Research Article

Li Ma*, Ru Wang, and Liangqing Yan

Khasminskii-type theorem for a class of stochastic functional differential equations

Abstract: This paper is concerned with the existence and uniqueness theorems for stochastic functional differential equations with Markovian switching and jump, where the linear growth condition is replaced by more general Khasminskii-type conditions in terms of a pair of Lyapunov-type functions.

1 Introduction

In general, in order for a stochastic differential equation to have a unique global solution for any given initial data, the coefficients of the equation are generally required to satisfy the linear growth condition (see, e.g., [1]) or a non-Lipschitz condition and linear growth condition (see, e.g., [2–4]), so the linear growth condition plays an important role in avoiding explosion in the finite time. However, many important equations in practice do not satisfy the linear growth condition, such as stochastic Lotka-Volterra systems (see, e.g., [5]). So it is necessary to establish more general existence-and-uniqueness theorems. There are many results of the solution of a stochastic functional differential equation (SFDE) without jumps. Xuerong and Rassias (see, e.g., [6]) examined the global solutions of SFDE under a more general condition, which has been introduced by Khasminskii. By this idea, Yi et al. (see, e.g., [7]) considered existence-and-uniqueness theorems of global solutions to SFDE. Qi et al. (see, e.g., [8]) established the existence-and-uniqueness theorems of global solutions to SFDE under local Lipschitz condition and Khasminskii-type conditions. Minghui et al. (see, e.g., [9]) established existence-and-uniqueness theorems for SFDE where the linear growth condition is replaced by more general Khasminskii-type conditions in terms of a pair of Lyapunov-type functions. Then, Fuke (see, e.g., [10]) considered the existence-and-uniqueness theorems of global solutions to neutral SFDE with the local Lipschitz condition but without the linear growth condition. Later, Fuke established the Khasminskii-type theorems for SFDE with finite delay. Quanxin (see, e.g., [11]) found the pth moment exponential stability of impulsive SFDEs with Markovian switching. Recently, Quanxin and his cooperators (see, e.g., [12]) studied the Razumikhin stability theorem for a class of impulsive stochastic delay differential systems.

---

* Corresponding author: Li Ma, Key Laboratory of Data Science and Intelligence Education, Ministry of Education, Hainan Normal University, Haikou 571158, China; Department of Mathematics and Statistics, Hainan Normal University, Haikou 571158, China, e-mail: malihnsd@163.com
Ru Wang: Department of Mathematics and Statistics, Hainan Normal University, Haikou 571158, China, e-mail: 1179719022@qq.com
Liangqing Yan: Department of Mathematics and Statistics, Hainan Normal University, Haikou 571158, China, e-mail: 1299953496@qq.com

For SFDE with jumps, Wei et al. (see, e.g., [13]) found the existence and uniqueness of solutions to a general neutral SFDE with infinite delay and Lévy jumps in the phase space C_g under the local Carathêodory-type conditions and gave the exponential estimation and almost surely asymptotic estimation of solutions. By using the Razumikhin method and Lyapunov functions, Quanxin (see, e.g., [14]) obtained several Razumikhin-type theorems to prove the pth moment exponential stability of the suggested system and further discussed the pth moment exponential stability of stochastic delay differential equations with Lévy noise and Markov switching.

For a class of nonlinear stochastic differential delay equations with Poisson jump and time-dependent delay, Haidan and Quanxin (see, e.g., [15]) proved that the considered stochastic system has a unique global solution and investigated the pth moment exponential stability and the almost surely exponential stability of solutions under the local Lipschitz condition and a new nonlinear growth condition, which are weaker than those in the previous literature, by virtue of the Lyapunov function and the semi-martingale convergence theorem.

In this paper, we consider the existence-and-uniqueness theorems of global solutions to neutral SFDE with Markovian switching and Lévy jumps and establish the Khasmiskii-type theorem in the spirit of Minghui et al. (see, e.g., [9]). The main difficulty comes from the neutral term and the Lévy jumps term. After placing some assumptions on the neutral term and jump term, we obtained the existence and uniqueness of the solution by elementary inequality, the Gronwall inequality, the Burkhölder-Davis-Gundy inequality, and the Itô formula.

This paper is organized as follows. We will establish the Khasminiskii-type existence-and-uniqueness theorems for neutral SFDEs with Markovian switching and Lévy jumps in Section 2. We will proceed to consider a special class of neutral SFDEs with Markovian switching and Lévy jumps, namely, neutral stochastic differential delay equations with variable delays in Section 3. An example is given in Section 4 to illustrate our results throughout the paper.

2 The Khasminskii-type theorem for SFDEs with Markovian switching and jumps

Throughout this paper, unless otherwise specified, we use the following notations. Let $|x|$ be the Euclidean norm of a vector $x \in R^n$. Let R_+ be the family of nonnegative real numbers. If A is a matrix, its trace norm is denoted by $|A| = \sqrt{\text{trace}(A^T A)}$. Let $\tau > 0$. Let $C([-\tau, 0]; R^n)$ be the family of continuous functions from $[-\tau, 0]$ to R^n with supremum norm $\|\varphi\| = \sup_{-\tau \le \theta \le 0}|\varphi(\theta)|$, which is a Banach space. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \ge 0}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t \ge 0}$ satisfying the usual conditions (i.e., it is increasing and right continuous while $\mathcal{F}_0$ contains all P-null sets). Let $p \ge 1$ and $L^p_{\mathcal{F}_t}([-\tau, 0]; R^n)$ be the family of $\mathcal{F}_t$-measurable $C([-\tau, 0], R^n)$-valued random variables ϕ such that $E\|\phi\|^p < \infty$. Let $W(t) = (W_1(t), \ldots, W_m(t))^T$ be an m-dimensional Brownian motion defined on the probability space. Let $\bar{p} = \{\bar{p}(t), t \ge 0\}$ be a stationary and R^n-valued Poisson point process. Then, for $A \in \mathcal{B}(R^n - \{0\})$, here $\mathcal{B}(R^n - \{0\})$ denotes the Borel σ-field on $R^n - \{0\}$, and we define the Poisson counting measure N associated with $\bar{p}$ by

$$N((0, t] \times A) = \sum_{0 < s \le t} I_A(\bar{p}(s)).$$

For simplicity, we denote $N(t, A) = N((0, t] \times A)$. It is well known that there exists a σ-finite measure π, such that

$$E[N(t, A)] = \pi(A)t, \quad P(N(t, A) = n) = \frac{\exp(-\pi(A)t)(\pi(A)t)^n}{n!}.$$

This measure π is called the Lévy measure. Moreover, by Doob-Meyer's decomposition theorem, there exists a unique $\{\mathcal{F}_t\}_{t \ge 0}$-adapted martingale $\tilde{N}(t, A)$ and a unique $\{\mathcal{F}_t\}_{t \ge 0}$-adapted natural increasing process $\hat{N}(t, A)$ such that

$$N(t, A) = \tilde{N}(t, A) + \hat{N}(t, A), \quad t > 0.$$

Here, $\tilde{N}(t, A)$ is called the compensated Lévy jumps and $\hat{N}(t, A) = \pi(A)t$ is called the compensator.

Let $r(t)$, $t \geq 0$, be a right-continuous Markovian chain on the probability space taking values in a finite state space $S = \{1, 2, \dots, N\}$ with generator $\Gamma = (r_{ij})_{N \times N}$ given by

$$P\{r(t + \Delta) = j | r(t) = i\} = \begin{cases} \gamma_{ij}\Delta + o(\Delta), & \text{if } i \neq j; \\ 1 + \gamma_{ij}\Delta + o(\Delta), & \text{if } i = j. \end{cases}$$

Here, $\Delta > 0$ and $\gamma_{ij} \geq 0$, $i \neq j$, is the transition rate of the Markovian chain from i to j, while $\gamma_{ii} = -\sum_{i \neq j}\gamma_{ij}$. We assume that the Markovian chain, Brownian motion, and Lévy jumps are independent. For $Z \in \mathcal{B}(R^n - \{0\})$, $\pi(Z) < \infty$, consider a nonlinear neutral SFDE with Markovian switching and Lévy jump,

$$d[x(t) - G(x_t, r(t))] = f(x_t, x(t), t, r(t))dt + g(x_t, x(t), t, r(t))dW(t)$$
$$+ \int_Z h(x_{t-}, x(t-), t, r(t-), v)N(dt, dv), \tag{1}$$

with the initial data $x_0 = \xi \in C([-\tau, 0]; R^n)$. Here, $G : C([-\tau, 0]; R^n) \times S \to R^n$, $f : C([-\tau, 0]; R^n) \times R^n \times R_+ \times S \to R^n$, $g : C([-\tau, 0]; R^n) \times R^n \times R_+ \times S \to R^{n \times m}$, $h : C([-\tau, 0]; R^n) \times R^n \times R_+ \times S \times Z \to R^n$, and for $\theta \in [-\tau, 0]$, $x_t(\theta) = x(t + \theta)$. Now we denote by $C^{1,2}([-\tau, \infty) \times R^n; R_+)$ the family of all continuous nonnegative function $V(t, x)$ defined on $[-\tau, \infty) \times R^n$, such that they are continuously twice differentiable in x and once in t. Given $V \in C^{1,2}([-\tau, \infty) \times R^n; R_+)$, define the function $LV : R_+ \times C([-\tau, 0]; R^n) \to R$ by

$$LV(t, \varphi) = V_t(t, \varphi(0) - G(\varphi, r(t))) + V_x(t, \varphi(0) - G(\varphi, r(t)))f(\varphi, \varphi(0), t, r(t))$$
$$+ \frac{1}{2}\text{trace}[g^T(\varphi, \varphi(0), t, r(t))V_{xx}(t, \varphi(0) - G(\varphi, r(t)))g(\varphi, \varphi(0), t, r(t))]$$
$$+ \int_Z [V(t, \varphi(0) - G(\varphi, r(t)) + h(\varphi, \varphi(0), t, r(t), v)) - V(t, \varphi(0) - G(\varphi, r(t)))]\pi(dv),$$

where

$$V_t(t, x) = \frac{\partial V(t, x)}{\partial t}, \quad V_x(t, x) = \left(\frac{\partial V(t, x)}{\partial x_1}, \dots, \frac{\partial V(t, x)}{\partial x_n}\right), \quad V_{xx}(t, x) = \left(\frac{\partial^2 V(t, x)}{\partial x_i \partial x_j}\right)_{n \times n}.$$

Assumption 2.1. (Local Lipschitz condition) For any integer $m \geq 1$, there exists a positive constant k_m, such that

$$|f(\varphi, x, t, i) - f(\phi, y, t, i)|^2 \vee |g(\varphi, x, t, i) - g(\phi, y, t, i)|^2 \vee \int_Z |h(\varphi, x, t, i, v) - h(\phi, y, t, i, v)|^2\pi(dv)$$

$$\leq k_m(\|\varphi - \phi\|^2 + |x - y|^2),$$

for any $\varphi, \phi \in C([-\tau, 0]; R^n)$, $x, y \in R^n$ with $\|\varphi\| \vee \|\phi\| \vee |x| \vee |y| \leq m$, $i \in S$ and any $t \in R_+$.

Assumption 2.2. (Contraction condition) For any $p \geq 1$, there exists a constant $\kappa \in (0, 1/2)$ such that for all $\varphi \in C([-\tau, 0]; R^n)$, $i \in S$,

$$E(|G(\varphi, i)|^p) \leq \kappa^p \sup_{-\tau \leq \theta \leq 0} E(|\varphi(\theta)|^p).$$

Assumption 2.3. (Khasminskii-type condition) Let $p \geq 1$. There are two functions $V \in C^{1,2}([-\tau, \infty) \times R^n; R_+)$ and $U \in C([-\tau, \infty) \times R^n; R_+)$, a probability measure $\mu(\cdot)$ on $[-\tau, 0]$ as well as a positive constant K, two positive constants c_1, c_2, such that for any $(t, x) \in R_+ \times R^n$,

$$c_1|x|^p \leq V(t, x) \leq c_2|x|^p, \tag{2}$$

and for any $(t, \varphi) \in R_+ \times C([-\tau, 0]; R^n)$,

$$LV(t, \varphi) \le K\left[1 + \sup_{-\tau \le \theta \le 0} V(t + \theta, \varphi(\theta))\right] - U(t, \varphi(0)) + \int_{-\tau}^{0} U(t + \theta, \varphi(\theta))\mathrm{d}\mu(\theta). \tag{3}$$

Assumption 2.4. For the function V stated in Assumption 2.3 and constant K, we have

$$|V_x(t, \varphi(0) - G(\varphi, r(t)))g(\varphi, \varphi(0), t, r(t))| \le K\left(1 + \sup_{-\tau \le \theta \le 0} V(t + \theta, \varphi(\theta))\right),$$

$$\int_Z [V(t, \varphi(0) - G(\varphi, r(t)) + h(\varphi, \varphi(0), t, r(t), v)) - V(t, \varphi(0) - G(\varphi, r(t)))]^2 \pi(\mathrm{d}v)$$

$$\le K^2\left(1 + \sup_{-\tau \le \theta \le 0} V(t + \theta, \varphi(\theta))\right)^2$$

for all $(t, \varphi) \in R_+ \times C([-\tau, 0]; R^n)$.

Theorem 2.1. *Under Assumptions 2.1–2.4, for any given initial data $x_0 = \xi \in C([-\tau, 0]; R^n)$, there is a unique global solution $x(t)$ to equation (1) on $t \in [-\tau, \infty)$. Moreover, for any $T \ge 0$, the solution has the property that*

$$E \sup_{-\tau \le t \le T} |V(t, x(t))| \le C_4 e^{C_3 T},$$

where

$$C_1 = E\left[V(0, x(0) - G(\xi, r(0))) + \int_{-\tau}^{0} U(s, x(s))\mathrm{d}s\right],$$

$$C_2 = 2E\|\xi\|^p + \frac{2^p C_1}{c_1} + \frac{1 - 2^p \kappa^p}{c_2} + \left[\frac{2^p K}{c_1} + \frac{2^{2p+5} c_2 K^2}{c_1^2(1 - 2^p \kappa^p)}\right]T, \quad C_3 = \frac{2^p k}{c_1} + \frac{2^{2p+5} K^2 c_2^2}{c_1^2(1 - 2^p \kappa^p)}, \quad C_4 = \frac{C_2}{c_2}.$$

Proof. Similar to [16] Theorem 3.15, there is a unique maximal local solution $x(t)$ on $t \in [-\tau, \sigma)$, where σ is the explosion time. To show that $x(t)$ is actually global, we need to show $\sigma = \infty$, a.s. Let $k_0 > 0$ be sufficiently large for $\|\xi\| < k_0$. For each integer $k \ge k_0$, define the stopping time

$$\sigma_k = \inf\{-\tau \le t < \sigma : |x(t)| \ge k\},$$

where, as usual, $\inf \varnothing = \infty$. Clearly, σ_k is nondecreasing and $\lim_{k \to \infty} \sigma_k = \sigma_\infty \le \sigma$. This proof can be completed if $\sigma_\infty = \infty$ a.s.. By the Itô formula ([17], Lemma 4.4.6),

$$V(t, x(t) - G(x_t, r(t))) = V(0, x(0) - G(x_0, r(0))) + \int_0^t LV(s, x_s)\mathrm{d}s + \int_0^t [V_x(s, x(s)$$

$$- G(x_s, r(s)))g(x_s, x(s), s, r(s))]\mathrm{d}W(s) + \int_0^t \int_Z [V(s, x(s) - G(x_s, r(s))$$

$$+ h(x_s, x(s), s, r(s), v)) - V(s, x(s) - G(x_s, r(s)))]\tilde{N}(\mathrm{d}s, \mathrm{d}v). \tag{4}$$

By (3), we have

$$V(t, x(t) - G(x_t, r(t))) \le V(0, x(0) - G(x_0, r(0))) + K\int_0^t \left(1 + \sup_{-\tau \le \theta \le 0} V(s + \theta, x(s + \theta))\right)\mathrm{d}s$$

$$- \int_0^t U(s, x(s))\mathrm{d} + \int_0^t \int_{-\tau}^0 U(s + \theta, x(s + \theta))\mathrm{d}\mu(\theta)\mathrm{d}s \tag{5}$$

$$+ \int_0^t [V_x(s, x(s) - G(x_s, r(s)))g(x_s, x(s), s, r(s))]dW(s)$$

$$+ \int_0^t \int_Z [V(s, x(s) - G(x_s, r(s)) + h(x_s, x(s), s, r(s), v))$$

$$- V(s, x(s) - G(x_s, r(s)))]\tilde{N}(ds, dv).$$

By the Fubini theorem, we have

$$\int_0^t \int_{-\tau}^0 U(s + \theta, x(s + \theta))d\mu(\theta)ds = \int_{-\tau}^0 \left(\int_0^t U(s + \theta, x(s + \theta))ds \right)d\mu(\theta)$$

$$\leq \int_{-\tau}^0 \left(\int_{-\tau}^t U(s, x(s))ds \right)d\mu(\theta) \leq \int_{-\tau}^t U(s, x(s))ds. \tag{6}$$

Substituting (6) into (5) yields

$$V(t, x(t) - G(x_t, r(t))) \leq \bar{C} + K \int_0^t \left(1 + \sup_{-\tau \leq \theta \leq 0} V(s + \theta, x(s + \theta)) \right)ds$$

$$+ \int_0^t V_x(s, x(s) - G(x_s, r(s)))g(x_s, x(s), s, r(s))dW(s)$$

$$+ \int_0^t \int_Z [V(s, x(s) - G(x_s, r(s)) + h(x_s, x(s), s, r(s), v))$$

$$- V(s, x(s) - G(x_s, r(s)))]\tilde{N}(ds, dv),$$

where $\bar{C} = V(0, x(0) - G(x_0, r(0))) + \int_{-\tau}^0 U(s, x(s))ds$. This implies that for any $k \geq k_0$, $t < T$, where T is an arbitrary positive constant,

$$V\left(t \wedge \sigma_k, x(t \wedge \sigma_k) - G\left(x_{t \wedge \sigma_k}, r(t \wedge \sigma_k)\right)\right)$$

$$\leq \bar{C} + K \int_0^{t \wedge \sigma_k} \left(1 + \sup_{-\tau \leq \theta \leq 0} V(s + \theta, x(s + \theta)) \right)ds + \left| \int_0^{t \wedge \sigma_k} V_x(s, x(s) - G(x_s, r(s)))g(x_s, x(s), s, r(s))dW(s) \right|$$

$$+ \left| \int_0^{t \wedge \sigma_k} \int_Z [V(s, x(s) - G(x_s, r(s)) + h(x_s, x(s), s, r(s), v)) - V(s, x(s) - G(x_s, r(s)))]\tilde{N}(ds, dv) \right|.$$

Taking upper bound and expectation on the above inequality, we obtain

$$E \sup_{0 \leq t \leq T} V\left(t \wedge \sigma_k, x(t \wedge \sigma_k) - G\left(x_{t \wedge \sigma_k}, r(t \wedge \sigma_k)\right)\right)$$

$$\leq \bar{C} + K \sup_{0 \leq t \leq T} \int_0^{t \wedge \sigma_k} (1 + \sup_{-\tau \leq \theta \leq 0} V(s + \theta, x(s + \theta)))ds$$

$$+ E \sup_{0 \leq t \leq T} \left| \int_0^t I_{[0,\sigma_k]}(s)V_x(s, x(s) - G(x_s, r(s)))g(x_s, x(s), s, r(s))dW(s) \right| \tag{7}$$

$$+ E \sup_{0 \leq t \leq T} \left| \int_0^t I_{[0,\sigma_k]}(s) \int_Z [V(s, x(s) - G(x_s, r(s)) + h(x_s, x(s), s, r(s), v)) \right.$$

$$\left. - V(s, x(s) - G(x_s, r(s)))]\tilde{N}(ds, dv) \right|.$$

By the Burkhölder-Davis-Gundy inequality ([5], Theorem 1.7.3) and Assumption 2.4,

$$E \sup_{0 \le t \le T} \left| \int_0^t I_{[0,\sigma_k]}(s) V_x(s, x(s) - G(x_s, r(s))) g(x_s, x(s), s, r(s)) \mathrm{d}W(s) \right|$$

$$\le \sqrt{32} E \left(\int_0^T I_{[0,\sigma_k]}(s) |V_x(s, x(s) - G(x_s, r(s))) g(x_s, x(s), s, r(s))|^2 \mathrm{d}s \right)^{\frac{1}{2}}$$

$$\le \sqrt{32} K E \left(\int_0^T \left[1 + \sup_{-\tau \le \theta \le 0} V(s \wedge \sigma_k + \theta, x(s \wedge \sigma_k + \theta)) \right]^2 \mathrm{d}s \right)^{\frac{1}{2}}$$

$$\le \sqrt{32} K E \left(\int_0^T \left[1 + \sup_{-\tau \le t \le s} V(t \wedge \sigma_k, x(t \wedge \sigma_k)) \right]^2 \mathrm{d}s \right)^{\frac{1}{2}} \qquad (8)$$

$$\le \sqrt{32} K E \left(\left[1 + \sup_{-\tau \le t \le T} V(t \wedge \sigma_k, x(t \wedge \sigma_k)) \right] \int_0^T \left[1 + \sup_{-\tau \le t \le s} V(t \wedge \sigma_k, x(t \wedge \sigma_k)) \right] \mathrm{d}s \right)^{\frac{1}{2}}$$

$$\le \frac{c_1(1 - 2^p \kappa^p)}{2^{p+1} c_2} E \left(1 + \sup_{-\tau \le t \le T} V(t \wedge \sigma_k, x(t \wedge \sigma_k)) \right)$$

$$+ \frac{2^{p+4} c_2 K^2}{c_1(1 - 2^p \kappa^p)} E \int_0^T \left[1 + \sup_{-\tau \le t \le s} V(t \wedge \sigma_k, x(t \wedge \sigma_k)) \right] \mathrm{d}s.$$

By Assumption 2.4, similar to (8) and [5], Theorem 1.7.3,

$$E \sup_{0 \le t \le T} \left| \int_0^t I_{[0,\sigma_k]}(s) \int_Z (V(s, x(s) - G(x_s, r(s)) + h(x_s, x(s), s, r(s), v)) - V(s, x(s) - G(x_s, r(s)))) \tilde{N}(\mathrm{d}s, \mathrm{d}v) \right|$$

$$\le \sqrt{32} E \left(\int_0^T I_{[0,\sigma_k]}(s) \int_Z |V(s, x(s) - G(x_s, r(s)) + h(x_s, x(s), s, r(s), v)) - V(s, x(s) - G(x_s, r(s)))|^2 \pi(\mathrm{d}v) \mathrm{d}s \right)^{\frac{1}{2}}$$

$$\le \sqrt{32} K \left[E \int_0^T \left(1 + \sup_{-\tau \le \theta \le 0} V(s \wedge \sigma_k + \theta, x(s \wedge \sigma_k + \theta))^2 \right) \mathrm{d}s \right]^{\frac{1}{2}} \qquad (9)$$

$$\le \frac{c_1(1 - 2^p \kappa^p)}{2^{p+1} c_2} E \left(1 + \sup_{-\tau \le t \le T} V(t \wedge \sigma_k, x(t \wedge \sigma_k)) \right) + \frac{2^{p+4} c_2 K^2}{c_1(1 - 2^p \kappa^p)} E \int_0^T \left[1 + \sup_{-\tau \le t \le s} V(t \wedge \sigma_k, x(t \wedge \sigma_k)) \right] \mathrm{d}s.$$

Substituting (8) and (9) into (7),

$$E \sup_{0 \le t \le T} V(t \wedge \sigma_k, x(t \wedge \sigma_k) - G(x_{t \wedge \sigma_k}, r(t \wedge \sigma_k)))$$

$$\le C_1 + \left[K + \frac{2^{p+5} c_2 K^2}{c_1(1 - 2^p \kappa^p)} \right] E \int_0^T \left[1 + \sup_{-\tau \le t \le s} V(t \wedge \sigma_k, x(t \wedge \sigma_k)) \right] \mathrm{d}s \qquad (10)$$

$$+ \frac{c_1(1 - 2^p \kappa^p)}{2^p c_2} E \left(1 + \sup_{-\tau \le t \le T} V(t \wedge \sigma_k, x(t \wedge \sigma_k)) \right),$$

where $C_1 = E \left(V(0, x(0) - G(\xi, r(0))) + \int_{-\tau}^0 U(s, x(s)) \mathrm{d}s \right)$.

Recall elementary inequality $|a + b|^p \le 2^{p-1}(|a|^p + |b|^p)$ for any $p \ge 1$, $a \in R^n$, $b \in R^n$, so $|a|^p \ge \frac{1}{2^{p-1}} |a + b|^p - |b|^p$. Note the relationship between sup and inf, so by Fatou's lemma, (2), and Assumptions 2.2–2.3, we have

$$
\begin{aligned}
E &\sup_{0 \le t \le T} V\left(t \wedge \sigma_k, x(t \wedge \sigma_k) - G\left(x_{t \wedge \sigma_k}, r(t \wedge \sigma_k)\right)\right) \\
&\ge c_1 E \sup_{0 \le t \le T} \left| x(t \wedge \sigma_k) - G\left(x_{t \wedge \sigma_k}, r(t \wedge \sigma_k)\right) \right|^p \\
&\ge c_1 \left[\frac{1}{2^{p-1}} E \sup_{0 \le t \le T} |x(t \wedge \sigma_k)|^p + E \sup_{0 \le t \le T} -\left| G\left(x_{t \wedge \sigma_k}, r(t \wedge \sigma_k)\right) \right|^p \right] \\
&= c_1 \left[\frac{1}{2^{p-1}} E \sup_{0 \le t \le T} |x(t \wedge \sigma_k)|^p - E \inf_{0 \le t \le T} \left| G\left(x_{t \wedge \sigma_k}, r(t \wedge \sigma_k)\right) \right|^p \right] \\
&\ge c_1 \left[\frac{1}{2^{p-1}} E \sup_{0 \le t \le T} |x(t \wedge \sigma_k)|^p - \inf_{0 \le t \le T} E \left| G\left(x_{t \wedge \sigma_k}, r(t \wedge \sigma_k)\right) \right|^p \right] \\
&\ge c_1 \left[\frac{1}{2^{p-1}} E \sup_{0 \le t \le T} |x(t \wedge \sigma_k)|^p - \sup_{0 \le t \le T} E \left| G\left(x_{t \wedge \sigma_k}, r(t \wedge \sigma_k)\right) \right|^p \right] \\
&\ge c_1 \left[\frac{1}{2^{p-1}} E \sup_{0 \le t \le T} |x(t \wedge \sigma_k)|^p - \kappa^p \sup_{0 \le t \le T} \sup_{-\tau \le \theta \le 0} E |x(t \wedge \sigma_k + \theta)|^p \right] \\
&\ge c_1 \left[\frac{1}{2^{p-1}} E \sup_{0 \le t \le T} |x(t \wedge \sigma_k)|^p - \kappa^p E \sup_{0 \le t \le T} \sup_{-\tau \le \theta \le 0} |x(t \wedge \sigma_k + \theta)|^p \right] \\
&= c_1 \left[\frac{1}{2^{p-1}} E \sup_{0 \le t \le T} |x(t \wedge \sigma_k)|^p - \kappa^p E \sup_{-\tau \le t \le T} |x(t \wedge \sigma_k)|^p \right],
\end{aligned}
\tag{11}
$$

and

$$
E \int_0^T \left[1 + \sup_{-\tau \le t \le s} V(t \wedge \sigma_k, x(t \wedge \sigma_k)) \right] ds \le E \int_0^T \left[1 + c_2 \sup_{-\tau \le t \le s} |x(t \wedge \sigma_k)|^p \right] ds.
\tag{12}
$$

Substituting (11) and (12) into (10) gives

$$
\begin{aligned}
c_1 &\left[\frac{1}{2^{p-1}} E \sup_{0 \le t \le T} |x(t \wedge \sigma_k)|^p - \kappa^p E \sup_{-\tau \le t \le T} |x(t \wedge \sigma_k)|^p \right] \\
&\le C_1 + \left[K + \frac{2^{p+5} c_2 K^2}{c_1(1 - 2^p \kappa^p)} \right] E \int_0^T \left[1 + c_2 \sup_{-\tau \le t \le s} |x(t \wedge \sigma_k)|^p \right] ds + \frac{c_1(1 - 2^p \kappa^p)}{2^p c_2} \\
&\quad + c_1 \left(\frac{1}{2^p} - \kappa^p \right) E \sup_{-\tau \le t \le T} |x(t \wedge \sigma_k)|^p.
\end{aligned}
\tag{13}
$$

By (13), we have

$$
\begin{aligned}
c_1 &\left(\frac{1}{2^{p-1}} - \kappa^p \right) E \sup_{-\tau \le t \le T} |x(t \wedge \sigma_k)|^p \\
&\le C_1 + \frac{c_1}{2^{p-1}} E \|\xi\|^p + \left[K + \frac{2^{p+5} c_2 K^2}{c_1(1 - 2^p \kappa^p)} \right] T + \left[K + \frac{2^{p+5} c_2^2 K^2}{c_1(1 - 2^p \kappa^p)} \right] E \int_0^T \sup_{-\tau \le t \le s} |x(t \wedge \sigma_k)|^p ds \\
&\quad + \frac{c_1(1 - 2^p \kappa^p)}{2^p c_2} + c_1 \left(\frac{1}{2^p} - \kappa^p \right) E \sup_{-\tau \le t \le T} |x(t \wedge \sigma_k)|^p.
\end{aligned}
\tag{14}
$$

So,

$$
\begin{aligned}
\frac{c_1}{2^p} E \sup_{-\tau \le t \le T} |x(t \wedge \sigma_k)|^p &\le C_1 + \frac{c_1}{2^{p-1}} E \|\xi\|^p + \left[K + \frac{2^{p+5} c_2 K^2}{c_1(1 - 2^p \kappa^p)} \right] T + \frac{c_1(1 - 2^p \kappa^p)}{2^p c_2} \\
&\quad + \left[K + \frac{2^{p+5} c_2^2 K^2}{c_1(1 - 2^p \kappa^p)} \right] E \int_0^T \sup_{-\tau \le t \le s} |x(t \wedge \sigma_k)|^p ds.
\end{aligned}
$$

That is,

$$E \sup_{-\tau \le t \le T} |x(t \wedge \sigma_k)|^p \le C_2 + C_3 \int_0^T E \sup_{-\tau \le t \le s} |x(t \wedge \sigma_k)|^p ds,$$

where

$$C_2 = 2E(\|\xi\|^p) + \frac{2^p C_1}{c_1} + \frac{1 - 2^p \kappa^p}{c_2} + \left[\frac{2^p K}{c_1} + \frac{2^{2p+5} c_2 K^2}{c_1^2(1 - 2^p \kappa^p)} \right] T, \quad C_3 = \frac{2^p K}{c_1} + \frac{2^{2p+5} K^2 c_2^2}{c_1^2(1 - 2^p \kappa^p)}.$$

By the Gronwall inequality ([18] Lemma 2), we therefore obtain

$$E \sup_{-\tau \le t \le T} |x(t \wedge \sigma_k)|^p \le C_2 e^{C_3 T}. \tag{15}$$

So

$$k^p P(\sigma_k \le t) \le E|x(t \wedge \sigma_k)|^p \le E \sup_{-\tau \le t \le T} |x(t \wedge \sigma_k)|^p \le C_2 e^{C_3 T}.$$

Let $k \to +\infty$, then $\lim_{k \to +\infty} P(\sigma_k \le t) = 0$, and hence, $P(\sigma_\infty \le t) = 0$ and $P(\sigma_\infty > t) = 1$. Since $T \ge t > -\tau$ and T is arbitrary, we must have that $\sigma_\infty = \infty$ a.s. By (2) and (15), we have

$$E \sup_{-\tau \le t \le T} |V(t \wedge \sigma_k, x(t \wedge \sigma_k))| \le \frac{C_2}{c_2} e^{C_3 T}. \tag{16}$$

Letting $k \to \infty$ in (16) yields

$$E \sup_{-\tau \le t \le T} |V(t, x(t))| \le \frac{C_2}{c_2} e^{C_3 T}. \tag{17}$$

The proof is therefore completed. □

Remark 2.1. From (15), we see that the pth moment will grow at most exponentially with exponent C_3. That is,

$$\limsup_{t \to \infty} \frac{1}{t} \log E|x(t)|^p \le C_3.$$

Here, $C_3 = \frac{2^p k}{c_1} + \frac{2^{2p+5} K^2 c_2^2}{c_1^2(1 - 2^p \kappa^p)}.$

The next theorem shows that the pth exponential estimations implies the almost surely asymptotic estimations, and we give an upper bound for the sample Lyapunov exponent.

Theorem 2.2. *Under Assumptions 2.1–2.4, for any given initial data $x_0 = \xi \in C([-\tau, 0]; R^n)$, we have*

$$\limsup_{t \to \infty} \frac{1}{t} \log |x(t)| \le \frac{2k}{c_1} + \frac{2^8 K^2 c_2^2}{c_1^2(1 - 4\kappa^2)} \quad a.s. \tag{18}$$

That is, the sample Lyapunov exponent of the solution should not be greater than $\frac{2k}{c_1} + \frac{2^8 K^2 c_2^2}{c_1^2(1 - 4\kappa^2)}$.

Proof. For each $n = 1, 2, \cdots$, it follows from (15) (taking $p = 2$) that

$$E \sup_{n-1 \le t \le n} |x(t)|^2 \le \beta_n e^{\gamma n},$$

where $\beta_n = 2E\|\xi\|^2 + \frac{4C_1}{c_1} + \frac{1 - 4\kappa^2}{c_2} + \left[\frac{4K}{c_1} + \frac{2^9 c_2 K^2}{c_1^2(1 - 4\kappa^2)} \right] n$, and $\gamma = \frac{4k}{c_1} + \frac{2^9 K^2 c_2^2}{c_1^2(1 - 4\kappa^2)}$. Hence, for any $\varepsilon > 0$, by the Chebyshev inequality, it follows that

$$P\left\{ \omega : \sup_{n-1 \le t \le n} |x(t)|^2 > e^{(\gamma + \varepsilon)n} \right\} \le \beta_n e^{-\varepsilon n}.$$

Since $\sum_n \beta_n e^{-\varepsilon n} \leq 2E\|\xi\|^2 + \frac{4C_1}{c_1} + \frac{1-4\kappa^2}{c_2} + \left[\frac{4K}{c_1} + \frac{2^9 c_2 K^2}{c_1^2(1-4\kappa^2)}\right]\sum_n ne^{-\varepsilon n} < \infty$, by the Borel-Cantelli Lemma, we deduce that there exists an integer n_0 such that

$$\sup_{n-1 \leq t \leq n} |x(t)|^2 \leq e^{(\gamma+\varepsilon)n}, \quad \text{a.s. } n \geq n_0.$$

Thus, for almost all $\omega \in \Omega$, if $n - 1 \leq t \leq n$ and $n \geq n_0$, then

$$\frac{1}{t}\log|x(t)| = \frac{1}{2t}\log|x(t)|^2 \leq \frac{(\gamma+\varepsilon)n}{2(n-1)} \quad \text{a.s.} \tag{19}$$

Taking the limsup in (19) leads to an almost surely exponential estimate, that is,

$$\limsup_{t\to\infty} \frac{1}{t}\log|x(t)| \leq \frac{\gamma+\varepsilon}{2} \quad \text{a.s.}$$

The required assertion (18) follows because $\varepsilon > 0$ is arbitrary. $\qquad\qquad\square$

3 Neutral SDDEs with variable delays

We now turn to considering neutral stochastic differential delay equations (SDDEs) with Markovian switching and Lévy jumps where the delays are time-dependent variables. That is, we consider the following equation.

$$d[x(t) - \bar{G}(x(t-\delta(t)), r(t))] = \bar{f}(x(t-\delta(t)), x(t), t, r(t))dt + \bar{g}(x(t-\delta(t)), x(t), t, r(t))dW(t)$$
$$+ \int_Z \bar{h}(x(t-\delta(t)-), x(t-), t, r(t-), v)N(dt, dv), \tag{20}$$

on $t \geq 0$ with the initial data $x_0 = \xi \in C([-\tau, 0]; R^n)$, where $\delta : R_+ \to [0, \tau]$, $\bar{G} : R^n \times S \to R^n$, $\bar{f} : R^n \times R^n \times R_+ \times S \to R^n$, $\bar{g} : R^n \times R^n \times R_+ \times S \to R^{n\times m}$, and $\bar{h} : R^n \times R^n \times R_+ \times S \times Z \to R^n$ are all Borel measurable. [7,9] have established the Khasminskii-type theorems for SDDEs with constant delay. But these results could not be applied to the SDDEs where the delay is time-variable. If we define $f : C([-\tau, 0]; R^n) \times R^n \times R_+ \times S \to R^n$, $g : C([-\tau, 0]; R^n) \times R^n \times R_+ \times S \to R^{n\times m}$, $h : C([-\tau, 0]; R^n) \times R^n \times R_+ \times S \times Z \to R^n$, and $G : C([-\tau, 0]; R^n) \times S \to R^n$ by

$$G(\varphi, r(t)) = \bar{G}(\varphi(-\delta(t)), r(t)), \quad f(\varphi, \varphi(0), t, r(t)) = \bar{f}(\varphi(-\delta(t)), \varphi(0), t, r(t)),$$
$$g(\varphi, \varphi(0), t, r(t)) = \bar{g}(\varphi(-\delta(t)), \varphi(0), t, r(t)), \quad h(\varphi, \varphi(0), t, r(t), v) = \bar{h}(\varphi(-\delta(t)), \varphi(0), t, r(t), v),$$

then we can apply the theory established in the previous sections to this neutral SDDE with Markovian switching. Let us proceed in this way to see what we can obtain. First, we can transfer Assumption 2.1 into the following one.

Assumption 3.1. For each integer $m \geq 1$, there is a positive constant k_m such that

$$|\bar{f}(y, x, t, i) - \bar{f}(\bar{y}, \bar{x}, t, i)|^2 \vee |\bar{g}(y, x, t, i) - \bar{g}(\bar{y}, \bar{x}, t, i)|^2 \vee \int_Z |\bar{h}(y, x, t, i, v) - \bar{h}(\bar{y}, \bar{x}, t, i, v)|^2\pi(dv) \tag{21}$$

$$\leq k_m(|y-\bar{y}|^2 + |x-\bar{x}|^2),$$

for those $y, x, \bar{y}, \bar{x} \in R^n$ with $|y| \vee |x| \vee |\bar{y}| \vee |\bar{x}| \leq m$ and any $t \in R_+$.

The following assumption is corresponding to Assumption 2.2.

Assumption 3.2. (Contraction condition) For any $p \geq 1$, there exists a constant $\kappa \in (0, 1/2)$ such that for all $\varphi \in C([-\tau, 0]; R^n)$, $i \in S$,

$$E|\bar{G}(\varphi(-\delta(t)), i)|^p \leq \kappa^p[E|\varphi(0)|^p + E|\varphi(-\delta(t))|^p].$$

Comparing with Assumption 2.3, we can obtain the following. For $V \in C^{1,2}([-\tau, \infty) \times R^n; R_+)$, the operator $LV : R_+ \times C([-\tau, 0]; R^n) \longrightarrow R$ takes the form as follows:

$$LV(t, \varphi) = \mathcal{L}V(\varphi(-\delta(t)), \varphi(0), t, r(t)),$$

where $\mathcal{L}V : R^n \times R^n \times R_+ \times S \longrightarrow R$ is defined by

$$
\begin{aligned}
\mathcal{L}V(y, x, t, i) = {} & V_t(t, x - \bar{G}(y, i)) + V_x(t, x - \bar{G}(y, i))\bar{f}(y, x, t, i) \\
& + \frac{1}{2}\text{trace}[\bar{g}(y, x, t, i)^T V_{xx}(t, x - \bar{G}(y, i))\bar{g}(y, x, t, i)] \\
& + \int_Z [V(t, x - \bar{G}(y, i)) + \bar{h}(y, x, t, i, v) - V(t, x - \bar{G}(y, i))]\pi(dv).
\end{aligned}
$$

And clearly, (3) should become

$$LV(t, \varphi) \leq K[1 + V(t, \varphi(0)) + V(t - \delta(t), \varphi(-\delta(t)))] - U(t, \varphi(0)) + \int_{-\tau}^{0} U(t + \theta, \varphi(\theta))d\mu(\theta), \quad (22)$$

with

$$\int_{-\tau}^{0} U(t + \theta, \varphi(\theta))d\mu(\theta) = U(t - \delta(t), \varphi(-\delta(t))).$$

This implies that $\mu(\cdot)$ should be a point probability measure at $-\delta(t)$, which means $\mu(\cdot)$ is a t-dependent probability measure. However, the theory established in the previous section requires essentially that it is t-independent. This forces the function $U \equiv 0$ in order for the previous theory to be applicable. So the condition corresponding to (3) is as follows.

Assumption 3.3. There is a function $V \in C^{1,2}([-\tau, \infty) \times R^n; R_+)$ and a positive constant K such that V obeys (2) and for all $(x, y, t, i) \in R^n \times R^n \times R_+ \times S$,

$$\mathcal{L}V(y, x, t, i) \leq K[1 + V(t, x) + V(t - \delta(t), y)]. \quad (23)$$

The condition corresponding to Assumption 2.4 is as follows.

Assumption 3.4. For the function V stated in Assumption 3.2 and constant K, we have

$$|V_x(t, x - G(y, i))g(y, x, t, i)| \leq K[1 + V(t, x) + V(t - \delta(t), y)], \quad (24)$$

$$\left| \int_Z [V(t, x - G(y, i) + h(y, x, t, i, v)) - V(t, x - G(y, i))]\pi(dv) \right|^2 \leq K^2[1 + V(t, x) + V(t - \delta(t), y)], \quad (25)$$

for all $(t, \varphi) \in R_+ \times C([-\tau, 0]; R^n)$.

By Theorem 2.1, we can obtain the following result.

Theorem 3.1. *Under Assumptions 3.1–3.4, we obtain that for any initial data $x_0 = \xi \in C([-\tau, 0]; R^n)$, there is a unique global solution $x(t)$ to equation (20) on $t \in [-\tau, \infty)$.*

Assumption 3.5. For any $t \geq 0$, assume that $\delta(t)$ is differentiable and obeys

$$\frac{\mathrm{d}\delta(t)}{\mathrm{d}t} \leq \bar{\delta} < 1, \tag{26}$$

where $\bar{\delta}$ is a constant. Assume also that there is a function $V \in C^{1,2}([-\tau, \infty) \times R^n; R_+)$ and a positive constant K such that V obeys (2) and for all $(x, y, t, i) \in R^n \times R^n \times R_+ \times S$,

$$\mathcal{L}V(y, x, t, i) \leq K[1 + V(t, x) + V(t - \delta(t), y)] - U(t, x) + (1 - \bar{\delta})U(t - \delta(t), y). \tag{27}$$

Now we could give an alternative Khasminskii-type theorem for the neutral SDDE (20) by removing Assumption 3.4 as follows.

Theorem 3.2. *Under Assumptions* 3.1, 3.2, *and* 3.5. *For any initial data* $x_0 = \xi \in C([-\tau, 0]; R^n)$, *there is a unique global solution* $x(t)$ *to equation* (3.1) *on* $t \in [-\tau, \infty)$.

Proof. For any initial data $x_0 = \xi \in C([-\tau, 0]; R^n)$, under Assumption 3.1 (the local Lipschitz condition), by the standard truncation technique, there is a unique maximal local solution $x(t)$ to equation (20) on $t \in [-\tau, \sigma)$, where σ is the explosion time. The following proof is similar to Theorem 2.1, so here we just give the key steps. In the following, we will follow the notations of the proof of Theorem 2.1. Define

$$\tilde{h}_k(t) = E(|x(t \wedge \sigma_k) - G(x(t \wedge \sigma_k - \delta(t) \wedge \sigma_k), r(t \wedge \sigma_k))|^p),$$
$$\tilde{H}_k(t) = \sup_{0 \leq s \leq t} \tilde{h}_k(s), \quad h_k(t) = E(|x(t \wedge \sigma_k)|^p), \quad H_k(t) = \sup_{-\tau \leq s \leq t} h_k(s).$$

Recall the elementary inequality: for $p \geq 1$, $x, y \in R^n$, $k \in (0, 1)$, then

$$|x + y|^p \leq (1 - k)^{1-p}|x|^p + k^{1-p}|y|^p.$$

For $t \geq 0$, by Assumption 3.2,

$$E|x(t)|^p \leq (1 - \kappa)^{1-p}E|x(t) - G(x(t - \delta(t)), r(t))|^p + \kappa^{1-p}E|G(x(t - \delta(t)), r(t))|^p$$
$$\leq (1 - \kappa)^{1-p}E|x(t) - G(x(t - \delta(t)), r(t))|^p + \kappa E|x(t)|^p + \kappa E|x(t - \delta(t))|^p$$
$$\leq (1 - \kappa)^{1-p}E|x(t) - G(x(t - \delta(t)), r(t))|^p + 2\kappa \sup_{-\tau \leq s \leq t} E|x(s)|^p.$$

Replacing t by $t \wedge \sigma_k$ gives $h_k(t) \leq (1 - \kappa)^{1-p}\tilde{h}_k(t) + 2\kappa H_k(t)$ on $t \geq 0$. So

$$H_k(t) \leq \sup_{-\tau \leq s \leq 0} E|\xi(s)|^p + \sup_{0 \leq s \leq t} h_k(s) \leq E\|\xi\|^p + (1 - \kappa)^{1-p}\tilde{H}_k(t) + 2\kappa H_k(t).$$

Hence,

$$H_k(t) \leq \frac{E\|\xi\|^p}{1 - 2\kappa} + \frac{\tilde{H}_k(t)}{(1 - \kappa)^{p-1}(1 - 2\kappa)}, \quad \text{on } t \geq 0. \tag{28}$$

By the Itô formula and Assumption 3.5, we can show for any sufficiently large k and any $t \geq 0$,

$$c_1\tilde{h}_k(t) \leq EV(t \wedge \sigma_k, x(t \wedge \sigma_k) - G(x(t \wedge \sigma_k - \delta(t) \wedge \sigma_k), r(t \wedge \sigma_k)))$$

$$\leq EV(0, x(0) - G(x(-\delta(0)), r(0))) + K \int_0^t \left[1 + 2 \sup_{-\tau \leq u \leq s} EV(u \wedge \sigma_k, x(u \wedge \sigma_k))\right] \mathrm{d}s$$

$$+ E \int_0^{t \wedge \sigma_k} [-U(s, x(s)) + (1 - \bar{\delta})U(s - \delta(s), x(s - \delta(s)))]\mathrm{d}s$$

$$\leq EV(0, x(0) - G(x(-\delta(0)), r(0))) + K \int_0^t [1 + 2c_2 H_k(s)]\mathrm{d}s$$

$$+ E \int_0^{t \wedge \sigma_k} [-U(s, x(s)) + (1 - \bar{\delta})U(s - \delta(s), x(s - \delta(s)))]\mathrm{d}s.$$

But

$$\int_0^{t\wedge\sigma_k} (1 - \bar{\delta})U(s - \delta(s), x(s - \delta(s)))\mathrm{d}s \leq \int_{-\delta(0)}^{t\wedge\sigma_k-\delta(t\wedge\sigma_k)} U(u, x(u))\mathrm{d}u \leq \int_{-\tau}^{t\wedge\sigma_k} U(u, x(u))\mathrm{d}u.$$

Recall $C_1 = E\left(V(0, x(0) - G(x(-\delta(0)), r(0))) + \int_{-\tau}^0 U(s, x(s))\mathrm{d}s\right)$, so

$$c_1\tilde{h}_k(t) \leq C_1 + K\int_0^t [1 + 2c_2H_k(s)]\mathrm{d}s.$$

Hence,

$$c_1\tilde{H}_k(t) \leq C_1 + Kt + 2c_2K\int_0^t H_k(s)\mathrm{d}s. \tag{29}$$

Plugging (29) into (28),

$$H_k(t) \leq \frac{E(\|\xi\|^p)}{1 - 2\kappa} + \frac{C_1 + Kt}{c_1(1 - \kappa)^{p-1}(1 - 2\kappa)} + \frac{2c_2K\int_0^t H_k(s)\mathrm{d}s}{c_1(1 - \kappa)^{p-1}(1 - 2\kappa)}, \quad \text{for } t \geq 0.$$

By Assumption 3.5 and the Gronwall inequality, we have

$$\sup_{-\tau\leq s\leq t} EV(s \wedge \sigma_k, x(s \wedge \sigma_k)) \leq c_2M(t)\exp\left\{\frac{2C_2Kt}{c_1(1 - \kappa)^{p-1}(1 - 2\kappa)}\right\},$$

where $M(t) = \frac{E(\|\xi\|^p)}{1 - 2\kappa} + \frac{C_1 + Kt}{c_1(1 - \kappa)^{p-1}(1 - 2\kappa)}$. Noting $|x(\sigma_k)| \geq k$, so we obtain that

$$c_1k^pP(\sigma_k \leq t) \leq c_1E|x(t \wedge \sigma_k)|^p \leq EV(t \wedge \sigma_k, x(t \wedge \sigma_k)) \leq c_2M(t)\exp\left\{\frac{2C_2Kt}{c_1(1 - \kappa)^{p-1}(1 - 2\kappa)}\right\}.$$

Letting $k \to \infty$,

$$\lim_{k\to\infty} P(\sigma_k \leq t) \leq \lim_{k\to\infty} \frac{c_2M(t)}{c_1k^p}\exp\left\{\frac{2C_2Kt}{c_1(1 - \kappa)^{p-1}(1 - 2\kappa)}\right\} = 0.$$

So $\sigma_\infty = \infty$ a.s., the proof is finished. □

Next, we present some useful asymptotic moment estimations for the solution of the neutral SDDE (20) by the following theorem.

Theorem 3.3. *Let Assumptions 3.1, 3.2, and 3.5 hold, but (27) is replaced by*

$$\mathcal{L}V(y, x, t, i) \leq \alpha_1 - \alpha_2V(t, x) + \alpha_3V(t - \delta(t), y) - U(t, x) + \alpha U(t - \delta(t), y), \tag{30}$$

where $\alpha_1 \geq 0$, $\alpha_2(1 - \bar{\delta}) > \alpha_3 \geq 0$ and $1 - \bar{\delta} > \alpha \geq 0$. Then for any given initial data $x_0 = \xi \in C([-\tau, 0]; R^n)$, the unique global solution $x(t)$ to equation (3.1) has the properties that

$$\limsup_{t\to\infty} EV(t, x(t) - G(x(t - \delta(t)), r(t))) \leq \left(C + \frac{\alpha_1}{\varepsilon}\right)e, \tag{31}$$

and

$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t EU(t, x(t))\mathrm{d}t \leq \frac{\alpha_1(1 - \bar{\delta})}{1 - \bar{\delta} - \alpha}, \tag{32}$$

where

$$C = EV(0, x(0) - G(x(-\delta(0)), r(0))) + \frac{\alpha_3 e^{\varepsilon\tau}}{1 - \bar{\delta}} \int\limits_{-\tau}^{0} e^{\varepsilon s}V(s, x(s))\mathrm{d}s + \frac{\alpha e^{\varepsilon\tau}}{1 - \bar{\delta}} \int\limits_{-\tau}^{0} e^{\varepsilon s}U(s, x(s))\mathrm{d}s,$$

$\varepsilon = \varepsilon_1 \wedge \varepsilon_2$, *while* $\varepsilon_1 > 0$ *and* $\varepsilon_2 > 0$ *are the unique roots to the following equations*:

$$\alpha_2 = \frac{\alpha_3 e^{\varepsilon_1 \tau}}{1 - \bar{\delta}} \tag{33}$$

and

$$1 = \frac{\alpha e^{\varepsilon_2 \tau}}{1 - \bar{\delta}}, \tag{34}$$

respectively. Moreover, if $\alpha_1 = 0$, *then*

$$\limsup_{t \to \infty} \frac{1}{t} \log EV(t, x(t) - G(x(t - \delta(t)), r(t))) \le 0, \tag{35}$$

and

$$\int\limits_{0}^{\infty} EU(t, x(t))\mathrm{d}t < \infty. \tag{36}$$

Proof. Clearly, (30) is stronger than (23). So, for any given initial data $x_0 = \xi \in C([-\tau, 0]; R^n)$, there is a unique global solution $x(t)$ to equation (20) on $[-\tau, \infty)$. By the Itô formula and (30), for any sufficiently large k and any $t \ge 0$, we can obtain

$$E(e^{\varepsilon(t \wedge \sigma_k)}V(t \wedge \sigma_k, x(t \wedge \sigma_k) - G(x(t \wedge \sigma_k - \delta(t) \wedge \sigma_k), r(t \wedge \sigma_k)))) - V(0, x(0) - G(x(-\delta(0)), r(0)))$$

$$= E \int\limits_{0}^{t \wedge \sigma_k} e^{\varepsilon s}[\varepsilon V(s, x(s) - G(x(s - \delta(s)), r(s))) + \mathcal{L}V(x(s - \delta(s)), x(s), s, r(s))]\mathrm{d}s$$

$$\le E \int\limits_{0}^{t \wedge \sigma_k} e^{\varepsilon s}[\varepsilon V(s, x(s) - G(x(s - \delta(s)), r(s))) + \alpha_1 - \alpha_2 V(s, x(s)) + \alpha_3 V(s - \delta(s), x(s - \delta(s)))$$

$$- U(s, x(s)) + \alpha U(s - \delta(s), x(s - \delta(s)))]\mathrm{d}s \tag{37}$$

$$= \varepsilon E \int\limits_{0}^{t \wedge \sigma_k} e^{\varepsilon s}V(s, x(s) - G(x(s - \delta(s)), r(s)))\mathrm{d}s + \alpha_1 E \int\limits_{0}^{t \wedge \sigma_k} e^{\varepsilon s}\mathrm{d}s$$

$$- \alpha_2 E \int\limits_{0}^{t \wedge \sigma_k} e^{\varepsilon s}V(s, x(s))\mathrm{d}s + \alpha_3 E \int\limits_{0}^{t \wedge \sigma_k} e^{\varepsilon s}V(s - \delta(s), x(s - \delta(s)))\mathrm{d}s$$

$$- E \int\limits_{0}^{t \wedge \sigma_k} e^{\varepsilon s}U(s, x(s))\mathrm{d}s + \alpha E \int\limits_{0}^{t \wedge \sigma_k} e^{\varepsilon s}U(s - \delta(s), x(s - \delta(s)))\mathrm{d}s.$$

We have

$$\int\limits_{0}^{t \wedge \sigma_k} e^{\varepsilon s}V(s - \delta(s), x(s - \delta(s)))\mathrm{d}s \le e^{\varepsilon\tau} \int\limits_{0}^{t \wedge \sigma_k} e^{\varepsilon(s - \delta(s))}V(s - \delta(s), x(s - \delta(s)))\mathrm{d}s$$

$$\le \frac{e^{\varepsilon\tau}}{1 - \bar{\delta}} \int\limits_{-\delta(0)}^{t \wedge \sigma_k - \delta(t \wedge \sigma_k)} e^{\varepsilon u}V(u, x(u))\mathrm{d}u \tag{38}$$

$$\le \frac{e^{\varepsilon\tau}}{1 - \bar{\delta}} \int\limits_{-\tau}^{t \wedge \sigma_k} e^{\varepsilon u}V(u, x(u))\mathrm{d}u.$$

Similarly,

$$\int_0^{t \wedge \sigma_k} e^{\varepsilon s} U(s - \delta(s), x(s - \delta(s))) ds \le \frac{e^{\varepsilon \tau}}{1 - \bar{\delta}} \int_{-\tau}^{t \wedge \sigma_k} e^{\varepsilon u} U(u, x(u)) du. \tag{39}$$

Substituting (38) and (39) into (37), we can obtain

$$E(e^{\varepsilon(t \wedge \sigma_k)} V(t \wedge \sigma_k, x(t \wedge \sigma_k) - G(x(t \wedge \sigma_k - \delta(t) \wedge \sigma_k), r(t \wedge \sigma_k))))$$

$$\le C + \frac{\alpha_1 e^{\varepsilon t}}{\varepsilon} - \left(\alpha_2 - \frac{\alpha_3 e^{\varepsilon \tau}}{1 - \bar{\delta}} \right) E \int_0^{t \wedge \sigma_k} e^{\varepsilon s} V(s, x(s)) ds - \left(1 - \frac{\alpha e^{\varepsilon \tau}}{1 - \bar{\delta}} \right) E \int_0^{t \wedge \sigma_k} e^{\varepsilon s} U(s, x(s)) ds$$

$$+ \varepsilon E \int_0^{t \wedge \sigma_k} e^{\varepsilon s} V(s, x(s) - G(x(s - \delta(s)), r(s))) ds,$$

where

$$C = EV(0, x(0) - G(x(-\delta(0)), r(0))) + \frac{\alpha_3 e^{\varepsilon \tau}}{1 - \bar{\delta}} \int_{-\tau}^0 e^{\varepsilon s} V(s, x(s)) ds + \frac{\alpha e^{\varepsilon \tau}}{1 - \bar{\delta}} \int_{-\tau}^0 e^{\varepsilon s} U(s, x(s)) ds.$$

However, for $\varepsilon = \varepsilon_1 \wedge \varepsilon_2$, by (33) and (34), we can obtain

$$\alpha_2 - \frac{\alpha_3 e^{\varepsilon \tau}}{1 - \bar{\delta}} \ge 0 \quad \text{and} \quad 1 - \frac{\alpha e^{\varepsilon \tau}}{1 - \bar{\delta}} \ge 0.$$

So we have

$$E(e^{\varepsilon(t \wedge \sigma_k)} V(t \wedge \sigma_k, x(t \wedge \sigma_k) - G(x(t \wedge \sigma_k - \delta(t) \wedge \sigma_k), r(t \wedge \sigma_k))))$$

$$\le C + \frac{\alpha_1 e^{\varepsilon t}}{\varepsilon} + \varepsilon E \int_0^{t \wedge \sigma_k} e^{\varepsilon s} V(s, x(s) - G(x(s - \delta(s)), r(s))) ds.$$

When $k \to \infty$, for any $t \ge 0$, we have

$$EV(t, x(t) - G(x(t - \delta(t)), r(t))) \le Ce^{-\varepsilon t} + \frac{\alpha_1}{\varepsilon} + \varepsilon E \int_0^t e^{\varepsilon(s-t)} V(s, x(s) - G(x(s - \delta(s)), r(s))) ds$$

$$\le C + \frac{\alpha_1}{\varepsilon} + \varepsilon E \int_0^t e^{\varepsilon(s-t)} V(s, x(s) - G(x(s - \delta(s)), r(s))) ds. \tag{40}$$

So by the Gronwall's inequality,

$$EV(t, x(t) - G(x(t - \delta(t)), r(t))) \le \left[C + \frac{\alpha_1}{\varepsilon} \right] e^{1 - e^{-\varepsilon t}}. \tag{41}$$

Let $t \to \infty$. Then the assertion (31) was proved.

To show assertion (32), by the Itô formula and (30), we have

$$0 \le EV(0, x(0) - G(x(-\delta(0)), r(0))) + \alpha_1 t + E \int_0^t [-\alpha_2 V(s, x(s)) + \alpha_3 V(s - \delta(s), x(s - \delta(s)))] ds$$

$$+ E \int_0^t [-U(s, x(s)) + \alpha U(s - \delta(s), x(s - \delta(s)))] ds \tag{42}$$

$$\leq EV(0, x(0) - G(x(-\delta(0)), r(0))) + \alpha_1 t - \alpha_2 E \int_0^t V(s, x(s))ds$$

$$+ \frac{\alpha_3}{1 - \bar{\delta}} E \int_{-\tau}^t V(s, x(s))ds - E \int_0^t U(s, x(s))ds + \frac{\alpha}{1 - \bar{\delta}} \int_{-\tau}^t EU(s, x(s))ds$$

$$\leq EV(0, x(0) - G(x(-\delta(0)), r(0))) + \alpha_1 t + \frac{\alpha_3}{1 - \bar{\delta}} \int_{-\tau}^0 EV(s, x(s))ds$$

$$+ \frac{\alpha}{1 - \bar{\delta}} \int_{-\tau}^0 EU(s, x(s))ds - \left(1 - \frac{\alpha}{1 - \bar{\delta}}\right) \int_0^t EU(s, x(s))ds.$$

Dividing both sides by t and then letting $t \to \infty$, we obtain assertion (32).

It is not difficult to show that if $\alpha_1 = 0$, for any $t \geq 0$, then (41) becomes

$$EV(t, x(t) - G(x(t - \delta(t)), r(t))) \leq Ce^{1-e^{-\varepsilon t}},$$

and so assertion (35) follows. Moreover, for all $t \geq 0$ from (42), we have

$$\int_0^t EU(s, x(s))ds \leq \frac{1 - \bar{\delta}}{1 - \bar{\delta} - \alpha} EV(0, x(0) - G(x(-\delta(0)), r(0))) + \frac{\alpha_3}{1 - \bar{\delta} - \alpha} \int_{-\tau}^0 EV(s, x(s))ds$$

$$+ \frac{\alpha}{1 - \bar{\delta} - \alpha} \int_{-\tau}^0 EU(s, x(s))ds.$$

Let $t \to \infty$, we obtain assertion (36). The proof is therefore completed. $\qquad\square$

4 Examples

Consider the following stochastic differential equation,

$$d[x(t) - 0.1x(t - \delta(t))] = x(t)[a + bx(t - \delta(t)) - x(t)^2]dt + cx(t)x(t - \delta(t))dW(t)$$
$$+ \int_Z \beta x(t-)x(t - \delta(t)-)vN(dt, dv) \tag{43}$$

on all $t \geq 0$ with initial data $\{x(t) : -\tau \leq t \leq 0\} \in C([-\tau, 0]; R)$, where $W(t)$ is an one dimensional Brownian motion and b and c are all positive constants. Compare (43) with (20), and let $\bar{G}(y, i) = 0.1y$, $\bar{f}(y, x, t, i) = x(a + by - x^2)$, $\bar{g}(y, x, t, i) = cxy$, $\bar{h}(y, x, t, i, v) = \beta xyv$, $V(t, x) = x^2$, and $U(t, x) = x^4$ for $(t, x) \in R_+ \times R$. In the following, we will verify the conditions of Theorems 3.2 and 3.3. By the elementary inequality $|u \pm w|^2 \leq 2|u|^2 + 2|w|^2$, we can show that for any positive constant N, if $|x| \vee |\bar{x}| \vee |y| \vee |\bar{y}| \leq N$, then

$$\begin{aligned}
|\bar{f}(y, x, t, i) - \bar{f}(\bar{y}, \bar{x}, t, i)|^2 &= |x(a + by - x^2) - \bar{x}(a + b\bar{y} - \bar{x}^2)|^2 \\
&= |bxy - b\bar{x}\bar{y} - (x^3 - \bar{x}^3) + a(x - \bar{x})|^2 \\
&= |b(xy - \bar{x}y + \bar{x}y - \bar{x}\bar{y}) - (x - \bar{x})(x + \bar{x} + x\bar{x}) + a(x - \bar{x})|^2 \\
&\leq 3|b[(x - \bar{x})y + \bar{x}(y - \bar{y})]|^2 + 9(|x|^2 + |\bar{x}|^2 + |x\bar{x}|^2)|x - \bar{x}|^2 + 3a^2|x - \bar{x}|^2 \\
&\leq 6(bN)^2(|x - \bar{x}|^2 + |y - \bar{y}|^2) + (18N^2 + 9N^4 + 3a^2)|x - \bar{x}|^2 \\
&\leq \max(6b^2N^2, 18N^2 + 9N^4 + 3a^2)(|x - \bar{x}|^2 + |y - \bar{y}|^2), \\
|\bar{g}(y, x, t, i) - \bar{g}(\bar{y}, \bar{x}, t, i)|^2 &= |cxy - c\bar{x}\bar{y}|^2 = |cxy - c\bar{x}y + c\bar{x}y - c\bar{x}\bar{y}|^2 \leq 2c^2N^2(|x - \bar{x}|^2 + |y - \bar{y}|^2).
\end{aligned}$$

We assume that $\int_Z v\pi(dv) < \infty$ and $\int_Z v^2\pi(dv) < \infty$, then

$$\int_Z |\bar{h}(y, x, t, i, v) - \bar{h}(\bar{y}, \bar{x}, t, i, v)|^2 \pi(dv) = \int_Z |\beta xyv - \beta\bar{y}\bar{x}v|^2 \pi(dv)$$

$$= \beta^2 |xy - \bar{x}y + \bar{x}y - \bar{x}\bar{y}|^2 \int_Z v^2\pi(dv)$$

$$= \beta^2 (|x - \bar{x}|y + |y - \bar{y}|\bar{x})^2 \int_Z v^2\pi(dv)$$

$$\leq 2\beta^2 N^2 \int_Z v^2\pi(dv)(|x - \bar{x}|^2 + |y - \bar{y}|^2).$$

Hence, Assumption 3.1 holds true. Obviously, Assumption 3.2 holds true too. Next,

$$LV(y, x, t, i) = 2(x - 0.1y)x(a + by - x^2) + (cxy)^2 + \beta^2 \int_Z [(x - 0.1y + xyv)^2 - (x - 0.1y)^2]\pi(dv)$$

$$= 2ax^2 + 2bx^2y - 2x^4 - 0.2axy - 0.2bxy^2 + 0.2x^3y + (cxy)^2 + \beta^2 \int_Z (2x^2y - 0.2xy^2)v\pi(dv)$$

$$+ \beta^2 \int_Z (xyv)^2\pi(dv)$$

$$= 2ax^2 + 2x^2y\left[b + \beta^2 \int_Z v\pi(dv)\right] - 2x^4 - 0.2axy + 0.2x^3y - 0.2xy^2\left[b + \beta^2 \int_Z v\pi(dv)\right]$$

$$+ (xy)^2\left[c^2 + \beta^2 \int_Z v^2\pi(dv)\right]$$

$$\leq 2ax^2 + 2x^2y\left[b + \beta^2 \int_Z v\pi(dv)\right] - 1.9x^4 - 0.2axy - 0.2xy^2\left[b + \beta^2 \int_Z v\pi(dv)\right]$$

$$+ (xy)^2\left[0.1 + c^2 + \beta^2 \int_Z v^2\pi(dv)\right]$$

$$\leq 2ax^2 + 0.4x^4 + 2.5y^2\left[b + \beta^2 \int_Z v\pi(dv)\right]^2 - 1.9x^4 + 0.1a^2x^2 + 0.1y^2 + 0.02x^2\left[b + \beta^2 \int_Z v\pi(dv)\right]^2$$

$$+ 0.5y^4 + 0.5x^4 + 0.5y^4\left[0.1 + c^2 + \beta^2 \int_Z v^2\pi(dv)\right]^2$$

$$= \left[2a + 0.1a^2 + 0.02\left(b + \beta^2 \int_Z v\pi(dv)\right)^2\right]x^2 + \left[2.5\left(b + \beta^2 \int_Z v\pi(dv)\right)^2 + 0.1\right]y^2 - x^4$$

$$+ \left[0.5 + 0.5\left(0.1 + c^2 + \beta^2 \int_Z v^2\pi(dv)\right)^2\right]y^4.$$

Assume $\delta(\cdot)$ obeys (26) and let $0.5 + 0.5\left(0.1 + c^2 + \beta^2 \int_Z v^2 \pi(\mathrm{d}v)\right)^2 < 1 - \bar{\delta}$, then by Theorem 3.2, for any initial data $x_0 = \xi \in C([-\tau, 0]; R^n)$, there is a unique global solution $x(t)$ to equation (43) on $t \in [-\tau, \infty)$. If we further assume that

$$2a + 0.1a^2 + 0.02\left(b + \beta^2 \int_Z v\pi(\mathrm{d}v)\right)^2 < 0$$

and

$$2a + 0.1a^2 + 0.02\left(b + \beta^2 \int_Z v\pi(\mathrm{d}v)\right)^2 (\bar{\delta} - 1) > 2.5\left(b + \beta^2 \int_Z v\pi(\mathrm{d}v)\right)^2 + 0.1,$$

then by Theorem 3.3, we obtain that for any given initial data $x_0 = \xi \in C([-\tau, 0]; R)$, the unique global solution $x(t)$ to equation (43) has the properties that

$$\limsup_{t \to \infty} \frac{1}{t} \log E(x(t) - 0.1x(t - \delta(t)))^2 \le 0$$

and

$$\int_0^{\infty} Ex^4(t)\mathrm{d}t < \infty.$$

5 Conclusion

In this paper, first, we obtained the existence and uniqueness theorems for SFDEs with Markovian switching, jump term, and neutral term under local Lipschitz condition and Khasminskii-type condition. Second, we considered the neutral stochastic differential delay equation with variable delays and proved similar results under some new conditions. Finally, we provided one example and verified whether the coefficients satisfied assumptions of Theorem 3.1, and so the equation had a unique global solution.

Conflict of interest: The authors confirm that this article content has no conflicts of interest.

References

[1]   M. Xuerong, *A note on the LaSalle-type theorems for stochastic differential delay equations*, J. Math. Anal. Appl. **268** (2002), 125–142.

[2]   N. Ikeda and S. Watanabe, *Stochastic Differential Equations and Diffusion Processes*, Amsterdam, North Holland, 1981.

[3]   M. Xuerong, *Exponential Stability of Stochastic Differential Equations*, Marcel Dekker, New York, 1994.

[4] I. I. Gikhman and A. V. Skorokhod, *Stochastic Differential Equations*, Springer, Berlin, 1972.

[5] M. Xuerong, *Stochastic Differential Equations and Applications*, 2nd Edition, Horwood Publishing, Chichester, 2007.

[6] M. Xuerong and M. J. Rassias, *Khasminskii-type theorems for stochastic differential delay equations*, Stoch. Anal. Appl. **23** (2005), 1045–1069.

[7] S. Yi, L. Qi, and M. Xuerong, *The improved LaSalle-type theorems for stochastic functional differential equations*, J. Math. Anal. Appl. **318** (2006), 134–154.

[8] L. Qi, M. Xuerong, and S. Yi, *Generalised theory on asymptotic stability and boundedness of stochastic functional differential equations*, Automatica **47** (2011), 2071–2081.

[9] S. Minghui, H. Liangjian, M. Xuerong, and Z. Liguo, *Khasminskii-type theorems for stochastic functional differential equations*, Discrete Contin. Dyn. Syst. Ser. B **18** (2013), no. 6, 1697–1714.

[10] W. Fuke, *Khasminskii-type theorems for neutral stochastic functional differential equations*, Math. Appl. (Wuhan) **21** (2008), no. 4, 794–799.

[11] Z. Quanxin, *pth Moment exponential stability of impulsive stochastic functional differential equations with Markovian switching*, J. Franklin Inst. **351** (2004), no. 7, 3965–3986.

[12] H. Wei, Z. Quanxin, and H. R. Karimi, *Some improved Razumikhin stability criteria for impulsive stochastic delay differential systems*, IEEE Trans. Automat. Control **64** (2019), no. 12, 5207–5213.

[13] M. Wei, H. Liangjian, and M. Xuerong, *Neutral stochastic functional differential equations with Lévy jumps under the local Lipschitz condition*, Adv. Differ. Equ. **24** (2017), 1–22.

[14] Z. Quanxin, *Razumikhin-type theorem for stochastic functional differential equations with Lévy noise and Markov switching*, Int. J. Control **90** (2017), no. 8, 1703–1712.

[15] L. Haidan and Z. Quanxin, *The pth moment exponential stability and almost surely exponential stability of stochastic differential delay equations with Poisson jump*, J. Math. Anal. Appl. **471** (2019), 197–210.

[16] M. Xuerong and Y. Chenggui, *Stochastic Differential Equations with Markovian Switching*, Imperial College Press, London, 2006.

[17] D. Applebaum, *Lévy processes and Stochastic Calculus*, 2nd Edition, Cambridge Studies in Advanced Mathematics, New York, 2009.

[18] W. Fuke and H. Shigeng, *Khasminskii-type theorems for stochastic functional differential equations with infinite delay*, Stat. Probab. Lett. **81** (2011), 1690–1694.