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Research Article

Shoubo Jin*

Existence and uniqueness of solutions for the stochastic Volterra-Levin equation with variable delays

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Abstract: The Picard iteration method is used to study the existence and uniqueness of solutions for the stochastic Volterra-Levin equation with variable delays. Several sufficient conditions are specified to ensure that the equation has a unique solution. First, the stochastic Volterra-Levin equation is transformed into an integral equation. Then, to obtain the solution of the integral equation, the successive approximation sequences are constructed, and the existence and uniqueness of solutions for the stochastic Volterra-Levin equation are derived by the convergence of the sequences. Finally, two examples are given to demonstrate the validity of the theoretical results.

Keywords: stochastic Volterra-Levin equations, variable delays, existence, uniqueness

MSC 2020: 34K50, 60H10

1 Introduction

As stochastic modeling is used in the fields such as physics, economics, chemistry, and scholars have paid more and more attention to stochastic differential equations. Therefore, the existence and uniqueness of solutions of the equation have become a hot topic in recent years. The Volterra equation is a significant differential equation, which has been applied to the circulating fuel nuclear reactor, the neural networks, the population projection and others. In 1928, Volterra [1] first proposed the Volterra equation, i.e.,

$$x'(t) = -\int_{t-L}^{t} p(s-t)f(x,s)\mathrm{d}s, \tag{1}$$

and Levin [2] obtained the asymptotical stability of (1). Burton investigated the stability of equation (1) by the contraction mapping principle in [3]. Zhao and Yuan [4] considered 3/2-stability of a generalized Volterra-Levin equation. The discrete Volterra equation describing the evolutionary process of the population was recently investigated in [5].

To analyze the Volterra equation, Levin [2] used the limited condition that is pretty hard to be checked in practical application, $\lim_{|x|\to\infty}\int_0^x f(x)\mathrm{d}x = \infty$, and the author also required that the function $p(t)$ has good properties, such as $\frac{\mathrm{d}p}{\mathrm{d}t} \le 0$, $\frac{\mathrm{d}^2p}{\mathrm{d}t^2} \ge 0$ and $\frac{\mathrm{d}^3p(t)}{\mathrm{d}t^3} \le 0$ for any $t \in (0, +\infty)$. Although the conditions of $f(x)$ were simplified by averages in [3], there were still more requirements for the function $f(x)$. In this paper, the constrains of $f(x)$ and $p(t)$ will be weaken in the stochastic Volterra-Levin equation with variable delays.

* **Corresponding author: Shoubo Jin,** School of Mathematics and Statistics, Suzhou University, Suzhou 234000, China, e-mail: jin_shoubo@163.com

Let $\{\Omega, \mathcal{F}, P\}$ be a complete probability space equipped with some filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions, that the filtration is right continuous and $\{\mathcal{F}_0\}$ contains all P-null sets. Let $\{W(t), t \geq 0\}$ denote a standard Brownian motion defined on $\{\Omega, \mathcal{F}, P\}$. We investigate the existence and uniqueness of solutions for the stochastic Volterra-Levin equations with variable delays, i.e.,

$$\begin{cases} \mathrm{d}x(t) = -\left(\int\limits_{t-L}^{t} p(s-t)f(x(s-\alpha(s)))\mathrm{d}s\right)\mathrm{d}t + g(x(t-\beta(t)))\mathrm{d}W(t), \quad t \in [0, T], \\ x(s) = \varphi(s) \in C([-L-\tau, 0]; R). \end{cases} \tag{2}$$

where $f(x)$ and $g(x)$ are known functions satisfying certain conditions, the constant $L > 0$, $p(s) \in C([-L, T]; R)$, and $R = (-\infty, +\infty)$, $\alpha(t)$ and $\beta(t)$ are the variable delays, satisfying $\alpha(t), \beta(t) \in [0, \tau]$.

Scholars have become increasingly interested in the stochastic Volterra-Levin equations. The equation has been applied to many special research fields, such as the population model of spatial heterogeneity [6], the predator-prey model [7], and the nonautonomous competitive model [8]. After reviewing and sorting out the literature, it is found that most scholars currently use the principle of contraction mapping to explore the equation. For example, Luo [9] analyzed the exponential stability of the classical stochastic Volterra-Levin equations. Zhao et al. [10] investigated the mean square asymptotic stability of the generalized stochastic Volterra-Levin equations, which improved the results in [9]. Li and Xu [11] demonstrated the existence and global attractiveness of periodic solutions for impulsive stochastic Volterra-Levin equations. In this paper, the Picard iteration method is directly used to prove the existence and uniqueness of solutions of the stochastic Volterra-Levin equations with variable delays, which can give a more intuitive approximate solution. Recently, for the case without delay, Jaber [12] proved the weak existence and uniqueness of affine stochastic Volterra equations. Dung [13] revealed Itô differential representation of the stochastic Volterra integral equations. For the case of constant delay, Guo and Zhu [14] used this approximate method to prove the existence of solutions of stochastic Volterra-Levin equations. Some delay Volterra integral problems on a half-line were analyzed in [15]. The qualitative properties of solutions of nonlinear Volterra equations without random disturbance were investigated in [16]. However, there are only a few results of the stochastic Volterra equations with variable delay.

Generally, a time delay is inevitable and variable in practical application, and the future state of an existing system depends not only on the current state of the systems but also on the past [17–19]. When the function $f(x) = x$ in equation (2), Benhadri and Zeghdoudi [20] applied the variable delays to the Volterra-Levin equation with Poisson jump and obtained the mean square stability by the fixed-point theory. The authors in [5] discussed the linear discrete Volterra equation with infinite delay when the function $g(x) = 0$, which means there are not any random noises. In this paper, we will investigate the Volterra-Levin equation with the variable delays and the standard Brownian motion in more general conditions. Moreover, the Picard successive approximation method is used to prove the existence and uniqueness of the solution in some sufficient conditions. Compared with [2] and [3], these conditions are easier to be verified.

The rest of this paper is organized as follows. In Section 2, some necessary conditions and lemmas are established. In Section 3, the existence and uniqueness of solutions are proved. In Section 4, two examples are given to demonstrate the validity of the main results.

2 Assumptions and lemmas

To obtain the existence and uniqueness of the solutions for equation (2), the following assumptions are given in this paper.

(H_1) $\lim_{x\to 0}\frac{f(x)}{x} = \beta > 0$.

(H_2) $g(0) = f(0) = 0$, and there exists a constant $\mu > 0$, such that $\frac{f(x)}{x} > \mu$.

(H_3) $\int_{-L}^{0} p(s)\mathrm{d}s = m > 0$, $\int_{-L}^{0} |p(s)s|\mathrm{d}s = m_1 > 0$ and $\max_{-L\le s\le 0}|p(s)| = m_2$.

(H_4) There is a positive constant $K > 0$, such that $|f(x) - f(y)| \vee |g(x) - g(y)| \le K|x - y|$ for all $x, y \in R$.

(H_5) $5K^2\left(\frac{2}{\mu K}(1 - \mathrm{e}^{-m\mu T}) + 2m_1^2 + \frac{1}{2m\mu}(1 - \mathrm{e}^{-2m\mu T})\right) \vee \left(K^2L^4m_2^2 + \frac{1}{2}K^2 + \frac{9}{10}\right)\mathrm{e}^{mKT} < 1.$

Remark 2.1. Assumptions H_1–H_3 are some common conditions for studying the Volterra-Levin equations. For instance, Luo [9] discussed the exponential stability for classical stochastic Volterra-Levin equations on Assumptions H_1–H_3. Zhao et al. [10] studied the mean square asymptotic stability of a class of generalized nonlinear stochastic Volterra-Levin equations on similar assumptions. Assumption H_4 is the Lipschitz condition, which is the core condition for ensuring the existence and uniqueness of solutions for the initial value problem.

Now, we transform (2) into the following form by using the properties of integrals.

Lemma 2.1. *Assuming that H_1–H_3 are established*, equation (2) *can be transformed into*

$$
\begin{aligned}
x(t) = {} & \mathrm{e}^{-\int_0^t ma(u-\alpha(u))\mathrm{d}u}\left(\varphi(0) - \int_{-L}^{0} p(s)\int_{s}^{0} f(\varphi(u-\alpha(u)))\mathrm{d}u\mathrm{d}s\right) \\
& + \int_0^t [x(v) - x(v-\alpha(v))]ma(v-\alpha(v))\mathrm{e}^{-\int_v^t ma(u-\alpha(u))\mathrm{d}u}\mathrm{d}v \\
& + \int_{-L}^{0} p(s)\int_{t+s}^{t} f(x(u-\alpha(u)))\mathrm{d}u\mathrm{d}s \\
& - \int_0^t \mathrm{e}^{-\int_v^t ma(u-\alpha(u))\mathrm{d}u} ma(v-\alpha(v))\left[\int_{-L}^{0} p(s)\int_{v+s}^{v} f(x(u-\alpha(u)))\mathrm{d}u\mathrm{d}s\right]\mathrm{d}v \\
& - \int_0^t \mathrm{e}^{-\int_s^t ma(u-\alpha(u))\mathrm{d}u} g(x(s-\beta(s)))\mathrm{d}W(s).
\end{aligned}
\tag{3}
$$

Proof. Let $a(t) = \begin{cases} \frac{f(x(t))}{x(t)} & x(t) \neq 0 \\ \beta & x(t) = 0 \end{cases}$, it is obtained that $a(t) \in C([-\tau, T]; R^+)$ from Assumptions H_1 and H_2. Using

$$
\begin{aligned}
\int_{t-L}^{t} p(s-t)f(x(s-\alpha(s)))\mathrm{d}s &= \int_{-L}^{0} p(s)f(x(s+t-\alpha(s+t)))\mathrm{d}s \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\int_{-L}^{0} p(s)\int_{0}^{t+s} f(x(u-\alpha(u)))\mathrm{d}u\mathrm{d}s \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\int_{-L}^{0} p(s)\int_{t}^{t+s} f(x(u-\alpha(u)))\mathrm{d}u\mathrm{d}s + mf(x(t-\alpha(t))) \\
&= -\frac{\mathrm{d}}{\mathrm{d}t}\int_{-L}^{0} p(s)\int_{t+s}^{t} f(x(u-\alpha(u)))\mathrm{d}u\mathrm{d}s + ma(t-\alpha(t))x(t-\alpha(t)),
\end{aligned}
\tag{4}
$$

Equation (2) can be transformed into

$$
\mathrm{d}x(t) = -ma(t-\alpha(t))x(t-\alpha(t))\mathrm{d}t + \mathrm{d}\left(\int_{-L}^{0} p(s)\int_{t+s}^{t} f(x(u-\alpha(u)))\mathrm{d}u\mathrm{d}s\right) + g(x(t-\beta(t)))\mathrm{d}W(t). \tag{5}
$$

The two sides of the aforementioned equation are multiplied by $e^{\int_s^t ma(u-\alpha(u))\mathrm{d}u}$, and then integral from 0 to t, using the distribution integral method and the following formula:

$$\mathrm{d}\left(x(t)e^{\int_s^t ma(u-\alpha(u))\mathrm{d}u}\right) = e^{\int_s^t ma(u-\alpha(u))\mathrm{d}u}\mathrm{d}x(t) + ma(t-\alpha(t))x(t)e^{\int_s^t ma(u-\alpha(u))\mathrm{d}u}\mathrm{d}t. \tag{6}$$

It obtains

$$\begin{aligned}
&x(t)e^{\int_s^t ma(u-\alpha(u))\mathrm{d}u} - x(0)e^{\int_s^0 ma(u-\alpha(u))\mathrm{d}u}\\
&= \int_0^t [x(v) - x(v-\alpha(v))]ma(v-\alpha(v))e^{\int_s^v ma(u-\alpha(u))\mathrm{d}u}\mathrm{d}v + e^{\int_s^t ma(u-\alpha(u))\mathrm{d}u}\int_{-L}^0 p(s)\int_{t+s}^t f(x(u-\alpha(u)))\mathrm{d}u\mathrm{d}s\\
&\quad - e^{\int_s^0 ma(u-\alpha(u))\mathrm{d}u}\int_{-L}^0 p(s)\int_s^0 f(\varphi(u-\alpha(u)))\mathrm{d}u\mathrm{d}s\\
&\quad - \int_0^t e^{\int_s^t ma(u-\alpha(u))\mathrm{d}u} ma(t-\alpha(t))\left(\int_{-L}^0 p(s)\int_{t+s}^t f(x(u-\alpha(u)))\mathrm{d}u\mathrm{d}s\right)\mathrm{d}t\\
&\quad + \int_0^t e^{\int_s^t ma(u-\alpha(u))\mathrm{d}u} g(x(t-\beta(t)))\mathrm{d}W(t).
\end{aligned} \tag{7}$$

Two sides of the aforementioned equality are multiplied by $e^{-\int_s^t ma(u-\alpha(u))\mathrm{d}u}$, and using

$$e^{-\int_s^t ma(u-\alpha(u))\mathrm{d}u}\int_0^t e^{\int_s^v ma(u-\alpha(u))\mathrm{d}u} g(x(v-\beta(v)))\mathrm{d}W(v) = \int_0^t e^{\int_t^s ma(u-\alpha(u))\mathrm{d}u} g(x(s-\beta(s)))\mathrm{d}W(s). \tag{8}$$

We can obtain equation (3). □

Remark 2.2. The method of transforming the stochastic differential equation into the integral equation, has been widely used. When the function $g(x)$ is independent of the variable x, Luo studied the exponential stability for a class of stochastic Volterra-Levin equations by using the method in [9]. Zhao et al. investigated the mean square asymptotic of the generalized nonlinear stochastic Volterra-Levin equations [10]. Based on the semigroup of operators, Yang et al. transformed the heat conduction equation into the fractional Volterra integral equation in [21]. In this lemma, due to the appearance of variable delays, we need to deal with it more precisely.

3 Existence and uniqueness

Picard iteration is the most commonly used method in the proof of the existence of solutions to the stochastic equations [20–23]. In this paper, the existence and uniqueness of solutions for equation (2) are proved by the Picard iteration method. An important characteristic of this method is that it is constructive, and the bounds on the difference between iterates and the solutions are easily available. Such bounds are not only useful for the approximation of solutions but also necessary in the study of qualitative properties of solutions.

Now, let's briefly summarize this idea of the Picard iteration method. To obtain the solution for a class of integral equation $y(t) = y_0 + \int_0^t f(\tau, y(\tau))\mathrm{d}\tau$, Picard successive approximation sequences are constructed as follows.

$$y_{m+1}(t) = y_0 + \int_0^t f(\tau, y_m(\tau))\mathrm{d}\tau.$$

If the sequences $\{y_m(t)\}$ converge uniformly to a continuous function $y(t)$ in some interval J, then we may past to the limit in both sides of the aforementioned equation to obtain

$$y(t) = \lim_{m\to\infty} y_{m+1}(t) = y_0 + \lim_{m\to\infty}\int_0^t f(\tau, y_m(\tau))\mathrm{d}\tau = y_0 + \int_0^t f(\tau, y(\tau))\mathrm{d}\tau.$$

So that $y(t)$ is the desired solution.

Theorem 3.1. *Suppose that assumptions H_1–H_5 hold, then equation* (2) *has a unique solution in* $[0, T]$.

Proof. The Picard iteration method is used in the proof of this theorem, and using Lemma 2.1, we construct the Picard iteration sequences.

$$\begin{cases} x_0^0 = \varphi(s),\ x^0(t) = \varphi(0),\ (0 \le t \le T), \\ x_0^n = \varphi(s),\ n \ge 1, \\ x^n(t) = I_0^{n-1}(t) + I_1^{n-1}(t) + I_2^{n-1}(t) + I_3^{n-1}(t) + I_4^{n-1}(t), \end{cases} \tag{9}$$

where

$$I_0^{n-1}(t) = \mathrm{e}^{-\int_0^t ma(u-\alpha(u))\mathrm{d}u}\left(\varphi(0) - \int_{-L}^0 p(s)\int_s^0 f(\varphi(u-\alpha(u)))\mathrm{d}u\mathrm{d}s\right),$$

$$I_1^{n-1}(t) = \int_0^t [x^{n-1}(v) - x^{n-1}(v-\alpha(v))]ma(v-\alpha(v))\mathrm{e}^{-\int_v^t ma(u-\alpha(u))\mathrm{d}u}\mathrm{d}v,$$

$$I_2^{n-1}(t) = \int_{-L}^0 p(s)\int_{t+s}^t f(x^{n-1}(u-\alpha(u)))\mathrm{d}u\mathrm{d}s,$$

$$I_3^{n-1}(t) = -\int_0^t \mathrm{e}^{-\int_v^t ma(u-\alpha(u))\mathrm{d}u} ma(v-\alpha(v))\left[\int_{-L}^0 p(s)\int_{v+s}^v f(x^{n-1}(u-\alpha(u)))\mathrm{d}u\mathrm{d}s\right]\mathrm{d}v,$$

$$I_4^{n-1}(t) = -\int_0^t \mathrm{e}^{-\int_s^t ma(u-\alpha(u))\mathrm{d}u} g(x^{n-1}(s-\beta(s)))\mathrm{d}W(s).$$

(1) We first verify the mean square boundness of $x^n(t)(n \ge 0)$, so we only need to prove $E\sup_{0\le t\le T}|x^n(t)|^2$ is bounded.

It is obvious that $E|x^0(t)|^2 = E|\varphi(0)|^2 < +\infty$ for $n = 0$. Suppose $E|x^{n-1}(t)|^2$ is bounded, we begin to prove $E|x^n(t)|^2$ is bounded. Using the formula (9), it obtains $E|x^n(t)|^2 \le 5\sum_{i=0}^4 E|I_i^{n-1}(t)|^2$.

Using Assumptions H_2 and H_4, it obtains

$$\begin{aligned} E\left[\sup_{0\le t\le T}|I_0^{n-1}(t)|^2\right] &= E\left[\sup_{0\le t\le T}\left|\mathrm{e}^{-\int_0^t ma(u-\alpha(u))\mathrm{d}u}\left(\varphi(0) - \int_{-L}^0 p(s)\int_s^0 f(\varphi(u-\alpha(u)))\mathrm{d}u\mathrm{d}s\right)\right|^2\right] \\ &\le 2E|\varphi(0)|^2 + 2E\left|\int_{-L}^0 p(s)\int_s^0 f(\varphi(u-\alpha(u)))\mathrm{d}u\mathrm{d}s\right|^2 \\ &\le 2E|\varphi(0)|^2 + 2K^2m_1^2E\left[\sup_{-L-\tau\le t\le 0}|\varphi(t)|^2\right] < +\infty \end{aligned} \tag{10}$$

and

$$
\begin{aligned}
E\left[\sup_{0\le t\le T}|I_1^{n-1}(t)|^2\right] &= E\left[\sup_{0\le t\le T}\left|\int_0^t [x^{n-1}(v)-x^{n-1}(v-\alpha(v))]ma(v-\alpha(v))\mathrm{e}^{-\int_v^t ma(u-\alpha(u))\mathrm{d}u}\mathrm{d}v\right|^2\right]\\
&\le 2E\left[\sup_{-L-\tau\le t\le T}|x^{n-1}(t)|^2\left|\int_0^t ma(v-\alpha(v))\mathrm{e}^{-\int_v^t ma(u-\alpha(u))\mathrm{d}u}\mathrm{d}v\right|^2\right]\\
&= 2E\left[\sup_{-L-\tau\le t\le T}|x^{n-1}(t)|^2\left|1-\mathrm{e}^{-\int_0^t ma(u-\alpha(u))\mathrm{d}u}\right|^2\right]\\
&\le 2(1-\mathrm{e}^{-mKT})^2E\left[\sup_{-L-\tau\le t\le T}|x^{n-1}(t)|^2\right] < +\infty.
\end{aligned}
\tag{11}
$$

Further, by using Hölder inequality, we obtain

$$
\begin{aligned}
E\left[\sup_{0\le t\le T}|I_2^{n-1}(t)|^2\right] &\le K^2E\left[\sup_{0\le t\le T}\left|\int_{-L}^0|p(s)|\int_{t+s}^t|x^{n-1}(u-\alpha(u))|\mathrm{d}u\mathrm{d}s\right|^2\right]\\
&\le K^2\max_{s\in[-L,0]}|p(s)|^2E\left[\sup_{0\le t\le T}\int_{-L}^0\int_{t+s}^t|x^{n-1}(u-\alpha(u))|\mathrm{d}u\mathrm{d}s\right]^2\\
&\le K^2\max_{s\in[-L,0]}|p(s)|^2E\left[\sup_{0\le t\le T}\int_{-L}^0\int_{t-L}^t|x^{n-1}(u-\alpha(u))|\mathrm{d}u\mathrm{d}s\right]^2\\
&\le K^2L^4\max_{s\in[-L,0]}|p(s)|^2E\left[\sup_{-L-\tau\le t\le T}|x^{n-1}(t)|^2\right]\\
&\le K^2L^4m_2^2\left[\sup_{-L-\tau\le s<0}E|\varphi(s)|^2+\sup_{0\le t\le T}E|x^{n-1}(t)|^2\right] < +\infty
\end{aligned}
\tag{12}
$$

and

$$
\begin{aligned}
E\sup_{0\le t\le T}|I_3^{n-1}(t)|^2 &\le K^2E\left[\sup_{0\le t\le T}\int_0^t \mathrm{e}^{-\int_v^t ma(u-\alpha(u))\mathrm{d}u}ma(v-\alpha(v))\left(\int_{-L}^0|p(s)|\int_{v+s}^v|x^{n-1}(u-\alpha(u))|\mathrm{d}u\mathrm{d}s\right)\mathrm{d}v\right]^2\\
&\le K^2L^2\max_{s\in[-L,0]}|p(s)|^2E\left[\sup_{0\le t\le T}\int_0^t \mathrm{e}^{-\int_v^t ma(u-\alpha(u))\mathrm{d}u}ma(v-\alpha(v))\left(\int_{v-L}^v|x^{n-1}(u-\alpha(u))|\mathrm{d}u\right)\mathrm{d}v\right]^2\\
&\le K^2L^2\max_{s\in[-L,0]}|p(s)|^2E\left[\sup_{0\le t\le T}\int_0^t \mathrm{e}^{-\int_v^t ma(u-\alpha(u))\mathrm{d}u}ma(v-\alpha(v))\mathrm{d}v\right.\\
&\quad\left.\times\int_0^t \mathrm{e}^{-\int_v^t ma(u-\alpha(u))\mathrm{d}u}ma(v-\alpha(v))\left(\int_{v-L}^v|x^{n-1}(u-\alpha(u))|\mathrm{d}u\right)^2\mathrm{d}v\right]\\
&\le K^2L^2\max_{s\in[-L,0]}|p(s)|^2E\sup_{0\le t\le T}\left[\left(1-\mathrm{e}^{-\int_0^t ma(u-\alpha(u))\mathrm{d}u}\right)\right.\\
&\quad\left.\times\int_0^t \mathrm{e}^{-\int_v^t ma(u-\alpha(u))\mathrm{d}u}ma(v-\alpha(v))\left(\int_{v-L}^v|x^{n-1}(u-\alpha(u))|\mathrm{d}u\right)^2\mathrm{d}v\right]
\end{aligned}
\tag{13}
$$

$$\le K^2L^3 \max_{s\in[-L,0]}|p(s)|^2 E\left[\sup_{0\le t\le T}\int_0^t e^{-\int_v^t ma(u-\alpha(u))\mathrm{d}u} ma(v-\alpha(v))\left(\int_{v-L}^{v}|x^{n-1}(u-\alpha(u))|^2\mathrm{d}u\right)\mathrm{d}v\right]$$
$$\le K^2L^4 \max_{s\in[-L,0]}|p(s)|^2 E\left[\sup_{-L-\tau\le t\le T}|x^{n-1}(t)|^2\int_0^t e^{-\int_v^t ma(u-\alpha(u))\mathrm{d}u} ma(v-\alpha(v))\mathrm{d}v\right]$$
$$\le K^2L^4 \max_{s\in[-L,0]}|p(s)|^2 E\left[\sup_{-L-\tau\le t\le T}|x^{n-1}(t)|^2\left(1-e^{-\int_0^t ma(u-\alpha(u))\mathrm{d}u}\right)\right]$$
$$\le K^2L^4m_2^2\left[\sup_{-L-\tau\le s<0}E|\varphi(s)|^2+\sup_{0\le t\le T}E|x^{n-1}(t)|^2\right]<+\infty.$$

From Assumptions H_1 and H_4, we know

$$\begin{aligned}E\sup_{0\le t\le T}|I_4^{n-1}(t)|^2 &\le E\sup_{0\le t\le T}\left|\int_0^t e^{-\int_s^t ma(u-\alpha(u))\mathrm{d}u}g(x^{n-1}(s-\beta(s)))\mathrm{d}W(s)\right|^2\\
&\le K^2\sup_{0\le t\le T}E\left[\int_0^t e^{-2\int_s^t ma(u-\alpha(u))\mathrm{d}u}|x^{n-1}(s-\beta(s))|^2\mathrm{d}s\right]\\
&\le K^2\sup_{0\le t\le T}E\left[\left(1-e^{-2\int_0^t ma(u-\alpha(u))\mathrm{d}u}\right)\sup_{-L-\tau\le t\le T}|x^{n-1}(t)|^2\right]\\
&\le K^2E\left[\sup_{-L-\tau\le s\le 0}|\varphi(s)|^2+\sup_{0\le t\le T}|x^{n-1}(t)|^2\right]<+\infty.\end{aligned}\tag{14}$$

So,

$$\begin{aligned}E\left[\sup_{-L-\tau\le t\le T}|x^n(t)|^2\right] &\le E\left[\sup_{-L-\tau\le t\le 0}|\varphi(s)|^2\right]+E\left[\sup_{0\le t\le T}|x^n(t)|^2\right]\\
&\le 10E|\varphi(0)|^2+(1+10K^2m_1^2)E\left[\sup_{-L-\tau\le s\le 0}|\varphi(s)|^2\right]\\
&\quad+[10K^2L^4m_2^2+5K^2+10(1-e^{-mKT})]\left(E\sup_{-L-\tau\le s\le 0}|\varphi(s)|^2+E\sup_{0\le s\le T}|x^{n-1}(t)|^2\right)<+\infty.\end{aligned}\tag{15}$$

(2) Verifying the mean square continuity of $x^n(t)$

Suppose $t_1>0$ and r is sufficiently small, we obtain the properties as follows.

$$E|x^n(t_1+r)-x^n(t_1)|^2\le 5\sum_{i=0}^{4}E|I_i^{n-1}(t_1+r)-I_i^{n-1}(t_1)|^2.$$

By Itô integration, we have

$$\begin{aligned}E|I_0^{n-1}(t_1+r)-I_0^{n-1}(t_1)|^2 &\le\left[e^{-\int_{t_1}^{t_1+r}ma(u-\alpha(u))\mathrm{d}u}-1\right]^2E\left|\varphi(0)-\int_{-L}^{0}p(s)\int_s^0 f(\varphi(u-\alpha(u)))\mathrm{d}u\mathrm{d}s\right|^2\\
&\to 0\ (r\to 0),\end{aligned}\tag{16}$$

$$\begin{aligned}E|I_1^{n-1}(t_1+r)-I_1^{n-1}(t_1)|^2 &\le 2\left(e^{-\int_{t_1}^{t_1+r}ma(u-\alpha(u))\mathrm{d}u}-1\right)^2E\left[\int_0^{t_1}[x^{n-1}(v)-x^{n-1}(v-\alpha(v))]ma(v-\alpha(v))\mathrm{d}v\right]^2\\
&\quad+2E\left[\int_{t_1}^{t_1+r}[x^{n-1}(v)-x^{n-1}(v-\alpha(v))]ma(v-\alpha(v))\mathrm{d}v\right]^2\to 0\ (r\to 0),\end{aligned}\tag{17}$$

$$
\begin{aligned}
E|I_2^{n-1}(t_1+r)-I_2^{n-1}(t_1)|^2 \le{}& 2E\left(\int_{-L}^{0}|p(s)|\int_{t_1+s}^{t_1+r+s}|f(x^{n-1}(u-\alpha(u)))|\mathrm{d}u\mathrm{d}s\right)^2 \\
&+2m^2E\left(\int_{t_1}^{t_1+r}|f(x^{n-1}(u-\alpha(u)))|\mathrm{d}u\right)^2 \to 0\ (r\to 0),
\end{aligned}
\tag{18}
$$

$$
\begin{aligned}
E|I_3^{n-1}(t_1+r)-I_3^{n-1}(t_1)|^2 \le{}& 2\left(\mathrm{e}^{-\int_{t_1}^{t_1+r}ma(u-\alpha(u))\mathrm{d}u}-1\right)^2E\left(\int_0^{t_1}\int_{-L}^{0}|p(s)|\int_{v+s}^{v}|f(x^{n-1}(u-\alpha(u)))|\mathrm{d}u\mathrm{d}s\mathrm{d}v\right)^2 \\
&+2E\left(\int_{t_1}^{t_1+r}ma(v-\alpha(v))\int_{-\tau}^{0}|p(s)|\int_{v+s}^{v}|f(x^{n-1}(u-\alpha(u)))|\mathrm{d}u\mathrm{d}s\mathrm{d}v\right)^2 \to 0\ (r\to 0),
\end{aligned}
\tag{19}
$$

and

$$
\begin{aligned}
E|I_4^{n-1}(t_1+r)-I_4^{n-1}(t_1)|^2 \le{}& 2\left(\mathrm{e}^{-\int_{t_1}^{t_1+r}ma(u-\alpha(u))\mathrm{d}u}-1\right)^2E\left(\int_0^{t_1}\mathrm{e}^{-\int_s^{t_1}ma(u-\alpha(u))\mathrm{d}u}g(x^{n-1}(s-\alpha(s)))\mathrm{d}W(s)\right) \\
&+2E\left(\int_{t_1}^{t_1+r}\mathrm{e}^{-\int_s^{t_1+r}ma(u-\alpha(u))\mathrm{d}u}g(x^{n-1}(s-\alpha(s)))\mathrm{d}W(s)\right)^2 \\
\le{}& 2\left(\mathrm{e}^{-\int_{t_1}^{t_1+r}ma(u-\alpha(u))\mathrm{d}u}-1\right)^2E\int_0^{t_1}\mathrm{e}^{-2\int_s^{t_1}ma(u-\alpha(u))\mathrm{d}u}g^2(x^{n-1}(s-\alpha(s)))\mathrm{d}s \\
&+2m^2E\int_{t_1}^{t_1+r}\mathrm{e}^{-2\int_s^{t_1+r}a^2(u-\alpha(u))\mathrm{d}u}g^2(x^{n-1}(s-\alpha(s)))\mathrm{d}s \to 0\quad (r\to 0).
\end{aligned}
\tag{20}
$$

So $E|x^n(t_1+r)-x^n(t_1)|^2 \le 5\sum_{i=0}^{4}E|I_i^{n-1}(t_1+r)-I_i^{n-1}(t_1)|^2 \to 0$, the mean square continuity of $x^n(t)$ is verified.

(3) This part proves the convergence of sequences $\{x^n(t)\}_{n\ge 0}$.
By using the similar method of Step (1), we have

$$
\begin{aligned}
E\sup_{0\le t\le T}|x^{n+1}(t)-x^n(t)|^2 \le{}& 5E\sup_{0\le t\le T}\left[\int_0^t(x^n(v)-x^{n-1}(v))ma(v-\alpha(v))\mathrm{e}^{-\int_v^t ma(u-\alpha(u))\mathrm{d}u}\mathrm{d}v\right]^2 \\
&+5E\sup_{0\le t\le T}\left[\int_0^t(x^n(v-\alpha(v))-x^{n-1}(v-\alpha(v)))ma(v-\alpha(v))\mathrm{e}^{-\int_v^t ma(u-\alpha(u))\mathrm{d}u}\mathrm{d}v\right]^2 \\
&+5K^2E\sup_{0\le t\le T}\left[\int_{-L}^{0}|p(s)|\left|\int_{t+s}^{t}|x^n(u-\alpha(u))-x^{n-1}(u-\alpha(u))|\mathrm{d}u\right|\mathrm{d}s\right]^2+5K^2E\sup_{0\le t\le T} \\
&\times\left[\int_0^t\mathrm{e}^{-\int_v^t ma(u-\alpha(u))\mathrm{d}u}ma(v-\alpha(v))\left(\int_{-L}^{0}|p(s)|\left|\int_{v+s}^{v}|x^n(u-\alpha(u))-x^{n-1}(u-\alpha(u))|\mathrm{d}u\right|\mathrm{d}s\right)\mathrm{d}v\right]^2 \\
&+5K^2E\sup_{0\le t\le T}\left[\int_0^t\mathrm{e}^{-\int_s^t ma(u-\alpha(u))\mathrm{d}u}|x^n(s-\beta(s))-x^{n-1}(s-\beta(s))|\mathrm{d}W(s)\right]^2
\end{aligned}
$$

$$
\begin{aligned}
&\leq 10E\sup_{-L-\tau\leq t\leq T}|x^n(t)-x^{n-1}(t)|^2E\sup_{0\leq t\leq T}\left[\int_0^t ma(v-\alpha(v))\mathrm{e}^{-\int_v^t ma(u-\alpha(u))\mathrm{d}u}\mathrm{d}v\right]^2\\
&\quad+5K^2E\left[\int_{-L}^0|p(s)s|\mathrm{d}sE\sup_{-L-\tau\leq t\leq T}|x^n(t)-x^{n-1}(t)|\right]^2\\
&\quad+5K^2E\left[\sup_{0\leq t\leq T}\int_0^t \mathrm{e}^{-\int_v^t ma(u-\alpha(u))\mathrm{d}u}ma(v-\alpha(v))\mathrm{d}v\int_{-L}^0|p(s)s|\mathrm{d}s\sup_{t\geq -L-\tau}|x^n(t)-x^{n-1}(t)|^2\right]^2\\
&\quad+5K^2E\sup_{0\leq t\leq T}\int_0^t \mathrm{e}^{-2\int_s^t ma(u-\alpha(u))\mathrm{d}u}|x^n(s-\beta(s))-x^{n-1}(s-\beta(s))|^2\mathrm{d}s\\
&\leq\frac{10K}{\mu}(1-\mathrm{e}^{-m\mu T})E\sup_{-L-\tau\leq t\leq T}|x^n(t)-x^{n-1}(t)|^2+5K^2\left(\int_{-L}^0|p(s)s|\mathrm{d}s\right)^2E[\sup_{0\leq t\leq T}|x^n(t)-x^{n-1}(t)|^2] \qquad (21)\\
&\quad+5K^2E\left[\sup_{0\leq t\leq T}\left(1-\mathrm{e}^{-\int_0^t ma(u-\alpha(u))\mathrm{d}u}\right)\int_{-L}^0|p(s)s|\mathrm{d}s\sup_{0\leq t\leq T}|x^n(t)-x^{n-1}(t)|\right]^2\\
&\quad+5K^2E\left[\sup_{0\leq t\leq T}\int_0^t \mathrm{e}^{-2m\mu(t-s)}\mathrm{d}s\sup_{0\leq t\leq T}|x^n(t)-x^{n-1}(t)|^2\right]\\
&\leq 5K^2\left[\frac{2}{\mu K}(1-\mathrm{e}^{-m\mu T})+2\left(\int_{-L}^0|p(s)s|\mathrm{d}s\right)^2+\sup_{0\leq t\leq T}\frac{1}{2m\mu}(1-\mathrm{e}^{-2m\mu t})\right]\times E\sup_{0\leq t\leq T}|x^n(t)-x^{n-1}(t)|^2\\
&\leq 5K^2\left[\frac{2}{\mu K}(1-\mathrm{e}^{-m\mu T})+2m_1^2+\frac{1}{2m\mu}(1-\mathrm{e}^{-2m\mu T})\right]E\sup_{0\leq t\leq T}|x^n(t)-x^{n-1}(t)|^2\\
&\leq \delta E\sup_{0\leq t\leq T}|x^n(t)-x^{n-1}(t)|^2.
\end{aligned}
$$

$$\delta=5K^2\left[\tfrac{2}{\mu K}(1-\mathrm{e}^{-m\mu T})+2m_1^2+\tfrac{1}{2m\mu}(1-\mathrm{e}^{-2m\mu T})\right],\text{ so}$$

$$E\sup_{0\leq t\leq T}|x^{n+1}(t)-x^n(t)|^2\leq M\delta^n.$$

By Chebyshev inequality, we obtain

$$P\left\{\sup_{0\leq t\leq T}|x^{n+1}(t)-x^n(t)|^2\geq\delta^{n/4}\right\}\leq\frac{M\delta^n}{\delta^{n/2}}=M\delta^{\frac{n}{2}}.$$

From Assumption H_5, we have $\delta<1$. By Borel-Cantelli lemma, it follows that there exists a positive integer $n_0=n_0(w)$ for almost all $w\in\Omega$, satisfying

$$\sup_{0\leq t\leq T}|x^{n+1}(t)-x^n(t)|\leq\delta^{n/4},$$

for any $n\geq n_0$.

Next we show that $x^n(t)$ are uniformly convergent on $[-L-\tau,+\infty]$. Since $x^n(t)=x^0(t)+\sum_{i=1}^n[x^i(t)-x^{i-1}(t)]$ can be regarded as the partial sum of function series $x^0(t)+\sum_{i=1}^\infty[x^i(t)-x^{i-1}(t)]$, as well as $\sup_{0\leq t\leq T}|x^i(t)-x^{i-1}(t)|\leq\delta^{(i-1)/4}(i=1,2,\ldots)$, it follows that $x^n(t)$ are uniformly convergent on $[-L-\tau,+\infty]$ by using the convergence of constant series $\sum_{i=1}^\infty\delta^{(i-1)/4}$ and Weierstrass' discriminance.

Let $x(t)$ be the sum function, it obtains the function sequences $\{x^n(t)\}_{n\geq0}$ converge uniformly to $x(t)$ on $[-L-\tau,+\infty)$. Considering $x^n(t)$ are continuous and F_t compatible, we obtain the sum fucntion $x(t)$ that is also continuous and F_t compatible. Using inequality (15), $\{x^n(t)\}_{n\geq0}$ are the Cauchy sequences in L^2, so

$$E|x^n(t) - x(t)|^2 \to 0 \quad (n \to \infty).$$

On the other hand, by inequality (15), we have

$$E\left[\sup_{-L-\tau\le t\le T}|x(t)|^2\right] \le \frac{10E|\varphi(0)|^2 + (10K^2m_1^2 + 2)E[\sup_{-L-\tau\le s\le 0}|\varphi(s)|^2]}{10\mathrm{e}^{-mKT} - 10K^2L^4m_2^2 - 5K^2 - 9}. \tag{22}$$

So by Assumption H_5, it obtains $E[\sup_{-L-\tau\le t\le T}|x(t)|^2] < +\infty$.

(4) This part proves that $x(t)$ is a solution for equation (2)
After simple calculation, we have

$$E\left[\int_0^t (x^n(v) - x(v))ma(v-\alpha(v))\mathrm{e}^{-\int_v^t ma(u-\alpha(u))\mathrm{d}u}\mathrm{d}v\right]^2 \le E\left[\sup_{0\le t\le T}|x^n(t) - x(t)|^2\right]\left(1 - \mathrm{e}^{-\int_v^t ma(u-\alpha(u))\mathrm{d}u}\right)^2 \to 0 \quad (n\to\infty) \tag{23}$$

and

$$E\left[\int_0^t (x^n(v-\alpha(v)) - x(v-\alpha(v)))ma(v-\alpha(v))\mathrm{e}^{-\int_v^t ma(u-\alpha(u))\mathrm{d}u}\mathrm{d}v\right]^2 \to 0 \quad (n\to\infty). \tag{24}$$

Using Hölder inequality, it follows

$$\begin{aligned} &E\left|\int_{-L}^0 p(s)\int_{t+s}^t f(x^n(u-\alpha(u)))\mathrm{d}u\mathrm{d}s - \int_{-L}^0 p(s)\int_{t+s}^t f(x(u-\alpha(u)))\mathrm{d}u\mathrm{d}s\right|^2 \\ &\le K^2E\left[\int_{-L}^0 |p(s)|\left|\int_{t+s}^t |x^n(u-\alpha(u)) - x(u-\alpha(u))|\mathrm{d}u\right|\mathrm{d}s\right]^2 \\ &\le K^2\left|\int_{-L}^0 p^2(s)\mathrm{d}s\right| E\left[\int_{-L}^0\left|s\int_{t+s}^t |x^n(u-\alpha(u)) - x(u-\alpha(u))|^2\mathrm{d}u\right|\mathrm{d}s\right] \\ &\le \frac{K^2L^3}{3}\left|\int_{-L}^0 p^2(s)\mathrm{d}s\right| E\left[\sup_{0\le t\le T}|x^n(t) - x(t)|^2\right] \to 0. \end{aligned} \tag{25}$$

By the similar method of Step (3), it obtains

$$\begin{aligned} &E\left[\int_0^t \mathrm{e}^{\int_v^t ma(u-\alpha(u))\mathrm{d}u} ma(v-\alpha(v))\left(\int_{-L}^0 p(s)\int_{v+s}^v f(x^n(u-\alpha(u)))\mathrm{d}u\mathrm{d}s\right)\mathrm{d}v\right. \\ &\left.- \int_0^t \mathrm{e}^{\int_v^t ma(u-\alpha(u))\mathrm{d}u} ma(v-\alpha(v))\left(\int_{-L}^0 p(s)\int_{v+s}^v f(x(u-\alpha(u)))\mathrm{d}u\mathrm{d}s\right)\mathrm{d}v\right]^2 \\ &\le K^2\left(\int_{-L}^0 |p(s)s|\mathrm{d}s\right)^2 E\left[\sup_{0\le t\le T}|x^n(t) - x(t)|^2\right] \to 0 \quad (n\to\infty) \end{aligned} \tag{26}$$

and

$$E\left[\int_0^t e^{-\int_s^t ma(u-\alpha(u))du} g(x^n(s-\beta(s)))dW(s) - \int_0^t e^{-\int_s^t ma(u-\alpha(u))du} g(x(s-\beta(s)))dW(s)\right]^2$$
$$\leq \frac{K^2}{2m\mu} E\left[\sup_{0\leq t\leq T} |x^n(t) - x(t)|^2\right] \to 0 \quad (n \to \infty). \tag{27}$$

Taking the limit on both sides on equation (9), it obtains $x(t)$ satisfies equation (2), so $x(t)$ is a solution of the stochastic differential equation.

(5) This part proves the uniqueness.

Let $\varsigma_n = T \wedge \inf\{t \in [0, T]; |x(t)| \geq n\}$ be the stopping time. Supposing $x(t)$ and $y(t)$ are the solutions to the stochastic differential equation with the same initial value. Obviously, it obtains $\varsigma_n \uparrow T$ for $n \to \infty$. Let $r_n = r \wedge \varsigma_n$, By the similar method of Step (1), we have

$$E \sup_{0\leq r\leq t} |x(r_n) - y(r_n)|^2 \leq 8(1 - e^{-mKT}) E\left[\sup_{0\leq s\leq t_n} |x(s) - y(s)|^2\right]$$
$$+ 4E \sup_{0\leq r\leq t} \left|\int_{-L}^0 p(s) \int_{r_n+s}^{r_n} (f(x(u-\alpha(u))) - f(y(u-\alpha(u))))duds\right|^2 + 4E \sup_{0\leq r\leq t}$$
$$\times \left|\int_0^{r_n} e^{\int_v^{r_n} ma(u-\alpha(u))du} ma(v-\alpha(v))\left[\int_{-L}^0 p(s) \int_{v+s}^{v} (f(x(u-\alpha(u))) - f(y(u-\alpha(u))))duds\right] dv\right|^2$$
$$+ 4E \sup_{0\leq r\leq t} \left|\int_0^{r_n} e^{-\int_s^{r_n} ma(u-\alpha(u))du} (g(x(s-\beta(s))) - g(y(s-\beta(s))))dW(s)\right|^2 \tag{28}$$
$$\leq [8(1 - e^{-mKT}) + 8K^2L^4m_2^2 + 4K^2] E\left[\sup_{0\leq s\leq t_n} |x(s) - y(s)|^2\right].$$

For $n \to \infty$, it obtains $E \sup_{0\leq r\leq t} |x(r) - y(r)|^2 \leq [8(1 - e^{-m\mu T}) + 8K^2L^4m_2^2 + 4K^2] E \sup_{0\leq s\leq t} |x(s) - y(s)|^2$, so

$$E \sup_{0\leq t\leq T} |x(t) - y(t)|^2 \leq [8(1 - e^{-m\mu T}) + 8K^2L^4m_2^2 + 4K^2] E \sup_{0\leq t\leq T} |x(t) - y(t)|^2. \tag{29}$$

By Assumption H_5, it obtains

$$\left(\frac{7}{8} + K^2L^4m_2^2 + \frac{K^2}{2}\right)e^{mKT} < \left(\frac{9}{10} + K^2L^4m_2^2 + \frac{K^2}{2}\right)e^{mKT} < 1,$$

so we have $8(1 - e^{-mKT}) + 8K^2L^4m_2^2 + 4K^2 < 1$ by some simple calculations, and it follows $E \sup_{0\leq t\leq T} |x(t) - y(t)|^2 = 0$. Hence, $x(t) = y(t)$ for $0 \leq t \leq T$, the uniqueness of the solution to the stochastic differential equation is proved. □

Remark 3.1. In the past, the contraction mapping principle has been the main method to prove the existence and uniqueness of solutions for the stochastic Volterra-Levin equation. However, Theorem 3.1 is proved by the Picard iteration method, which is more intuitive and easier to understand than the other methods. The preconditions in Theorem 3.1 are simpler than those in [2] and [3]. More importantly, the Picard iteration method is applied to the case of variable delay, which can increase the application scope of related problems, especially for the stochastic Volterra-Levin equation.

4 Examples

Next, we give two examples to illustrate the application of Theorem 1.

Example 4.1. Considering equation (2) with $f(x) = g(x) = \frac{x}{10}$, $\alpha(t) = \beta(t) = \frac{1}{10\pi}\text{arccot}(t)$, $p(s) = s^2$ and $L = 1$, it becomes

$$\mathrm{d}x(t) = -\left(\int_{t-1}^{t} (s-t)^2 \frac{x(s-\alpha(s))}{10}\mathrm{d}s\right)\mathrm{d}t + \frac{x(t-\beta(t))}{10}\mathrm{d}W(t). \tag{30}$$

Then there exists a unique solution $x(t)$ for any $0 \le t \le T$, where $T = -60\ln\frac{400}{401}$.

Proof. By calculation, we have

(1) $\lim\limits_{x\to 0}\frac{f(x)}{x} = \frac{1}{10} = \beta$.

(2) $f(0) = g(0) = 0$, there exists $\mu = \frac{1}{20}$, such that $\frac{f(x)}{x} = \frac{1}{10} > \mu$.

(3) $m = \int_{-1}^{0} s^2\mathrm{d}s = \frac{1}{3}$, $m_1 = \int_{-1}^{0}|s^3|\mathrm{d}s = \frac{1}{4}$ and $m_2 = \max_{-1\le s\le 0} s^2 = 1$.

(4) There is a positive constant $K = \frac{1}{5}$, such that $|f(x) - f(y)| \vee |g(x) - g(y)| = \frac{1}{10}|x-y| \le K|x-y|$ for all $x, y \in R$.

(5) If $T = -60\ln\frac{400}{401}$, then $5K^2\left(\frac{2}{\mu K}(1-\mathrm{e}^{-m\mu T}) + 2m_1^2 + \frac{1}{2m\mu}(1-\mathrm{e}^{-2m\mu T})\right) \approx 0.77 < 1$ and $\left(K^2L^4m_2^2 + \frac{1}{2}K^2 + \frac{9}{10}\right)\mathrm{e}^{mKT} \approx 0.989 < 1$. □

So equation (30) satisfies Assumptions H_1–H_5, it follows from Theorem 1 that the equation has a unique solution in $[0, T]$.

Example 4.2. Considering equation (2) with $f(x) = \frac{x}{10}$, $g(x) = \frac{x}{6}$, $p(s) = 1$, $L = 1$, $\alpha(t) = \frac{1}{7\pi}\text{arccot}(t)$, and $\beta(t) = \frac{1}{8\pi}\text{arccot}(t)$, it becomes

$$\mathrm{d}x(t) = -\left(\int_{t-1}^{t} \frac{x(s-\alpha(s))}{10}\mathrm{d}s\right)\mathrm{d}t + \frac{x(t-\beta(t))}{6}\mathrm{d}W(t). \tag{31}$$

Then there exists a unique solution $x(t)$ for any $0 \le t \le T$, where $T = -20\ln\frac{800}{801}$.

Proof. By calculation, we have

(1) $\lim_{x\to 0}\frac{f(x)}{x} = \frac{1}{10} = \beta$.

(2) $f(0) = g(0) = 0$, there exists $\mu = \frac{1}{20}$, such that $\frac{f(x)}{x} = \frac{1}{10} > \mu$.

(3) $m = \int_{-1}^{0} 1\mathrm{d}s = 1$, $m_1 = \int_{-1}^{0}|s|\mathrm{d}s = \frac{1}{2}$, and $m_2 = \max_{-1\le s\le 0} 1 = 1$.

(4) There is a positive constant $K = \frac{1}{5}$, such that $|f(x) - f(y)| \vee |g(x) - g(y)| \le \frac{1}{5}|x-y|$ for all $x, y \in R$.

(5) If $T = -20\ln\frac{800}{801}$, then $5K^2\left(\frac{2}{\mu K}(1-\mathrm{e}^{-m\mu T}) + 2m_1^2 + \frac{1}{2m\mu}(1-\mathrm{e}^{-2m\mu T})\right) \approx 0.77 < 1$ and $\left(K^2L^4m_2^2 + \frac{1}{2}K^2 + \frac{9}{10}\right)\mathrm{e}^{mKT} \approx 0.965 < 1$.

So equation (31) satisfies Assumptions H_1–H_5, it follows from Theorem 1 that the equation has a unique solutions in $[0, T]$. □

5 Conclusion

This work studies the existence and uniqueness of solutions for the stochastic Volterra-Levin equation with variable delays. The Picard iteration approach is utilized as the major technique to obtain the results. The simpler sufficient conditions for the existence and uniqueness of solutions are constructed as the study's key conclusions. Finally, two examples are given to illustrate the validity of the theorem.

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