Research Article

Zenonas Navickas, Inga Timofejeva, Tadas Telksnys*, Romas Marcinkevicius, and Minvydas Ragulskis

Construction of special soliton solutions to the stochastic Riccati equation

https://doi.org/10.1515/math-2022-0051
received December 19, 2020; accepted April 8, 2022

Abstract: A scheme for the analytical stochastization of ordinary differential equations (ODEs) is presented in this article. Using Itô calculus, an ODE is transformed into a stochastic differential equation (SDE) in such a way that the analytical solutions of the obtained equation can be constructed. Furthermore, the constructed stochastic trajectories remain bounded in the same interval as the deterministic solutions. The proposed approach is in a stark contrast to methods based on the randomization of solution trajectories and is not focused on the analysis of martingales. This article extends the theory of Itô calculus by directly implementing it into analytical schemes for the solution of differential equations based on the generalized operator of differentiation. The efficacy of the presented analytical stochastization techniques is demonstrated by deriving stochastic soliton solutions to the Riccati differential equation. The presented semi-analytical stochastization scheme is relevant for the investigation of the global dynamics of different biological and biomedical processes where the variation interval of the stochastic solution is predetermined by the rationale of the model.

Keywords: stochastic differential equation, analytical solution, Itô calculus

MSC 2020: 60H10, 35R60

1 Introduction and motivation

Ever since the seminal work by Einstein [1], studying the effects of noise on dynamical systems has been an important area of research in physics and other applied sciences. With the introduction of the Wiener process, the derivative of which is white noise, [2] and Itô calculus [3], formal mathematical formulation of stochastic differential equations (SDEs) became possible.

Since then, many areas of applications for SDEs have been the focus of research. One of the first and foremost historical areas is economics and finance. SDEs in this area range from the now-classical Black-Scholes equation for call and put option pricing [4] to more recent works. The price change of assets obeying more sophisticated factors than simple supply and demand is modeled via SDEs in [5].

*** Corresponding author: Tadas Telksnys,** Center for Nonlinear Systems, Kaunas University of Technology, Studentu 50-147, Kaunas LT-51368, Lithuania, e-mail: tadas.telksnys@ktu.lt
Zenonas Navickas: Center for Nonlinear Systems, Kaunas University of Technology, Studentu 50-147, Kaunas LT-51368, Lithuania, e-mail: zenonas.navickas@ktu.lt
Inga Timofejeva: Center for Nonlinear Systems, Kaunas University of Technology, Studentu 50-147, Kaunas LT-51368, Lithuania, e-mail: inga.timofejeva@ktu.edu
Romas Marcinkevicius: Department of Software Engineering, Kaunas University of Technology, Studentu 50-415, Kaunas LT-51368, Lithuania, e-mail: minvydas.ragulskis@ktu.lt
Minvydas Ragulskis: Center for Nonlinear Systems, Kaunas University of Technology, Studentu 50-147, Kaunas LT-51368, Lithuania, e-mail: romas.marcinkevicius@ktu.lt

A stochastic differential game of two insurers investing into the same financial market is posed using SDEs in [6]. A duopolistic competition model with sticky prices is considered in [7]. A stochastic control model of investment based on SDEs is presented in [8]. And agent-based model of a financial market is presented in [9].

In recent years, SDE models have been applied to a wide array of biological phenomena [10]. An SDE model for the evolution of the MCF-7 breast cancer cell line treated by radiotherapy is developed in [11]. Tumor-immune responses to chemotherapy are studied using SDEs in [12]. Plant growth is modeled via a Gompertz-type SDE in [13]. Biochemical reaction systems are studied using SDEs in [14].

With the recent outbreak of the novel coronavirus COVID-19, the study of stochastic models for the modeling of this phenomenon has emerged. Most are based on the well-known deterministic differential equation model: susceptible-exposed-infected-recovered model (SEIR) [15,16]. Adak et al. used Brownian motion to induce stochasticity into a SLIR (susceptible-latent-infected-recovered) model by adding stochastic differential directly to a system of ordinary differential equations (ODE) in [17]. An adaptation of an SIR model to include stochastic transition is discussed in [18]. A SIRS epidemic model including fractional white noise is presented in [19]. A variant of stochastization is used in the generalized logistic equation to model COVID-19 evolution in [20].

It is clear that the analysis of SDEs is currently a particularly relevant topic. Furthermore, there arises a requirement to induce stochastization into previously deterministic models described via ODEs. While there are many ways to approach this problem, the aim of this article is to provide a technique for stochastization in such a way that if an analytical solution to the ODE can be constructed, an analytical solution to the SDE can also be constructed. Consider the following ODE of the form:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = P(t, y), \tag{1}$$

where $P(t, y)$ is a continuously differentiable function. Let $\omega(t)$ denote a Wiener process [21]. The objective is to construct an SDE with respect to function $\tilde{\xi}(t, \omega(t)|\alpha)$ (where α is a scalar parameter) of the form:

$$\mathrm{d}\tilde{\xi} = h_1(t, \tilde{\xi})\mathrm{d}t + h_2(t, \tilde{\xi})\mathrm{d}\omega. \tag{2}$$

Equation (2) and its solution possess the following properties:

$$\lim_{\alpha\to 0} \tilde{\xi}(t, \omega(t)|\alpha) = y(t); \tag{3}$$

and as α tends to 0, the SDE (2) tends to (1) and conversely the solution to (2) tends to the solution of (1). Itô calculus is a classical yet powerful mathematical theory of SDEs. Introduced almost seven decades ago it is still widely used in a variety of technical, biomedical, and economical applications.

Note that other approaches to this problem also exist. The most straightforward approach would consist of adding noise to (1) in every step of integration of a numerical integrator. While this technique does enable the randomization of the solution trajectory, a solution of (1) bounded to an interval $I \subset \mathbb{R}$ becomes no longer bounded, which presents a problem for many applications.

Another option for inputting randomness into an ODE is based on random differential equations, discussed in detail in [22]. Here the time variable t is replaced by a stochastic Wiener process $\omega(t)$, yielding the following random differential equation:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = P(\omega(t), y). \tag{4}$$

The latter approach is an improvement boundedness-wise over the former; however, the menagerie of trajectories obtained from (4) would be much smaller than those from (2).

The main objective of this article is to construct a semi-analytical scheme for solving (2) and to apply this scheme to the paradigmatic Riccati equation [23]. It is well known that the Riccati equation does admit the first-order solitary solution (the kink-type solution) [24]. In its turn, soliton solutions (and the Riccati equation in general) do play an important role in defining the global dynamics of different models, including the prostate cancer model [25], the Hepatitis C treatment model [26], and COVID-19 within host model with immune response [27]. Thus, the construction and analysis of solitary solutions to various differential equations have become a very active field of research in recent years. An overview of the latest studies conducted in this area is presented further. The dynamics of ion acoustic solitary wave solutions of two-dimensional nonlinear Kadomtsev-Petviashvili-Burgers equation in quantum plasma have been investigated in [28]. Solitary wave solutions to the nonlinear Schrödinger equation were constructed in [29–34]. In [35], the authors have proposed novel methods for the construction of exact traveling wave solutions of the modified Liouville equation and the symmetric regularized long wave equation. The stability analysis of solitary wave solutions for the fourth-order nonlinear Boussinesq water wave equation was performed in [36]. Dispersive traveling wave solutions of the equal-width and modified equal-width equations were obtained in [37]. The problem formulations of models for internal solitary waves in a stratified shear flow with a free surface were presented in [38]. New traveling wave solutions for the fourth-order nonlinear Ablowitz-Kaup-Newell-Segur water wave dynamical equation were constructed and investigated in [39]. In [40], the authors have developed a novel approach for the construction of solitary wave solutions to the nonlinear Nizhnik-Novikov-Vesselov equation. Exact traveling and solitary wave solutions of the Kudryashov-Sinelshchikov equation were obtained in [41]. The propagation of soliton-like solutions to the coupled nonlinear $(2 + 1)$-dimensional Maccari system applied in plasma physics was investigated in [42]. The authors of [43] have developed a modification of the variational iteration algorithm for the study of the numerical solution to the dispersive water wave phenomena. The interaction properties of soliton molecules were investigated and Painleve analysis for nano bioelectronics transmission model was performed in [44].

Stochastization of such models describing different biological and biomedical processes is an important research direction [17,45]. Stochastic models allow a better description of real-life processes and do help to represent the effect of local unpredictability caused by the noise and different uncertainties. However, computational analysis of SDEs containing a variable that represents random white noise (usually calculated as the derivative of Brownian motion or the Wiener process) results in the investigation of martingales [46]. As a matter of fact is that a particular solution of an SDE can wander far away from its non-stochastic counterpart solution as time moves away from the initial conditions [47].

SDE analysis is transformed into analytical or computational investigation of martingales. In some situations, such wandering out of a predetermined variation interval is not tolerable. The ability to control the variation range while allowing the stochastic wandering around the deterministic counterpart solution is an open problem in mathematical sciences up to the best of our knowledge.

However, one needs to keep in mind that the range of values of the solution (especially if it is a soliton solution) is predetermined by the structure of the model (e.g., the concentration of the infected cells cannot be negative). Therefore, it is important to construct such stochastization schemes for ODEs which would guarantee that the stochastic soliton solution would remain in the predetermined range of values. The derivation of such soliton solutions to the stochastic Riccati equation is the main objective of this article. That will be achieved by extending the theory of Itô calculus by directly implementing it into analytical schemes for the solution of differential equations based on the generalized operator of differentiation.

This article is organized as follows. Preliminary results of Itô calculus and some operator methods for the solution of differential equations are presented in Section 2; the inverse balancing technique is adapted to SDEs in Section 3; the scheme for the stochastization of first-order ODEs is constructed in Section 4; the developed scheme is applied to obtain the stochastization of the Riccati equation in Section 5; concluding remarks are given in Section 6.

2 Preliminaries

A short review of the concepts of Itô calculus required for the article's main results is provided in this section.

The following notations are used throughout the text:
- $\omega(t)$ – Wiener process;
- $\xi, \tilde{\xi}, \eta$ – random processes;
- $I(t, \omega(t))$ – Itô integral;
- $a_\xi(t, \omega(t)), \sigma_\xi(t, \omega(t))$ – functions describing an SDE;
- Φ_ξ – Itô function with respect to random process ξ;
- $\mathbf{D}_\lambda$ – partial differentiation operator with respect to variable λ;
- $\mathbf{L}_\lambda$ – integration operator with respect to variable λ;
- $\mathbf{D}$ – generalized differential operator;
- $\mathbf{G}_t$ – multiplicative operator.

2.1 Wiener process and Itô integral

Let us consider a non-differentiable Wiener process $\omega(t)$ with the following properties [21]:

$$\lim_{\Delta t \to 0} \Delta\omega(t) = 0; \tag{5}$$

$$\lim_{\Delta t \to 0} \frac{(\Delta\omega(t))^n}{\Delta t} = \begin{cases} 1, & n = 2 \\ 0, & n = 3, 4, \ldots \end{cases} \tag{6}$$

Let $\sigma(t, x)$ be continuous and satisfy a global Lipschitz condition. The integral of $\sigma(t, \omega(t))$ with respect to the Wiener process is defined as the Itô integral [22]:

$$\begin{aligned} I(t, \omega(t)) &= \int_0^t a(s, \omega(s))\mathrm{d}t + \int_0^t \sigma(s, \omega(s))\mathrm{d}\omega(s) \\ &= \lim_{N \to +\infty} \sum_{j=0}^{N-1} a(t_j^{(N)}, \omega(t_j^{(N)}))(t_{j+1}^{(N)} - t_j^{(N)}) \\ &\quad + \lim_{N \to +\infty} \sum_{j=0}^{N-1} \sigma(t_j^{(N)}, \omega(t_j^{(N)}))(\omega(t_{j+1}^{(N)}) - \omega(t_j^{(N)})), \end{aligned} \tag{7}$$

where $0 = t_0^{(N)} < t_1^{(N)} < \cdots < t_N^{(N)} = t$ is a partition of the interval $[0, t]$. The aforementioned limit is taken in the mean-square sense, ensuring that $\max_{0 \le j \le N-1}(t_{j+1}^{(N)} - t_j^{(N)}) \to 0$.

2.2 Itô's lemma

Itô's lemma is a fundamental result in the SDE theory. Suppose a process $\xi(t)$ is given that has the following differential:

$$\mathrm{d}\xi(t) = a_\xi(t, \xi)\mathrm{d}t + \sigma_\xi(t, \xi)\mathrm{d}\omega(t), \tag{8}$$

where a_ξ, σ_ξ and an additional function $f(t, x)$ satisfy conditions detailed in the previous section. Then, the differential of process $f(t, \xi)$ is given by:

$$\mathrm{d}f(t, \xi) = \left(\frac{\partial f}{\partial t} + a_\xi \frac{\partial f}{\partial x} + \frac{1}{2}\sigma_\xi^2 \frac{\partial^2 f}{\partial x^2}\right)\Bigg|_{x=\xi} \mathrm{d}t + \sigma_\xi \frac{\partial f}{\partial x}\Bigg|_{x=\xi} \mathrm{d}\omega(t). \tag{9}$$

The aforementioned equation is an analogy of the chain rule of differentiation for stochastic processes.

2.3 SDEs and their solutions

First, consider the following stochastic equation where the coefficients depend only on t:

$$d\eta = a_\eta(t)dt + \sigma_\eta(t)d\omega(t). \tag{10}$$

In this case, it is possible to compute the solution $\eta(t)$ directly by applying the Itô integral (7):

$$\eta(t) = \eta_0 + \int_0^t a_\eta(s)ds + \int_0^t \sigma_\eta(s)d\omega(s). \tag{11}$$

Now let us consider a generalized version of (10) that takes the form (8).

Suppose that it is possible to determine such a function $f(t,x)$ that taking $\eta = f(t,\xi)$ and using (9) to compute its differential yields (10). Then, η is given by (11). Furthermore, if some function $g(t,x)$ satisfies the condition $g(t, f(t,x)) = x$, then the solution to (8) is given by:

$$\xi = g(t, \eta(t)) = g\left(t, \eta_0 + \int_0^t a_\eta(s)ds + \int_0^t \sigma_\eta(s)d\omega(s)\right). \tag{12}$$

The substitutions described above are possible only if Itô's condition holds true [48]

$$\frac{d}{dt}\ln\sigma_\eta(t) = \sigma_\xi(t,x)\left(\frac{1}{\sigma_\xi^2(t,x)}\frac{\partial\sigma_\xi(t,x)}{\partial t} - \frac{\partial}{\partial x}\frac{a_\xi(t,x)}{\sigma_\xi(t,x)} + \frac{1}{2}\frac{\partial^2\sigma_\xi(t,x)}{\partial x^2}\right). \tag{13}$$

Thus, given (8), it is possible to apply Itô's condition to compute $\sigma_\eta(t)$. Then, Itô's lemma yields:

$$\frac{\partial f}{\partial x} = \frac{\sigma_\eta(t)}{\sigma_\xi(t,x)}, \tag{14}$$

from which the function $f(t,x)$ can be determined as follows:

$$f(t,x) = \sigma_\eta(t)\int_0^x \frac{du}{\sigma_\xi(t,u)}. \tag{15}$$

In the subsequent parts of this article the following notation is used:

$$\Phi_\xi(t) = \frac{d}{dt}\ln\sigma_\eta(t), \tag{16}$$

and the only time-dependent function Φ_ξ is referred to as the Itô function of stochastic process ξ.

2.4 Operator solutions to ODEs

In order to obtain stochastization of ODEs, the analytical solutions of these ODEs must be considered. In this section, a short review of the generalized differential operator method for the construction of analytical solutions to ODEs is presented. The method was first presented in [49] and later developed in [50,51].

Consider the following ODE:

$$\frac{dy}{dt} = \Lambda(t,y); \quad y = y(t,s); \quad y(0,s) = s. \tag{17}$$

The generalized differential operator for (17) reads

$$\mathbf{D} = \Lambda(t,s)\mathbf{D}_s, \tag{18}$$

where $\mathbf{D}_s$ denotes the partial differentiation operator with respect to s. Using the generalized differential operator (18), the following multiplicative operator is constructed:

$$\mathbf{G}_t = \sum_{j=0}^{+\infty} \frac{t^j}{j!} \mathbf{D}^j. \tag{19}$$

Operator $\mathbf{G}_t$ possesses the following property:

$$\mathbf{G}_t f(x, s) = f(\mathbf{G}_t x, \mathbf{G}_t s), \tag{20}$$

for any analytic bivariate function $f(x, s)$. Furthermore, the solution to (17) can be written as:

$$y = y(t, s) = \mathbf{G}_t s = \sum_{j=0}^{+\infty} \frac{t^j}{j!} \mathbf{D}^j s. \tag{21}$$

2.5 Operator solution to a particular partial differential equation

In subsequent sections, a partial differential equation arises that possesses the following form:

$$\frac{\partial u}{\partial t} - f_0(t, x)\frac{\partial u}{\partial x} = f_1(t, x)u(t, x), \tag{22}$$

where $f_0(t, x), f_1(t, x)$ are analytical functions. The following boundary condition is posed:

$$u(0, x) = \varphi_0(x), \tag{23}$$

where $\varphi_0(x)$ is also analytic.

An operator solution to (22) and (23) is presented in [49]. Let $\mathbf{L}_\lambda$ denote the integration operator with respect to variable λ.

Then, the solution to (22) and (23) is given by:

$$u(t, x) = \exp(A(t, x)), \tag{24}$$

where $A(t, x)$ is the solution to the following operator problem:

$$(\mathbf{D}_t - f_0(t, x)\mathbf{D}_x)A = f_1(t, x); \tag{25}$$

$$A(0, x) = \ln \varphi_0(x) = \psi_0(x). \tag{26}$$

The solution to (25) and (26) reads

$$A(t, x) = \left(\sum_{k=0}^{+\infty} (\mathbf{L}_t f_0(t, x)\mathbf{D}_x)^k \right) (\mathbf{L}_t f_1(t, x) + \psi_0(x)). \tag{27}$$

3 Inverse balancing technique for SDEs

The main idea of the inverse balancing technique is as follows: given a differential equation, and an analytical form of its solution to assume that the solution is known, and determine the parameters of the differential equation in terms of the solution parameters. This yields a robust approach to determining the necessary conditions of the existence of a particular solution to a class of differential equations. It has been applied in a variety of fields, ranging from astrophysics [52], and population dynamics [53] to medicine [25].

In this section, the method is extended to include SDEs.

Theorem 3.1. *Suppose that the Itô function* $\Phi_\xi(t)$, $\sigma_\xi(t,x)$ *and a constant* $0 < \alpha \le 1$ are given.

Then, the Itô partial differential equation with respect to function $a_\xi(t,x|\alpha)$

$$\sigma_\xi(t,x)\left(\frac{1}{\sigma_\xi^2(t,x)}\frac{\partial \sigma_\xi(t,x)}{\partial t} - \frac{\partial}{\partial x}\frac{a_\xi(t,x|\alpha)}{\sigma_\xi(t,x)} + \frac{\alpha^2}{2}\frac{\partial^2 \sigma_\xi(t,x)}{\partial x^2}\right) = \Phi_\xi(t) \tag{28}$$

has the following general solution:

$$a_\xi(t,x|\alpha) = \sigma_\xi(t,x)\left(-\int_0^x \frac{\partial}{\partial t}\left(\frac{1}{\sigma_\xi(t,u)}\right)du - \Phi_\xi(t)\int_0^x \frac{du}{\sigma_\xi(t,u)} + \frac{\alpha^2}{2}\frac{\partial \sigma_\xi(t,x)}{\partial x} + \frac{C(t)}{\sigma_\xi(t,0)} - \frac{\alpha^2}{2}\frac{\partial \sigma_\xi(t,x)}{\partial x}\bigg|_{x=0}\right), \tag{29}$$

where $C(t)$ *is an arbitrary function.*

Proof. Equation (28) can be rewritten as:

$$\frac{\partial a_\xi(t,x|\alpha)}{\partial x} = \frac{a_\xi(t,x|\alpha)}{\sigma_\xi(t,x)}\frac{\partial \sigma_\xi(t,x)}{\partial x} + \frac{\alpha^2}{2}\sigma_\xi(t,x)\frac{\partial^2 \sigma_\xi(t,x)}{\partial x^2} - \Phi_\xi(t) + \frac{1}{\sigma_\xi(t,x)}\frac{\partial \sigma_\xi(t,x)}{\partial t}. \tag{30}$$

Denote

$$M(t,x) = \frac{1}{\sigma_\xi(t,x)}\frac{\partial \sigma_\xi(t,x)}{\partial x}; \tag{31}$$

$$N(t,x) = \frac{\alpha^2}{2}\sigma_\xi(t,x)\frac{\partial^2 \sigma_\xi(t,x)}{\partial x^2} - \Phi_\xi(t) + \frac{1}{\sigma_\xi(t,x)}\frac{\partial \sigma_\xi(t,x)}{\partial t}. \tag{32}$$

Then, the solution to (30) takes the following form:

$$a_\xi(t,x|\alpha) \;=\; \exp\left(\int_0^x M(t,u)du\right)\left(C(t) + \int_0^x N(t,v)\exp\left(-\int_0^v M(t,u)du\right)dv\right), \tag{33}$$

where $C(t)$ is an arbitrary function. Note that

$$\exp\left(\int_0^x M(t,u)du\right) = \frac{\sigma_\xi(t,x)}{\sigma_\xi(t,0)} > 0 \tag{34}$$

and

$$\begin{aligned}\int_0^x N(t,v)\exp\left(-\int_0^v M(t,u)du\right)dv &= \sigma_\xi(t,0)\times\left(\frac{\alpha^2}{2}\int_0^x \frac{\partial^2\sigma_\xi(t,v)}{\partial v^2}dv - \Phi_\xi(t)\int_0^x \frac{dv}{\sigma_\xi(t,v)} + \int_0^x \frac{1}{\sigma_\xi^2(t,v)}\frac{\partial \sigma_\xi(t,v)}{\partial t}dv\right)\\ &= \sigma_\xi(t,0)\left(\frac{\alpha^2}{2}\frac{\partial \sigma_\xi(t,x)}{\partial x} - \frac{\alpha^2}{2}\frac{\partial \sigma_\xi(t,x)}{\partial x}\bigg|_{x=0} - \Phi_\xi(t)\int_0^x \frac{dv}{\sigma_\xi(t,v)} - \int_0^x \frac{\partial}{\partial t}\frac{dv}{\sigma_\xi(t,v)}\right).\end{aligned} \tag{35}$$

Combining (34) and (35) yields (29). □

For clarity in further derivations, (29) will be rewritten as:

$$a_\xi(t,x) = P(t,x) + \alpha Q(t,x|\alpha), \tag{36}$$

where

$$P(t,x) = \sigma_\xi(t,x)\left(S_\xi(t) - \int_0^x \frac{\partial}{\partial t}\left(\frac{1}{\sigma_\xi(t,u)}\right)du - \Phi_\xi(t)\int_0^x \frac{du}{\sigma_\xi(t,u)}\right); \tag{37}$$

$$Q(t, x|\alpha) = \sigma_\xi(t, x)\left(\frac{\alpha}{2}\frac{\partial\sigma_\xi(t, x)}{\partial x} + S_\xi^{(+)}(t)\right), \tag{38}$$

with

$$S_\xi(t) = \frac{C(t)}{\sigma_\xi(t, 0)}; \quad S_\xi^{(+)}(t) = -\frac{\alpha}{2}\frac{\partial\sigma_\xi(t, x)}{\partial x}\bigg|_{x=0}. \tag{39}$$

Theorem 3.2. *The SDE with respect to process $\tilde{\xi} = \tilde{\xi}(t, \omega(t)|\alpha)$:*

$$\mathrm{d}\tilde{\xi} = (P(t, \tilde{\xi}) + \alpha Q(t, \tilde{\xi}|\alpha))\mathrm{d}t + \alpha\sigma_\xi(t, \tilde{\xi})\mathrm{d}\omega(t); \tag{40}$$

satisfies the Itô condition (13) *with Itô function $\Phi_{\tilde{\xi}}(t) = \Phi_\xi(t)$, which is defined by* (28).

Proof. Note that

$$a_{\tilde{\xi}}(t, \tilde{\xi}|\alpha) = P(t, \tilde{\xi}) + \alpha Q(t, \tilde{\xi}|\alpha); \tag{41}$$

$$\sigma_{\tilde{\xi}}(t, \tilde{\xi}|\alpha) = \alpha\sigma_\xi(t, \tilde{\xi}). \tag{42}$$

The Itô condition for these functions is given by (28), which results in the proof of the theorem. □

Taking the limit as $\alpha \to 0$ in the SDE (40) results in the ODE (1). Note that the solution of the SDE also tends to the deterministic solution of the ODE:

$$\lim_{\alpha\to 0}\tilde{\xi}(t, \omega(t)|\alpha) = \tilde{\xi}(t, 0) = y(t). \tag{43}$$

4 Stochastization of first-order ODEs

4.1 Construction of analytical solutions to (40)

In order to construct the analytical solutions to (40) the algorithm described in Section 2.3 is applied. Let (40) be given. A transformation $\tilde{\eta}(t) = f(t, \tilde{\xi})$ of process $\tilde{\xi}$ must be determined in order to transform (40) into:

$$\mathrm{d}\tilde{\eta} = a_{\tilde{\eta}}(t|\alpha)\mathrm{d}t + \sigma_{\tilde{\eta}}(t)\mathrm{d}\omega(t). \tag{44}$$

The solution to (44) is given by (11), which results in $\tilde{\xi} = g(t, \tilde{\eta}(t))$, where $g(t, x)$ is the inverse transformation to $f(t, x)$ with respect to x.

Theorem 4.1. *Functions $a_{\tilde{\eta}}(t)$, $\sigma_{\tilde{\eta}}(t)$ are given by:*

$$a_{\tilde{\eta}}(t|\alpha) = \sigma_{\tilde{\eta}}(t)\left(\frac{1}{\alpha}S_\xi(t) + S_\xi^{(+)}(t)\right); \tag{45}$$

$$\sigma_{\tilde{\eta}}(t) = \gamma\exp\left(\int_0^t \Phi_{\tilde{\xi}}(s)\mathrm{d}s\right); \quad \gamma \in \mathbb{R}\backslash\{0\}, \tag{46}$$

where $S_\xi(t)$, $S_\xi^{(+)}(t)$ are defined by (39).

Proof. Using (16) directly yields (46). Then, Itô lemma (9) yields

$$a_{\tilde{\eta}}(t|\alpha) = \frac{\partial f}{\partial t} + a_{\tilde{\xi}}(t, x|\alpha)\frac{\partial f}{\partial x} + \frac{1}{2}\sigma_{\tilde{\xi}}(t, x|\alpha)^2\frac{\partial^2 f}{\partial x^2}. \tag{47}$$

Inserting (15) into (47) results in

$$a_{\tilde{\eta}}(t|\alpha) = \frac{\partial}{\partial t}\left(\sigma_{\tilde{\eta}}(t)\int_0^x \frac{\mathrm{d}u}{\sigma_{\tilde{\xi}}(t,u|\alpha)}\right) + a_{\tilde{\xi}}(t,x|\alpha)\frac{\sigma_{\tilde{\eta}}(t)}{\sigma_{\tilde{\xi}}(t,x|\alpha)} - \frac{\sigma_{\tilde{\eta}}(t)}{2}\frac{\partial\sigma_{\tilde{\xi}}(t,x|\alpha)}{\partial x}. \tag{48}$$

Using (46), (41), and (42) transforms (48) into:

$$\begin{aligned} a_{\tilde{\eta}}(t|\alpha) = \frac{\gamma}{\alpha}\Bigg(\frac{\partial}{\partial t}\left(\exp\left(\int_0^t \Phi_{\tilde{\xi}}(s)\mathrm{d}s\right)\cdot\int_0^x \frac{\mathrm{d}u}{\sigma_{\xi}(t,u)}\right) + \exp\left(\int_0^t \Phi_{\tilde{\xi}}(s)\mathrm{d}s\right) \\ \times\left(S_{\xi}(t) + \alpha S_{\xi}^{(+)}(t) + \frac{\alpha^2}{2}\frac{\partial\sigma_{\xi}(t,x)}{\partial x} - \int_0^x\left(\frac{\partial}{\partial t}\frac{1}{\sigma_{\xi}(t,u)} + \frac{\Phi_{\xi}(t)}{\sigma_{\xi}(t,u)}\right)\mathrm{d}u\right) \\ -\frac{\alpha^2}{2}\frac{\partial\sigma_{\xi}(t,x)}{\partial x}\exp\left(\int_0^t \Phi_{\tilde{\xi}}(s)\mathrm{d}s\right)\Bigg). \end{aligned} \tag{49}$$

Simplifying and cancelling like terms in (49) yield (45). □

Using the results of Theorem 4.1, SDE (44) is rewritten as:

$$\mathrm{d}\tilde{\eta} = \sigma_{\tilde{\eta}}(t)\left(\left(\frac{1}{\alpha}S_{\xi}(t) + S_{\xi}^{(+)}(t)\right)\mathrm{d}t + \mathrm{d}\omega(t)\right), \tag{50}$$

which leads to the solution:

$$\tilde{\eta}(t|\alpha) = \tilde{\eta}_0 + \int_0^t \sigma_{\tilde{\eta}}(s)\left(\frac{1}{\alpha}S_{\xi}(s) + S_{\xi}^{(+)}(s)\right)\mathrm{d}s + \int_0^t \sigma_{\tilde{\eta}}(s)\mathrm{d}\omega(s); \quad \tilde{\eta}_0 \in \mathbb{R}. \tag{51}$$

After the transformed SDE (44) is determined, the solution to (40) can be obtained if the function:

$$f(t,x) = \frac{\gamma}{\alpha}\exp\left(\int_0^t \Phi_{\xi}(s)\mathrm{d}s\right)\int_0^x \frac{\mathrm{d}u}{\sigma_{\xi}(t,u)}; \tag{52}$$

can be inverted with respect to x. The following theorem addresses this problem.

Theorem 4.2. *The inverse function to* (52) *with respect to* x *reads*

$$g(t,x) = \Psi\left(t, \frac{\alpha}{\gamma}x\exp\left(-\int_0^t \Phi_{\xi}(s)\mathrm{d}s\right)\right), \tag{53}$$

where $\Psi(t,z)$ *satisfies the condition* $\Psi(t,0) = 0$ *and the following differential equation:*

$$\frac{\partial\Psi}{\partial z} = \sigma_{\xi}(t,\Psi). \tag{54}$$

Proof. The function $g(t,x)$ is inverse to $f(t,x)$ with respect to x if the following relation holds true:

$$f(t,g(t,x)) = x. \tag{55}$$

Using (52) in the aforementioned equation yields

$$\frac{\gamma}{\alpha}\exp\left(\int_0^t \Phi_{\xi}(s)\mathrm{d}s\right)\int_0^{g(t,x)} \frac{\mathrm{d}u}{\sigma_{\xi}(t,u)} = x, \tag{56}$$

which can be rearranged as:

$$\int_{0}^{g(t,x)} \frac{\mathrm{d}u}{\sigma_{\xi}(t,u)} = x\frac{\alpha}{\gamma}\exp\left(-\int_{0}^{t}\Phi_{\xi}(s)\mathrm{d}s\right). \tag{57}$$

Let $\Psi(t, z)$ satisfy the following:

$$\int_{0}^{\Psi} \frac{\mathrm{d}u}{\sigma_{\xi}(t,u)} = z. \tag{58}$$

Differentiating (58) with respect to Ψ and rearranging yields

$$\frac{\partial\Psi}{\partial z} = \sigma_{\xi}(t,\Psi); \quad \Psi(t,0) = 0. \tag{59}$$

Applying (59) to (57) yields (53). □

Corollary 4.1. *The solution to SDE* (40) *is given by:*

$$\tilde{\xi}(t,\omega(t)|\alpha) = g(t,\tilde{\eta}(t|\alpha)), \tag{60}$$

where $g, \tilde{\eta}$ *are given by* (53) *and* (51), *respectively.*

4.2 Scheme for stochastization of first-order ODEs

Suppose that the ODE (1) and an Itô function $\Phi_{\tilde{\xi}}(t)$ are given. Thus, the function $P(t, x)$ is known and (37) yields

$$S_{\xi}(t) = \frac{P(t,x)}{\sigma_{\xi}(t,x)} + \int_{0}^{x}\left(\frac{\Phi_{\tilde{\xi}}(t)}{\sigma_{\xi}(t,u)} + \frac{\partial}{\partial t}\frac{1}{\sigma_{\xi}(t,u)}\right)\mathrm{d}u. \tag{61}$$

Note that the left-hand side of (61) does not depend on x, thus differentiating (61) with respect to x results in a partial differential equation with respect to the unknown function $\sigma_{\xi}(t, x)$:

$$\frac{\partial\sigma_{\xi}}{\partial t} + P\frac{\partial\sigma_{\xi}}{\partial x} = \sigma_{\xi}(t,x)\left(\Phi_{\tilde{\xi}}(t) + \frac{\partial P}{\partial x}\right). \tag{62}$$

The aforementioned differential equation can be used to determine such $\sigma_{\xi}(t, x)$ that (61) holds true. Then, $Q(t, x)$ can be computed via (38), leading to (40), which is the stochastization of (1).

5 Stochastization of the Riccati equation

5.1 General case

Consider the Riccati differential equation [24]:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \delta(y - y_1)(y - y_2); \quad \delta, y_1, y_2 \in \mathbb{R}. \tag{63}$$

The aim of this section is to provide a stochastization of (63) in the form (40).

Let $P(x) = \delta(x - y_1)(x - y_2)$. Note that (62) has the form (22) with functions $f_0(t, x) = -P(x)$, $f_1(t, x) = \Phi_{\tilde{\xi}}(t) + \frac{\mathrm{d}P}{\mathrm{d}x}$. Then, following from Section 2.5, the solution to (62) reads:

$$\sigma_{\tilde{\xi}}(t, x) = \exp(A(t, x)), \tag{64}$$

where, by (27):

$$A(t, x) = \left(\sum_{k=0}^{+\infty}(-\mathbf{L}_t P(x)\mathbf{D}_x)^k\right)\left(\mathbf{L}_t\left(\Phi_{\tilde{\xi}}(t) + \frac{\mathrm{d}P}{\mathrm{d}x}\right) + \psi_0(x)\right), \tag{65}$$

and the function $\psi_0(x)$ satisfies $\sigma_{\tilde{\xi}}(0, x) = \exp\psi_0(x)$. Note that when considering the stochastization of an ODE, $\psi_0(x)$ can be selected arbitrarily.

Denoting $\widehat{\Phi}_{\tilde{\xi}}(t) = \mathbf{L}_t\Phi_{\tilde{\xi}}(s) = \int_0^t \Phi_{\tilde{\xi}}(s)\mathrm{d}s$ and rearranging (65) lead to:

$$A(t, x) = \mathbf{L}_t \sum_{k=0}^{+\infty}(\mathbf{L}_t(-P(x)\mathbf{D}_x))^k \frac{\mathrm{d}P}{\mathrm{d}x} + \widehat{\Phi}_{\tilde{\xi}}(t) + \sum_{k=0}^{+\infty}(\mathbf{L}_t(-P(x)\mathbf{D}_x))^k\psi_0(x). \tag{66}$$

Note that the term $\mathbf{L}_t^k$ can be written as a factor $\frac{t^k}{k!}$ when the operands do not depend on t. Thus, (66) is transformed to:

$$A(t, x) = \mathbf{L}_t \sum_{k=0}^{+\infty}\frac{t^k}{k!}(-P(x)\mathbf{D}_x)^k\delta(2x - y_1 - y_2) + \widehat{\Phi}_{\tilde{\xi}}(t) + \sum_{k=0}^{+\infty}\frac{t^k}{k!}(-P(x)\mathbf{D}_x)^k\psi_0(x). \tag{67}$$

Note that the operator $\mathbf{G}_t = \sum_{k=0}^{+\infty}\frac{t^k}{k!}(-P(x)\mathbf{D}_x)^k$ is the multiplicative operator defined in (19). Applying property (20), (67) is simplified as:

$$A(t, x) = \int_0^t \delta((\mathbf{G}_s x) - y_1 - y_2)\mathrm{d}s + \widehat{\Phi}_{\tilde{\xi}}(t) + \psi_0(\mathbf{G}_t x). \tag{68}$$

By (21), $\mathbf{G}_t x$ gives the solution to ODE (63) with initial condition $y(0) = x$. The analytical form of this solution reads [50]

$$y = y(t, s) = y_2\frac{\exp(\mu t) - \frac{y_1(s - y_2)}{y_2(s - y_1)}}{\exp(\mu t) - \frac{s - y_2}{s - y_1}}; \quad y(0, s) = s, \tag{69}$$

where $\mu = \delta(y_2 - y_1)$.

Thus, the final form of $A(t, x)$ reads

$$A(t, x) = \delta\int_0^t (2y(s, x) - y_1 - y_2)\mathrm{d}s + \widehat{\Phi}_{\tilde{\xi}}(t) + \psi_0(y(t, x)). \tag{70}$$

From (64) and (40) it follows that the stochastization of (63) reads

$$\mathrm{d}\tilde{\xi} = \left(P(\tilde{\xi}) + \frac{\alpha^2}{2}\exp(A(t, \tilde{\xi}))\left(\exp(A(t, \tilde{\xi}))\frac{\mathrm{d}A}{\mathrm{d}x}\bigg|_{x=\tilde{\xi}} - \exp(A(t, 0))\frac{\mathrm{d}A}{\mathrm{d}x}\bigg|_{x=0}\right)\right)\mathrm{d}t + \alpha\exp(A(t, \tilde{\xi}))\mathrm{d}\omega(t). \tag{71}$$

Note that the stochastization (71) depends on the solution of the non-stochastic equation (63).

5.2 Stochastization of equation (63): special case $\sigma_{\tilde{\xi}} = \sigma_{\tilde{\xi}}(x)$

In this section, a stochastization for (63) is obtained such that the function $\sigma_{\tilde{\xi}}$ depends only on x and the solution to the obtained stochastic equation is constructed.

Suppose that $A(t, x) = B(x)$, then:

$$\sigma_{\xi}(t, x) = \exp(B(x)). \tag{72}$$

Since the right-hand side does not depend on t, $\sigma_{\xi}(t, x)$ is only dependent on x. Furthermore, inserting $t = 0$ yields

$$\sigma_{\xi}(0, x) = \exp(B(x)) = \exp(\psi_0(x)). \tag{73}$$

Then, (70) can be rewritten as:

$$\exp(B(x) - B(y(t, x))) = \exp\left(\delta \int_0^t (2y(s, x) - y_1 - y_2)\mathrm{d}s + \widehat{\Phi}_{\tilde{\xi}}(t)\right), \tag{74}$$

where $y(t, x)$ is given by (69). The solution of the functional equation (74) with respect to $B(x)$ reads:

$$B(x) = \ln(\nu\delta(x - y_1)(x - y_2)) = \ln(\nu P(x)); \quad \nu \in \mathbb{R}\backslash\{0\}, \tag{75}$$

when $\widehat{\Phi}_{\tilde{\xi}}(t) = \Phi_{\tilde{\xi}}(t) \equiv 0$. In that case, $\sigma_{\tilde{\xi}}(t, x)$ has the following form:

$$\sigma_{\tilde{\xi}}(t, x) = \nu\delta(x - y_1)(x - y_2) = \nu P(x). \tag{76}$$

Inserting (76) into (71) yields the stochastization of (63) in the special case:

$$\mathrm{d}\tilde{\xi} = P(\tilde{\xi})\left(1 + \frac{\nu\alpha^2}{2}P(\tilde{\xi})(\nu\delta(2\tilde{\xi} - y_1 - y_2) + S_{\tilde{\xi}}^{(+)})\mathrm{d}t + \alpha\nu\mathrm{d}\omega(t)\right), \tag{77}$$

where $S_{\tilde{\xi}}^{(+)} = \ln(\nu\delta y_1 y_2)\frac{y_1 + y_2}{y_1 y_2}$.

5.3 Analytical solution of (77)

Following the algorithm outlined in Theorem 4.1, the functions (45) and (46) read

$$a_{\tilde{\eta}}(t|\alpha) = \gamma\left(\frac{\nu}{\alpha} + S_{\tilde{\xi}}^{(+)}\right); \tag{78}$$

$$\sigma_{\tilde{\eta}}(t) = \gamma; \quad \gamma \in \mathbb{R}\backslash\{0\}. \tag{79}$$

Thus, (77) is transformed into:

$$\mathrm{d}\tilde{\eta} = \gamma\left(\frac{\nu}{\alpha} + S_{\tilde{\xi}}^{(+)}\right)\mathrm{d}t + \gamma\mathrm{d}\omega(t), \tag{80}$$

with the solution

$$\tilde{\eta}(t|\alpha) = \tilde{\eta}_0 + \gamma\left(\frac{\nu}{\alpha} + S_{\tilde{\xi}}^{(+)}\right)t + \gamma\omega(t). \tag{81}$$

By Theorem 4.2, the function $\Psi(z)$ must be derived from the differential equation:

$$\frac{\mathrm{d}\Psi}{\mathrm{d}z} = \nu\delta(z - y_1)(z - y_2); \quad \Psi(0) = 0. \tag{82}$$

The solution to the aforementioned equation reads

$$\Psi(z) = y_2\frac{\exp(\kappa z) - 1}{\exp(\kappa z) - \frac{y_2}{y_1}}, \quad \kappa = \delta\nu(y_1 - y_2). \tag{83}$$

By (53) we obtain

$$g(t, x) = g(x) = \Psi\left(\frac{\alpha}{\gamma}x\right), \tag{84}$$

and the analytical solution to (77) is given by:

$$\widetilde{\xi}(t|\alpha) = g(\widetilde{\eta}(t|\alpha)) = y_2 \frac{\exp\left(\kappa\frac{\alpha\nu}{\gamma}\left(\widetilde{\eta}_0 + \gamma\left(\frac{\nu}{\alpha} + S_{\widetilde{\xi}}^{(+)}\right)t + \gamma\omega(t)\right)\right) - 1}{\exp\left(\kappa\frac{\alpha\nu}{\gamma}\left(\widetilde{\eta}_0 + \gamma\left(\frac{\nu}{\alpha} + S_{\widetilde{\xi}}^{(+)}\right)t + \gamma\omega(t)\right)\right) - \frac{y_2}{y_1}}. \tag{85}$$

Since it must hold that $\lim_{\alpha\to 0}\widetilde{\xi}(t|\alpha) = y(t)$, where $y(t)$ is given by (69), parameters $\widetilde{\eta}_0 = 0$, $\nu = 1$. Inserting these values into (85) yields

$$\widetilde{\xi}(t|\alpha) = y_2 \frac{\exp(\kappa t + \kappa\alpha(S_{\widetilde{\xi}}^{(+)}t + \omega(t))) - 1}{\exp(\kappa t + \kappa\alpha(S_{\widetilde{\xi}}^{(+)}t + \omega(t))) - \frac{y_2}{y_1}}. \tag{86}$$

Note that comparing the above solution to (69) it can be seen that only the variable within the exp function has the Wiener process $\omega(t)$. This means that the solution of the stochastic Riccati equation belongs to the same set of values as the deterministic Riccati equation.

5.4 Numerical comparison: stochastization and randomization of the Riccati equation

In this section, two different approaches to induce randomness into the Riccati equation are compared.

The first approach is described in Sections 5.2 and 5.3, which leads to a special case of the stochastic Riccati equation (77) and its analytical solution (86).

The randomization procedure is described as follows. Let the Riccati equation (63), a scaling variable $\varepsilon > 0$, and a sample $\theta_0, \dots, \theta_n$ of a Gaussian random variable with mean zero and unit variance be given. Consider any constant-step time-forward numerical integrator with step size $h > 0$. The randomized solution at points $t_k = kh$, $k = 0, \dots, n$ is denoted as $\widehat{\xi}_k = \widehat{\xi}(t_k)$.

We initialize the process by setting the first point equal to the initial condition: $\widehat{\xi}_0 = y_0$. In the kth step, the value $\widehat{\xi}_k$ is computed by performing one-time forward integration step for following the differential equation:

$$\frac{\mathrm{d}\widehat{\xi}}{\mathrm{d}t} = \delta(\widehat{\xi} - y_1)(\widehat{\xi} - y_2) + \varepsilon\theta_k; \quad \widehat{\xi}(t_{k-1}) = \widehat{\xi}_{k-1}, \quad k = 1, \dots, n. \tag{87}$$

The process of randomization described above yields a random solution trajectory with the mean close to the deterministic solution (Figure 1(a)). The non-stochastic solution represented by the thick black line in Figure 1 is validated in the article [53]. However, note that the randomized solution $\widehat{\xi}$ leaves the interval to which the deterministic solution $y(t)$ is bound. This is not the case for the stochastic solution, which remains bounded to the same interval as $y(t)$ (Figure 1(b)).

Note that the statistical mean of function (86) is not equal to the solution (69) of the deterministic Riccati equation. As mentioned before, the deterministic Riccati equation solution is obtained when $\alpha \to 0$.

6 Concluding remarks

A scheme for the analytical stochastization of ODEs is presented in this article. Given an ODE, its SDE counterpart is constructed in such a way that it satisfies the Itô condition. This ensures that it is possible to construct an analytical solution to the obtained SDE via the application of Itô calculus.

The described technique of stochastization has two important properties: as the parameter α that governs the influence of randomness on the SDE solution tends to zero, the solution tends to the

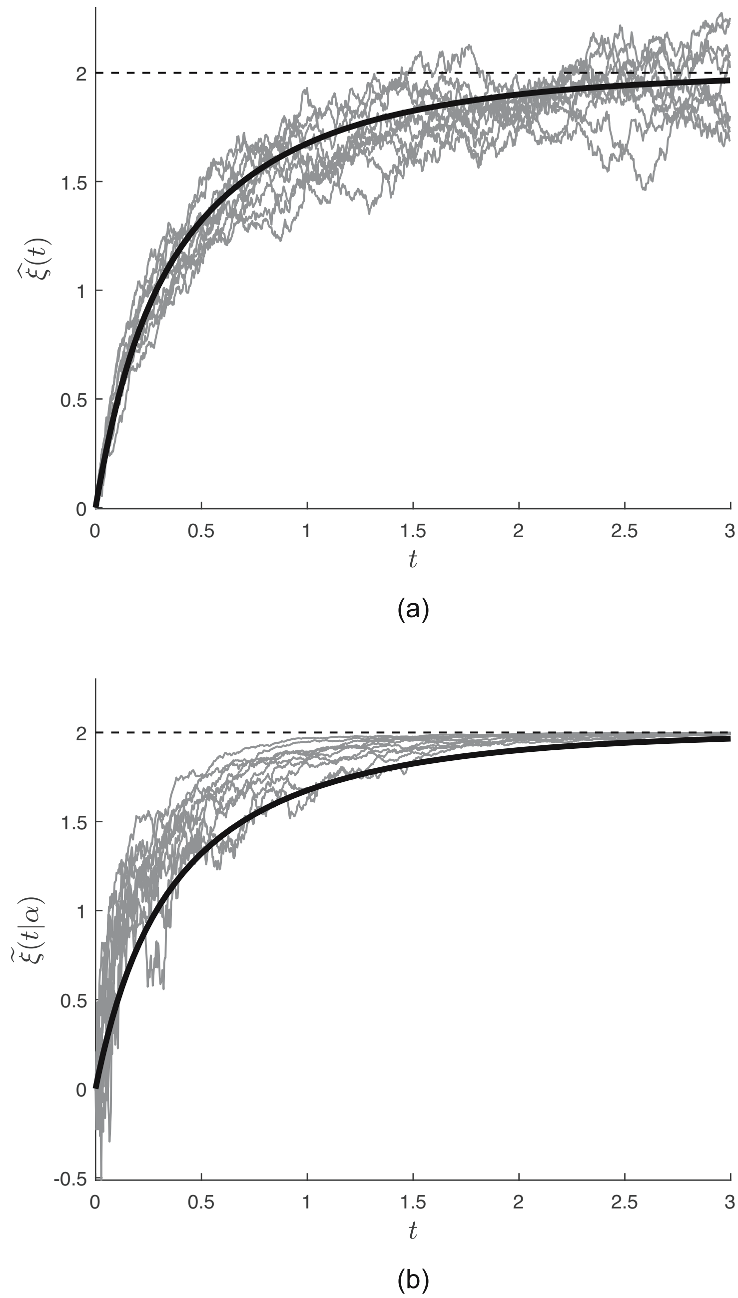

Figure 1: The randomized solution $\widehat{\xi}(t)$ of the Riccati equation (a) and the solution $\widetilde{\xi}(t|\alpha)$ of the stochastic Riccati equation (b). Parameters of the Riccati equation are set to $\delta = 1$, $y_1 = 2$, $y_2 = 3$; the initial conditions are set to zero at $t = 0$. The scaling variable ε is set to 5 in (a); α is set to 0.5 in (b). Thin grey lines denote randomized and stochastic solution trajectories in (a) and (b), respectively. Thick black lines depict the solution of the deterministic Riccati equation. The dashed black line denotes the upper bound for the deterministic solution for $t > 0$.

deterministic ODE solution. Furthermore, if the ODE solution is bounded to an interval, the constructed stochastic trajectories can also only belong to that interval – which is not true for most other stochastization schemes.

Due to the fact that the presented semi-analytical stochastization scheme allows us to confine a stochastic solution to a particular variation interval, the scheme is especially relevant for the investigation of the global dynamics of different biological and biomedical processes where the variation interval of the stochastic solution is predetermined by the rationale of the model.

The scheme is applied to the paradigmatic Riccati equation which possesses kink soliton solutions. It is shown that the general analytical form of the deterministic solution is preserved in the stochastic solution after the transformation of the ODE to the SDE. Stochastic trajectories obtained in this manner are a generalization of kink soliton solutions in the stochastic sense.

The extension of the presented stochastization scheme to higher-order ODEs and systems of ODEs, as well as applications to real-world models, remains a definite objective of future research.

Funding information: This research was funded by a grant (No. S-COV-20-8) from the Research Council of Lithuania.

Conflict of interest: The authors declare no conflict of interest.

References

[1] A. Einstein, *Über die von der molekularkinetischen Theorie der Wärme geforderte Bewegung von in ruhenden Flüssigkeiten suspendierten Teilchen*, Ann. Phys. **4** (1905), 549–560.
[2] N. Wiener, *Collected Works*, vol. 1, P. Masani, MIT Press, Cambridge, MA, 1976.
[3] K. Itô, *Stochastic integral*, Proc. Imp. Acad. **20** (1944), no. 8, 519–524.
[4] J. C. Hull, *Options Futures and Other Derivatives*, Pearson Education, India, 2003.
[5] C. Caginalp and G. Caginalp, *Derivation of non-classical stochastic price dynamics equations*, Phys. A **560** (2020), 125–118.
[6] S. Chen, H. Yang, and Y. Zeng, *Stochastic differential games between two insurers with generalized mean-variance premium principle*, Astin Bull. **48** (2018), no. 1, 413–434.
[7] L. Colombo and P. Labrecciosa, *A stochastic differential game of duopolistic competition with sticky prices*, J. Econom. Dynam. Control **122** (2020), 104030.
[8] Md. A. Baten and R. Khalid, *A stochastic control model of investment and consumption with applications to financial economics*, Stoch. Qual. Control **35** (2020), no. 2, 43–55.
[9] F. Westerhoff, *A simple agent-based financial market model: direct interactions and comparisons of trading profits*, Nonlinear Dynamics in Economics, Finance and Social Sciences, Springer, Berlin/Heidelberg, Germany, 2010, p. 313–332.
[10] C. A. Braumann, *Introduction to Stochastic Differential Equations with Applications to Modelling in Biology and Finance*, John Wiley and Sons, New Jersey, US, 2019.
[11] A. Oroji, M. Omar, and S. Yarahmadian, *An Ito stochastic differential equations model for the dynamics of the MCF-7 breast cancer cell line treated by radiotherapy*, J. Theoret. Biol. **407** (2016), 128–137.
[12] X. Liu, Q. Li, and J. Pan, *A deterministic and stochastic model for the system dynamics of tumor-immune responses to chemotherapy*, Phys. A **500** (2018), 162–176.
[13] Q. Lv, M. K. Schneider, and J. W. Pitchford, *Individualism in plant populations: using stochastic differential equations to model individual neighbourhood-dependent plant growth*, Theor. Popul. Biol. **74** (2008), no. 1, 74–83.
[14] Y. Niu, K. Burrage, and L. Chen, *Modelling biochemical reaction systems by stochastic differential equations with reflection*, J. Theoret. Biol. **396** (2016), 90–104.
[15] S. He, Y. Peng, and K. Sun, *SEIR modeling of the COVID-19 and its dynamics*, Nonlinear Dyn. **101** (2020), no. 3, 1667–1680.
[16] A. Godio, F. Pace, and A. Vergnano, *SEIR modeling of the Italian epidemic of SARS-CoV-2 using computational swarm intelligence*, Int. J. Environ. Res. Public Health **17** (2020), no. 10, 3535.
[17] D. Adak, A. Majumder, and N. Bairagi, *Mathematical perspective of COVID-19 pandemic: disease extinction criteria in deterministic and stochastic models*, Chaos Solitons Fractals **142** (2020), 110381.
[18] K. Karako, P. Song, Y. Chen, and W. Tang, *Analysis of COVID-19 infection spread in Japan based on stochastic transition model*, Biosci. Trends **14** (2020), 134–138.
[19] M. A. Akinlar, M. Inc, J. F. Gómez-Aguilar, and B. Boutarfa, *Solutions of a disease model with fractional white noise*, Chaos Solitons Fractals **137** (2020), 109840.
[20] E. Pelinovsky, A. Kurkin, O. Kurkina, M. Kokoulina, and A. Epifanova, *Logistic equation and COVID-19*, Chaos Solitons Fractals **140** (2020), 110241.
[21] R. Durrett, *Stochastic Calculus: A Practical Introduction*, vol. 6, CRC press, Boca Raton, Florida, 1996.
[22] X. Han and P. E. Kloeden, *Random Ordinary Differential Equations and Their Numerical Solution*, Springer, 2017.
[23] W. T. Reid, *Riccati Differential Equations*, Elsevier, Amsterdam, Netherlands, 1972.
[24] V. F. Zaitsev and A. D. Polyanin, *Handbook of Exact Solutions for Ordinary Differential Equations*, CRC Press, Boca Raton, Florida, 2002.
[25] T. Telksnys, I. Timofejeva, Z. Navickas, R. Marcinkevicius, R. Mickevicius, and M. Ragulskis, *Solitary solutions to an androgen-deprivation prostate cancer treatment model*, Math. Methods Appl. Sci. **43** (2020), no. 7, 3995–4006.

[26] T. Telksnys, Z. Navickas, M. A. F. Sanjuán, R. Marcinkevicius, and M. Ragulskis, *Kink solitary solutions to a hepatitis C evolution model*, Discrete Contin. Dynam. Syst. B **22** (2017), no. 11.

[27] A. E. S. Almocera, G. Quiroz, and E. A. Hernandez-Vargas, *Stability analysis in COVID-19 within-host model with immune response*, Commun. Nonlinear Sci. Numer. Simul. **95** (2020), 105584.

[28] A. R. Seadawy, *Ion acoustic solitary wave solutions of two-dimensional nonlinear Kadomtsev-Petviashvili-Burgers equation in quantum plasma*, Math. Methods Appl. Sci. **40** (2017), no. 5, 1598–1607.

[29] M. Arshad, A. R. Seadawy, and D. Lu, *Modulation stability and optical soliton solutions of nonlinear Schrödinger equation with higher order dispersion and nonlinear terms and its applications*, Superlattices Microstruct. **112** (2017), 422–434.

[30] M. Arshad, A. R. Seadawy, and D. Lu, *Elliptic function and solitary wave solutions of the higher-order nonlinear Schrödinger dynamical equation with fourth-order dispersion and cubic-quintic nonlinearity and its stability*, Eur. Phys. J. Plus **132** (2017), no. 8, 1–11.

[31] N. Cheemaa, A. R. Seadawy, and S. Chen, *More general families of exact solitary wave solutions of the nonlinear Schrödinger equation with their applications in nonlinear optics*, Eur. Phys. J. Plus **133** (2018), no. 12, 1–9.

[32] N. Nasreen, A. R. Seadawy, D. Lu, and M. Arshad, *Solitons and elliptic function solutions of higher-order dispersive and perturbed nonlinear Schrödinger equations with the power-law nonlinearities in non-Kerr medium*, Eur. Phys. J. Plus **134** (2019), no. 10, 1–12.

[33] A. R. Seadawy and N. Cheemaa, *Propagation of nonlinear complex waves for the coupled nonlinear Schrödinger equations in two core optical fibers*, Phys. A **529** (2019), 121330.

[34] Y. S. Özkan, E. Yaşar, and A. R. Seadawy, *A third-order nonlinear Schrödinger equation: the exact solutions, group in invariant solutions and conservation laws*, J. Taibah Univ. Sci. **14** (2020), no. 1, 585–597.

[35] D. Lu, A. R. Seadawy, and A. Ali, *Applications of exact traveling wave solutions of modified Liouville and the symmetric regularized long wave equations via two new techniques*, Results Phys. **9** (2018), 1403–1410.

[36] M. A. Helal, A. R. Seadawy, and M. H. Zekry, *Stability analysis of solitary wave solutions for the fourth-order nonlinear Boussinesq water wave equation*, Appl. Math. Comput. **232** (2014), 1094–1103.

[37] D. Lu, A. R. Seadawy, and A. Ali, *Dispersive traveling wave solutions of the equal-width and modified equal-width equations via mathematical methods and its applications*, Results Phys. **9** (2018), 313–320.

[38] A. R. Seadawy, *Two-dimensional interaction of a shear flow with a free surface in a stratified fluid and its solitary-wave solutions via mathematical method*, Eur. Phys. J. Plus **132** (2017), no. 12, 1–11.

[39] A. Ali, A. R. Seadawy, and D. Lu, *Computational methods and traveling wave solutions for the fourth-order nonlinear Ablowitz-Kaup-Newell-Segur water wave dynamical equation via two methods and its applications*, Open Phys. **16** (2018), no. 1, 219–226.

[40] M. Iqbal, A. R. Seadawy, O. H. Khalil, and D. Lu, *Propagation of long internal waves in density stratified ocean for the $(2+1)$-dimensional nonlinear Nizhnik-Novikov-Vesselov dynamical equation*, Results Phys. **16** (2020), 102838.

[41] A. R. Seadawy, M. Iqbal, and D. Lu, *Nonlinear wave solutions of the Kudryashov-Sinelshchikov dynamical equation in mixtures liquid-gas bubbles under the consideration of heat transfer and viscosity*, J. Taibah Univ. Sci. **13** (2019), no. 1, 1060–1072.

[42] N. Cheemaa, S. Chen, and A. R. Seadawy, *Propagation of isolated waves of coupled nonlinear $(2+1)$-dimensional Maccari system in plasma physics*, Results Phys. **17** (2020), 102987.

[43] H. Ahmad, A. R. Seadawy, and T. A. Khan, *Study on numerical solution of dispersive water wave phenomena by using a reliable modification of variational iteration algorithm*, Math. Comput. Simulat. **177** (2020), 13–23.

[44] N. Farah, A. R. Seadawy, S. Ahmad, S. T. R. Rizvi, and M. Younis, *Interaction properties of soliton molecules and Painleve analysis for nano bioelectronics transmission model*, Opt. Quantum Electron. **52** (2020), no. 7, 1–15.

[45] G. H.-M. Wu, A. Auvinen, A. M.-F. Yen, M. Hakama, S. D. Walter, and H.-H. Chen, *A stochastic model for survival of early prostate cancer with adjustments for leadtime, length bias, and over-detection*, Biom. J. **54** (2012), no. 1, 20–44.

[46] O. Kallenberg, Stochastic differential equations and martingale problems, *Foundations of Modern Probability*, Springer, 2002, pp. 412–427.

[47] E. G. Eferina, A. V. Korolkova, M. N. Gevorkyan, D. S. Kulyabov, and L. A. Sevastyanov, *One-step stochastic processes simulation software package*, Discrete Contin. Models **3** (2014), 46–59.

[48] D. Bartl, M. Kupper, and A. Neufeld, *Stochastic integration and differential equations for typical paths*, Electron. J. Probab. **24** (2019), 1–21.

[49] Z. Navickas, *Operator method of solving nonlinear differential equations*, Lith. Math. J. **42** (2002), no. 4, 387–393.

[50] Z. Navickas and M. Ragulskis, *How far one can go with the Exp-function method?*, Appl. Math. Comput. **211** (2009), no. 2, 522–530.

[51] Z. Navickas, L. Bikulciene, and M. Ragulskis, *Generalization of Exp-function and other standard function methods*, Appl. Math. Comput. **216** (2010), no. 8, 2380–2393.

[52] R. Marcinkevicius, Z. Navickas, M. Ragulskis, and T. Telksnys, *Solitary solutions to a relativistic two-body problem*, Astrophys. Space Sci. **361** (2016), no. 6, 1–7.

[53] Z. Navickas, R. Marcinkevicius, T. Telksnys, and M. Ragulskis, *Existence of second order solitary solutions to Riccati differential equations coupled with a multiplicative term*, IMA J. Appl. Math. **81** (2016), no. 6, 1163–1190.