Research Article

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Analysis of a deteriorating system with delayed repair and unreliable repair equipment

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Abstract: In this article, we mainly consider a repairable degradation system consisting of a single component, a repairman, and repair equipment. Suppose that the system cannot be repaired immediately after failure and cannot be repaired "as good as new." Herein, the repair equipment may fail during repair and the system will replace a new one after failures. In particular, the repair time follows the general distribution. Under the above assumptions, a partial differential equation model is established through the geometric process and supplementary variable technique. By Laplace transform, we obtain the availability of the system, and from the expression one can see that the availability of the system will tend to zero after running for a long time. Therefore, we further study a replacement policy N based on the failed times of the system. We give the explicit expression of the system average cost $C(N)$ and obtain the optimal replacement policy N^* by minimizing the average cost rate $C(N^*)$. That is, the system will be replaced when the failure number of the system reaches N^*. Furthermore, the extended degenerate system is proposed by assuming that the system is not always successive degenerative after repair, and then the optimal replacement policy is studied. Finally, the numerical analysis is given to illustrate the theoretical results.

Keywords: deteriorating system, Laplace transform, availability, extended geometric process, replacement policy

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1 Introduction

With the development of modern technology and the wide application of electronic products, the reliability problem of a system attracts great attention. In addition, in order to improve the reliability of the system, the strategy of system maintenance and repair becomes a necessary study content. Therefore, a repairable system is an important system in reliability theory. Many authors have worked in this field, for instance, see literature [1–4].

In recent years, many scholars have studied repairable systems from the following two aspects: one is that the repair equipment will not fail during repair or can be repaired after failure; another is that the system cannot be repaired "as good as new" after failures, which is a special kind of repairable system called a degenerate repairable system. In general, the geometric process repair model is used to describe a

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degenerate repairable system (see [5–12]), which was proposed by Lam in [13]. For a degenerate repairable system, Lam mainly considered two kinds of replacement policies: N policy and T policy, where T policy is based on the working age of the system and N policy is based on the number of system failures. That is, the explicit expression of the system average cost $C(N)$ is given under the two policies, and by minimizing the average cost rate $C(N^*)$ to obtain the optimal replacement policy N^*. Lam also proved that under some conditions, the optimal replacement policy N^* is better than the optimal replacement policy T^*. Based on this result, Zhang combined the two types of policies and further proposed a bivariate replacement policy (T, N) (see [14]). Similarly, Zhang et al. showed that under some conditions, the optimal policy $(T, N)^*$ is better than the optimal policy T^* and N^*, where we refer to the literature [14–18].

In the aforementioned study, there is one essential assumption that the system is always successive degenerative (see [7–12]). However, it is not always the case in practice. In fact, the system failure is divided into different types, such as serious faults and minor faults. If there is a serious fault in the system, the system will deteriorate after repair. If there is a minor fault in the system, the system will be as good as new after appropriate repair and the system will not deteriorate. So when the system is repaired, it sometimes deteriorates and sometimes does not deteriorate. In general, this kind of repair model is called an extended geometric process repair model. It is also a generalization of the geometric process repair model. So recently, some scholars have extended the degenerate system and put forward the extended degenerate system (see [17,19]).

In this article, we propose a more general repairable system that is consisting of a component, a repairman and repair equipment. Suppose that the repair equipment may fail during repair and the equipment will be replaced by a new one after failure. In addition, the repair of the system will be delayed due to the repairman is not on duty or needs to determine the cause of the failure. Different from earlier literature in which the repair time satisfies the exponential distribution (see [20]), we assume that the repair time satisfies the general distribution, and the replacement time of the repair equipment is considered. Under these assumptions, we establish the mathematical model of the system, and then give the state availability of the system. Due to degeneration of the system, the availability of the system will tend to be zero after a long period of operation. In order to ensure the reliability of the system, we further study a replacement policy N based on the failure times of the system, considering the system cost.

The rest of the article is organized as follows. In Section 2, a mathematical model of the deteriorating system is established under the geometric process. In Section 3, the steady-state availability of the deteriorating system is given by using Laplace transform. In Section 4, a replacement policy N is studied based on the failed times of the system, and further the replacement policy N of the extended deteriorate system is put forward. In Section 5, the numerical result is given to illustrate the existence and uniqueness of the optimal replacement policy N^*, and the influence of the parameters on the optimal replacement policy N^* and the minimum cost rate $C(N^*)$. Section 6 gives a conclusion of this article.

2 Mathematical modelling

In this section, we establish the dynamic equation of the degenerate system by means of supplementary variable technique and probability analysis.

First, we give the following basic assumptions on the system:

Assumption 1. The system consists of a component, a repairman and repair equipment. Initially, the system is new and in a working state.

Assumption 2. Suppose that the system cannot be repaired "as good as new" after failures. The time interval between the completion of the $(n-1)$th repair and the completion of the nth repair of component is called the nth $(n = 1, 2, \ldots)$ cycle of the system. Let X_n and Y_n be the working time and the repair time of the system in the nth cycle, respectively. We define the cumulative distribution functions of X_n and Y_n as the following $F_n(t)$ and $G(t)$, respectively.

$$F_n(t) = F(a^{n-1}t) = 1 - \exp(-a^{n-1}\lambda t), \quad n \geq 1,$$

where $a \geq 1, \lambda > 0$, and

$$G(t) = \int_0^t g(y)\mathrm{d}y = 1 - \exp\left(-\int_0^t \mu(y)\mathrm{d}y\right),$$

where the expected value $\int_0^\infty tg(t)\mathrm{d}t = \frac{1}{\mu}$ and $t \geq 0$.

Assumption 3. Assume that the system cannot be repaired immediately after failure. Let W_n be the delayed repair time of the system in the nth cycle and the distribution of W_n satisfies the exponential distribution, that is, $H(t) = 1 - \exp(-\theta t)(\theta > 0)$.

Assumption 4. Suppose that the repair equipment will also fail during repair. If the repair equipment fails, the system will replace a new repair equipment, and then continue to complete the repair work. During the replacement of repair equipment, the system is waiting for repair. The repair equipment neither fails nor deteriorates during the system working time or the delayed repairing time. Herein, let U_n be the working time of the repair equipment in the nth cycle, and its distribution function is $U(t) = 1 - \exp(-\alpha t)(\alpha > 0)$. Let V_n be the replacement time of repair equipment in the n-th cycle, and its distribution function is

$$V(t) = \int_0^t v(y)\mathrm{d}y = 1 - \exp\left(-\int_0^t \beta(z)\mathrm{d}z\right)$$

where $\int_0^\infty tv(t)\mathrm{d}t = \frac{1}{\beta}$.

Assumption 5. X_n, Y_n, W_n, U_n and V_n, $n \in N$ are independent.

Assumption 6. Each switchover of the states is perfect and each switchover is completed instantaneously.

Under the aforementioned assumptions, we can divide the system into the following states:

State 0: The system is working at time t and the repair equipment is normal;

State 1: The system fails at time t and the repair is delayed;

State 2: The system is being repaired at time t;

State 3: Repair equipment failure at time t and the system is waiting for repair.

Now we introduce some random variables as follows:

The random variable $S(t)$ at time t valued in state set $\{0, 1, 2, 3\}$, $S(t) = k$ means that the system is in state k at time t;

The random variable $\{I(t), t \geq 0\}$ valued in state space $\mathbb{N}$, $I(t) = j$ means that the system is in the jth cycle;

The random variable $Y(t)$ indicates the elapsed time of repairing system at time t;

The random variable $Z(t)$ valued in $[0, \infty)$, $Z(t)$ means the elapsed time of replacement repair equipment at time t.

Using these random variables we can describe the probability or probability density functions of the system in all states:

$p_{ij}(t) = P\{S(t) = i, I(t) = j\}$ $(i = 0, 1.\ j \in \mathbb{N})$ represents the probability of the system in state i and in the jth cycle at time t.

$p_{2j}(t, y)\mathrm{d}y = P\{S(t) = 2, y \leq Y(t) < y + \mathrm{d}y, I(t) = j\}(j \in \mathbb{N})$ represents the probability of the system in state 2 and the repairman is dealing with the fault system with the elapsed time lying in $[y, y + \mathrm{d}y)$ at time t.

$p_{3j}(t, y, z)\mathrm{d}y = P\{S(t) = 3, z \leq Z(t) < z + \mathrm{d}z, I(t) = j, Y(t) = y\}(j \in \mathbb{N})$ represents the probability of the system in state 3 and the system is replacing repair equipment with the elapsed time lying in $[z, z + \mathrm{d}z)$ at time t.

In what follows, we will derive the differential equation of the system. We mainly consider the state change of the system in a relatively small time interval $(t, t + \Delta t]$.

(1) $p_{0j}(t + \Delta t) = p\{S(t + \Delta t) = 0, I(t + \Delta t) = j\} = \sum_{i=0}^3 p\{S(t + \Delta t) = 0, I(t + \Delta t) = j, N(t) = i\} = \sum_{i=0}^3 p\{S(t + \Delta t) = 0, I(t + \Delta t) = j|N(t) = i, I(t) = j\}p\{N(t) = i, I(t) = j\} + p\{S(t + \Delta t) = 0, I(t + \Delta t) = j|N(t) = 2, I(t) = j - 1\}p\{N(t) = 2, I(t) = j - 1\}$.

Item $p\{S(t + \Delta t) = 0, I(t + \Delta t) = j|N(t) = 0, I(t) = j\}$ indicates that the system at time t is in working state and in the jth cycle, and at time $t + \Delta t$ the system is still in working state and jth cycle. In other words, the system did not fail during the Δt period. Thus, we have

$$p\{S(t+\Delta t)=0, I(t+\Delta t)=j|N(t)=0, I(t)=j\}=1-a^{j-1}\lambda\Delta t+o(\Delta t), \quad j\geq 1.$$

Item $p\{S(t+\Delta t)=0, I(t+\Delta t)=j|N(t)=1, I(t)=j\}$ indicates that the system at time t is in the state of delayed repair and in the jth cycle. After a small time interval Δt, the system is in the state of working at time $t+\Delta t$. Obviously, this is impossible, because the system will not have two state transitions in a small period of time Δt. Therefore, we have

$$p\{S(t+\Delta t)=0, I(t+\Delta t)=j|N(t)=1, I(t)=j\}=0.$$

Item $p\{S(t+\Delta t)=0, I(t+\Delta t)=j|N(t)=2, I(t)=j\}$ indicates that the system is being repaired at time t. At time $t+\Delta t$, the system is in the working state. We can see that the system should be in different cycles at time t and time $t+\Delta t$. So we can obtain the following:

$$p\{S(t+\Delta t)=0, I(t+\Delta t)=j|N(t)=2, I(t)=j\}=0.$$

Item $p\{S(t+\Delta t)=0, I(t+\Delta t)=j|N(t)=2, I(t)=j-1\}$ means that at time t the repairman is repairing and in the $(j-1)$th cycle, and at time $t+\Delta t$ the repairman is working and is in the jth cycle. During this period of time Δt, the repairman completed the repair. Thus, we have

$$p\{S(t+\Delta t)=0, I(t+\Delta t)=j|N(t)=2, I(t)=j-1\}=\int_0^\infty \mu(y)p_{2,j-1}(t,y)\mathrm{d}y\Delta t+o(\Delta t) \quad j\geq 2.$$

Item $p\{S(t+\Delta t)=0, I(t+\Delta t)=j|N(t)=3, I(t)=j\}$ means that at time t the repair equipment is being replaced and the system is waiting for repair, while at time $t+\Delta t$ the system is in the working state. It is impossible. So we obtain the following result:

$$p\{S(t+\Delta t)=0, I(t+\Delta t)=j|N(t)=3, I(t)=j\}=0.$$

Now, we can further obtain the following differential equation through the above analysis.

$$p_{01}(t+\Delta t)=(1-\lambda\Delta t)p_{01}(t), \quad j=1,$$

$$p_{0j}(t+\Delta t)=(1-a^{j-1}\lambda\Delta t)p_{0j}(t)+\int_0^\infty \mu(y)p_{2,j-1}(t,y)\mathrm{d}y\Delta t+o(\Delta t), \quad j>1$$

and

$$\frac{\mathrm{d}p_{0,1}(t)}{\mathrm{d}t}=-\lambda p_{0,1}(t), \quad j=1,$$

$$\frac{\mathrm{d}p_{0,j}(t)}{\mathrm{d}t}=-a^{j-1}\lambda p_{0,j}(t)+\int_0^\infty \mu(y)p_{2,j-1}(t,y)\mathrm{d}y, \quad j>1.$$

(2) $p_{1j}(t+\Delta t)=p\{S(t+\Delta t)=1, I(t+\Delta t)=j\}=\sum_{i=0}^{3}p\{S(t+\Delta t)=1, I(t+\Delta t)=j, N(t)=i\}$ $=\sum_{i=0}^{3}p\{S(t+\Delta t)=1, I(t+\Delta t)=j|N(t)=i, I(t)=j\}p\{N(t)=i, I(t)=j\}, (j\geq 1)$.

Item $p\{S(t+\Delta t)=1, I(t+\Delta t)=j|N(t)=0, I(t)=j\}$ indicates that the system is in working state and jth cycle at time t, and is in the repair delay state at time $t+\Delta t$. That is to say, the system component fails in time interval Δt, then we have

$$p\{S(t+\Delta t)=1, I(t+\Delta t)=j|N(t)=0, I(t)=j\}=a^{j-1}\lambda\Delta t p_{0,j}(t)+o(\Delta t).$$

Item $p\{S(t+\Delta t)=1, I(t+\Delta t)=j|N(t)=1, I(t)=j\}$ indicates that the system is in the state of delayed repair at time t and is also in the state of delayed repair at time $t+\Delta t$. We can see that the delayed repair is not finished within the time interval Δt, so we have

$p\{S(t+\Delta t)=1, I(t+\Delta t)=j|N(t)=1, I(t)=j\}=(1-\theta\Delta t)p_{1,j}(t)+o(\Delta t)$.

From the analysis of situation (1), and we can easily know the following two terms:

$$p\{S(t+\Delta t)=1, I(t+\Delta t)=j|N(t)=2, I(t)=j\}=0$$

and

$$p\{S(t+\Delta t)=1, I(t+\Delta t)=j|N(t)=3, I(t)=j\}=0.$$

Again, we have the following differential equations:

$$\frac{\mathrm{d}p_{1,j}(t)}{\mathrm{d}t}=a^{j-1}\lambda p_{0,j}(t)-\theta p_{1,j}(t), \quad j\geq 1.$$

(3) In this part, we divide it into the following two cases to analyze.

One is that $S(t)=1$, this means that the system is in a delayed repair state at time t, if the delayed repair of the system has just been completed at time $t+\Delta t$, and the system repair has not started yet. That is, the repair time is zero at $t+\Delta t$. The probability of this state is also called the boundary value. So it has

$$p_{2,j}(t,0)=\theta p_{1,j}(t), \quad (j\geq 1).$$

The other case is that the system is being repaired at time $t+\Delta t$. When the repair time $y>0$, let $\Delta t<y$, we have

$$\begin{aligned}p_{2j}(t+\Delta t, y+\Delta t)\mathrm{d}y &= p\{S(t+\Delta t)=2, y+\Delta t\leq Y(t+\Delta t)\leq y+\Delta t+\mathrm{d}y, I(t+\Delta t)=j\}\\ &=\sum_{i=0}^{3}p\{S(t+\Delta t)=2, y+\Delta t\leq Y(t+\Delta t)\leq y+\Delta t+\mathrm{d}y, I(t+\Delta t)=j, S(t)=i\}, \quad (j\geq 1).\end{aligned}$$

If $S(t)=0$ and $S(t)=1$, it is obviously impossible, because the system is either working or the repair is delayed at time t, and the system is being repaired at time $t+\Delta t$. Therefore, the repair time cannot be greater than $y+\Delta t$ at time $t+\Delta t$, so there is

$$p\{S(t+\Delta t)=2, y+\Delta t\leq Y(t+\Delta t)\leq y+\Delta t+\mathrm{d}y, I(t+\Delta t)=j, S(t)=0\}=0.$$

$$p\{S(t+\Delta t)=2, y+\Delta t\leq Y(t+\Delta t)\leq y+\Delta t+\mathrm{d}y, I(t+\Delta t)=j, S(t)=1\}=0.$$

If $S(t)=2$ and $S(t)=3$, which means that the system is being repaired or the repair equipment is being replaced and the system is waiting to be repaired at time t. At time $t+\Delta t$ the system is under repair. That is to say, in the period Δt, either the system repair is not completed and the equipment is not damaged, or the replacement of the equipment is completed. So we also have

$$\begin{aligned}&p\{S(t+\Delta t)=2, y+\Delta t\leq Y(t+\Delta t)\leq y+\Delta t+\mathrm{d}y, I(t+\Delta t)=j, S(t)=2\}\\ &\qquad=(1-\mu(y)\Delta t-\alpha\Delta t)p_{2,j}(t,y)+0(\Delta t),\end{aligned}$$

$$p\{S(t+\Delta t)=2, y+\Delta t\leq Y(t+\Delta t)\leq y+\Delta t+\mathrm{d}y, I(t+\Delta t)=j, S(t)=3\}=\int_0^\infty \beta(z)p_{3,j}(t,y,z)\mathrm{d}z\Delta t+o(\Delta t)$$

Hence, we have the partial differential equations

$$\frac{\partial p_{2,j}(t,y)}{\partial t}+\frac{\partial p_{2,j}(t,y)}{\partial y}=-(\alpha+\mu(y))p_{2,j}(t,y)+\int_0^\infty \beta(z)p_{3,j}(t,y,z)\mathrm{d}z, \quad (j\geq 1).$$

(4) In this part, we also discuss the following two cases.

One is that $S(t)=2$, this means that the system is being repaired at time t. If the repair equipment fails and has not been replaced yet at time $t+\Delta t$. That is, the replacement time of equipment is zero at $t+\Delta t$. So we have

$$p_{3,j}(t,y,0)=\alpha p_{2,j}(t,y), \quad (j\geq 1).$$

The other case is that the system repair equipment is being replaced at time $t+\Delta t$. When the replacement time $z>0$, let $\Delta t<z$, we have

$$\begin{aligned}p_{3j}(t+\Delta t, z+\Delta t, y)\mathrm{d}z &= p\{S(t+\Delta t)=3, z+\Delta t\leq Z(t+\Delta t)\leq z+\Delta t+\mathrm{d}z, I(t+\Delta t)=j, Y(t+\Delta t)=y\}\\ &=\sum_{i=0}^{3}p\{S(t+\Delta t)=3, z+\Delta t\leq Z(t+\Delta t)\leq z+\Delta t+\mathrm{d}z, I(t+\Delta t)=j,\\ &\qquad Y(t+\Delta t)=y, S(t)=i\}, \quad (j\geq 1).\end{aligned}$$

By a similar analysis to case (3), we can easily obtain the following formula.

$$p\{S(t+\Delta t)=3, z+\Delta t \le Z(t+\Delta t) \le z+\Delta t+\mathrm{d}z, I(t+\Delta t)=j, Y(t+\Delta t)=y, S(t)=0\}=0$$

$$p\{S(t+\Delta t)=3, z+\Delta t \le Z(t+\Delta t) \le z+\Delta t+\mathrm{d}z, I(t+\Delta t)=j, Y(t+\Delta t)=y, S(t)=1\}=0$$

$$p\{S(t+\Delta t)=3, z+\Delta t \le Z(t+\Delta t) \le z+\Delta t+\mathrm{d}z, I(t+\Delta t)=j, Y(t+\Delta t)=y, S(t)=2\}=0$$

and

$$\begin{aligned} &p\{S(t+\Delta t)=3, z+\Delta t \le Z(t+\Delta t) \le z+\Delta t+\mathrm{d}z, I(t+\Delta t)=j, Y(t+\Delta t)=y, S(t)=3\} \\ &\quad = (1-\beta(z)\Delta t)p_{3,j}(t,z,y)+0(\Delta t). \end{aligned}$$

Therefore, the following differential equation holds

$$\frac{\partial p_{3,j}(t,z,y)}{\partial t}+\frac{\partial p_{3,j}(t,z,y)}{\partial z}=-\beta(z)p_{3,j}(t,z,y).$$

Thus, the dynamic behavior of the system is governed by the partial differential equations
when $j=1$

$$\begin{cases} \dfrac{\mathrm{d}p_{0,1}(t)}{\mathrm{d}t}=-\lambda p_{0,1}(t) \\ \dfrac{\mathrm{d}p_{1,1}(t)}{\mathrm{d}t}=\lambda p_{0,1}(t)-\theta p_{1,1}(t) \\ \dfrac{\partial p_{2,1}(t,y)}{\partial t}+\dfrac{\partial p_{2,1}(t,y)}{\partial y}=-(\alpha+\mu(y))p_{2,1}(t,y)+\displaystyle\int_0^\infty \beta(z)p_{3,1}(t,z,y)\mathrm{d}z \\ \dfrac{\partial p_{3,1}(t,z,y)}{\partial t}+\dfrac{\partial p_{3,1}(t,z,y)}{\partial z}=-\beta(z)p_{3,1}(t,z,y) \\ p_{2,1}(t,0)=\theta p_{1,1}(t) \\ p_{3,1}(t,0,y)=\alpha p_{2,1}(t,y), \end{cases} \tag{2.1}$$

when $j>1$

$$\begin{cases} \dfrac{\mathrm{d}p_{0,j}(t)}{\mathrm{d}t}=-a^{j-1}\lambda p_{0,j}(t)+\displaystyle\int_0^\infty \mu(y)p_{2,j-1}(t,y)\mathrm{d}y \\ \dfrac{\mathrm{d}p_{1,j}(t)}{\mathrm{d}t}=a^{j-1}\lambda p_{0,j}(t)-\theta p_{1,j}(t) \\ \dfrac{\partial p_{2,j}(t,y)}{\partial t}+\dfrac{\partial p_{2,j}(t,y)}{\partial y}=-(\alpha+\mu(y))p_{2,j}(t,y)+\displaystyle\int_0^\infty \beta(z)p_{3,j}(t,z,y)\mathrm{d}z \\ \dfrac{\partial p_{3,j}(t,z,y)}{\partial t}+\dfrac{\partial p_{3,j}(t,z,y)}{\partial z}=-\beta(z)p_{3,j}(t,z,y) \\ p_{2,j}(t,0)=\theta p_{1,j}(t) \\ p_{3,j}(t,0,y)=\alpha p_{2,j}(t,y), \end{cases} \tag{2.2}$$

with the initial conditions

$$\begin{cases} p_{0,1}(0)=1, p_{0,j}(0)=0 \quad (j\ge 2) \\ p_{1,j}(0)=p_{2,j}(0,y)=p_{3,j}(0,z,y)=0, y\in(0,\infty) \quad (j\ge 1). \end{cases} \tag{2.3}$$

Furthermore, from (2.2) we can obtain the sum of all state probabilities of the system is equal to 1, which satisfies the normalization condition:

$$\sum_{j=1}^{\infty}p_{0,j}(t)+\sum_{j=1}^{\infty}p_{1,j}(t)+\sum_{j=1}^{\infty}\int_0^\infty p_{2,j}(t,y)\mathrm{d}y+\sum_{j=1}^{\infty}\int_0^\infty\int_0^\infty p_{3,j}(t,z,y)\mathrm{d}z\mathrm{d}y=1, \quad (t\ge 0).$$

3 Availability of the system

In this section, we mainly discuss the steady-state availability of the system. The availability of the system at time t is the probability of the system in working state, which is defined by

$$A(t) = p\{S(t) = 0\} = p_0(t) = \sum_{j=1}^{\infty} p_{0,j}(t).$$

The Laplace transform of $A(t)$ is given by

$$A^*(s) = \sum_{j=1}^{\infty} p_{0,j}^*(s) = p_{0,1}^*(s) + \sum_{j=2}^{\infty} p_{0,j}^*(s).$$

According to the Tauberian theorem, the steady-state availability of the system is given by

$$A = \lim_{t\to\infty} A(t) = \lim_{t\to\infty} sA^*(s).$$

Therefore, we can obtain the steady-state availability of the system by the Laplace transform.

First, denote the Laplace transform of the functions by

$$p_{ij}^*(s) = \int_0^{\infty} p_{ij}(t)e^{-st}\mathrm{d}t, \quad i = 0, 1, \ j = 1, 2, \ldots,$$

$$p_{2j}^*(s, y) = \int_0^{\infty} p_{2j}(t, y)e^{-st}\mathrm{d}t, \quad j = 1, 2, \ldots.,$$

$$p_{3j}^*(s, y, z) = \int_0^{\infty} p_{3j}(t, y, z)e^{-st}\mathrm{d}t, \quad j = 1, 2, \ldots.$$

Next, taking the Laplace transform of equations (2.1) and (2.2) and using the initial conditions (2.3), we obtain the following equations:

$$\begin{cases}
sp_{0,1}^*(s) + \lambda p_{0,1}^*(s) = 1 \\
sp_{1,1}^*(s) + \theta p_{1,1}^*(s) = \lambda p_{0,1}^*(s) \\
sp_{2,1}^*(s, y) + \dfrac{\partial}{\partial y} p_{2,1}^*(s, y) + \mu(y)p_{2,1}^*(s, y) + \alpha p_{2,1}^*(s, y) = \displaystyle\int_0^{\infty} \beta(z)p_{3,1}^*(s, y, z)\mathrm{d}z \\
sp_{3,1}^*(s, y, z) + \dfrac{\partial}{\partial y} p_{3,1}^*(s, y, z) + \beta(z)p_{3,1}^*(s, y, z) = 0 \\
sp_{0,2}^*(s) + a\lambda p_{0,2}^*(s) = \displaystyle\int_0^{\infty} \mu(y)p_{2,1}^*(s, y)\mathrm{d}y \\
sp_{1,2}^*(s) + \theta p_{1,2}^*(s) = a\lambda p_{0,2}^*(s) \\
sp_{2,2}^*(s, y) + \dfrac{\partial}{\partial y} p_{2,2}^*(s, y) + \mu(y)p_{2,2}^*(s, y) + \alpha p_{2,2}^*(s, y) = \displaystyle\int_0^{\infty} \beta(z)p_{3,2}^*(s, y, z)\mathrm{d}z \\
sp_{3,2}^*(s, y, z) + \dfrac{\partial}{\partial y} p_{3,2}^*(s, y, z) + \beta(z)p_{3,2}^*(s, y, z) = 0 \\
\ldots\ldots \\
sp_{0,k}^*(s) + a^{k-1}\lambda p_{0,k}^*(s) = \displaystyle\int_0^{\infty} \mu(y)p_{2,k-1}^*(s, y)\mathrm{d}y \\
sp_{1,k}^*(s) + \theta p_{1,k}^*(s) = a^{k-1}\lambda p_{0,k}^*(s) \\
sp_{2,j}^*(s, y) + \dfrac{\partial}{\partial y} p_{2,j}^*(s, y) + \mu(y)p_{2,j}^*(s, y) + \alpha p_{2,j}^*(s, y) = \displaystyle\int_0^{\infty} \beta(z)p_{3,j}^*(s, y, z)\mathrm{d}z \\
sp_{3,j}^*(s, y, z) + \dfrac{\partial}{\partial y} p_{3,j}^*(s, y, z) + \beta(z)p_{3,j}^*(s, y, z) = 0 \\
\ldots\ldots
\end{cases} \tag{3.1}$$

solving the above equations, we have

$$\begin{cases} p_{0,1}^*(s) = \dfrac{1}{s+\lambda} \\ p_{1,1}^*(s) = \dfrac{\lambda}{s+\theta} p_{0,1}^*(s) \\ p_{2,1}^*(s,y) = \dfrac{\lambda\theta}{s+\theta} p_{0,1}^*(s) e^{-\int_0^y (s+\mu(\tau)+\alpha s v^*(s))\mathrm{d}\tau} \\ p_{3,1}^*(s,y,z) = \alpha p_{2,1}^*(s,y) e^{-\int_0^z (s+\beta(\tau))\mathrm{d}\tau} \\ p_{0,2}^*(s) = \dfrac{\lambda\theta}{(s+\theta)(s+\lambda a)} p_{0,1}^*(s)[1-(s+\alpha s v^*(s))G^*(s+\alpha s v^*(s))] \\ p_{1,2}^*(s) = \dfrac{\lambda a}{s+\theta} p_{0,2}^*(s) \\ p_{2,2}^*(s,y) = \dfrac{\lambda\theta a}{s+\theta} p_{0,2}^*(s) e^{-\int_0^y (s+\mu(\tau)+\alpha s v^*(s))\mathrm{d}\tau} \\ p_{3,2}^*(s,y,z) = \alpha p_{2,2}^*(s,y) e^{-\int_0^z (s+\beta(\tau))\mathrm{d}\tau} \\ \cdots \\ p_{0,j}^*(s) = \dfrac{\lambda\theta a^{j-2}}{(s+\theta)(s+\lambda a^{j-1})} p_{0,j-1}^*(s)[1-(s+\alpha s v^*(s))G^*(s+\alpha s v^*(s))] \\ p_{1,j}^*(s) = \dfrac{\lambda a^{j-1}}{s+\theta} p_{0,j}^*(s) \\ p_{2,j}^*(s,y) = \dfrac{\lambda\theta a^{j-1}}{s+\theta} p_{0,j}^*(s) e^{-\int_0^y (s+\mu(\tau)+\alpha s v^*(s))\mathrm{d}\tau} \\ p_{3,j}^*(s,y,z) = \alpha p_{2,j}^*(s,y) e^{-\int_0^z (s+\beta(\tau))\mathrm{d}\tau} \\ \cdots, \end{cases} \tag{3.2}$$

where $v^*(s) = \int_0^\infty e^{-\int_0^z (s+\beta(\tau))\mathrm{d}\tau} \mathrm{d}z$, $G^*(s+\alpha s v^*(s)) = \int_0^\infty e^{-\int_0^y (s+\alpha s v^*(s)+\mu(\tau))\mathrm{d}\tau} \mathrm{d}y$.

From the Laplace transform of (3.2), we can obtain

$$\begin{aligned} A^*(s) &= p_{0,1}^*(s) + \sum_{j=2}^{\infty} p_{0,j}^*(s) \\ &= \frac{1}{s+\lambda} + \sum_{j=2}^{\infty} \frac{\lambda\theta a^{j-2}}{(s+\theta)(s+\lambda a^{j-1})} p_{0,j-1}^*(s)[1-(s+\alpha s v^*(s))G^*(s+\alpha s v^*(s))] \\ &= \frac{1}{s+\lambda} + \frac{\lambda\theta}{(s+\theta)(s+\lambda a)} \cdot \frac{1}{(s+\lambda)}[1-(s+\alpha s v^*(s))G^*(s+\alpha s v^*(s))] \\ &\quad + \frac{\lambda a\theta}{(s+\theta)(s+\lambda a^2)} \cdot \frac{\lambda\theta}{(s+\theta)(s+\lambda a)} \cdot \frac{1}{(s+\lambda)}[1-(s+\alpha s v^*(s))G^*(s+\alpha s v^*(s))]^2 + \cdots \end{aligned} \tag{3.3}$$

and hence

$$\lim_{s\to 0} A^*(s) = \left(\frac{1}{\lambda} + \frac{1}{\lambda a} + \frac{1}{\lambda a^2} + \cdots\right) = \frac{1}{\lambda(a-1)} \quad (a > 1).$$

Therefore, the steady-state availability of the system is given by

$$A = \lim_{t\to\infty} A(t) = \lim_{t\to\infty} sA^*(s) = 0.$$

This shows that when $a > 1$, the availability of the system will tend to zero, which means that the deteriorating system will become completely unavailable after a long period of operation. In this case, the cost of the system may be higher, because the cost of repairing the degraded system will be more than the reward of system after multiple repairs. So, from the economic point of view, we have to study a replacement policy N based on the failed times of the system under consideration of the system cost. In the next section, we give the explicit expression of the system average cost $C(N)$ and obtain the optimal replacement policy N^* by minimizing the average cost rate $C(N^*)$. That is to say, when the failure number of the system reaches N^*, the system will be replaced. Furthermore, the extended degenerate system is proposed by assuming that the system is not always successive degenerative, and the optimal replacement policy is studied.

4 Average cost rate under policy *N*

In this section, we study a replacement policy N based on the failed times of the system. First, on the basis of the previous assumptions, we add a few hypothetical conditions.

Assumption 7. A replacement policy N based on the failed times of the system is used. It is to say, when the failure number of system reaches the optimal replacement times N^*, the system will be replaced by a new and identical one. The replacement time is negligible.

Assumption 8. The repair cost of system, the reward rate of system working normally, the cost of system replacement and the cost of replacing system repair equipment are denoted by c_r, c_w, r and c_m, respectively.

Let τ_1 be the first replacement time of the system, and τ_n be the time between the $(n-1)$th replacement and the nth replacement of the system under policy N. Obviously, $\{\tau_1, \tau_2, \ldots\}$ forms a renewal process. And the time interval between two consecutive replacements is called a renew cycle.

Next, we will give the explicit expression of the system average cost rate $C(N)$ under the policy N. Since the adjacent replacement time $\{\tau_1, \tau_2, \ldots\}$ forms a renewal process, so according to the renewal reward theorem, we have

$$C(N) = \frac{\text{the expected cost incurred in a renewal cycle}}{\text{the expected length of a renewal cycle}} = \frac{E[D]}{E[W]}, \tag{4.1}$$

where the length and the cost of a renewal cycle are denoted by W and D. According to the running process of system, we can know the system only be in four states: working state, delayed repair state, being repaired state and waiting for repair state. Therefore, the expression of the length W is given by

$$W = \sum_{n=1}^{N} X_n + \sum_{n=1}^{N-1} W_n + \sum_{n=1}^{N-1} Y_n + \sum_{i=1}^{N-1}\sum_{n=0}^{p_i} V_n$$

and the expected length of a renewal cycle is as follows:

$$E(W) = \sum_{n=1}^{N} E(X_n) + \sum_{n=1}^{N-1} E(W_n) + \sum_{n=1}^{N-1} E(Y_n) + \sum_{i=1}^{N-1}\sum_{n=0}^{p_i} E(V_n), \tag{4.2}$$

where p_i represents the number of failures of system repair equipment within a repair cycle. According to literature [21], the probability distribution of p_i is that

$$P\{p_i = m\} = \int_0^\infty e^{-\alpha x}\frac{(\alpha m)^m}{m!}\mathrm{d}G_i(x).$$

Then, the average number of failures for repairing equipment is that

$$E(p_i) = \sum_{m=1}^{\infty} mP\{p_i = m\} = \frac{\alpha}{\mu}.$$

Therefore, the waiting time for repair of the system is that

$$E\left(\sum_{n=0}^{p_i} V_n\right) = \sum_{m=0}^{\infty}\left[\sum_{n=0}^{m} E(V_n)\right]P\{p_i = m\} = \sum_{m=0}^{\infty}\left[\sum_{n=0}^{m}\frac{1}{\beta}\right]P\{p_i = m\} = \frac{\alpha}{\mu\beta}. \tag{4.3}$$

In addition, we can see from the system hypothesis

$$EX_n = \frac{1}{\lambda a^{n-1}}, \quad EW_n = \frac{1}{\theta}, \quad EY_n = \frac{1}{\mu}. \tag{4.4}$$

Note that, substituting (4.3) and (4.4) into (4.2), we obtain the expected length of a renewal cycle of the system

$$E(W) = \sum_{n=1}^{N}\frac{1}{\lambda a^{n-1}} + \frac{N-1}{\theta} + \frac{N-1}{\mu} + \frac{\alpha(N-1)}{\mu\beta}. \tag{4.5}$$

Furthermore, we consider the cost of the system in a renewal cycle, denote it by D.

From the above analysis, it is easy to know the following results:

$$\begin{aligned} E(D) &= c_r\sum_{n=1}^{N-1}E(Y_n) + c_f\sum_{i=1}^{N-1}\sum_{n=0}^{p_i}E(V_n) + r - c_w\sum_{n=1}^{N}E(X_n) \\ &= c_r\frac{N-1}{\mu} + c_f\frac{\alpha(N-1)}{\mu\beta} + r - c_w\sum_{n=1}^{N}\frac{1}{\lambda a^{n-1}}. \end{aligned} \tag{4.6}$$

Finally, the explicit expression of the system average cost rate $C(N)$ under the policy N is obtained by substituting equations (4.5) and (4.6) into equation (4.1)

$$C(N) = \frac{E(D)}{E(W)} = \frac{c_r\frac{N-1}{\mu} + c_f\frac{\alpha(N-1)}{\mu\beta} + r - c_w\sum_{n=1}^{N}\frac{1}{\lambda a^{n-1}}}{\sum_{n=1}^{N}\frac{1}{\lambda a^{n-1}} + \frac{N-1}{\theta} + \frac{N-1}{\mu} + \frac{\alpha(N-1)}{\mu\beta}}. \tag{4.7}$$

In the above discussion, we always assume that the system always degenerate after repair, so that the optimal replacement number N^* of the system will be relatively small. But this is not always the case in practice; this is because the system failure is divided into different types, such as serious faults and minor faults. If there is a slight fault in the system, the system will be as good as new after appropriate repair and the system will not deteriorate. And then the optimal replacement times of the system will not be too small. It is also more realistic. Therefore, we further propose an extended degenerate system and study the optimal replacement policy of the system.

For ease of understanding, we give the definition of the extended geometric process and new assumptions.

Definition 1. Assume that $\{\xi_n, n = 1, 2, \ldots\}$ is a sequence of independent non-negative random variables. If the distribution function of ξ_{n+1} is $H_{n+1}(t) = pH_n(t) + qH_n(at)$, $(n = 1, 2, \ldots)$, where a, p, q are all positive constants, and $p + q = 1$, then $\{\xi_n, n = 1, 2, \ldots\}$ is called an extended geometrical process and p is called the extended factor.

Obviously, if $p = 0$, then $H_n(t) = H(a^{n-1}t)(n = 1, 2, \ldots)$, that is, the extended geometrical process will become a geometrical process. It is to say the case where the system after repair is always successive degenerative. Therefore, the degenerate system discussed above is a special case of extended degenerate system. If $p = 1$, then $H_{n+1}(t) = H_n(t)$, $(n = 1, 2, \ldots)$, that is, the extended geometrical process will become a renewal process.

Assumption 2'. Assume that the system is not degenerative at each repair with probability p and is geometrically degenerative with probability $1 - p$.

Let X_n, Y_n be, respectively, the working time and the repair time of the system in the jth cycle, $j = 1, 2, \ldots$. We define the cumulative distribution functions of X_n and Y_n as $F_n(t)$ and $G(t)$, respectively. Assume that the system satisfies extended geometrical process, and they are given by:

$$F_n(t) = pF_{n-1}(t) + qF_{n-1}(at)$$

and

$$G(t) = \int_0^t g(y)\mathrm{d}y = 1 - \exp\left(\int_0^t \mu(y)\mathrm{d}y\right),$$

where the expected value $\int_0^{\infty} tg(t)\mathrm{d}t = \frac{1}{\mu}$ and $t \geq 0$, $a \geq 1$, $0 \leq p \leq 1$ and $p + q = 1$ for $n = 1, 2, \ldots$.

Similarly, we can obtain the explicit expression of the system average cost rate $C_1(N)$ under the policy N

$$C_1(N) = \frac{E(D)}{E(W)} = \frac{C_r \frac{N-1}{\mu} + C_f \frac{\alpha(N-1)}{\mu\beta} + r - C_w \sum_{n=1}^{N} \frac{1}{\lambda}\left(p + \frac{q}{a}\right)^{n-1}}{\sum_{n=1}^{N} \frac{1}{\lambda}\left(p + \frac{q}{a}\right)^{n-1} + \frac{N-1}{\theta} + \frac{N-1}{\mu} + \frac{\alpha(N-1)}{\mu\beta}}. \tag{4.8}$$

When $\alpha = 0$, it means that the repair equipment of the system will not fail during the repair process and does not need to be replaced. In this case, the explicit expression of the system average cost rate $C_2(N)$ under the policy N is as follows:

$$C_2(N) = \frac{E(D)}{E(W)} = \frac{C_r \frac{N-1}{\mu} + r - C_w \sum_{n=1}^{N} \frac{1}{\lambda}\left(p + \frac{q}{a}\right)^{n-1}}{\sum_{n=1}^{N} \frac{1}{\lambda}\left(p + \frac{q}{a}\right)^{n-1} + \frac{N-1}{\theta} + \frac{N-1}{\mu}}. \tag{4.9}$$

The next major goal is to find the optimal replacement policy N^* to minimize $C(N)$. We mainly use numerical simulation to verify the existence and uniqueness of the optimal replacement policy N^*.

5 Numerical analysis

5.1 Optimal replacement policy

In this section, we provide numerical analysis to illustrate the existence and uniqueness of the optimal replacement policy N^*. It makes $C(N)$ minimal.

We give some parameter values of the system as follows:

$$\begin{aligned} &a = 1.15, \quad \lambda = 0.3, \quad \mu = 0.3, \quad \alpha = 0.06, \\ &\beta = 0.2, \quad \theta = 0.4, \quad p = 0.4, \quad q = 0.6, \\ &C_r = 20, \quad C_f = 10, \quad C_w = 300, \quad r = 2{,}500. \end{aligned}$$

Substitute these parameter values into equations (4.7) ($p = 0$), (4.8) and (4.9) ($\alpha = 0$). Then the relationship between the number of failures N of system and the average cost $C(N)$, ($C_1(N)$, $C_2(N)$) of the system is shown in Table 1, Figures 1 and 2. It is also easy to see the optimal replacement number N^*, (N_1^*, N_2^*) and the minimum cost $C(N^*)$ from the table and figure.

From Table 1 and Figure 1 we can obtain a lot of useful information as follows:

(a) In Table 1, it is easy to find that the optimal replacement number of the system is $N^* = 8$, and the corresponding long run expected cost per unit time is $C(8) = -32.7$. In other words, when the number of failures of system reaches $N^* = 8$, we should replace a new system, and the system cost is the least at this time. From Figure 1, we can see the change trend of $C(N)$ with the number of failures N. It is obvious that $C(N)$ is decreasing when $N < 8$ and increasing when $N > 8$. Thus, $C(8)$ is the minimum value of the expected cost per unit time and the optimal replacement number $N^* = 8$ of the system is unique. Similarly, when $p = 0$ or $\alpha = 0$, the optimal replacement times of the system are $N_1^* = 10$ and $N_2^* = 10$, respectively. We can see from the table and figure that they are all unique.

(b) In Figure 1, comparing the numerical results of $C(N)$ and $C_2(N)$, we can see that the average cost of a system with reliable repair equipment is lower than that of a system with replaceable repair equipment. Because a reliable system for repairing equipment is a perfect system, the repair equipment will not fail, so there will be no cost of repair and replacement. Therefore, the average cost of a reliable system for repairing equipment is lower than a system for repairing replaceable equipment.

Table 1: The system average cost

N	C(N)	$C_1(N)$	$C_2(N)$	N	C(N)	$C_1(N)$	$C_2(N)$	N	C(N)	$C_1(N)$	$C_2(N)$
1	450	450	450	13	−29	−44.9	−51.6	25	−16.4	−33.7	−39.6
2	54.1	49.5	52.7	14	−27.9	−44.1	−50.8	26	−15.6	−32.8	−38.6
3	1.2	−5.2	−6.6	15	−26.7	−43.3	−50	27	−14.8	−31.9	−37.6
4	−17.6	−25.5	−29.1	16	−25.5	−42.4	−49	28	−14	−31	−36.6
5	−26.1	−35.3	−40.2	17	−24.4	−41.4	−48	29	−13.3	−30.1	−35.7
6	−30.2	−40.6	−46.2	18	−23.2	−40.5	−47	30	−12.7	−29.3	−34.8
7	−32	−43.6	−49.6	19	−22.1	−39.5	−46	31	−12	−28.5	−33.9
8	−32.7	−45.2	−51.5	20	−21.1	−38.5	−44.9	32	−11.4	−27.7	−33
9	−32.5	−45.9	−52.4	21	−20.1	−37.5	−43.8	33	−10.8	−26.9	−32.1
10	−32	−46.1	−52.7	22	−19.1	−36.5	−42.7	34	−10.2	−26.2	−31.3
11	−31.1	−45.9	−52.6	23	−18.1	−35.6	−41.7	35	−9.7	−25.4	−30.5
12	−30.1	−45.5	−52.2	24	−17.2	−34.6	−40.6	36	−9.2	−24.7	−29.7

(c) Since p represents the probability that the system will not be degenerative in the nth repair cycle. So when $p = 0$, the extended geometrical process will become a geometrical process. According to Figure 2 and Table 1, we can clearly see that the average cost of the degenerate system that follows an extended geometric repair process is lower than the degenerate system that satisfies geometrical process. This result is consistent with the practical situations.

5.2 Parametric sensitivity analysis

This subsection mainly studies the influence of system parameters on the optimal replacement policy N^* and the optimal average cost rate $C(N^*)$.

Now let us consider the influence of parameter (α, β) on N^* and $C(N^*)$. We take

$$a = 1.15,\quad \lambda = 0.3,\quad \mu = 0.3,\quad \beta = 0.2,$$
$$\theta = 0.4,\quad p = 0.4,\quad q = 0.6,\quad C_r = 20,$$
$$C_f = 10,\quad C_w = 300,\quad r = 2{,}500.$$

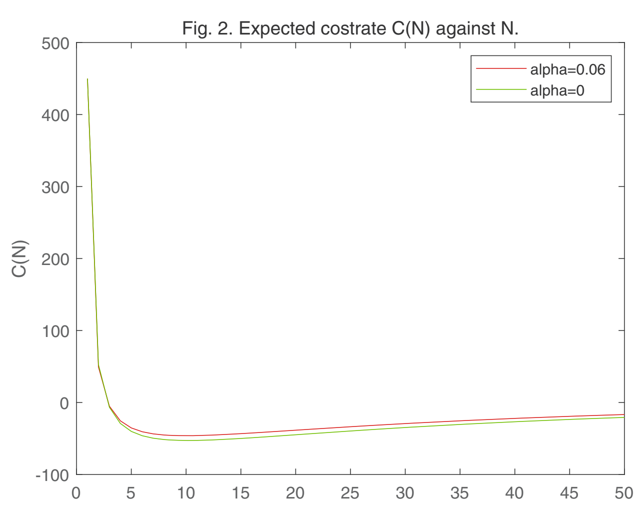

Figure 1: Change trend of $C(N)$ with the number of failures N.

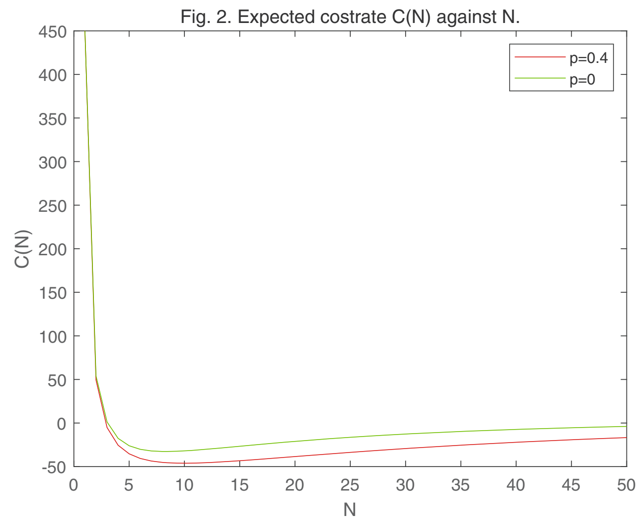

Figure 2: Comparison of the numerical results of $C(N)$ and $C_2(N)$.

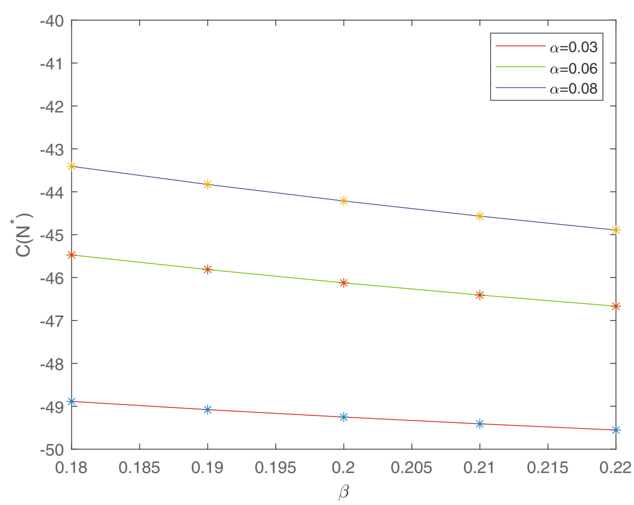

Figure 3: The effect of parameter $(\alpha, \beta)N^*$ and $C(N^*)$.

Herein α take 0.03, 0.06, 0.08, respectively, and the parameter β is valued in [0.18, 0.22]. We can easily obtain Figure 3 and Table 2. As can be seen from the figure and table, when the parameter α is fixed, $C(N^*)$ decreases with the increase in β; when β is fixed, $C(N^*)$ increases with the increase in α. But in general, (α, β) has a small effect on the optimal replacement policy N^*.

Table 2: The expected cost

(α, β)	N^*	$C(N^*)$	(α, β)	N^*	$C(N^*)$	(α, β)	N^*	$C(N^*)$
(0.03,0.18)	10	−48.9	(0.06,0.18)	10	−45.5	(0.08,0.18)	10	−43.4
(0.03,0.19)	10	−49.1	(0.06,0.19)	10	−45.8	(0.08,0.19)	10	−43.8
(0.03,0.20)	10	−49.3	(0.06,0.20)	10	−46.1	(0.08,0.20)	10	−44.2
(0.03,0.21)	10	−49.4	(0.06,0.21)	10	−46.4	(0.08,0.21)	10	−44.6
(0.03,0.22)	10	−49.6	(0.06,0.22)	10	−46.7	(0.08,0.22)	10	−44.9

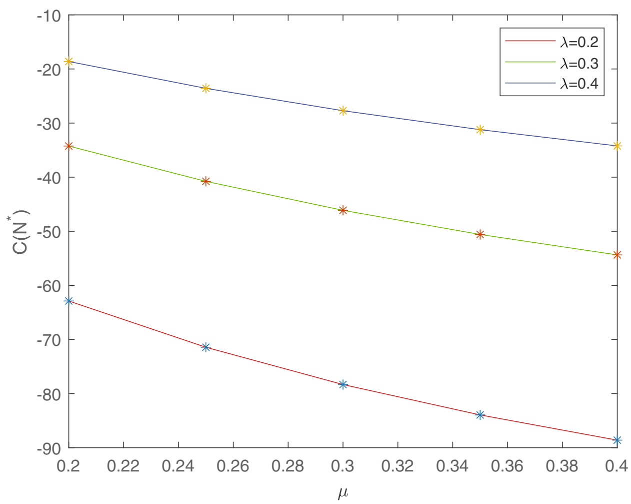

Figure 4: The effect of parameter $(\lambda, \mu)N^*$ and $C(N^*)$.

Table 3: The expected cost

(λ, μ)	N^*	$C(N^*)$	(λ, μ)	N^*	$C(N^*)$	(λ, μ)	N^*	$C(N^*)$
(0.2,0.20)	7	−62.9	(0.3,0.20)	10	−34.2	(0.4,0.20)	12	−18.6
(0.2,0.25)	8	−71.4	(0.3,0.25)	10	−40.8	(0.4,0.25)	12	−23.6
(0.2,0.30)	8	−78.3	(0.3,0.30)	10	−46.1	(0.4,0.30)	12	−27.7
(0.2,0.35)	8	−83.9	(0.3,0.35)	10	−50.6	(0.4,0.35)	12	−31.2
(0.2,0.40)	8	−88.6	(0.3,0.40)	10	−54.4	(0.4,0.40)	12	−34.2

And then we will discuss the effect of parameter (λ, μ) on N^* and $C(N^*)$. Similarly, λ take 0.2, 0.3 and 0.4, respectively. The parameter μ is valued in $[0.20, 0.40]$ and the other parameters are the same as above. Furthermore, Figure 4 and Table 3 are obtained. The figure and table show that when the parameter λ is fixed, $C(N^*)$ decreases with the increase in μ and N^* increases with the increase in μ; when μ is fixed, $C(N^*)$ increases with the increase in λ and N^* increases with the increase in μ. As a result, (λ, μ) has great influence on the optimal average cost rate $C(N^*)$.

6 Conclusion

In this article, we mainly study a single component deteriorating system with one repairman and the repair equipment that can be replaced. The repair time follows the general distribution. In this article, we establish the partial differential equation of the system, and then the steady-state availability of the system is discussed by Laplace transformation. We found that the availability of the system tends to zero, that is, the deteriorating system will become completely unavailable after a long period of operation. Therefore, in order to consider the system cost, we study a replacement policy N based on the failed times of the system and further discuss the extended degenerate system. At last, the numerical analysis is given to illustrate the theoretical results. We can see that our results are realistic.

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