Research Article

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A posteriori error estimates of characteristic mixed finite elements for convection-diffusion control problems

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Abstract: In this article, we consider fully discrete characteristic mixed finite elements for convection-diffusion optimal control problems. We use the characteristic line method to treat the hyperbolic part of the state equation as a directional derivative. The state and the co-state are discretized by the lowest order Raviart-Thomas mixed finite element spaces and the control is approximated by piecewise constant functions. Using some proper duality problems, we derive *a posteriori* error estimates for the scalar functions. Such estimates are not available in the literature. A numerical example is presented to validate the theoretical results.

Keywords: *A posteriori* error estimates, characteristic mixed finite element methods, convection-diffusion equations, optimal control problems

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1 Introduction

In the recent 30 years, optimal control problems (OCPs) have been extensively utilized in many aspects of modern life such as social, economic, scientific, and engineering numerical simulation [1]. Thus, they must be solved successfully with efficient numerical methods. Among these numerical methods, the finite element method (FEM) is a good choice for solving OCPs governed by partial differential equations (PDEs). There have been extensive studies on convergence or superconvergence of FEMs for OCPs, see [2–9]. A systematic introduction of FEM for PDE OCPs can be found in [10–13].

Adaptive FEM has been investigated extensively. It has become one of the most popular methods in cientific computation and numerical modeling. It ensures a higher density of nodes in a certain area of the given domain, where the solution is more difficult to approximate, indicated by *a posteriori* error estimators. Hence, it is an important approach to boost the accuracy and efficiency of finite element discretization. There is a lot of work concentrating on the adaptivity of various OCPs. For example, [14–20].

In flow control problem or temperature control problem, their objective functional contains not only the state variable but also its gradient. At this moment, the accuracy of the gradient is important in the numerical discretization of the coupled state equations. The mixed finite element method (MFEM) is appropriate for the state equations in such cases since both the scalar variable and its flux variable can be approximated to the same accuracy, for example, [21–25].

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Recently, more and more attention has been paid to elliptic convection-diffusion OCPs [26–28]. Fu and Rui have studied *a priori* and *a posteriori* error estimates of characteristic finite element methods (CFEMs) for time-dependent convection-diffusion OCPs in [29] and [30], respectively. They also considered *a priori* error estimates of characteristic mixed finite element method (CMFEM) for time-dependent convection-diffusion OCPs in [31]. In [32], Sun and Ma investigated a nonoverlapping domain decomposition method combined with the characteristic method for OCPs governed by parabolic convection-diffusion equations.

In this work, we consider *a posteriori* error estimates of fully discrete CMFEM for OCPs governed by convection-diffusion equations. The characteristic approximation is applied to handle the convection term, and the lowest order RT mixed finite element spatial approximation is adopted to deal with the diffusion term. The OCPs that we are interested in are as follows:

$$\min_{u\in K}\left\{\frac{1}{2}\int_0^T(\|\boldsymbol{p}-\boldsymbol{p}_d\|^2+\|y-y_d\|^2+\|u\|^2)\mathrm{d}t\right\},\tag{1}$$

$$y_t+\boldsymbol{b}\cdot\nabla y+\operatorname{div}\boldsymbol{p}+cy=f+u,\quad x\in\Omega,\ t\in J,\tag{2}$$

$$\boldsymbol{p}=-A\nabla y,\quad x\in\Omega,\ t\in J,\tag{3}$$

$$y=0,\quad x\in\partial\Omega,\ t\in J,\tag{4}$$

$$y(x,0)=y_0(x),\quad x\in\Omega,\tag{5}$$

where $\Omega\subset\mathbf{R}^2$ is a bounded convex polygon with Lipschitz continuous boundary $\partial\Omega$, $J=[0,T]$. Let $\boldsymbol{p}_d\in(L^2(J;\,H^1(\Omega)))^2$, $y_d\in L^2(J;\,H^1(\Omega))$, $f\in L^2\,(J;\,L^2(\Omega))$, $y_0(x)\in H_0^1(\Omega)$, $c\in W^{1,\infty}(\Omega)$, $A=(a_{ij}(x))_{2\times2}\in C^\infty(\bar{\Omega};\,\mathbf{R}^{2\times2})$ such that $\mathbf{X}^TA\mathbf{X}\ge c_1\|\mathbf{X}\|^2_{\mathbf{R}^2}$, $\forall\mathbf{X}\in\mathbf{R}^2$, where $c_1>0$. Usually, $\boldsymbol{b}=(b_1(x,t),b_2(x,t))^T$ denotes a velocity field in the flow control, we assume that $\boldsymbol{b}\in(C(J;\,C_0^1(\bar{\Omega})))^2$ and the flow is incompressible, i.e.,

$$\nabla\cdot\boldsymbol{b}=0,\quad\forall x\in\Omega,\quad t\in J.$$

Moreover, we assume that the constraint on the control is an obstacle such that

$$K=\{u\in L^2(J;\,L^2(\Omega)):u(x,t)\ge0,\quad\text{a.e. in }\Omega\times J\}.$$

In this article, we adopt the standard notation $W^{m,p}(\Omega)$ for Sobolev spaces on Ω with a norm $\|\cdot\|_{m,p}$ given by $\|v\|^p_{m,p}=\sum_{|\alpha|\le m}\|D^\alpha v\|^p_{L^p(\Omega)}$, a semi-norm $|\cdot|_{m,p}$ given by $|v|^p_{m,p}=\sum_{|\alpha|=m}\|D^\alpha v\|^p_{L^p(\Omega)}$. We set $W_0^{m,p}(\Omega)=\{v\in W^{m,p}(\Omega):v|_{\partial\Omega}=0\}$. For $p=2$, we denote $H^m(\Omega)=W^{m,2}(\Omega)$, $H_0^m(\Omega)=W_0^{m,2}(\Omega)$, and $\|\cdot\|_m=\|\cdot\|_{m,2}$, $\|\cdot\|=\|\cdot\|_{0,2}$.

We denote by $L^s(0,T;\,W^{m,p}(\Omega))$ the Banach space of all L^s integrable functions from J into $W^{m,p}(\Omega)$ with norm $\|v\|_{L^s(J;\,W^{m,p}(\Omega))}=\left(\int_0^T\|v\|^s_{W^{m,p}(\Omega)}\mathrm{d}t\right)^{\frac{1}{s}}$ for $s\in[1,\infty)$, and the standard modification for $s=\infty$. Similarly, one can define the spaces $H^1\,(J;\,W^{m,p}(\Omega))$ and $C^k\,(J;\,W^{m,p}(\Omega))$. In addition, C denotes a general positive constant independent of h and k, where h is the spatial mesh size for the control and state discretization and k is the time increment.

The plan of this article is as follows. In Section 2, we shall construct a CMFEM approximation for the OCPs (1)–(5). In Section 3, we derive *a posteriori* $L^2(0,T;\,L^2(\Omega))$ error estimates by using two duality problems. A numerical example is provided in Section 4.

2 CMFEM approximation of convection-diffusion OCPs

In this section, we shall construct a CMFEM and backward Euler discretization approximation of convection-diffusion OCPs (1)–(5). To fix the idea, we shall take the state spaces $\boldsymbol{L}=L^2\,(J;\,\boldsymbol{V})$ and $Q=H^1\,(J;\,W)$, where $\boldsymbol{V}$ and W are defined as follows:

$$\boldsymbol{V}=H(\operatorname{div};\,\Omega)=\{\boldsymbol{v}\in(L^2(\Omega))^2,\ \operatorname{div}\boldsymbol{v}\in L^2(\Omega)\},\ W=L^2(\Omega).$$

The Hilbert space $\boldsymbol{V}$ is equipped with the following norm:

$$\|\boldsymbol{v}\|_{H(\mathrm{div};\,\Omega)} = (\|\boldsymbol{v}\|_{0,\Omega}^2 + \|\mathrm{div}\boldsymbol{v}\|_{0,\Omega}^2)^{1/2}.$$

Now, we take account of the hyperbolic part of (2), namely $y_t + \boldsymbol{b}\cdot\nabla y$, as a directional derivative. Let s denote the unit vector in the direction of $(b_1, b_2, 1)$ in $\Omega\times J$, and set

$$\lambda = (|\boldsymbol{b}|^2 + 1)^{1/2} = (b_1^2 + b_2^2 + 1)^{1/2}.$$

Then equation (2) is equivalent to the following form:

$$\lambda y_s + \mathrm{div}\boldsymbol{p} + cy = f + u.$$

We recast (1)–(5) as the following weak form: find $(\boldsymbol{p}, y, u) \in \boldsymbol{L}\times Q\times K$ such that

$$\min_{u\in K}\left\{\frac{1}{2}\int_0^T (\|\boldsymbol{p}-\boldsymbol{p}_d\|^2 + \|y-y_d\|^2 + \|u\|^2)\mathrm{d}t\right\} \tag{6}$$

$$(\lambda y_s, w) + (\mathrm{div}\boldsymbol{p}, w) + (cy, w) = (f+u, w), \quad \forall w\in W,\ t\in J, \tag{7}$$

$$(A^{-1}\boldsymbol{p}, \boldsymbol{v}) - (y, \mathrm{div}\boldsymbol{v}) = 0, \quad \forall \boldsymbol{v}\in \boldsymbol{V},\ t\in J, \tag{8}$$

$$y(x,0) = y_0(x), \quad \forall x\in\Omega. \tag{9}$$

It follows from [1] that the OCPs (6)–(9) have a unique solution $(\boldsymbol{p}, y, u)\in \boldsymbol{L}\times Q\times K$ and that a triplet $(\boldsymbol{p}, y, u)$ is the solution of (6)–(9) if and only if there is a co-state $(\boldsymbol{q}, z)\in\boldsymbol{L}\times Q$ such that $(\boldsymbol{p}, y, \boldsymbol{q}, z, u)$ satisfies the following optimality conditions:

$$(\lambda y_s, w) + (\mathrm{div}\boldsymbol{p}, w) + (cy, w) = (f+u, w), \quad \forall w\in W, \quad t\in J, \tag{10}$$

$$(A^{-1}\boldsymbol{p}, \boldsymbol{v}) - (y, \mathrm{div}\boldsymbol{v}) = 0, \quad \forall \boldsymbol{v}\in \boldsymbol{V},\ t\in J, \tag{11}$$

$$y(x,0) = y_0(x), \quad \forall x\in\Omega, \tag{12}$$

$$-(\lambda z_s, w) + (\mathrm{div}\boldsymbol{q}, w) + (cz, w) = (y-y_d, w), \quad \forall w\in W,\ t\in J, \tag{13}$$

$$(A^{-1}\boldsymbol{q}, \boldsymbol{v}) - (z, \mathrm{div}\boldsymbol{v}) = -(\boldsymbol{p}-\boldsymbol{p}_d, \boldsymbol{v}), \quad \forall \boldsymbol{v}\in\boldsymbol{V},\ t\in J, \tag{14}$$

$$z(x,T) = 0, \quad \forall x\in\Omega, \tag{15}$$

$$(u+z, \tilde{u}-u) \geq 0, \quad \forall \tilde{u}\in K, \tag{16}$$

where $(\cdot,\cdot)$ is the inner product of $L^2(\Omega)$.

We now consider the time discretization. Let $N\in\mathbb{Z}^+$, $k = T/N$, and $t_i = ik$, $i = 0, 1, \dots, N$. Approximation $y_s^i(x) = y_s(x, t_i)$ by a backward-Euler difference quotient in the s-direction, that is,

$$y_s^i(x) \simeq \frac{y^i(x) - y^{i-1}(x - \boldsymbol{b}(x,t_i)k)}{k\sqrt{1+|\boldsymbol{b}(x,t_i)|^2}}.$$

Let $G(x^*, t^*;\ t)$ be the approximate characteristic curve passing through point x^* at time t^*, which is defined by

$$G(x^*, t^*;\ t) := x^* - \boldsymbol{b}(x^*, t^*)(t^* - t).$$

We denote by $\bar{x} = G(x, t_i;\ t_{i-1})$ the foot at time t_{i-1} of the characteristic curve with head x at time t_i, and $\bar{f}(x) = f(\bar{x})$, then

$$\lambda^i y_s^i \simeq \frac{y^i - \bar{y}^{i-1}}{k}.$$

Next, we use the MFEM for spatial discretization. Let $\mathcal{T}_h$ be regular triangulations of Ω. h_τ is the diameter of element τ and $h = \max_{\tau\in\mathcal{T}_h}\{h_\tau\}$. Let $\boldsymbol{V}_h\times W_h\subset \boldsymbol{V}\times W$ denote the lowest order Raviart-Thomas space [33] associated with the triangulations $\mathcal{T}_h$ of Ω. P_k denotes the space of polynomials of total degree at most k ($k \geq 0$). Let $\boldsymbol{V}(\tau) = \{\boldsymbol{v} \in P_0^2(\tau) + x \cdot P_0(\tau)\}$, $W(\tau) = P_0(\tau)$. We define

$$\begin{aligned}
\boldsymbol{V}_h &:= \{\boldsymbol{v}_h \in \boldsymbol{V} : \boldsymbol{v}_h|_\tau \in \boldsymbol{V}(\tau), \forall \tau \in \mathcal{T}_h\},\\
W_h &:= \{w_h \in W : w_h|_\tau \in W(\tau), \forall \tau \in \mathcal{T}_h\},\\
K_h &:= \{w_h \in W_h : w_h|_\tau = \max\{0, w_h\}, \forall \tau \in \mathcal{T}_h\}.
\end{aligned}$$

Then a fully discrete CMFEM approximation of (6)–(9) is to find $(\boldsymbol{p}_h^i, y_h^i, u_h^i) \in \boldsymbol{V}_h \times W_h \times K_h$, $i = 1, 2, \ldots, N$, such that

$$\min_{u_h^i \in K_h} \left\{ \frac{1}{2} \sum_{i=1}^{N} k(\|\boldsymbol{p}_h^i - \boldsymbol{p}_d^i\|^2 + \|y_h^i - y_d^i\|^2 + \|u_h^i\|^2) \right\} \tag{17}$$

$$\left(\frac{y_h^i - \bar{y}_h^{i-1}}{k}, w_h \right) + (\operatorname{div} \boldsymbol{p}_h^i, w_h) + (c y_h^i, w_h) = (f^i + u_h^i, w_h), \quad \forall w_h \in W_h, \tag{18}$$

$$(A^{-1} \boldsymbol{p}_h^i, \boldsymbol{v}_h) - (y_h^i, \operatorname{div} \boldsymbol{v}_h) = 0, \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, \tag{19}$$

$$y_h^0(x) = y_0^h(x), \quad \forall x \in \Omega, \tag{20}$$

where $y_0^h(x) = R_h y_0(x)$ and R_h is the $L^2(\Omega)$-projection operator, which will be specific later.

It follows from [31] that the control problem (17)–(20) has a unique solution $(\boldsymbol{p}_h^i, y_h^i, u_h^i)$, $i = 1, 2, \ldots, N$, and that a triplet $(\boldsymbol{p}_h^i, y_h^i, u_h^i) \in \boldsymbol{V}_h \times W_h \times K_h$ is the solution of (17)–(20) if and only if there is a co-state $(\boldsymbol{q}_h^{i-1}, z_h^{i-1}) \in \boldsymbol{V}_h \times W_h$, $i = N, N-1, \ldots, 1$ such that $(\boldsymbol{p}_h^i, y_h^i, \boldsymbol{q}_h^{i-1}, z_h^{i-1}, u_h^i) \in (\boldsymbol{V}_h \times W_h)^2 \times K_h$ satisfies the following optimality conditions:

$$\left(\frac{y_h^i - \bar{y}_h^{i-1}}{k}, w_h \right) + (\operatorname{div} \boldsymbol{p}_h^i, w_h) + (c y_h^i, w_h) = (f^i + u_h^i, w_h), \quad \forall w_h \in W_h, \tag{21}$$

$$(A^{-1} \boldsymbol{p}_h^i, \boldsymbol{v}_h) - (y_h^i, \operatorname{div} \boldsymbol{v}_h) = 0, \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, \tag{22}$$

$$y_h^0(x) = y_0^h(x), \quad \forall x \in \Omega, \tag{23}$$

$$\left(\frac{z_h^{i-1} - \bar{\bar{z}}_h^i \cdot J^i}{k}, w_h \right) + (\operatorname{div} \boldsymbol{q}_h^{i-1}, w_h) + (c z_h^{i-1}, w_h) = (y_h^i - y_d^i, w_h), \quad \forall w_h \in W_h, \tag{24}$$

$$(A^{-1} \boldsymbol{q}_h^{i-1}, \boldsymbol{v}_h) - (z_h^{i-1}, \operatorname{div} \boldsymbol{v}_h) = -(\boldsymbol{p}_h^i - \boldsymbol{p}_d^i, \boldsymbol{v}_h), \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, \tag{25}$$

$$z_h^N(x) = 0, \quad \forall x \in \Omega, \tag{26}$$

$$(u_h^i + z_h^{i-1}, \tilde{u}_h - u_h^i) \geq 0, \quad \forall \tilde{u}_h \in K_h, \tag{27}$$

where $\bar{\bar{z}}_h^i = z_h^i(\bar{\bar{x}})$, and $\bar{\bar{x}}$ represents the head of the characteristic curve with foot x at time t_{i-1}, namely,

$$x = G(\bar{\bar{x}}, t_i; t_{i-1}).$$

We denote by $J^i := |\det \mathcal{D}G(x, t_i; t_{i-1})^{-1}|$ the determinant of the Jacobian transformation from G to x. Similar to the discussion in [29], we know that

$$\det \mathcal{D}G(\bar{\bar{x}}, t_i; t_{i-1}) = 1 - (\nabla \cdot \boldsymbol{b}^i)k + O(k^2),$$

since the flow is incompressible, i.e., $\nabla \cdot \boldsymbol{b} = 0$, $\forall x \in \Omega$, $t \in (0, T)$. Thus, $J^i = 1$ for sufficiently small k.

For $i = 0$ and $i = N$, we let

$$(A^{-1} \boldsymbol{p}_h^0, \boldsymbol{v}_h) - (y_h^0, \operatorname{div} \boldsymbol{v}_h) = 0, \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, \tag{28}$$

$$(A^{-1} \boldsymbol{q}_h^N, \boldsymbol{v}_h) - (z_h^N, \operatorname{div} \boldsymbol{v}_h) = -(\boldsymbol{p}_h^N - \boldsymbol{p}_d^N, \boldsymbol{v}_h), \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h. \tag{29}$$

For $i = 1, 2, \ldots, N$, let

$$\begin{aligned} Y_h|_{(t_{i-1},t_i]} &= ((t_i - t)y_h^{i-1} + (t - t_{i-1})y_h^i)/k, \\ Z_h|_{(t_{i-1},t_i]} &= ((t_i - t)z_h^{i-1} + (t - t_{i-1})z_h^i)/k, \\ \boldsymbol{P}_h|_{(t_{i-1},t_i]} &= ((t_i - t)\boldsymbol{p}_h^{i-1} + (t - t_{i-1})\boldsymbol{p}_h^i)/k, \\ \boldsymbol{Q}_h|_{(t_{i-1},t_i]} &= ((t_i - t)\boldsymbol{q}_h^{i-1} + (t - t_{i-1})\boldsymbol{q}_h^i)/k, \\ U_h|_{(t_{i-1},t_i]} &= u_h^i. \end{aligned}$$

For any function $w \in C(J; L^2(\Omega))$ and $i = 1, 2, \ldots, N$, let

$$\hat{w}(x,t)|_{t\in(t_{i-1},t_i]} = w(x,t_i), \quad \tilde{w}(x,t)|_{t\in(t_{i-1},t_i]} = w(x,t_{i-1}).$$

Moreover, we let

$$\begin{aligned} \bar{\boldsymbol{p}}_d|_{(t_{i-1},t_i]} &= ((t_i - t)\boldsymbol{p}_d^i + (t - t_{i-1})\boldsymbol{p}_d^{i+1})/k, \ i = 1, 2, \ldots, N-1, \ \bar{\boldsymbol{p}}_d|_{(t_{N-1},t_N]} = \boldsymbol{p}_d^N, \\ \bar{\boldsymbol{P}}_h|_{(t_{i-1},t_i]} &= ((t_i - t)\boldsymbol{p}_h^i + (t - t_{i-1})\boldsymbol{p}_h^{i+1})/k, \ i = 1, 2, \ldots, N-1, \ \bar{\boldsymbol{P}}_h|_{(t_{N-1},t_N]} = \boldsymbol{p}_h^N. \end{aligned}$$

Then the optimality conditions (21)–(27) satisfying

$$(Y_{h,t}, w_h) + (\mathrm{div}\hat{\boldsymbol{P}}_h, w_h) + (c\hat{Y}_h, w_h) = \left(\hat{f} + U_h - \frac{\tilde{Y}_h - \tilde{Y}_h(\bar{x})}{k}, w_h\right), \quad \forall w_h \in W_h, \tag{30}$$

$$(A^{-1}\hat{\boldsymbol{P}}_h, \boldsymbol{v}_h) - (\hat{Y}_h, \mathrm{div}\boldsymbol{v}_h) = 0, \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, \tag{31}$$

$$Y_h(x,0) = y_0^h(x), \quad \forall x \in \Omega, \tag{32}$$

$$-(Z_{h,t}, w_h) + (\mathrm{div}\tilde{\boldsymbol{Q}}_h, w_h) + (c\tilde{Z}_h, w_h) = \left(\hat{Y}_h - \hat{y}_d + \frac{\hat{Z}_h(\bar{\bar{x}})\cdot \hat{J} - \hat{Z}_h}{k}, w_h\right), \quad \forall w_h \in W_h, \tag{33}$$

$$(A^{-1}\tilde{\boldsymbol{Q}}_h, \boldsymbol{v}_h) - (\tilde{Z}_h, \mathrm{div}\boldsymbol{v}_h) = -(\hat{\boldsymbol{P}}_h - \hat{\boldsymbol{p}}_d, \boldsymbol{v}_h), \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, \tag{34}$$

$$Z_h(x,T) = 0, \quad \forall x \in \Omega, \tag{35}$$

$$(U_h + \tilde{Z}_h, \tilde{u}_h - U_h) \geq 0, \quad \forall \tilde{u}_h \in K_h. \tag{36}$$

In the rest of the article, we shall use some intermediate variables. For any control function $U_h \in K_h$, we first define the state solution $(\boldsymbol{p}(U_h), y(U_h), \boldsymbol{q}(U_h), z(U_h))$ satisfies

$$(y_t(U_h) + \boldsymbol{b}\cdot\nabla y(U_h), w) + (\mathrm{div}\boldsymbol{p}(U_h), w) + (cy(U_h), w) = (f + U_h, w), \quad \forall w \in W, \tag{37}$$

$$(A^{-1}\boldsymbol{p}(U_h), \boldsymbol{v}) - (y(U_h), \mathrm{div}\boldsymbol{v}) = 0, \quad \forall \boldsymbol{v} \in \boldsymbol{V}, \tag{38}$$

$$y(U_h)(x,0) = y_0(x), \quad \forall x \in \Omega, \tag{39}$$

$$-(z_t(U_h) + \boldsymbol{b}\cdot\nabla z(U_h), w) + (\mathrm{div}\boldsymbol{q}(U_h), w) + (cz(U_h), w) = (y(U_h) - y_d, w), \quad \forall w \in W, \tag{40}$$

$$(A^{-1}\boldsymbol{q}(U_h), \boldsymbol{v}) - (z(U_h), \mathrm{div}\boldsymbol{v}) = -(\boldsymbol{p}(U_h) - \boldsymbol{p}_d, \boldsymbol{v}), \quad \forall \boldsymbol{v} \in \boldsymbol{V}, \tag{41}$$

$$z(U_h)(x,T) = 0, \quad \forall x \in \Omega. \tag{42}$$

Let $R_h : W \to W_h$ be the $L^2(\Omega)$-projection into W_h [34], which satisfies for any $w \in W$

$$(R_h w - w, \chi) = 0, \quad \forall w \in W, \quad \forall \chi \in W_h, \tag{43}$$

$$\|R_h w - w\|_{0,q} \leq Ch\|w\|_{1,q}, \quad \forall w \in W \cap W^{1,q}(\Omega). \tag{44}$$

Let $\Pi_h : \boldsymbol{V} \to \boldsymbol{V}_h$ be the Raviart-Thomas projection operator [35], which satisfies: for any $\boldsymbol{v} \in \boldsymbol{V}$

$$\int_E w_h(\boldsymbol{v} - \Pi_h\boldsymbol{v})\cdot \boldsymbol{\nu}_E \mathrm{d}s = 0, \quad \forall w_h \in W_h, \ E \in \mathcal{E}_h, \tag{45}$$

$$\int_\tau (\boldsymbol{v} - \Pi_h \boldsymbol{v})\cdot \boldsymbol{v}_h \mathrm{d}x\mathrm{d}y = 0, \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h,\ \tau \in \mathcal{T}_h, \tag{46}$$

where $\mathcal{E}_h$ denotes the set of element sides in $\mathcal{T}_h$.

We have the commuting diagram property

$$\mathrm{div} \circ \Pi_h = R_h \circ \mathrm{div}: \boldsymbol{V} \to W_h \quad \text{and} \quad \mathrm{div}(I - \Pi_h)\boldsymbol{V} \perp W_h, \tag{47}$$

where and after, I denotes the identity operator.

Furthermore, the interpolation operator Π_h satisfies a local error estimate:

$$\|\boldsymbol{v} - \Pi_h \boldsymbol{v}\|_{0,\Omega} \le Ch|\boldsymbol{v}|_{1,\mathcal{T}_h}, \quad \forall \boldsymbol{v} \in \boldsymbol{V} \cap H^1(\mathcal{T}_h). \tag{48}$$

3 *A posteriori* error estimates

In this section, we derive *a posteriori* error estimates of fully discrete CMFEM for parabolic convection-diffusion OCPs. In order to the following analysis, we divide the domain Ω into three parts:

$$\begin{aligned}
\Omega_- &= \{x \in \Omega : \tilde{Z}_h(x) \le 0\},\\
\Omega_0 &= \{x \in \Omega : \tilde{Z}_h(x) > 0, U_h(x) = 0\},\\
\Omega_+ &= \{x \in \Omega : \tilde{Z}_h(x) > 0, U_h(x) > 0\}.
\end{aligned}$$

It is easy to see that the partition of the above three subsets is dependent on t. For all t, the three subsets are not intersected each other, and

$$\bar{\Omega} = \bar{\Omega}_- \cup \bar{\Omega}_0 \cup \bar{\Omega}_+.$$

First, let us derive the *a posteriori* error estimates for the control u.

Theorem 3.1. *Let* $(y, \boldsymbol{p}, z, \boldsymbol{q}, u)$ *and* $(Y_h, \boldsymbol{P}_h, Z_h, \boldsymbol{Q}_h, U_h)$ *be the solutions of* (10)–(16) *and* (30)–(36), *respectively. Then we have*

$$\|u - U_h\|^2_{L^2(J;\, L^2(\Omega))} \le C\eta_1^2 + \|\tilde{Z}_h - z(U_h)\|^2_{L^2(J;\, L^2(\Omega))}, \tag{49}$$

where

$$\eta_1^2 = \|U_h + \tilde{Z}_h\|^2_{L^2(J;\, L^2(\Omega_- \cup \Omega_+))}.$$

Proof. It follows from (19) that

$$\begin{aligned}
\|u - U_h\|^2_{L^2(J;\, L^2(\Omega))} &= \int_0^T (u - U_h, u - U_h)\mathrm{d}t\\
&= \int_0^T (u + z, u - U_h)\mathrm{d}t + \int_0^T (U_h + \tilde{Z}_h, U_h - u)\mathrm{d}t + \int_0^T (\tilde{Z}_h - z(U_h), u - U_h)\mathrm{d}t\\
&\quad + \int_0^T (z(U_h) - z, u - U_h)\mathrm{d}t\\
&\le \int_0^T (U_h + \tilde{Z}_h, U_h - u)\mathrm{d}t + \int_0^T (\tilde{Z}_h - z(U_h), u - U_h)\mathrm{d}t + \int_0^T (z(U_h) - z, u - U_h)\mathrm{d}t\\
&:= I_1 + I_2 + I_3.
\end{aligned} \tag{50}$$

We first estimate I_1. Note that

$$I_1 = \int_0^T (U_h + \tilde{Z}_h, U_h - u)\mathrm{d}t = \int_0^T \int_{\Omega_- \cup \Omega_+} (U_h + \tilde{Z}_h)(U_h - u)\mathrm{d}x\mathrm{d}t + \int_0^T \int_{\Omega_0} (U_h + \tilde{Z}_h)(U_h - u)\mathrm{d}x\mathrm{d}t. \tag{51}$$

It follows from Hölder's inequality and Young's inequality that

$$\begin{aligned}\int_0^T \int_{\Omega_- \cup \Omega_+} (U_h + \tilde{Z}_h)(U_h - u)\mathrm{d}x\mathrm{d}t &\le C(\delta)\|U_h + \tilde{Z}_h\|^2_{L^2(J;\, L^2(\Omega_- \cup \Omega_+))} + \delta\|u - U_h\|^2_{L^2(J;\, L^2(\Omega_- \cup \Omega_+))} \\ &= C(\delta)\eta_1^2 + \delta\|u - U_h\|^2_{L^2(J;\, L^2(\Omega))},\end{aligned} \tag{52}$$

Furthermore, we have that

$$U_h + \tilde{Z}_h \ge \tilde{Z}_h > 0, \quad U_h - u = 0 - u \le 0 \quad \text{on } \Omega_0.$$

It yields that

$$\int_0^T \int_{\Omega_0} (U_h + \tilde{Z}_h)(U_h - u)\mathrm{d}x\mathrm{d}t \le 0. \tag{53}$$

Then (51)–(53) imply that

$$I_1 \le C(\delta)\eta_1^2 + \delta\|u - U_h\|^2_{L^2(J;\, L^2(\Omega))}. \tag{54}$$

Moreover, it is clear that

$$I_2 = \int_0^T (\tilde{Z}_h - z(U_h), u - U_h)\mathrm{d}t \le C(\delta)\|\tilde{Z}_h - z(U_h)\|^2_{L^2(J;\, L^2(\Omega))} + \delta\|u - U_h\|^2_{L^2(J;\, L^2(\Omega))}. \tag{55}$$

Now we turn to I_3. Note that

$$y(x, 0) = y(U_h)(x, 0) = y_0(x) \quad \text{and} \quad z(x, T) = z(U_h)(x, T) = 0.$$

Then from (10)–(15) and (37)–(42), we have

$$I_3 = \int_0^T (z(U_h) - z, u - U_h)\mathrm{d}t = \int_0^T ((y(U_h) - y, y - y(U_h)) + (\boldsymbol{p}(U_h) - \boldsymbol{p}, \boldsymbol{p} - \boldsymbol{p}(U_h)))\mathrm{d}t \le 0. \tag{56}$$

It follows from (50) and (54)–(56), we obtain (49). □

In order to estimate the error $\|\tilde{Z}_h - z(U_h)\|^2_{L^2(J;\, L^2(\Omega))}$, we need the following well-known stability results (see [36] for the details) for the following dual equations:

$$\begin{cases} \varphi_t + \boldsymbol{b} \cdot \nabla\varphi - \operatorname{div}(A\nabla\varphi) + c\varphi = F, & x \in \Omega,\ t \in J, \\ \varphi|_{\partial\Omega} = 0, & t \in J, \\ \varphi(x, 0) = 0, & x \in \Omega, \end{cases} \tag{57}$$

and

$$\begin{cases} -\psi_t - \boldsymbol{b} \cdot \nabla\psi - \operatorname{div}(A\nabla\psi) + c\psi = F, & x \in \Omega,\ t \in J, \\ \psi|_{\partial\Omega} = 0, & t \in J, \\ \psi(x, T) = 0, & x \in \Omega. \end{cases} \tag{58}$$

Lemma 3.1. [36] *Let φ and ψ be the solutions of* (57) *and* (58), *respectively. Let Ω be a convex domain. Then, for $\phi = \varphi$ or $\phi = \psi$*

$$\int_0^T\int_\Omega |\phi(x,t)|^2 \mathrm{d}x \le C\|F\|^2_{L^2(J;\, L^2(\Omega))},$$
$$\int_0^T\int_\Omega |\nabla\phi|^2 \mathrm{d}x\mathrm{d}t \le C\|F\|^2_{L^2(J;\, L^2(\Omega))},$$
$$\int_0^T\int_\Omega |D^2\phi|^2 \mathrm{d}x\mathrm{d}t \le C\|F\|^2_{L^2(J;\, L^2(\Omega))},$$
$$\int_0^T\int_\Omega |\phi_t|^2 \mathrm{d}x\mathrm{d}t \le C\|F\|^2_{L^2(J;\, L^2(\Omega))},$$

where $|D^2\phi| = \max\{|\partial^2\phi/\partial x_i\partial x_j|,\ 1 \le i, j \le 2\}$.

Lemma 3.2. [37] *Let f and g be piecewise continuous nonnegative functions defined on $0 \le t \le T$, g being nondecreasing. If for each $t \in J$,*

$$f(t) \le g(t) + \int_0^t f(s)\mathrm{d}s, \tag{59}$$

then $f(t) \le e^t g(t)$.

In the following two theorems, we shall estimate the error $\|\tilde{Z}_h - z(U_h)\|_{L^2(J;L^2(\Omega))}$.

Theorem 3.2. *Let $(Y_h, \boldsymbol{P}_h, Z_h, \boldsymbol{Q}_h, U_h)$ and $(y(U_h), \boldsymbol{p}(U_h),\ z(U_h), \boldsymbol{q}(U_h), U_h)$ be the solutions of* (30)–(36) *and* (37)–(42), *respectively. Then we have*

$$\|Y_h - y(U_h)\|^2_{L^2\,(J;\, L^2(\Omega))} \le C\sum_{i=2}^{8}\eta_i^2, \tag{60}$$

where

$$\eta_2^2 = \int_0^T \sum_\tau h_\tau^2 \int_\tau \left(Y_{h,t} + \mathrm{div}\hat{\boldsymbol{P}}_h + c\tilde{Y}_h - \hat{f} - U_h + \frac{\tilde{Y}_h - \tilde{Y}_h(\bar{x})}{k}\right)^2 \mathrm{d}x\mathrm{d}t;$$
$$\eta_3^2 = \int_0^T \int_\Omega \left(Y_{h,t} - \boldsymbol{b}\cdot A^{-1}\boldsymbol{P}_h - \frac{\hat{Y}_h - \tilde{Y}_h(\bar{x})}{k}\right)^2 \mathrm{d}x\mathrm{d}t;$$
$$\eta_4^2 = \int_0^T \sum_\tau h_\tau^2 \int_\tau (A^{-1}\boldsymbol{P}_h)^2 \mathrm{d}x\mathrm{d}t; \quad \eta_5^2 = \|\hat{\boldsymbol{P}}_h - \boldsymbol{P}_h\|^2_{L^2\,(J;\, L^2(\Omega))};$$
$$\eta_6^2 = \|\hat{f} - f\|^2_{L^2\,(J;\, L^2(\Omega))}; \quad \eta_7^2 = \|\hat{Y}_h - Y_h\|^2_{L^2(J;\, L^2(\Omega))}; \quad \eta_8^2 = \|y_0^h(x) - y_0(x)\|^2_{L^2(\Omega)}.$$

Proof. From (31), we can obtain the following equality:

$$(A^{-1}\boldsymbol{P}_h, \boldsymbol{v}_h) - (Y_h, \mathrm{div}\boldsymbol{v}_h) = 0, \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h. \tag{61}$$

Let ψ be the solution of (58) with $F = Y_h - y(U_h)$, using (30)–(32), (37)–(39), and (45)–(48), we infer that

$$\begin{aligned}
\|Y_h - y(U_h)\|^2_{L^2(J;\,L^2(\Omega))} &= \int_0^T (Y_h - y(U_h), F)\mathrm{d}t \\
&= \int_0^T (Y_h - y(U_h), -\psi_t - \operatorname{div}(A\nabla\psi) - \boldsymbol{b}\cdot\nabla\psi + c\psi)\mathrm{d}t \\
&= \int_0^T (((Y_h - y(U_h))_t, \psi) - (Y_h, \operatorname{div}(\Pi_h(A\nabla\psi))) + (\boldsymbol{p}(U_h), \nabla\psi))\mathrm{d}t \\
&\quad + \int_0^T ((\boldsymbol{b}(y(U_h) - Y_h), \nabla\psi) + (c(Y_h - y(U_h)), \psi))\mathrm{d}t \\
&\quad + ((Y_h - y(U_h))(x,0), \psi(x,0)) \\
&= \int_0^T (((Y_h - y(U_h))_t, \psi) - (A^{-1}\boldsymbol{P}_h, \Pi_h(A\nabla\psi)) - (\boldsymbol{b}\cdot\nabla y(U_h), \psi) - (\operatorname{div}\boldsymbol{p}(U_h), \psi))\mathrm{d}t \\
&\quad + \int_0^T (-(Y_h, \nabla\cdot(\boldsymbol{b}\psi)) + (c(Y_h - y(U_h)), \psi))\mathrm{d}t + (y_0^h(x) - y_0(x), \psi(x,0)) \\
&= \int_0^T ((Y_{h,t}, \psi) + (A^{-1}\boldsymbol{P}_h, A\nabla\psi - \Pi_h(A\nabla\psi)) - (\hat{\boldsymbol{P}}_h - \boldsymbol{P}_h, \nabla\psi) - (\operatorname{div}\hat{\boldsymbol{P}}_h, \psi))\mathrm{d}t \\
&\quad + \int_0^T (-(Y_h, \nabla\cdot(\Pi_h(\boldsymbol{b}\psi)) + (cY_h - f - U_h, \psi))\mathrm{d}t + (y_0^h(x) - y_0(x), \psi(x,0)) \\
&= \int_0^T \left(Y_{h,t} + \operatorname{div}\hat{\boldsymbol{P}}_h + c\hat{Y}_h - \hat{f} - U_h + \frac{\tilde{Y}_h - \tilde{Y}_h(\bar{x})}{k}, \psi\right)\mathrm{d}t \\
&\quad + \int_0^T ((\hat{f} - f, \psi) + (c(Y_h - \hat{Y}_h), \psi) + (\hat{\boldsymbol{P}}_h - \boldsymbol{P}_h, \nabla\psi))\mathrm{d}t \\
&\quad + \int_0^T (A^{-1}P_h, A\nabla\psi - \Pi_h(A\nabla\psi))\mathrm{d}t + (y_0^h(x) - y_0(x), \psi(x,0)) \\
&\quad - \int_0^T (Y_h, \nabla\cdot(\Pi_h(\boldsymbol{b}\psi)))\mathrm{d}t - \int_0^T \left(\frac{\tilde{Y}_h - \tilde{Y}_h(\bar{x})}{k}, \psi\right)\mathrm{d}t \\
&:= L_1 + L_2 + \cdots + L_6.
\end{aligned} \tag{62}$$

Using (30), (43), Cauchy inequality, and Lemma 3.1, we have

$$\begin{aligned}
L_1 &= \int_0^T \left(Y_{h,t} + \operatorname{div}\hat{\boldsymbol{P}}_h + c\hat{Y}_h - \hat{f} - U_h + \frac{\tilde{Y}_h - \tilde{Y}_h(\bar{x})}{k}, \psi - R_h\psi\right)\mathrm{d}t \\
&\le C(\delta)\eta_2^2 + \delta\|\psi\|^2_{L^2(J;\,H^1(\Omega))} \\
&\le C\eta_2^2 + \frac{1}{6}\|Y_h - y(U_h)\|^2_{L^2(J;L^2(\Omega))}.
\end{aligned} \tag{63}$$

Similarly, using Cauchy inequality, and Lemma 3.1, we derive

$$L_2 \le C(\eta_5^2 + \eta_6^2 + \eta_7^2) + \frac{1}{6}\|Y_h - y(U_h)\|^2_{L^2(J;L^2(\Omega))}, \tag{64}$$

$$L_3 \le C\eta_4^2 + \frac{1}{6}\|Y_h - y(U_h)\|^2_{L^2(J;\, L^2(\Omega))}, \tag{65}$$

$$L_4 \le C\eta_8^2 + \frac{1}{6}\|Y_h - y(U_h)\|^2_{L^2(J;\, L^2(\Omega))}. \tag{66}$$

Finally, for L_5 and L_6, using (61), Cauchy inequality, and Lemma 3.1, we find that

$$\begin{aligned} L_5 + L_6 &= -\int_0^T (A^{-1}\boldsymbol{P}_h, \Pi_h(\boldsymbol{b}\psi))\mathrm{d}t - \int_0^T \left(\frac{\tilde{Y}_h - \tilde{Y}_h(\bar{x})}{k}, \psi \right)\mathrm{d}t \\ &= \int_0^T \left(Y_{h,t} - \frac{\hat{Y}_h - \tilde{Y}_h(\bar{x})}{k} - \boldsymbol{b}\cdot A^{-1}\boldsymbol{P}_h, \psi \right)\mathrm{d}t + \int_0^T (A^{-1}\boldsymbol{P}_h, \boldsymbol{b}\psi - \Pi_h(\boldsymbol{b}\psi))\mathrm{d}t \\ &\le C\eta_3^2 + C\eta_4^2 + \frac{1}{6}\|Y_h - y(U_h)\|^2_{L^2(J;\, L^2(\Omega))}. \end{aligned} \tag{67}$$

Hence, using (61)–(67), we obtain (60). ◻

Theorem 3.3. *Let* $(y, \boldsymbol{p}, z, \boldsymbol{q}, u)$ *and* $(Y_h, \boldsymbol{P}_h, Z_h, \boldsymbol{Q}_h, U_h)$ *be the solutions of* (10)–(16) *and* (30)–(36), *respectively. Let* $(y(U_h), \boldsymbol{p}(U_h), z(U_h), \boldsymbol{q}(U_h), U_h)$ *be defined as in* (37)–(42). *Then we have the following error estimate*:

$$\|\tilde{Z}_h - z(U_h)\|^2_{L^2(J;\, L^2(\Omega))} \le C\left(\eta_4^2 + \eta_7^2 + \sum_{i=9}^{16}\eta_i^2\right) + C\|Y_h - y(U_h)\|^2_{L^2(J;\, L^2(\Omega))}, \tag{68}$$

where

$$\begin{aligned} \eta_9^2 &= \int_0^T \sum_\tau h_\tau^2 \int_\tau \left(-Z_{h,t} + \operatorname{div}\tilde{\boldsymbol{Q}}_h - \frac{\hat{Z}_h(\bar{\bar{x}})\cdot \hat{J} - \hat{Z}_h}{k} + c\tilde{Z}_h - \hat{Y}_h + \hat{y}_d \right)^2 \mathrm{d}x\mathrm{d}t; \\ \eta_{10}^2 &= \int_0^T \int_\Omega \left(Z_{h,t} - \boldsymbol{b}\cdot A^{-1}\boldsymbol{Q}_h - \boldsymbol{b}\cdot \bar{\boldsymbol{P}}_h + \boldsymbol{b}\cdot \bar{\boldsymbol{p}}_d - \frac{\hat{Z}_h(\bar{\bar{x}})\cdot \hat{J} - \tilde{Z}_h}{k} \right)^2 \mathrm{d}x\mathrm{d}t; \\ \eta_{11}^2 &= \int_0^T \sum_\tau h_\tau^2 \int_\tau (A^{-1}Q_h + \bar{\boldsymbol{P}}_h - \bar{\boldsymbol{p}}_d)^2 \mathrm{d}x\mathrm{d}t; \\ \eta_{12}^2 &= \|\tilde{\boldsymbol{Q}}_h - \boldsymbol{Q}_h\|^2_{L^2(J;\, L^2(\Omega))}; \eta_{13}^2 = \|\bar{\boldsymbol{P}}_h - \boldsymbol{P}_h\|^2_{L^2(J;\, L^2(\Omega))}; \\ \eta_{14}^2 &= \|\tilde{Z}_h - Z_h\|^2_{L^2(J;\, L^2(\Omega))}; \eta_{15}^2 = \|\bar{\boldsymbol{p}}_d - \boldsymbol{p}_d\|^2_{L^2(J;\, L^2(\Omega))}; \\ \eta_{16}^2 &= \|\hat{y}_d - y_d\|^2_{L^2(J;\, L^2(\Omega))}, \end{aligned}$$

η_4 *and* η_7 *are defined in* Theorem 3.2.

Proof. Similar to (61), using (34) and the definitions of Z_h, $\boldsymbol{Q}_h$, $\bar{\boldsymbol{P}}_h$, and $\bar{\boldsymbol{p}}_d$, we obtain

$$(A^{-1}\boldsymbol{Q}_h, \boldsymbol{v}_h) - (Z_h, \operatorname{div}\boldsymbol{v}_h) = -(\bar{\boldsymbol{P}}_h - \bar{\boldsymbol{p}}_d, \boldsymbol{v}_h), \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h. \tag{69}$$

Let φ be the solution of (57) with $F = Z_h - z(U_h)$. It follows from (33)–(35), (40)–(42), and (45)–(48) that

$$\begin{aligned} \|Z_h - z(U_h)\|^2_{L^2(J;\, L^2(\Omega))} &= \int_0^T (Z_h - z(U_h), F)\mathrm{d}t \\ &= \int_0^T (Z_h - z(U_h), \varphi_t - \operatorname{div}(A\nabla\varphi) + \boldsymbol{b}\cdot\nabla\varphi + c\varphi)\mathrm{d}t \end{aligned} \tag{70}$$

$$
\begin{aligned}
&= \int_0^T ((-(Z_h - z(U_h))_t, \varphi) - (Z_h, \operatorname{div}(\Pi_h(A\nabla\varphi - \boldsymbol{b}\varphi))))\mathrm{d}t - \int_0^T (z(U_h), \boldsymbol{b}\cdot\nabla\varphi)\mathrm{d}t \\
&\quad + \int_0^T (A^{-1}\boldsymbol{q}(U_h) + \boldsymbol{p}(U_h) - \boldsymbol{p}_d, A\nabla\varphi)\mathrm{d}t + \int_0^T (c(Z_h - z(U_h)), \phi)\mathrm{d}t \\
&= \int_0^T ((-(Z_h - z(U_h))_t, \varphi) - (A^{-1}\boldsymbol{Q}_h + \bar{\boldsymbol{P}}_h - \bar{\boldsymbol{p}}_d, \Pi_h(A\nabla\varphi - \boldsymbol{b}\varphi)))\mathrm{d}t \\
&\quad + \int_0^T ((\boldsymbol{p}(U_h) - \boldsymbol{p}_d, A\nabla\varphi) - (\operatorname{div}\boldsymbol{q}(U_h), \varphi) + (\boldsymbol{b}\cdot\nabla z(U_h), \varphi) + (c(Z_h - z(U_h)), \varphi))\mathrm{d}t \\
&= \int_0^T \left(-Z_{h,t} + \operatorname{div}\tilde{\boldsymbol{Q}}_h + c\tilde{Z}_h - \hat{Y}_h + \hat{y}_d - \frac{\hat{Z}_h(\bar{\bar{x}})\cdot\hat{J} - \hat{Z}_h}{k}, \varphi\right)\mathrm{d}t + \int_0^T (c(Z_h - \tilde{Z}_h), \varphi)\mathrm{d}t \\
&\quad + \int_0^T (A^{-1}\boldsymbol{Q}_h + \bar{\boldsymbol{P}}_h - \bar{\boldsymbol{p}}_d, A\nabla\phi - \boldsymbol{b}\varphi - \Pi_h(A\nabla\varphi - \boldsymbol{b}\varphi))\mathrm{d}t + \int_0^T (\tilde{\boldsymbol{Q}}_h - \boldsymbol{Q}_h, \nabla\varphi)\mathrm{d}t \\
&\quad + \int_0^T (y_d - \hat{y}_d + \hat{Y}_h - y(U_h), \varphi)\mathrm{d}t + \int_0^T (\boldsymbol{p}(U_h) - \bar{\boldsymbol{P}}_h + \bar{\boldsymbol{p}}_d - \boldsymbol{p}_d, A\nabla\varphi)\mathrm{d}t \\
&\quad - \int_0^T \left(Z_{h,t} - \boldsymbol{b}\cdot A^{-1}\boldsymbol{Q}_h - \boldsymbol{b}\cdot\bar{\boldsymbol{P}}_h + \boldsymbol{b}\cdot\bar{\boldsymbol{p}}_d - \frac{\hat{Z}_h(\bar{\bar{x}})\cdot\hat{J} - \tilde{Z}_h}{k}, \varphi\right)\mathrm{d}t \\
&\coloneqq J_1 + J_2 + \cdots + J_7.
\end{aligned}
$$

First, using the same estimates as (63)–(66), we have

$$J_1 \le C\eta_9^2 + \frac{1}{8}\|Z_h - z(U_h)\|^2_{L^2(J;\,L^2(\Omega))}, \tag{71}$$

$$J_2 \le C\eta_{14}^2 + \frac{1}{8}\|Z_h - z(U_h)\|^2_{L^2(J;\,L^2(\Omega))}, \tag{72}$$

$$J_3 \le C\eta_{11}^2 + \frac{1}{8}\|Z_h - z(U_h)\|^2_{L^2(J;\,L^2(\Omega))}, \tag{73}$$

$$J_4 \le C\eta_{12}^2 + \frac{1}{8}\|Z_h - z(U_h)\|^2_{L^2(J;\,L^2(\Omega))}, \tag{74}$$

$$J_7 \le C\eta_{10}^2 + \frac{1}{8}\|Z_h - z(U_h)\|^2_{L^2(J;\,L^2(\Omega))}. \tag{75}$$

For J_5, using Cauchy inequality, and Lemma 3.1, we have

$$
\begin{aligned}
J_5 &= \int_0^T (\hat{Y}_h - Y_h + Y_h - y(U_h) + y_d - \hat{y}_d, \phi)\mathrm{d}t \\
&\le C(\eta_7^2 + \eta_{16}^2) + C\|Y_h - y(U_h)\|^2_{L^2(J;\,L^2(\Omega))} + \frac{1}{8}\|Z_h - z(U_h)\|^2_{L^2(J;\,L^2(\Omega))}.
\end{aligned}
\tag{76}
$$

Finally, for J_6, using (31), (38), Cauchy inequality, and Lemma 3.1, we derive

$$J_6 = \int_0^T (A^{-1}(\boldsymbol{p}(U_h) - \boldsymbol{P}_h), A^2\nabla\varphi)\mathrm{d}t + \int_0^T (\boldsymbol{P}_h - \bar{\boldsymbol{P}}_h + \bar{\boldsymbol{p}}_d - \boldsymbol{p}_d, A\nabla\varphi)\mathrm{d}t \tag{77}$$

$$= \int_0^T ((y(U_h) - Y_h, \operatorname{div}(A^2\nabla\varphi)) + (A^{-1}\boldsymbol{P}_h, \Pi_h(A^2\nabla\varphi) - A^2\nabla\varphi))dt + \int_0^T (\boldsymbol{P}_h - \bar{\boldsymbol{P}}_h + \bar{\boldsymbol{p}}_d - \boldsymbol{p}_d, A\nabla\varphi)dt$$
$$\leq C(\eta_4^2 + \eta_{13}^2 + \eta_{15}^2) + C\|Y_h - y(U_h)\|^2_{L^2(J;\, L^2(\Omega))} + \frac{1}{8}\|Z_h - z(U_h)\|^2_{L^2(J;\, L^2(\Omega))}.$$

Therefore, the above estimates and the triangle inequality yield (68). □

Let $(\boldsymbol{p}, y, \boldsymbol{q}, z, u)$ and $(\boldsymbol{P}_h, Y_h, \boldsymbol{Q}_h, Z_h, U_h)$ be the solutions of (10)–(16) and (30)–(36), respectively. We decompose the errors as follows:

$$\begin{aligned}
\boldsymbol{p} - \boldsymbol{P}_h &= \boldsymbol{p} - \boldsymbol{p}(U_h) + \boldsymbol{p}(U_h) - \boldsymbol{P}_h := \boldsymbol{\varepsilon}_1 + \boldsymbol{\varepsilon}_1,\\
y - Y_h &= y - y(U_h) + y(U_h) - Y_h := r_1 + e_1,\\
\boldsymbol{q} - \boldsymbol{Q}_h &= \boldsymbol{q} - \boldsymbol{q}(U_h) + \boldsymbol{q}(U_h) - \boldsymbol{Q}_h := \boldsymbol{\varepsilon}_2 + \boldsymbol{\varepsilon}_2,\\
z - Z_h &= z - z(U_h) + z(U_h) - Z_h := r_2 + e_2.
\end{aligned}$$

From (10)–(16) and (37)–(42), we derive the following error equations:

$$(A^{-1}\boldsymbol{\varepsilon}_1, \boldsymbol{v}) - (r_1, \operatorname{div}\boldsymbol{v}) = 0, \quad \forall \boldsymbol{v} \in \boldsymbol{V}, \tag{78}$$
$$(r_{1,t}, w) + (\boldsymbol{b} \cdot A^{-1}\boldsymbol{\varepsilon}_1, w) + (\operatorname{div}\boldsymbol{\varepsilon}_1, w) + (cr_1, w) = (u - U_h, w), \quad \forall w \in W, \tag{79}$$
$$(A^{-1}\boldsymbol{\varepsilon}_2, \boldsymbol{v}) - (r_2, \operatorname{div}\boldsymbol{v}) = -(\boldsymbol{\varepsilon}_1, \boldsymbol{v}), \quad \forall \boldsymbol{v} \in \boldsymbol{V}, \tag{80}$$
$$-(r_{2,t}, w) + (\boldsymbol{b} \cdot(\boldsymbol{\varepsilon}_1 + A^{-1}\boldsymbol{\varepsilon}_2), w) + (\operatorname{div}\boldsymbol{\varepsilon}_2, w) + (cr_2, w) = (r_1, w), \quad \forall w \in W. \tag{81}$$

Theorem 3.4. *There is a constant $C > 0$, independent of h, such that*

$$\|\boldsymbol{\varepsilon}_1\|_{L^2(J;\, L^2(\Omega))} + \|r_1\|_{L^2(J;\, L^2(\Omega))} \leq C\|u - U_h\|_{L^2(J;\, L^2(\Omega))}, \tag{82}$$
$$\|\boldsymbol{\varepsilon}_2\|_{L^2(J;\, L^2(\Omega))} + \|r_2\|_{L^2(J;\, L^2(\Omega))} \leq C\|u - U_h\|_{L^2(J;\, L^2(\Omega))}. \tag{83}$$

Proof. Choosing $\boldsymbol{v} = \boldsymbol{\varepsilon}_1$ and $w = r_1$ as the test functions and adding the two relations of (78)–(79), we have

$$(A^{-1}\boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_1) + (r_{1,t}, r_1) = (u - U_h, r_1) + (\boldsymbol{b} \cdot A^{-1}\boldsymbol{\varepsilon}_1, r_1) - (cr_1, r_1). \tag{84}$$

Then, using ε-Cauchy inequality, we can find an estimate as follows:

$$(A^{-1}\boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_1) + (r_{1,t}, r_1) \leq C(\|r_1\|^2_{L^2(\Omega)} + \|u - U_h\|^2_{L^2(\Omega)}) + \frac{1}{2}(A^{-1}\boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_1). \tag{85}$$

Note that

$$(r_{1,t}, r_1) = \frac{1}{2}\frac{\partial}{\partial t}\|r_1\|^2_{L^2(\Omega)},$$

then, using the assumption on A, we can obtain that

$$\frac{1}{2}c_1\|\boldsymbol{\varepsilon}_1\|^2_{L^2(\Omega)} + \frac{1}{2}\frac{\partial}{\partial t}\|r_1\|^2_{L^2(\Omega)} \leq C(\|r_1\|^2_{L^2(\Omega)} + \|u - U_h\|^2_{L^2(\Omega)}). \tag{86}$$

Integrating (86) in time and since $r_1(0) = 0$, using Lemma 3.2 to obtain

$$\|\boldsymbol{\varepsilon}_1\|^2_{L^2(J;\, L^2(\Omega))} + \|r_1\|^2_{L^\infty(J;\, L^2(\Omega))} \leq C\|u - U_h\|^2_{L^2(J;\, L^2(\Omega))}, \tag{87}$$

this implies (82).

Similarly, we can obtain

$$\|\boldsymbol{\varepsilon}_2\|^2_{L^2(J;\, L^2(\Omega))} + \|r_2\|^2_{L^\infty(J;\, L^2(\Omega))} \leq C\Big(\|\boldsymbol{\varepsilon}_1\|^2_{L^2(J;\, L^2(\Omega))} + \|r_1\|^2_{L^2(J;\, L^2(\Omega))}\Big). \tag{88}$$

Using (88) and (82), we complete the proof of Theorem 3.4. □

Collecting Theorems 3.1–3.4, we can derive the following results:

Theorem 3.5. *Let* $(\boldsymbol{p}, y, \boldsymbol{q}, z, u)$ *and* $(\boldsymbol{P}_h, Y_h, \boldsymbol{Q}_h, Z_h, U_h)$ *be the solutions of* (10)–(16) *and* (30)–(36), *respectively. Then we have*

$$\|u - U_h\|^2_{L^2(J;\, L^2(\Omega))} + \|y - Y_h\|^2_{L^2\,(J;\, L^2(\Omega))} + \|z - Z_h\|^2_{L^2\,(J;\, L^2(\Omega))} \le C\sum_{i=1}^{16}\eta_i^2, \tag{89}$$

where η_1 *is defined in* Theorem 3.1, $\eta_2, \dots, \eta_8$ *are defined in* Theorem 3.2, *and* $\eta_9, \dots, \eta_{16}$ *are defined in* Theorem 3.3, *respectively.*

4 Numerical experiments

In this section, we illustrate our theoretical results by a numerical example.

For a constrained optimization problem:

$$\min_{u\in K} J(u),$$

where $J(u)$ is a convex functional on U and K is a close convex subset of U, the iterative scheme reads ($n = 0, 1, 2, \dots$):

$$\begin{cases} b(u_{n+\frac{1}{2}}, v) = b(u_n, v) - \rho_n(J'(u_n), v), \quad \forall v \in U, \\ u_{n+1} = P_K^b(u_{n+\frac{1}{2}}), \end{cases} \tag{90}$$

where ρ_n is a step size of iteration, $b(\cdot,\cdot) = \int_0^T(\cdot,\cdot)$ is a symmetric positive definite bilinear form, and the projection operator P_K^b can be obtained like in [38].

By applying (90) to the fully discretized problem (21)–(27), for an acceptable error Tol, we present the following algorithm in which we have omitted the subscript h just for ease of exposition.

Algorithm 4.1. Projection gradient algorithm

Step 1. Initialize u_0.

Step 2. Solve the following equations:

$$\begin{cases} b\Big(u_{n+\frac{1}{2}}, v\Big) = b(u_n, v) - \rho_n \displaystyle\int_0^T (u_n + z_n, v), \quad \forall v \in W_h, \\ \left(\dfrac{y_n^i - \bar{y}_n^{i-1}}{k}, w\right) + (\mathrm{div}\boldsymbol{p}_n^i, w) + (cy_n^i, w) = (f^i + u_n^i, w), \quad \forall w \in W_h, \\ (A^{-1}\boldsymbol{p}_n^i, \boldsymbol{v}) - (y_h^i, \mathrm{div}\boldsymbol{v}) = 0, \quad \forall \boldsymbol{v} \in \boldsymbol{V}_h, \\ \left(\dfrac{z_n^{i-1} - \bar{\bar{z}}_n^i}{k}, w\right) + (\mathrm{div}\boldsymbol{q}_n^{i-1}, w) + (cz_n^{i-1}, w) = (y_n^i - y_d^i, w), \quad \forall w \in W_h, \\ (A^{-1}\boldsymbol{q}_n^{i-1}, \boldsymbol{v}) - (z_n^{i-1}, \mathrm{div}\boldsymbol{v}) = -(\boldsymbol{p}_n^i - \boldsymbol{p}_d^i, \boldsymbol{v}), \quad \forall \boldsymbol{v} \in \boldsymbol{V}_h, \\ u_{n+1} = P_K^b\Big(u_{n+\frac{1}{2}}\Big). \end{cases} \tag{91}$$

Step 3. Calculate the iterative error:

$$E_{n+1} = \|u_{n+1} - u_n\|_{L^2(J;\, L^2(\Omega))}.$$

Step 4. If $E_{n+1} \le Tol$, stop; else go to Step 2.

Algorithm 4.2. Adaptive projection gradient algorithm

Step 1. Solve the discretized optimization problem (21)–(27) with Algorithm 4.1 on the current mesh (the same mesh is used for the control, state, and co-state variables), obtain numerical solution u_n', and calculate the error estimator $\eta = \sqrt{\sum_{i=1}^{16}\eta_i^2}$;

Step 2. Adjust the mesh by η, then update the numerical solution u_n' and obtain u_{n+1}' on new mesh;

Step 3. Calculate the iterative error:

$$E_{n+1}' = \|u_{n+1}' - u_n'\|_{L^2(J;\, L^2(\Omega))};$$

Step 4. If $E_{n+1}' > Tol'$, go to Step 1; else stop.

The following numerical example was solved with the C++ software package: AFEPack which is freely available. Let $\Omega = [0, 1] \times [0, 1]$, $T = 1$, $\boldsymbol{b} = (1, 1)^T$, $c(x) = 1$, and $A(x)$ be the 2×2 identity matrix. The discretization is described in Section 2. We use the Algorithms 4.1 and 4.2 to solve the following OCPs. The same mesh is used for the control, state, and dual state variables in Algorithm 4.2.

$$\begin{cases} \min\limits_{u\in K}\left\{\dfrac{1}{2}\displaystyle\int_0^T (\|\boldsymbol{p}(x,t) - \boldsymbol{p}_d(x,t)\|^2 + \|y(x,t) - y_d(x,t)\|^2 + \|u(x,t) - u_d(x,t)\|^2)\mathrm{d}t\right\}, \\ y_t(x,t) + \boldsymbol{b}(x,t)\cdot\nabla y(x,t) + \operatorname{div}\boldsymbol{p}(x,t) + c(x)y(x,t) = f(x,t) + u(x,t), \quad x \in \Omega,\ t \in J, \\ \boldsymbol{p}(x,t) = -A(x)\nabla y(x,t), \quad x \in \Omega,\ t \in J, \\ y(x,t) = 0, \quad x \in \partial\Omega,\ t \in J, \\ y(x,0) = y_0(x), \quad x \in \Omega. \end{cases}$$

Example 1. The data are as follows:

$$\begin{aligned} &y(x,t) = tx_1(x_1 - 1)x_2(x_2 - 1), \\ &\boldsymbol{p}(x,t) = -(t(2x_1 - 1)x_2(x_2 - 1), tx_1(x_1 - 1)(2x_2 - 1))^T, \\ &z(x,t) = (1 - t)x_1(x_1 - 1)x_2(x_2 - 1), \\ &\boldsymbol{q}(x,t) = -((1 - t)(2x_1 - 1)x_2(x_2 - 1), (1 - t)x_1(x_1 - 1)(2x_2 - 1))^T, \\ &u_d(x,t) = \begin{cases} (1 - t)\left(2.0 - \sin\dfrac{\pi x_1}{2} - \sin\dfrac{\pi x_2}{2}\right), & x_1 + x_2 > 1, \\ (1 - t)\left(-\sin\dfrac{\pi x_1}{2} - \sin\dfrac{\pi x_2}{2}\right), & x_1 + x_2 \le 1, \end{cases} \\ &u(x,t) = \max\{0, u_d(x,t) - z(x,t)\}, \\ &f(x,t) = y_t(x,t) + \boldsymbol{b}\cdot\nabla y(x,t) + \operatorname{div}\boldsymbol{p}(x,t) + cy(x,t) - u(x,t), \\ &\boldsymbol{p}_d(x,t) = \boldsymbol{p}(x,t) + \boldsymbol{q}(x,t) + \nabla z(x,t), \\ &y_d(x,t) = y(x,t) + z_t(x,t) + \boldsymbol{b}\cdot\nabla z(x,t) - \operatorname{div}\boldsymbol{q}(x,t) - cz(x,t). \end{aligned}$$

It is clear that the optimal control has a strong discontinuity along the diagonal $x_1 + x_2 = 1$ of Ω. We select $k = 10^{-2}$ and use $\eta = \sqrt{\sum_{i=1}^{16}\eta_i^2}$ as the indicator to construct adaptive meshes.

Table 1: Numerical results based on uniform and adaptive meshes

Mesh info	Uniform mesh			Adaptive mesh		
	u_h	y_h	z_h	u_h	y_h	z_h
Nodes	7,617	7,617	7,617	1,737	1,737	1,737
Sides	22,528	22,528	22,528	4,597	4,597	4,597
Elements	14,912	14,912	14,912	2,861	2,861	2,861
Dofs	14,912	14,912	14,912	2,861	2,861	2,861
$L^2(J;\, L^2(\Omega))$ error	2.8635×10^{-3}	2.2684×10^{-4}	2.2035×10^{-4}	2.7856×10^{-3}	2.4875×10^{-4}	2.3816×10^{-4}

Some numerical results are shown in Table 1, where the numerical results based on uniform refined mesh and adaptively refined mesh are obtained by Algorithms 4.1 and 4.2, respectively. The errors $\|u - u_h\|_{L^2(J;\, L^2(\Omega))} = 2.8635 \times 10^{-3}$ and $\|u - u_h\|_{L^2(J;\, L^2(\Omega))} = 2.7856 \times 10^{-3}$ are based on uniformly refined mesh with 7,617 nodes and adaptively refined mesh with 1,737 nodes, respectively. It obviously indicates that Algorithm 4.2 can save substantial computing work and *a posteriori* error estimates in Theorem 3.5 are efficient.

We plot the profile of the exact solution u at $t = 0.5$ and the profile of the numerical solution u_h at $t = 0.5$ on adaptive mesh with nodes = 1,737 in Figures 1 and 2, respectively. The adaptive mesh with nodes = 1,737 is shown in Figure 3. It can be found that a higher density of node points is indeed distributed along the neighborhood of the diagonal $x_1 + x_2 = 1$, where the control u has a strong jump. Therefore, the numerical results are consistent with our theoretical results.

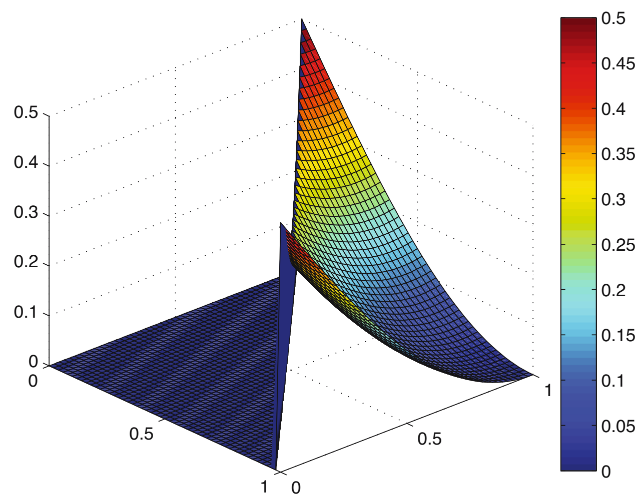

Figure 1: The exact solution u at $t = 0.5$.

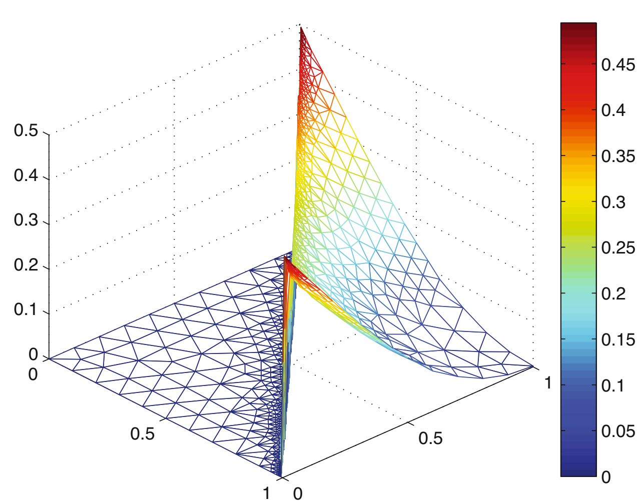

Figure 2: The numerical solution u_h at $t = 0.5$.

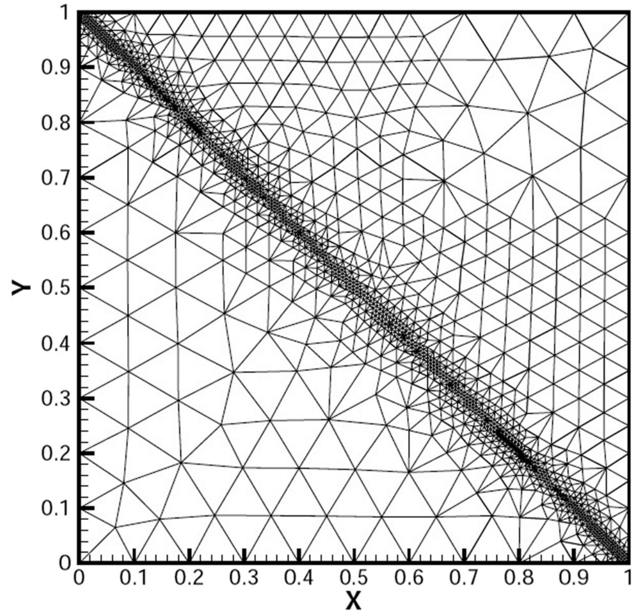

Figure 3: The adaptive mesh for u_h, y_h, z_h with nodes = 1,737.

5 Conclusion

In this article, we consider a fully discrete CMFEM for parabolic convection-diffusion OCPs. We derive *a posteriori* $L^2(J;\, L^2(\Omega))$ error estimates for the scalar functions. A numerical example demonstrates its reliability. Our results seem to be new. Furthermore, we shall investigate *a posteriori* error estimates for the vector-valued functions.

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