where $b : \mathbb{R} \to \mathbb{R}$ is a smooth function, which satisfies $b(s) = s$ for $|s| \le I - 1$, $b(-s) = -b(s)$; $b'(s) = 0$ for $s \ge I$ and $b'(s)$ is decreasing in $[I-1, I]$. This implies that

$$\begin{cases} \tilde{u}_m^I = u_m, & \text{for } |u_m| \le I-1, \\ |\tilde{u}_m^I| = |b(u_m)| \le |u_m|, & \text{for } I-1 \le |u_m| \le I, \\ |\tilde{u}_m^I| = C_I > 0, & \text{for } |u_m| \ge I, \end{cases}$$

where $I - 1 \le C_I \le I$. Moreover, one can easily have

$$0 \le \frac{sb'(s)}{b(s)} \le 1, \quad \forall s \ne 0.$$

Let $\psi = u_m|\tilde{u}_m^I|^{2r}$. Then $\psi \in X$, if one takes ψ as the test function, one can have

$$\begin{aligned} &\int_{\mathbb{R}^3} (I_\alpha * |u_m|^p)|u_m|^{p-2}u_m\psi \mathrm{d}x + m \int_{\mathbb{R}^3} \varphi(u_m)\psi \\ &= a\int_{\mathbb{R}^3} \nabla u_m \nabla\psi \mathrm{d}x + b\int_{\mathbb{R}^3} |\nabla u_m|^2 \mathrm{d}x \int_{\mathbb{R}^3} \nabla u_m\nabla\psi \mathrm{d}x + \int_{\mathbb{R}^3} V(x)u_m\psi \mathrm{d}x. \end{aligned} \tag{17}$$

Note that

$$\begin{aligned} \int_{\mathbb{R}^3} \nabla u_m \nabla \psi \mathrm{d}x \ge& \int_{|u_m|\le I-1} (1+r)|\tilde{u}_m^I|^{2r}|\nabla u_m|^2 \mathrm{d}x + \int_{|u_m|\ge I} |\tilde{u}_m^I|^{2r}|\nabla u_m|^2 \mathrm{d}x \\ &+ \int_{I-1<|u_m|<I} [|\tilde{u}_m^I|^{2r} + 2ru_m b(u_m)b'(u_m)|\tilde{u}_m^I|^{2r-2}]|\nabla u_m|^2 \mathrm{d}x \\ \ge& \int_{|u_m|\le I-1} |\tilde{u}_m^I|^{2r}|\nabla u_m|^2 \mathrm{d}x + \int_{|u_m|\ge I} |\tilde{u}_m^I|^{2r}|\nabla u_m|^2 \mathrm{d}x \\ &+ \int_{I-1<|u_m|<I} [|\tilde{u}_m^I|^{2r} + 2ru_m b(u_m)b'(u_m)|\tilde{u}_m^I|^{2r-2}]|\nabla u_m|^2 \mathrm{d}x \\ \ge& \frac{1}{(1+r)^2}\int_{|u_m|\le I-1} |\nabla[u_m(\tilde{u}_m^I)^r]|^2 \mathrm{d}x + \int_{|u_m|\ge I} |\nabla[u_m(\tilde{u}_m^I)^r]|^2 \mathrm{d}x \\ &+ \int_{I-1<|u_m|<I} [|\tilde{u}_m^I|^{2r} + 2ru_m^2(b'(u_m))^2|\tilde{u}_m^I|^{2r-2}]|\nabla u_m|^2 \mathrm{d}x \\ \ge& \frac{1}{(1+r)^2}\int_{|u_m|\le I-1} |\nabla[u_m(\tilde{u}_m^I)^r]|^2 \mathrm{d}x + \int_{|u_m|\ge I} |\nabla[u_m(\tilde{u}_m^I)^r]|^2 \mathrm{d}x \\ &+ \int_{I-1<|u_m|<I} \left[\frac{1}{(1+r)^2}|\tilde{u}_m^I|^{2r} + \frac{r}{(1+r)^2}2ru_m^2(b'(u_m))^2|\tilde{u}_m^I|^{2r-2}\right]|\nabla u_m|^2 \mathrm{d}x \\ =& \frac{1}{(1+r)^2}\int_{|u_m|\le I-1} |\nabla[u_m(\tilde{u}_m^I)^r]|^2 \mathrm{d}x + \int_{|u_m|\ge I} |\nabla[u_m(\tilde{u}_m^I)^r]|^2 \mathrm{d}x \\ &+ \int_{I-1<|u_m|<I} \left[\frac{1}{(1+r)^2}b^{2r}(u_m)|\nabla u_m|^2 + \frac{2}{(1+r)^2}u_m^2|\nabla b^r(u_m)|^2\right] \mathrm{d}x \\ \ge& \frac{1}{(1+r)^2}\int_{|u_m|\le I-1} |\nabla[u_m(\tilde{u}_m^I)^r]|^2 \mathrm{d}x + \int_{|u_m|\ge I} |\nabla[u_m(\tilde{u}_m^I)^r]|^2 \mathrm{d}x \\ &+ \frac{2C_1}{(1+r)^2}\int_{I-1<|u_m|<I} [b^{2r}(u_m)|\nabla u_m|^2 + u_m^2|\nabla b^r(u_m)|^2] \mathrm{d}x \\ \ge& \frac{1}{(1+r)^2}\int_{|u_m|\le I-1} |\nabla[u_m(\tilde{u}_m^I)^r]|^2 \mathrm{d}x + \int_{|u_m|\ge I} |\nabla[u_m(\tilde{u}_m^I)^r]|^2 \mathrm{d}x \\ &+ \frac{C_1}{(1+r)^2}\int_{I-1<|u_m|<I} |\nabla[u_m(\tilde{u}_m^I)^r]|^2 \mathrm{d}x \\ \ge& \frac{C_1}{(1+r)^2}\int_{\mathbb{R}^3} |\nabla[u_m(\tilde{u}_m^I)^r]|^2 \mathrm{d}x. \end{aligned}$$

Hence, by (17), we obtain

$$\int_{\mathbb{R}^3}(I_\alpha * |u_m|^p)|u_m|^p|\tilde{u}_m^I|^{2r}\,\mathrm{d}x + m\int_{\mathbb{R}^3}\varphi(u_m)u_m|\tilde{u}_m^I|^{2r}\,\mathrm{d}x \geq \frac{C_1}{(1+r)^2}\int_{\mathbb{R}^3}|\nabla[u_m(\tilde{u}_m^I)^r]|^2\mathrm{d}x + \int_{\mathbb{R}^3}V(x)|u_m|^2|\tilde{u}_m^I|^{2r}\,\mathrm{d}x.$$

For any $\varepsilon > 0$, by Lemma 5.1, properties of $\tilde{u}_m^I$ and φ, there exists $C_\varepsilon > 0$ such that

$$\int_{\mathbb{R}^3}(I_\alpha * |u_m|^p)|u_m|^p|\tilde{u}_m^I|^{2r}\,\mathrm{d}x \leq I_1\int_{\mathbb{R}^3}(I_\alpha * |u_m|^p)|u_m|^p\,\mathrm{d}x \leq I_2|u_m|^{2p}_{\frac{6p}{3+\alpha}} < C|\nabla u_m|_2^{2p} \leq M,$$

where I_1, I_2, and M are positive constants, and

$$|\varphi(t)| \leq \varepsilon|t| + C_\varepsilon|t|^{2^*-1}$$

for all $t \in \mathbb{R}$, where $2^* = \frac{2N}{N-2}$ if $N \geq 3$ and $2^* = \infty$ if $N = 1$ or 2. Thus, for fixed $m > 0$ and small $\varepsilon > 0$, we have

$$\begin{aligned}
&\frac{C_1}{(1+r)^2}\int_{\mathbb{R}^3}|\nabla[u_m(\tilde{u}_m^I)^r]|^2\mathrm{d}x \\
&\leq \int_{\mathbb{R}^3}(I_\alpha * |u_m|^p)|u_m|^p|\tilde{u}_m^I|^{2r}\,\mathrm{d}x + m\int_{\mathbb{R}^3}\varphi(u_m)u_m|\tilde{u}_m^I|^{2r}\,\mathrm{d}x - \int_{\mathbb{R}^3}V(x)|u_m|^2|\tilde{u}_m^I|^{2r}\,\mathrm{d}x \\
&\leq M + \int_{\mathbb{R}^3}V_0|u_m|^2|\tilde{u}_m^I|^{2r}\,\mathrm{d}x + mC\int_{\mathbb{R}^3}u_m^{2^*}|\tilde{u}_m^I|^{2r}\,\mathrm{d}x - \int_{\mathbb{R}^3}V_0|u_m|^2|\tilde{u}_m^I|^{2r}\,\mathrm{d}x \\
&\leq (1+m)C\int_{\mathbb{R}^3}u_m^{2^*}|\tilde{u}_m^I|^{2r}\,\mathrm{d}x.
\end{aligned}$$

Notice that

$$\frac{C_2}{(1+r)^2}\left[\int_{\mathbb{R}^3}|u_m|^{2^*}|\tilde{u}_m^I|^{2r\cdot\frac{2^*}{2}}\,\mathrm{d}x\right]^{\frac{2}{2^*}} \leq \frac{C_1}{(1+r)^2}\int_{\mathbb{R}^3}|\nabla[u_m(\tilde{u}_m^I)^r]|^2\mathrm{d}x.$$

Consequently,

$$\left[\int_{\mathbb{R}^3}|u_m|^{2^*}|\tilde{u}_m^I|^{2r\cdot\frac{2^*}{2}}\,\mathrm{d}x\right]^{\frac{2}{2^*}} \leq (1+m)C(r+1)^2\int_{\mathbb{R}^3}u_m^{2^*}|\tilde{u}_m^I|^{2r}\,\mathrm{d}x.$$

Take $r_0 > 0$ and $r_k = r_0\left(\frac{2^*}{2}\right)^k = r_{k-1}\cdot\frac{2^*}{2}$. Then

$$\begin{aligned}
\left[\int_{\mathbb{R}^3}|u_m|^{2^*}|\tilde{u}_m^I|^{2r_k}\,\mathrm{d}x\right]^{\frac{1}{2r_k}} &\leq [\sqrt{1+m}\sqrt{C}(r_{k-1}+1)]^{\frac{1}{r_{k-1}}}\left[\int_{\mathbb{R}^3}|u_m|^{2^*}|\tilde{u}_m^I|^{2r_{k-1}}\,\mathrm{d}x\right]^{\frac{1}{2r_{k-1}}} \\
&\leq \prod_{i=0}^{k-1}[\sqrt{1+m}\sqrt{C}(r_i+1)]^{\frac{1}{r_i}}\left[\int_{\mathbb{R}^3}|u_m|^{2^*}|\tilde{u}_m^I|^{2r_0}\,\mathrm{d}x\right]^{\frac{1}{2r_0}} \\
&= \prod_{i=0}^{k-1}(1+m)^{\frac{1}{2r_i}}\prod_{i=0}^{k-1}[\sqrt{C}(r_i+1)]^{\frac{1}{r_i}}\left[\int_{\mathbb{R}^3}|u_m|^{2^*}|\tilde{u}_m^I|^{2r_0}\,\mathrm{d}x\right]^{\frac{1}{2r_0}} \\
&= \prod_{i=0}^{k-1}(1+m)^{\frac{1}{2r_i}}\exp\left\{\sum_{i=0}^{k-1}\frac{1}{r_i}\ln[\sqrt{C}(r_i+1)]\right\}\left[\int_{\mathbb{R}^3}|u_m|^{2^*}|\tilde{u}_m^I|^{2r_0}\,\mathrm{d}x\right]^{\frac{1}{2r_0}}.
\end{aligned} \tag{18}$$

Notice that

$$\begin{aligned}\left[\int_{\mathbb{R}^3}|u_m|^{2^*}|\tilde{u}_m^I|^{2r_0\cdot\frac{N}{N-2}}\,\mathrm{d}x\right]^{\frac{N-2}{N}} &\le C(r_0+1)^2\int_{\mathbb{R}^3}|u_m|^{2^*}|\tilde{u}_m^I|^{2r_0}\,\mathrm{d}x\\ &\le C(r_0+1)^2\int_{|u_m(x)|<\rho}|u_m|^{2^*}|\tilde{u}_m^I|^{2r_0}\,\mathrm{d}x\\ &\quad + C(r_0+1)^2\left(\int_{|u_m(x)|\ge\rho}|u_m|^{2^*}\,\mathrm{d}x\right)^{\frac{2}{N}}\left(\int_{\mathbb{R}^3}|u_m|^{2^*}|\tilde{u}_m^I|^{2r_0\cdot\frac{N}{N-2}}\,\mathrm{d}x\right)^{\frac{N-2}{N}}.\end{aligned}$$

Take $\rho > 0$ be such that

$$C(r_0+1)^2\left(\int_{|u_m(x)|\ge\rho}|u_m|^{2^*}\,\mathrm{d}x\right)^{\frac{2}{N}} < \frac{1}{2}.$$

Then

$$\left[\int_{\mathbb{R}^3}|u_m|^{2^*}|\tilde{u}_m^I|^{2r_0\cdot\frac{N}{N-2}}\,\mathrm{d}x\right]^{\frac{N-2}{N}} \le C(r_0+1)^2\int_{|u_m(x)|<\rho}|u_m|^{2^*}|\tilde{u}_m^I|^{2r_0}\,\mathrm{d}x \le C.$$

Set

$$d_k = \prod_{i=0}^{k-1}[\sqrt{C}(r_i+1)]^{\frac{1}{r_i}} = \exp\left\{\sum_{i=0}^{k-1}\frac{1}{r_i}\ln[\sqrt{C}(r_i+1)]\right\}$$

and

$$e_k = \prod_{i=0}^{k-1}(1+m)^{\frac{1}{2r_i}} = (1+m)^{\frac{2^*}{(2^*-2)2r_0}\left[1-\left(\frac{2}{2^*}\right)^k\right]}.$$

Then $d_k \to d_\infty$ as $k\to\infty$ and $e_k \to e_\infty = (1+m)^{\frac{2^*}{(2^*-2)2r_0}}$ as $k\to\infty$. By (18) and Lemma 5.1, we have

$$\begin{aligned}\left[\int_{\mathbb{R}^3}|u_m|^{2^*}|\tilde{u}_m^I|^{2r_k}\,\mathrm{d}x\right]^{\frac{1}{2r_k}} &\le d_ke_k\left[\int_{\mathbb{R}^3}|u_m|^{2^*}|\tilde{u}_m^I|^{2r_0}\,\mathrm{d}x\right]^{\frac{1}{2r_0}}\\ &\le d_ke_k\left[\left(\int_{\mathbb{R}^3}|u_m|^{2^*}\,\mathrm{d}x\right)^{\frac{2}{N}}\left(\int_{\mathbb{R}^3}|u_m|^{2^*}|\tilde{u}_m^I|^{2r_0\cdot\frac{N}{N-2}}\,\mathrm{d}x\right)^{\frac{N-2}{N}}\right]^{\frac{1}{2r_0}} \qquad (19)\\ &\le Cd_ke_k\left(\int_{\mathbb{R}^3}|u_m|^{2^*}\,\mathrm{d}x\right)^{\frac{1}{Nr_0}} \le Cd_ke_k.\end{aligned}$$

From (19), by Fatou Lemma with $T\to+\infty$, one has

$$|u_m|_{\frac{2^*+2r_k}{2^*+2r_k}}^{\frac{2^*+2r_k}{2r_k}} \le Cd_ke_k.$$

Consequence, let $k\to\infty$, we obtain

$$\|u_m\|_{L^\infty} \le Cd_\infty e_\infty = Cd_\infty(1+m)^{\frac{2^*}{(2^*-2)2r_0}} := B(1+m)^D,$$

where $B > 0$ and $D > 0$. Now we complete the proof. □

Proof of Theorem 1.2. By Lemma 5.2, for large $M > 0$, we can choose small $m_0 > 0$ such that $\|u_m\|_{L^\infty} \leq B(1+m)^D \leq M$ for all $m \in (0, m_0]$. Consequently, u_m is a ground state solution of equation (2) with $m \in (0, m_0]$. Finally, we finish the proof. □

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References

[1] Y. Li, Z. Q. Wang, and J. Zeng, *Ground states of nonlinear Schrödinger equations with potentials*, Ann. Inst. H. Poincaré C Anal. Non Linéare **23** (2006), 829–837.

[2] Z. Liu and Z. Q. Wang, *On the Ambrosetti-Rabinowitz superlinear condition*, Adv. Nonlinear Stud. **4** (2004), 561–572.

[3] Z. Guo, *Ground states for Kirchhoff equations without compact condition*, J. Differential Equations **259** (2015), 2884–2902.

[4] D. Ruiz, *The Schrdinger-Poisson equation under the effect of a nonlinear local term*, J. Funct. Anal. **237** (2006), no. 2, 655–674.

[5] I. M. Moroz, R. Penrose, and P. Tod, *Spherically-symmetric solutions of Schrödinger-Newton equations*, Classical Quantum Gravity **15** (1998), 2733–2742.

[6] V. Moroz and J. Van Schaftingen, *Ground states of nonlinear Choquard equations: existence, qualitative properties and decay asymptotics*, J. Funct. Anal. **265** (2013), 153–184.

[7] V. Moroz and J. Van Schaftingen, *Existence of ground states for a class of nonlinear Choquard equations*, Trans. Amer. Math. Soc. **367** (2015), 6557–6579.

[8] V. Moroz and J. Van Schaftingen, *Nonexistence and optimal decay of supersolutions to Choquard equations in exterior domains*, J. Differential Equations **254** (2013), 3089–3145.

[9] V. Moroz and J. Van Schaftingen, *Ground states of nonlinear Choquard equations: Hardy-Littlewood-Sobolev critical exponent*, Commun. Contemp. Math. **17** (2015), 1550005.

[10] V. Moroz and J. Van Schaftingen, *A guide to the Choquard equation*, J. Fixed Point Theory Appl. **19** (2017), 773–813.

[11] M. Chimenti and J. Van Schaftingen, *Nodal solutions for the Choquard equation*, J. Funct. Anal. **271** (2016), 107–135.

[12] L. Ma and Z. Lin, *Classification of positive solitary solutions of the nonlinear Choquard equation*, Arch. Ration. Mech. Anal. **195** (2010), 455–467.

[13] P. Chen and X. C. Liu, *Ground states for Kirchhoff equation with Hartree-type nonlinearities*, J. Math. Anal. Appl. **473** (2019), 587–608.

[14] Q. Li, K. M. Teng, and X. Wu, *Ground states for Kirchhoff-type equations with critical or supercritical growth*, Math. Methods Appl. Sci. **40** (2017), 6732–6746.

[15] Q. Li, K. M. Teng, and X. Wu, *Existence of nontrivial solutions for Schrödinger-Kirchhoff-type equations with critical or supercritical growth*, Math. Methods Appl. Sci. **41** (2018), 1136–1144.

[16] E. H. Lieb and M. Loss, *Analysis*, 2nd ed, Grad. Stud. Math, vol. 14, American Mathematical Society, Province, RL, 2001.

[17] H. Brézis and E. H. Lieb, *A relation between pointwise convergence of functions and convergence of functionals*, Proc. Amer. Math. Soc. **88** (1983), 486–490.

[18] H. Berestycki and P. L. Lions, *Nonlinear scalar field equations I*, Arch. Ration. Mech. Anal. **82** (1983), 313–346.

[19] M. Willem, *Minimax theorems*, Progress in Nonlinear Differential Equations and Their Applications, vol. 24, Birkhäuser, Boston, MA, 1996.

[20] L. Jeanjean, *On the existence of bounded Palais-Snale sequences and application to a Landsman-Lazer-type problem set on* $\mathbb{R}^N$, Proc. Edinb. Math. Soc. **129** (1999), no. 2, 787–809.