**Research Article**

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# Global weak solution of 3D-NSE with exponential damping

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**Abstract:** In this paper, we prove the global existence of incompressible Navier-Stokes equations with damping $\alpha(e^{\beta|u|^2}-1)u$, where we use the Friedrich method and some new tools. The delicate problem in the construction of a global solution is the passage to the limit in exponential nonlinear term. To solve this problem, we use a polynomial approximation of the damping part and a new type of interpolation between $L^\infty(\mathbb{R}^+, L^2(\mathbb{R}^3))$ and the space of functions $f$ such that $(e^{\beta|f|^2}-1)|f|^2 \in L^1(\mathbb{R}^+\times\mathbb{R}^3)$. Fourier analysis and standard techniques are used.

**Keywords:** Navier-Stokes equations, Friedrich method, weak solution

**MSC 2020:** 35-XX, 35Q30, 76D05, 76N10

## 1 Introduction

In this paper, we study the global existence of weak solution to the modified incompressible Navier-Stokes equations (NSE) in $\mathbb{R}^3$:

$$(S)\begin{cases}\partial_t u - \nu\Delta u + u.\nabla u + \alpha\left(e^{\beta|u|^2}-1\right)u = -\nabla p & \text{in } \mathbb{R}^+\times\mathbb{R}^3,\\ \operatorname{div} u = 0 & \text{in } \mathbb{R}^+\times\mathbb{R}^3,\\ u(0,x) = u^0(x) & \text{in } \mathbb{R}^3,\end{cases}$$

where $u = u(t,x) = (u_1, u_2, u_3)$ and $p = p(t,x)$ denote, respectively, the unknown velocity and the unknown pressure of the fluid at the point $(t,x)\in\mathbb{R}^+\times\mathbb{R}^3$, and the viscosity of fluid $\nu > 0$. Also $\alpha, \beta > 0$ denote the parameters of damping term. The fact that $\operatorname{div} u = 0$, allows to write the term $(u.\nabla u) := u_1\partial_1 u + u_2\partial_2 u + u_3\partial_3 u$ in the following form:

$$\operatorname{div}(u\otimes u) := (\operatorname{div}(u_1u), \operatorname{div}(u_2u), \operatorname{div}(u_3u)).$$

While $u^0 = (u_1^0(x), u_2^0(x), u_3^0(x))$ is an initial given velocity. If $u^0$ is quite regular, the divergence-free condition determines the pressure $p$. We recall in our case it was assumed that the viscosity is unitary ($\nu = 1$) to simplify the calculations and the proofs of our results. The global existence of the weak solution of the initial value problem of classical incompressible Navier-Stokes was proved by Leray and Hopf (see [1–4]) long before. Uniqueness remains an open problem for the dimensions $d \geq 3$. The damping is from the resistance to the motion of the flow. It describes various physical situations such as porous media flow, drag or friction effects, and some dissipative mechanisms (see [2,3,5,6] and references therein).

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The polynomial damping $\alpha|u|^{\beta-1}u$ is studied in [7] by Cai and Jiu, where they proved the global existence of weak solution in

$$L^\infty(\mathbb{R}^+, L^2(\mathbb{R}^3)) \cap L^2(\mathbb{R}^+, \dot{H}^1(\mathbb{R}^3)) \cap L^{\beta+1}(\mathbb{R}^+, L^{\beta+1}(\mathbb{R}^3)).$$

The purpose of this paper is to study the well posedness of the incompressible NSEs with damping $\alpha(e^{\beta|u|^2} - 1)u$. We will show that the Cauchy problem $(S)$ has global weak solutions for any $\alpha, \beta \in (0, \infty)$. We apply the Friedrich method to construct the approximate solutions and make more delicate estimates to proceed to compactness arguments. In particular, we obtain new more *a priori* estimates:

$$\|u(t)\|_{L^2}^2 + 2\int_0^t \|\nabla u(z)\|_{L^2}^2 dz + 2\alpha\int_0^t \|\Big(e^{\beta|u(z)|^2} - 1\Big)|u(z)|^2\|_{L^1} dz \le \|u^0\|_{L^2}^2,$$

comparing with the NSEs, to guarantee that solution $u$ belongs to

$$L^\infty(\mathbb{R}^+, L^2(\mathbb{R}^3)) \cap L^2(\mathbb{R}^+, \dot{H}^1(\mathbb{R}^3)) \cap \mathcal{E}_\beta,$$

where

$$\mathcal{E}_\beta = \{f : \mathbb{R}^+ \times \mathbb{R}^3 \to \mathbb{R} \text{ measurable; } \Big(e^{\beta|f|^2} - 1\Big)|f|^2 \in L^1(\mathbb{R}^+ \times \mathbb{R}^3)\Big\}.$$

Before treating the global existence, we give the definition of solution of $(S)$.

**Definition 1.1.** The function pair $(u, p)$ is called a weak solution of problem $(S)$ if for any $T > 0$, the following conditions are satisfied:

(1) $u \in L^\infty([0, T]; L^2_\sigma(\mathbb{R}^3)) \cap L^2([0, T]; \dot{H}^1(\mathbb{R}^3))$ and $(e^{\beta|u|^2} - 1)|u|^2 \in L^1([0, T], L^1(\mathbb{R}^3))$.

(2) $\partial_t u - \Delta u + u.\nabla u + \alpha(e^{\beta|u|^2} - 1)u = -\nabla p$ in $\mathcal{D}'([0, T] \times \mathbb{R}^3)$: for any $\Phi \in C_0^\infty([0, T] \times \mathbb{R}^3)$ such that $\operatorname{div} \Phi(t, x) = 0$ for all $(t, x) \in [0, T] \times \mathbb{R}^3$ and $\Phi(T) = 0$, we have

$$-\int_0^T (u; \partial_t\Phi)_{L^2} + \int_0^T (\nabla u; \nabla\Phi)_{L^2} + \int_0^T ((u.\nabla)u; \Phi)_{L^2} + \alpha\int_0^T \Big(\Big(e^{\beta|u|^2} - 1\Big)u; \Phi\Big)_{L^2} = (u^0; \Phi(0))_{L^2}.$$

(3) $\operatorname{div} u(x, t) = 0$ for a.e. $(t, x) \in [0, T] \times \mathbb{R}^3$. $((;)_{L^2}$ is the inner product in $L^2(\mathbb{R}^3)$.)

**Remark 1.2.**

(1) If $(u, p)$ is a solution of the system $(S)$, then $(e^{\beta|u|^2} - 1)u \in L^1_{\mathrm{loc}}(\mathbb{R}^+, L^1(\mathbb{R}^3))$. Indeed: for $T > 0$, consider the subsets $A_T^1 = \{(t, x) \in [0, T] \times \mathbb{R}^3;\ 0 < |u(t, x)| < 1\}$ and $A_T^2 = \{(t, x) \in [0, T] \times \mathbb{R}^3;\ |u(t, x)| \ge 1\}$. We have

$$\begin{aligned}
\int_{[0,T]\times\mathbb{R}^3} \Big(e^{\beta|u|^2} - 1\Big)|u| &= \int_{A_T^1 \cup A_T^2} \Big(e^{\beta|u|^2} - 1\Big)|u| \\
&\le \int_{A_T^1} \Big(e^{\beta|u|^2} - 1\Big)|u| + \int_{A_T^2} \Big(e^{\beta|u|^2} - 1\Big)|u| \\
&\le \int_{A_T^1} \frac{(e^{\beta|u|^2} - 1)}{|u|}|u|^2 + \int_{A_T^2} \Big(e^{\beta|u|^2} - 1\Big)\frac{1}{|u|}|u|^2 \\
&\le M_\beta \int_{A_T^1} |u|^2 + \int_{A_T^2} \Big(e^{\beta|u|^2} - 1\Big)|u|^2, \quad \Bigg(M_\beta = \max_{0<X\le 1} \frac{(e^{\beta X^2} - 1)}{X}\Bigg) \\
&\le M_\beta \int_0^T \int_{\mathbb{R}^3} |u|^2 + \int_0^T \int_{\mathbb{R}^3} \Big(e^{\beta|u|^2} - 1\Big)|u|^2
\end{aligned}$$

$$\leq M_\beta T \|u\|_{L^\infty([0,T],L^2(\mathbb{R}^3))} + \int\limits_{[0,T]\times\mathbb{R}^3} \Big(e^{\beta|u|^2} - 1\Big)|u|^2$$
$$\leq M_\beta T \|u\|_{L^\infty([0,T],L^2(\mathbb{R}^3))} + \Big\|\Big(e^{\beta|u|^2} - 1\Big)|u|^2\Big\|_{L^1([0,T],L^1(\mathbb{R}^3))},$$

which implies the desired result.

(2) By this definition, the scalar pressure function $p$ is well defined in $L^1_{\mathrm{loc}}(\mathbb{R}^+, H^{-s}(\mathbb{R}^3))$, for all $s > 3/2$. Indeed: Formally, if $p$ exists, then

$$p = (-\Delta)^{-1}(\operatorname{div} u.\nabla u + \alpha \operatorname{div}\Big(e^{\beta|u|^2} - 1\Big)u).$$

So, just prove that $(-\Delta)^{-1}(\operatorname{div}(u.\nabla u))$, $(-\Delta)^{-1}(\alpha\operatorname{div}(e^{\beta|u|^2} - 1)u) \in L^1_{\mathrm{loc}}(\mathbb{R}^+, H^{-s}(\mathbb{R}^3))$, for all $s > 3/2$. For $s > 3/2$, we have

$$\begin{aligned}
\|(-\Delta)^{-1}(\operatorname{div}(u.\nabla u))(t)\|_{H^{-s}(\mathbb{R}^3)} &= \|(-\Delta)^{-1}(\operatorname{div}(\operatorname{div}(u\otimes u)))(t)\|_{H^{-s}(\mathbb{R}^3)}\\
&\leq \|u\otimes u(t)\|_{H^{-s}(\mathbb{R}^3)}\\
&\leq \left(\int\limits_{\mathbb{R}^3}(1+|\xi|^2)^{-s}|\mathcal{F}(u\otimes u)(t,\xi)|^2 d\xi\right)^{1/2}\\
&\leq c_s\|\mathcal{F}(u\otimes u)(t)\|_{L^\infty(\mathbb{R}^3)}, \quad \left(c_s = \left(\int\limits_{\mathbb{R}^3}(1+|\xi|^2)^{-s}d\xi\right)^{1/2}\right)\\
&\leq c_s\|u\otimes u(t)\|_{L^1(\mathbb{R}^3)}\\
&\leq c_s\|u(t)\|^2_{L^2(\mathbb{R}^3)} \in L^1_{\mathrm{loc}}(\mathbb{R}^+).
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\|(-\Delta)^{-1}\Big(\operatorname{div}\Big(e^{\beta|u|^2} - 1\Big)u\Big)(t)\|_{H^{-s}(\mathbb{R}^3)} &\leq \left(\int\limits_{\mathbb{R}^3}(1+|\xi|^2)^{-s}|\xi|^{-2}\Big|\mathcal{F}\Big(\Big(e^{\beta|u|^2}-1\Big)u\Big)(t,\xi)\Big|^2 d\xi\right)^{1/2}\\
&\leq C_s\Big\|\mathcal{F}\Big(\Big(e^{\beta|u|^2}-1\Big)u(t)\Big)\Big\|_{L^\infty(\mathbb{R}^3)}, \quad C_s = \left(\int\limits_{\mathbb{R}^3}(1+|\xi|^2)^{-s}|\xi|^{-2}d\xi\right)^{1/2}\\
&\leq C_s\Big\|\Big(e^{\beta|u|^2}-1\Big)u(t)\Big\|_{L^1(\mathbb{R}^3)} \in L^1_{\mathrm{loc}}(\mathbb{R}^+).
\end{aligned}$$

Then, $p$ is well defined in $L^1_{\mathrm{loc}}(\mathbb{R}^+, H^{-s}(\mathbb{R}^3))$, $s > 3/2$.

(3) If $(u(t,x), p(t,x))$ is a solution of $(S)$, then $(u(t,x), p(t,x) + h(t))$ is also the solution of $(S)$. Then, if we look for $p$ in the space $L^1_{\mathrm{loc}}(\mathbb{R}^+, H^{-2}(\mathbb{R}^3))$, we obtain $h = 0$ and we can express $p$ in terms of $u$.

In our case of exponential damping, we are trying to find more regularity of Leray solution in $\cap_p L^p(\mathbb{R}^+, L^p(\mathbb{R}^3))$. In particular, we give a new energy estimate. Our main result is the following.

**Theorem 1.3.** *Let $u^0 \in L^2(\mathbb{R}^3)$ be a divergence-free vector fields, then there is a global solution of $(S)$ $u \in L^\infty(\mathbb{R}^+, L^2(\mathbb{R}^3)) \cap C(\mathbb{R}^+, H^{-2}(\mathbb{R}^3)) \cap L^2(\mathbb{R}^+, \dot{H}^1(\mathbb{R}^3)) \cap \mathcal{E}_\beta$. Moreover, for all $t \geq 0$,*

$$\|u(t)\|^2_{L^2} + 2\int_0^t \|\nabla u(z)\|^2_{L^2}dz + 2\alpha\int_0^t \Big\|\Big(e^{\beta|u(z)|^2} - 1\Big)|u(z)|^2\Big\|_{L^1}dz \leq \|u^0\|^2_{L^2}. \tag{1.1}$$

**Remark 1.4.**

(1) If $(u, p)$ is a solution of $(S)$ given by Theorem 1.3, then by (1.1) and Proposition 2.1(3), we obtain

$$\lim_{t\to 0}\|u(t) - u^0\|_{L^2} = 0.$$

(2) The fact $(e^{\beta|u|^2}-1)|u|^2 \in L^1(\mathbb{R}^+, L^1(\mathbb{R}^3))$ implies that $u \in \cap_{4\le p<\infty} L^p(\mathbb{R}^+, L^p(\mathbb{R}^3))$. Indeed, we have

$$\int_0^\infty \|\Big(e^{\beta|u(t)|^2}-1\Big)|u(t)|^2\|_{L^1}\mathrm{d}t = \sum_{k=1}^\infty \frac{\beta^k}{k!}\int_0^\infty \|u(t)\|_{L^{2k+2}}^{2k+2}\mathrm{d}t.$$

(3) The fact $u \in L^\infty(\mathbb{R}^+, L^2(\mathbb{R}^3)) \cap C(\mathbb{R}^+, H^{-2}(\mathbb{R}^3))$ implies that $u \in C_b(\mathbb{R}^+, H^{-r}(\mathbb{R}^3))$, for all $r>0$.

The remainder of our paper is organized as follows. In Section 2, we provide some notations, definitions, and preliminary results. Section 3 is devoted to prove Theorem 1.3, and this proof is done in two steps. In the first, we give a general result of equicontinuity, and in the second, we apply the Friedritch method to construct a global solution of $(S)$, and the only delicate point is to show the convergence of the nonlinear part $(e^{\beta|u_{\varphi(n)}|^2}-1)u_{\varphi(n)}$ to $(e^{\beta|u|^2}-1)u$. To do this, we approximate $(e^{\beta|u_{\varphi(n)}|^2}-1)u_{\varphi(n)}$ by $\sum_{k=1}^m \frac{\beta^k}{k!}|u_{\varphi(n)}|^{2k}u_{\varphi(n)}$, and we use the convergence in $L^p_{\mathrm{loc}}(\mathbb{R}^+, L^p_{\mathrm{loc}}(\mathbb{R}^3))$ given by technical lemmas.

# 2 Notations and preliminary results

## 2.1 Notations

- For $R>0$: $B_R = \{x \in \mathbb{R}^3;\ |x|<R\}$.
- For $m \in \mathbb{N}$: $P_m(X) = \sum_{k=1}^m \frac{X^k}{k!}$. Clearly, for each $R>0$, we have $P_m(\beta R^2) \nearrow (e^{\beta R^2}-1)$, as $m \to +\infty$.
- For $R>0$, the Friedritch operator $J_R$ is defined by

$$J_R(D)f = \mathcal{F}^{-1}\Big(\mathbf{1}_{B_R}(\xi)\widehat{f}\Big).$$

- The Leray projector $\mathbb{P} : (L^2(\mathbb{R}^3))^3 \to (L^2(\mathbb{R}^3))^3$ is defined by

$$\mathcal{F}(\mathbb{P}f) = \widehat{f}(\xi) - \left(\widehat{f}(\xi)\cdot \frac{\xi}{|\xi|}\right)\frac{\xi}{|\xi|} = M(\xi)\widehat{f}(\xi);\ M(\xi) = \left(\delta_{k,l} - \frac{\xi_k\xi_l}{|\xi|^2}\right)_{1\le k,l\le 3}.$$

- For $R>0$, we define the operator $A_R(D)$ by

$$A_R(D)f = \mathbb{P}J_R(D)f = \mathcal{F}^{-1}\Big(M(\xi)\mathbf{1}_{B_R}(\xi)\widehat{f}\Big).$$

- If $p, q \in (1,\infty)$ and $\Omega_1 \subset \mathbb{R}^n, \Omega_2 \subset \mathbb{R}^m$ are two open subsets, and we define $L^p(\Omega_1, L^q(\Omega_2))$ by: $f \in L^p(\Omega_1, L^q(\Omega_2))$ if $f : \Omega_1 \times \Omega_2 \to \mathbb{C},\ (x_1,x_2) \to f(x_1,x_2)$ is measurable and

$$\|f\|_{L^p(\Omega_1,L^q(\Omega_2))} = \|\|f\|_{L^q(\Omega_2)}\|_{L^p(\Omega_1)}$$

  is well defined and finite. Particularly, if $p=q$, then $L^p(\Omega_1, L^q(\Omega_2)) = L^p(\Omega_1\times\Omega_2)$ and

$$\|f\|_{L^p(\Omega_1,L^p(\Omega_2))} = \|f\|_{L^p(\Omega_1\times\Omega_2)}.$$

## 2.2 Preliminary results

In this section, we collect some classical results, and we provide some lemmas, which are well suited to the study of the system $(S)$.

**Proposition 2.1.** [8] *Let $H$ be Hilbert space.*

(1) *If $(x_n)$ is a bounded sequence of elements in $H$, then there is a subsequence $(x_{\varphi(n)})$ such that*

$$(x_{\varphi(n)}|y) \to (x|y),\ \forall y \in H.$$

(2) *If $x \in H$ and $(x_n)$ is a bounded sequence of elements in $H$ such that*

$$(x_n|y) \to (x|y), \quad \forall y \in H,$$

*then* $\|x\| \leq \liminf_{n\to\infty}\|x_n\|$.

(3) *If $x \in H$ and $(x_n)$ is a bounded sequence of elements in $H$ such that*

$$(x_n|y) \to (x|y), \quad \forall y \in H$$

*and*

$$\limsup_{n\to\infty}\|x_n\| \leq \|x\|,$$

*then* $\lim_{n\to\infty}\|x_n - x\| = 0$.

**Lemma 2.2.** [9] *Let $s_1$ and $s_2$ be two real numbers and $d \in \mathbb{N}$.*

(1) *If $s_1 < d/2$ and $s_1 + s_2 > 0$, there exists a constant $C_1 = C_1(d, s_1, s_2)$, such that: if $f, g \in \dot{H}^{s_1}(\mathbb{R}^d) \cap \dot{H}^{s_2}(\mathbb{R}^d)$, then $f \cdot g \in \dot{H}^{s_1+s_2-\frac{d}{2}}(\mathbb{R}^d)$ and*

$$\|fg\|_{\dot{H}^{s_1+s_2-\frac{d}{2}}} \leq C_1\Big(\|f\|_{\dot{H}^{s_1}}\|g\|_{\dot{H}^{s_2}} + \|f\|_{\dot{H}^{s_2}}\|g\|_{\dot{H}^{s_1}}\Big).$$

(2) *If $s_1, s_2 < d/2$ and $s_1 + s_2 > 0$, there exists a constant $C_2 = C_2(d, s_1, s_2)$ such that: if $f \in \dot{H}^{s_1}(\mathbb{R}^d)$ and $g \in \dot{H}^{s_2}(\mathbb{R}^d)$, then $f \cdot g \in \dot{H}^{s_1+s_2-\frac{d}{2}}(\mathbb{R}^d)$ and*

$$\|fg\|_{\dot{H}^{s_1+s_2-\frac{d}{2}}} \leq C_2\|f\|_{\dot{H}^{s_1}}\|g\|_{\dot{H}^{s_2}}.$$

**Lemma 2.3.** *For $m \in \mathbb{N}$ and $\beta > 0$, there is $c_{m,\beta} > 0$ such that*

$$(P_m(\beta z^2))^2 \leq c_{m,\beta}\Big(e^{\beta z^2} - 1\Big)z^2, \quad \forall z \in \mathbb{R}.$$

**Proof.** It suffices to prove that $(P_m(\beta z^2))^2(e^{\beta z^2} - 1)^{-1}z^{-2}$ is bounded on $(0, \infty)$. □

**Lemma 2.4.** *For $m \in \mathbb{N}$ and $\beta > 0$, there is $C_{m,\beta} > 0$ such that*

$$|P_m(\beta|x|^2)x - P_m(\beta|y|^2)y| \leq C_{m,\beta}(P_m(\beta|x|^2) + P_m(\beta|y|^2))|x - y|, \quad \forall x, y \in \mathbb{R}^3.$$

**Proof.** Suppose that $|y| \leq |x|$. We have

$$P_m(\beta|x|^2)x - P_m(\beta|y|^2)y = P_m(\beta|x|^2)(x - y) + \sum_{k=1}^{m}\frac{\beta^k}{k!}(|x|^{2k} - |y|^{2k})y.$$

For $k \in \{1, \ldots, m\}$, we have

$$\begin{aligned}||x|^{2k} - |y|^{2k}|\cdot|y| &\leq 2k|x - y|\cdot(|x|^{2k-1} + |y|^{2k-1})\cdot|y| \\ &\leq 2m|x - y|\cdot(|x|^{2k-1}|y| + |y|^{2k}) \\ &\leq 2m|x - y|\cdot(|x|^{2k} + |y|^{2k}),\end{aligned}$$

which implies the desired result. □

**Lemma 2.5.** *For all $s > d/2$: $L^1(\mathbb{R}^d) \hookrightarrow H^{-s}(\mathbb{R}^d)$. Moreover, we have*

$$\|f\|_{H^{-s}(\mathbb{R}^d)} \leq \sigma_{s,d}\|f\|_{L^1(\mathbb{R}^d)}, \quad \forall f \in H^{-s}(\mathbb{R}^d),$$

*where* $\sigma_{s,d} = \left(\int_{\mathbb{R}^d}(1 + |\xi|^2)^{-s}\mathrm{d}\xi\right)^{1/2}$.

**Proof.** Using the fact $s > d/2$, we obtain

$$\int_{\mathbb{R}^d} (1+|\xi|^2)^{-s} |\widehat{f}(\xi)|^2 d\xi \leq \left( \int_{\mathbb{R}^d} (1+|\xi|^2)^{-s} d\xi \right) \|\widehat{f}\|^2_{L^\infty(\mathbb{R}^d)} \leq \left( \int_{\mathbb{R}^d} (1+|\xi|^2)^{-s} d\xi \right) \|f\|^2_{L^1(\mathbb{R}^d)},$$

which ends the proof. □

**Lemma 2.6.** *Let $\varepsilon \in (0,1)$ and $d \in \mathbb{N}$ such that $d \geq 2$, then there is a constant $C > 0$ such that If $(f,g) \in H^{-\frac{\varepsilon}{2}}(\mathbb{R}^d) \times H^{1-\frac{\varepsilon}{2}}(\mathbb{R}^d)$, then $fg \in H^{1-\varepsilon-\frac{d}{2}}(\mathbb{R}^d)$ and*

$$\|fg\|_{H^{1-\varepsilon-\frac{d}{2}}(\mathbb{R}^d)} \leq C\|f\|_{H^{-\frac{\varepsilon}{2}}(\mathbb{R}^d)} \|g\|_{H^{1-\frac{\varepsilon}{2}}(\mathbb{R}^d)}.$$

**Proof.** We have

$$\begin{aligned}
\|fg\|^2_{H^{1-\varepsilon-\frac{d}{2}}} &\leq \|fg\|^2_{\dot{H}^{1-\varepsilon-\frac{d}{2}}} \\
&\leq \int_\xi |\xi|^{2\left(1-\varepsilon-\frac{d}{2}\right)} \left| \int_\eta \widehat{f}(\eta)\widehat{g}(\xi-\eta) \right|^2 \\
&\leq \int_\xi |\xi|^{2\left(1-\varepsilon-\frac{d}{2}\right)} \left( \int_\eta |\widehat{f}(\xi-\eta)||\widehat{g}(\eta)| \right)^2 \\
&\leq \int_\xi |\xi|^{2\left(1-\varepsilon-\frac{d}{2}\right)} \left( \int_\eta (1+|\xi-\eta|^2)^{-\varepsilon/4} |\widehat{f}(\xi-\eta)|(1+|\xi-\eta|^2)^{\varepsilon/4}|\widehat{g}(\eta)| \right)^2.
\end{aligned}$$

By using the elementary inequality

$$(1+|\xi-\eta|^2)^{\frac{\varepsilon}{4}} \leq (1+2|\xi|^2+2|\eta|^2)^{\frac{\varepsilon}{4}} \leq 6^{\frac{\varepsilon}{4}}(1+|\xi|^{\frac{\varepsilon}{2}}+|\eta|^{\frac{\varepsilon}{2}}),$$

we obtain

$$\begin{aligned}
\|fg\|^2_{H^{1-\varepsilon-\frac{d}{2}}} &\leq C \int_\xi |\xi|^{2\left(1-\varepsilon-\frac{d}{2}\right)} \left( \int_\eta (1+|\xi-\eta|^2)^{-\frac{\varepsilon}{4}} |\widehat{f}(\xi-\eta)|(1+|\xi|^{\frac{\varepsilon}{2}}+|\eta|^{\varepsilon/2})|\widehat{g}(\eta)| \right)^2 \\
&\leq C(I_1+I_2+I_3),
\end{aligned}$$

with

$$\begin{aligned}
I_1 &= \int_\xi |\xi|^{2\left(1-\varepsilon-\frac{d}{2}\right)} \left( \int_\eta (1+|\xi-\eta|^2)^{-\frac{\varepsilon}{4}} |\widehat{f}(\xi-\eta)||\widehat{g}(\eta)| \right)^2 = \|u_1 v_1\|^2_{\dot{H}^{1-\varepsilon-\frac{d}{2}}}, \\
u_1 &= \mathcal{F}^{-1}((1+|\xi|^2)^{-\frac{\varepsilon}{4}}|\widehat{f}(\xi)|), \\
v_1 &= \mathcal{F}^{-1}(|\widehat{g}(\xi)|), \\
I_2 &= \int_\xi |\xi|^{2\left(1-\frac{\varepsilon}{2}-\frac{d}{2}\right)} \left( \int_\eta (1+|\xi-\eta|^2)^{-\frac{\varepsilon}{4}} |\widehat{f}(\xi-\eta)||\widehat{g}(\eta)| d\eta \right)^2 d\xi = \|u_2 v_2\|^2_{\dot{H}^{1-\frac{\varepsilon}{2}-\frac{d}{2}}}, \\
u_2 &= \mathcal{F}^{-1}((1+|\xi|^2)^{-\frac{\varepsilon}{4}}|\widehat{f}(\xi)|), \\
v_2 &= \mathcal{F}^{-1}(|\widehat{g}(\xi)|),
\end{aligned}$$

and

$$I_3 = \int_\xi |\xi|^{2\left(1-\varepsilon-\frac{d}{2}\right)} \left( \int_\eta (1+|\xi-\eta|^2)^{-\frac{\varepsilon}{4}} |\widehat{f}(\xi-\eta)| |\eta|^{\frac{\varepsilon}{2}} |\widehat{g}(\eta)| \right)^2 = \|u_3 v_3\|^2_{\dot{H}^{1-\varepsilon-\frac{d}{2}}},$$
$$u_3 = \mathcal{F}^{-1}\Big((1+|\xi|^2)^{-\frac{\varepsilon}{4}} \widehat{f}(\xi)\Big),$$
$$v_3 = \mathcal{F}^{-1}\Big(|\xi|^{\frac{\varepsilon}{2}} |\widehat{g}(\xi)|\Big).$$

By applying Lemma 2.2 with the following choices:

$$(s_1, t_1) = (0, 1-\varepsilon), \ (s_2, t_2) = \left(0, 1-\frac{\varepsilon}{2}\right), \ (s_3, t_3) = (0, 1-\varepsilon),$$

we obtain

$$\|u_i v_i\|_{\dot{H}^{s_i+t_i-\frac{d}{2}}(\mathbb{R}^d)} \leq C\|u_i\|_{\dot{H}^{s_i}(\mathbb{R}^d)} \|v_i\|_{\dot{H}^{t_i}(\mathbb{R}^d)}, \ i \in \{1,2,3\}.$$

Then

$$I_1 \leq C\|u_1\|^2_{\dot{H}^0} \|v_1\|^2_{\dot{H}^{1-\varepsilon}} \leq C\|f\|^2_{H^{-\frac{\varepsilon}{2}}} \|g\|^2_{\dot{H}^{1-\varepsilon}} \leq C\|f\|^2_{H^{-\frac{\varepsilon}{2}}} \|g\|^2_{H^{1-\varepsilon}} \leq C\|f\|^2_{H^{-\frac{\varepsilon}{2}}} \|g\|^2_{H^{1-\frac{\varepsilon}{2}}},$$
$$I_2 \leq C\|u_2\|^2_{\dot{H}^0} \|v_2\|^2_{\dot{H}^{1-\frac{\varepsilon}{2}}} \leq C\|f\|^2_{H^{-\frac{\varepsilon}{2}}} \|g\|^2_{H^{1-\frac{\varepsilon}{2}}},$$
$$I_3 \leq C\|u_3\|^2_{\dot{H}^0} \|v_3\|^2_{\dot{H}^{1-\varepsilon}} \leq C\|f\|^2_{H^{-\frac{\varepsilon}{2}}} \|g\|^2_{\dot{H}^{1-\frac{\varepsilon}{2}}} \leq C\|f\|^2_{H^{-\frac{\varepsilon}{2}}} \|g\|^2_{H^{1-\frac{\varepsilon}{2}}},$$

which ends the proof of the Lemma. □

**Lemma 2.7.** *Let $p \in (1, +\infty)$ and $\Omega \neq \emptyset$ be an open subset of $\mathbb{R}^4$. If $(f_n)$ is a bounded sequence in $L^2(\Omega) \cap L^p(\Omega)$, then there is a subsequence $(f_{h(n)})$ and $f \in L^2(\Omega) \cap L^p(\Omega)$ such that*

$$\lim_{n\to\infty} (f_{h(n)}|g)_{L^2} = (f|g)_{L^2}, \ \forall g \in L^2(\Omega),$$
$$\|f\|_{L^p} \leq \liminf_{n\to\infty} \|f_{h(n)}\|_{L^p},$$
$$\|f\|_{L^2} \leq \liminf_{n\to\infty} \|f_{h(n)}\|_{L^2}.$$

**Proof.** Let $q$ the conjugate exponent of Hölder inequality

$$\frac{1}{p} + \frac{1}{q} = 1.$$

The sequence $(f_n)$ is bounded in the Hilbert space $L^2(\Omega)$, and then there is a subsequence $(f_{h(n)})$ and $f \in L^2(\Omega)$ such that

$$\lim_{n\to\infty} (f_{h(n)}|g)_{L^2} = (f|g)_{L^2}, \ g \in L^2(\Omega).$$

By Hölder inequality, we obtain

$$|(f_{h(n)}|g)_{L^2}| \leq \|f_{h(n)}\|_{L^p} \|g\|_{L^q}, \ \forall n \in \mathbb{N},$$

which implies

$$|(f|g)_{L^2}| \leq \left(\liminf_{n\to\infty} \|f_{h(n)}\|_{L^p}\right) \cdot \|g\|_{L^q}.$$

Particularly, for $g \in L^1(\Omega) \cap L^\infty(\Omega)$, we have

$$|(f|g)_{L^2}| \leq \left(\liminf_{n\to\infty} \|f_{h(n)}\|_{L^p}\right) \cdot \|g\|_{L^q}. \tag{2.1}$$

For $k \in \mathbb{N}$, put the following functions:

$$g(x) = \begin{cases} \dfrac{\overline{f(x)}}{|f(x)|}|f(x)|^{q-1}, & \text{if } f(x) \neq 0, \\ 0, & \text{if } F(x) = 0 \end{cases}$$

and

$$g_k(x) = \mathbf{1}_{A_k}(x) g(x),$$

where

$$A_k = \{x \in \Omega;\ |g(x)| \le k\} \cap B_k.$$

Clearly, $g_k \in L^1(\Omega) \cap L^\infty(\Omega)$ and $\mathbf{1}_{A_k} \longrightarrow \mathbf{1}_\Omega$ almost every where. Applying inequality (2.1) to $g_k$, we obtain

$$\int_\Omega \mathbf{1}_{A_k}(x)|f(x)|^p \mathrm{d}x \le \big(\liminf_{n\to\infty} \|f_{h(n)}\|_{L^p}\big)\Bigg(\int_\Omega \mathbf{1}_{A_k}(x)|f(x)|^p \mathrm{d}x\Bigg)^{1/q}$$

and

$$\Bigg(\int_\Omega \mathbf{1}_{A_k}(x)|f(x)|^p \mathrm{d}x\Bigg)^{1/p} \le \liminf_{n\to\infty} \|f_{h(n)}\|_{L^p}.$$

As $A_k \subset A_{k+1}$ and

$$\mathbf{1}_{A_k}(x)|f(x)|^p \longrightarrow |f(x)|^p,\ \ a.e\ x \in \Omega,$$

then Monotone Convergence Theorem implies that $f \in L^p(\Omega)$ and

$$\|f\|_{L^p} \le \liminf_{n\to\infty} \|f_{h(n)}\|_{L^p},$$

which ends the proof of the first inequality.

The last result is given by Proposition 2.1. □

**Remark 2.8.** Let $p \in (1, +\infty)$ and $\Omega \neq \varnothing$ be an open subset of $\mathbb{R}^4$. If $(f_n)$ is a bounded sequence in $L^2(\Omega) \cap L^p(\Omega)$ and $f \in L^2(\Omega)$ such that

$$\lim_{n\to\infty} (f_n|g)_{L^2} = (f|g)_{L^2},\ \ \forall g \in L^2(\Omega).$$

Then $f \in L^p(\Omega)$ and

$$\|f\|_{L^2} \le \liminf_{n\to\infty} \|f_n\|_{L^2},$$
$$\|f\|_{L^p} \le \liminf_{n\to\infty} \|f_n\|_{L^p}.$$

**Lemma 2.9.** *Let $T > 0$ and $(f_n)$ be a bounded sequence in $L^2_T(H^1(\mathbb{R}^3))$ such that*

$$f_n \to f \ \text{ strongly in } C([0,T], H^{-4}_{\mathrm{loc}}(\mathbb{R}^3)).$$

*Then*

$$f_n \longrightarrow f \ \text{ strongly in } L^2([0,T], L^2_{\mathrm{loc}}(\mathbb{R}^3)).$$

**Proof.** Combining the inclusion $C([0,T], H^{-4}_{\mathrm{loc}}(\mathbb{R}^3)) \hookrightarrow L^2([0,T], H^{-4}_{\mathrm{loc}}(\mathbb{R}^3))$ and the interpolation

$$L^2([0,T], H^{-4}_{\mathrm{loc}}(\mathbb{R}^3) \cap H^1(\mathbb{R}^3)) \hookrightarrow L^2([0,T], L^2_{\mathrm{loc}}(\mathbb{R}^3)),$$

we obtain the desired result. □

**Lemma 2.10.** *Let $T > 0$ and $p_0 \in (2, +\infty)$. If $(f_n)$ is a bounded sequence in $L^2_T(H^1(\mathbb{R}^3)) \cap L^{p_0}_T(L^{p_0}(\mathbb{R}^3))$ such that*

$$f_n \to f \text{ strongly in } C([0, T], H^{-4}_{\mathrm{loc}}(\mathbb{R}^3)),$$

*then $f \in L^{p_0}_T(L^{p_0}(\mathbb{R}^3))$ and*

$$f_n \longrightarrow f \text{ strongly in } L^p([0, T], L^p_{\mathrm{loc}}(\mathbb{R}^3)), \ \forall 2 \le p < p_0.$$

**Proof.** By Lemma 2.9, we obtain

$$f_n \longrightarrow f \text{ strongly in } L^2([0, T], L^2_{\mathrm{loc}}(\mathbb{R}^3)).$$

By interpolation, between $L^{p_0}([0, T], L^{p_0}(\mathbb{R}^3))$ and $L^2([0, T], L^2_{\mathrm{loc}}(\mathbb{R}^3))$, we obtain

$$f_n \to f \text{ strongly in } L^p([0, T], L^p_{\mathrm{loc}}(\mathbb{R}^3)), \ \forall p \in [2, p_0).$$

It remains to show that $f \in L^{p_0}_T(L^{p_0}(\mathbb{R}^3))$. For $m \in \mathbb{N}$, put

$$\Omega_m = ]0, T[ \times B_m \text{ and } g_{m,n} = \mathbf{1}_{\Omega_m} f_n.$$

Clearly, $\Omega_m$ is an open subset of $\mathbb{R}^4$ and the sequence $(g_{m,n})_n$ is bounded in

$$L^2([0, T], L^2(B_m)) \cap L^{p_0}\big([0, T], L^{p_0}(B_m)\big) = L^2(\Omega_m) \cap L^{p_0}(\Omega_m).$$

Then, by Lemma 2.7 and Remark 2.8, we obtain $\mathbf{1}_{\Omega_m} f \in L^2(\Omega_m) \cap L^{p_0}(\Omega_m)$ and

$$\|\mathbf{1}_{\Omega_m} f\|_{L^{p_0}(\Omega_m)} \le \liminf_{n\to\infty} \|g_{n,m}\|_{L^{p_0}(\Omega_m)}.$$

But $\|g_{n,m}\|_{L^{p_0}(\Omega_m)} = \|f_n\|_{L^{p_0}(\Omega_m)} \le \|f_n\|_{L^{p_0}_T(L^{p_0}(\mathbb{R}^3))}$, then

$$\|\mathbf{1}_{\Omega_m} f\|_{L^{p_0}(\Omega_m)} \le \liminf_{n\to\infty} \|f_n\|_{L^{p_0}_T\left(L^{p_0}(\mathbb{R}^3)\right)}.$$

By applying the monotonic convergence theorem, we obtain $f \in L^{p_0}_T(L^{p_0}(\mathbb{R}^3))$ and

$$\|f\|_{L^{p_0}_T\left(L^{p_0}(\mathbb{R}^3)\right)} \le \liminf_{n\to\infty} \|f_n\|_{L^{p_0}_T\left(L^{p_0}(\mathbb{R}^3)\right)}.$$

# 3 Proof of Theorem 1.3

This proof is done in two steps:

## 3.1 Step 1

In this step, we prove a general result for bounded sequence in energy space of the system $(S)$.

**Proposition 3.1.** *Let $\nu_1, \nu_2, \nu_3 \in [0, \infty)$, $r_1, r_2, r_3 \in (0, \infty)$, and $f_0 \in L^2_\sigma(\mathbb{R}^3)$. For $n \in \mathbb{N}$, let $F_n : \mathbb{R}^+ \times \mathbb{R}^3 \to \mathbb{R}^3$ be a measurable function in $C^1(\mathbb{R}^+, L^2(\mathbb{R}^3))$ such that*

$$A_n(D)F_n = F_n, \ F_n(0, x) = A_n(D)f_0(x)$$

*and*

$$(E1)\ \ \partial_t F_n + \sum_{k=1}^{3} \nu_k |D_k|^{2r_k} F_n + A_n(D)\,\mathrm{div}(F_n \otimes F_n) + \alpha A_n(D)\Big[\Big(e^{\beta|F_n|^2} - 1\Big)F_n\Big] = 0,$$

$$(E2)\ \ \|F_n(t)\|^2_{L^2} + 2\sum_{k=1}^{3} \nu_k \int_0^t \||D_k|^{r_k} F_n\|^2_{L^2} + 2\alpha \int_0^t \|\Big(e^{\beta|F_n|^2} - 1\Big)|F_n|^2\|_{L^1} \le \|f_0\|^2_{L^2}.$$

*Then*:
- *For $m \in \mathbb{N}$, there is a constant $C = C(m, \alpha, \beta)$ such that*

$$\int_0^\infty \int_{\mathbb{R}^3} (P_m(\beta|F_n(t,x)|^2))^2 dt dx \le C\|f_0\|_{L^2}^2. \tag{3.1}$$

- *For all $T, R > 0$ and $s > 3/2$, we have*

$$\int_0^T \|\left(e^{\beta|F_n|^2} - 1\right)|F_n|\|_{H^{-s}} \le \sigma_{3,s}\left(M_{\beta,R}T + \frac{1}{2R\alpha}\right)\|f_0\|_{L^2}^2, \quad \forall n \in \mathbb{N}, \tag{3.2}$$

*where* $M_{\beta,R} = \sup_{0<r\le R}\frac{e^{\beta r^2}-1}{r}$.
- *For every $\varepsilon > 0$, there is $\delta = \delta(\varepsilon, \alpha, \beta, \nu_1, \nu_2, \nu_3, r_1, r_2, r_3, \|f_0\|_{L^2}) > 0$ such that for all $t_1, t_2 \in \mathbb{R}^+$, we have*

$$(|t_2 - t_1| < \delta \ \Rightarrow\ \|F_n(t_2) - F_n(t_1)\|_{H^{-s_0}} < \varepsilon), \quad \forall n \in \mathbb{N}, \tag{3.3}$$

*with* $s_0 = \max(3, 2\max_{1\le i\le 3} r_i)$.

**Remark 3.2.** As $\|F_n(t)\|_{L^2} \le \|f_0\|_{L^2}$ for all $(t,n) \in \mathbb{R}^+ \times \mathbb{N}$, then by interpolation between $H^{-s_0}(\mathbb{R}^3)$ and $H^0(\mathbb{R}^3) = L^2(\mathbb{R}^3)$, we obtain for all $\sigma > 0$, for each $\varepsilon > 0$, there is $\delta > 0$ such that

$$(|t_2 - t_1| < \delta \ \Rightarrow\ \|F_n(t_2) - F_n(t_1)\|_{H^{-\sigma}} < \varepsilon), \quad \forall n \in \mathbb{N}.$$

**Proof of Proposition 3.1.**
- First, (3.1) is given by $(E2)$ and Lemma 2.3.
- To prove (3.2), beginning by noting that the function $h_\beta : [0, +\infty) \longrightarrow \mathbb{R}^+,\ y \longmapsto \begin{cases} \frac{e^{\beta y^2}-1}{y} & \text{if } t > 0 \\ 0 & \text{if } t = 0 \end{cases}$ is continuous, and $M_{\beta,R} = \sup_{y\in[0,R]} h_\beta(y)$ is well defined and finite.

For $t \in [0, T]$, put the following subset of $\mathbb{R}^3$

$$X_{n,R,t} = \{x \in \mathbb{R}^3;\ |F_n(t,x)| \le R\}.$$

Then, for $x \in \mathbb{R}^3$, we have

$$\begin{aligned} x \in X_{n,R,t} &\Rightarrow \left(e^{\beta|F_n(t,x)|^2} - 1\right)|F_n(t,x)| = h_\beta(|F_n(z,x)|)|F_n(t,x)|^2 \le M_{\beta,R}|F_n(t,x)|^2, \\ x \in X_{n,R,t}^c &\Rightarrow \left(e^{\beta|F_n(t,x)|^2} - 1\right)|F_n(t,x)| \le \frac{1}{R}\left(e^{\beta|F_n(t,x)|^2} - 1\right)|F_n(t,x)|^2. \end{aligned}$$

By Lemma 2.5, we obtain

$$D_n(T) := \int_0^T \|\left(e^{\beta|F_n(z)|^2} - 1\right)|F_n(z)|\|_{H^{-s}} \le \sigma_{3,s}\int_0^T \|\left(e^{\beta|F_n(z)|^2} - 1\right)|F_n(z)|\|_{L^1} dz.$$

Combining this inequality with the aforementioned inequalities, we obtain

$$\begin{aligned} \frac{1}{\sigma_{3,s}} D_n(T) &= \int_0^T \int_{\mathbb{R}^3} \left(e^{\beta|F_n(z,x)|^2} - 1\right)|F_n(z,x)| \\ &= \int_0^T \int_{X_{n,R,z}} \left(e^{\beta|F_n(z,x)|^2} - 1\right)|F_n(z,x)| + \int_0^T \int_{X_{n,R,z}^c} \left(e^{\beta|F_n(z,x)|^2} - 1\right)|F_n(z,x)| \\ &\le M_{\beta,R}\int_0^T \int_{X_{n,R,z}} |F_n(z,x)|^2 + \frac{1}{R}\int_0^T \int_{X_{n,R,z}^c} \left(e^{\beta|F_n(z,x)|^2} - 1\right)|F_n(z,x)|^2 \end{aligned}$$

$$\leq M_{\beta,R}\int_0^T\int_{\mathbb{R}^3}|F_n(z,x)|^2+\frac{1}{R}\int_0^T\int_{\mathbb{R}^3}\Big(e^{\beta|F_n(z,x)|^2}-1\Big)|F_n(z,x)|^2$$
$$\leq M_{\beta,R}\int_0^T\|F_n(z)\|_{L^2}^2\mathrm{d}z+\frac{1}{R}\int_0^T\|\Big(e^{\beta|F_n(z)|^2}-1\Big)|F_n(z)|^2\|_{L^1(\mathbb{R}^3)}.$$

By using $(E_2)$, we obtain the desired result.

- To prove (3.3), integrate $(E1)$ over $[t_1,t_2]\subset\mathbb{R}^+$, we obtain

$$\|F_n(t_2)-F_n(t_1)\|_{H^{-s_0}}\leq I_{n,1}(t_1,t_2)+I_{n,2}(t_1,t_2)+I_{n,3}(t_1,t_2),$$

with

$$I_{n,1}(t_1,t_2)=\sum_{k=1}^{3}\nu_k\int_{t_1}^{t_2}\||D_k|^{2r_k}F_n\|_{H^{-s_0}},$$
$$I_{n,2}(t_1,t_2)=\int_{t_1}^{t_2}\|A_n(D)\,\mathrm{div}(F_n\otimes F_n)\|_{H^{-s_0}},$$
$$I_{n,3}(t_1,t_2)=\int_{t_1}^{t_2}\|\alpha A_n(D)\Big[\Big(e^{\beta|F_n|^2}-1\Big)F_n\Big]\|_{H^{-s_0}}.$$

Let $\varepsilon>0$ be a positive real, let us find a positive real $\delta>0$ such that if $|t_2-t_1|<\delta$, we obtain

$$I_{n,k}(t_1,t_2)<\varepsilon/3,\ \ k\in\{1,2,3\}.$$

- To estimate $I_{n,1}(t_1,t_2)$, we write

$$I_{n,1}(t_1,t_2)\leq\sum_{k=1}^{3}\nu_k\int_{t_1}^{t_2}\|F_n\|_{H^{2r_k-s_0}}\leq\left(\sum_{k=1}^{3}\nu_k\right)\int_{t_1}^{t_2}\|F_n(z)\|_{H^0}\mathrm{d}z\leq\left(\sum_{k=1}^{3}\nu_k\right)\|\widehat{f_0}\|_{L^2}(t_2-t_1).$$

Then if $|t_2-t_1|<\delta_1:=\frac{\varepsilon}{3(\sum_{k=1}^{3}\nu_k)\,\|\widehat{f_0}\|_{L^2}+3}$, we obtain $I_{n,1}(t_1,t_2)<\varepsilon/3$.

- Estimate $I_{n,2}(t_1,t_2)$: By Lemma 2.5, we obtain

$$\begin{aligned}I_{n,2}(t_1,t_2)&\leq\int_{t_1}^{t_2}\|\mathrm{div}(F_n\otimes F_n)(z)\|_{H^{-3}}\mathrm{d}z\\&\leq\int_{t_1}^{t_2}\|(F_n\otimes F_n)(z)\|_{H^{-2}}\mathrm{d}z\\&\leq\sigma_{2,3}\int_{t_1}^{t_2}\|(F_n\otimes F_n)(z)\|_{L^1}\mathrm{d}z\\&\leq\sigma_{2,3}\int_{t_1}^{t_2}\|F_n(z)\|_{L^2}^2\mathrm{d}z\\&\leq\sigma_{2,3}\|f_0\|_{L^2}^2(t_2-t_1).\end{aligned}$$

Then if $|t_2-t_1|<\delta_2:=\frac{\varepsilon}{3\sigma_{2,3}\,\|f_0\|_{L^2}^2+3}$, we obtain $I_{n,2}(t_1,t_2)<\varepsilon/3$.

- Estimate of $I_{n,3}(t_1,t_2)$: Let $R>0$ (to fixed later) and $t_1<t_2\in\mathbb{R}^+$. By Lemma 2.5, we obtain

$$\int_{t_1}^{t_2}\|\left(e^{\beta|F_n(z)|^2}-1\right)|F_n(z)|\|_{H^{-3}}\leq\sigma_{3,3}\int_{t_1}^{t_2}\|\left(e^{\beta|f(z)|^2}-1\right)|f(z)|\|_{L^1}dz.$$

Combining this inequality with the above inequalities, we obtain

$$\begin{aligned}\frac{I_3(t_1,t_2)}{\alpha\sigma_{3,3}}&=\int_{t_1}^{t_2}\int_{\mathbb{R}^3}\left(e^{\beta|F_n(z,x)|^2}-1\right)|F_n(z,x)|\\&=\int_{t_1}^{t_2}\int_{X_{n,R,z}}\left(e^{\beta|F_n(z,x)|^2}-1\right)|F_n(z,x)|+\int_{t_1}^{t_2}\int_{X_{n,R,z}^c}\left(e^{\beta|F_n(z,x)|^2}-1\right)|F_n(z,x)|\\&\leq M_{\beta,R}\int_{t_1}^{t_2}\int_{X_{n,R,z}}|F_n(z,x)|^2+\frac{1}{R}\int_{t_1}^{t_2}\int_{X_{n,R,z}^c}\left(e^{\beta|F_n(z,x)|^2}-1\right)|F_n(z,x)|^2\\&\leq M_{\beta,R}\int_{t_1}^{t_2}\int_{\mathbb{R}^3}|F_n(z,x)|^2+\frac{1}{R}\int_{t_1}^{t_2}\int_{\mathbb{R}^3}\left(e^{\beta|F_n(z,x)|^2}-1\right)|F_n(z,x)|^2\\&\leq M_{\beta,R}\int_{t_1}^{t_2}\|F_n(z)\|_{L^2}^2dz+\frac{1}{R}\int_{t_1}^{t_2}\|\left(e^{\beta|F_n(z)|^2}-1\right)|F_n(z)|^2\|_{L^1(\mathbb{R}^3)}.\end{aligned}$$

By using $(E_2)$, we obtain

$$\frac{I_3(t_1,t_2)}{\alpha\sigma_{3,3}}\leq M_{\beta,R}\|f_0\|_{L^2}^2(t_2-t_1)+\frac{1}{2R\alpha}\|f_0\|_{L^2}^2. \tag{3.4}$$

Then, with the choices $R=R_\varepsilon=\frac{3\sigma_{3,3}\|f_0\|_{L^2}^2+3}{\varepsilon}$ and $\delta_3=\frac{\varepsilon}{6\alpha\sigma_{3,3}M_{\beta,R_\varepsilon}\|f_0\|_{L^2}^2+6}$, we obtain

$$|t_1-t_2|<\delta_3\ \Rightarrow\ I_{n,3}(t_1,t_2)<\varepsilon/3.$$

To conclude, it suffices to take $\delta=\min\{\delta_1,\delta_2,\delta_3\}$. □

## 3.2 Step 2

In this step, we construct a global solution of $(S)$, where we use a method inspired by [9]. For this, consider the approximate system with the parameter $n\in\mathbb{N}$:

$$(S_n)\begin{cases}\partial_t u-\Delta J_n u+J_n(J_nu.\nabla J_nu+\alpha J_n\left[\left(e^{\beta|J_nu|^2}-1\right)J_nu\right]=-\nabla p_n\quad\text{in }\mathbb{R}^+\times\mathbb{R}^3,\\ p_n=(-\Delta)^{-1}(\operatorname{div}J_n(J_nu.\nabla J_nu+\alpha\operatorname{div}J_n\left[\left(e^{\beta|J_nu|^2}-1\right)J_nu\right]),\\ \operatorname{div}u=0\quad\text{in }\mathbb{R}^+\times\mathbb{R}^3,\\ u(0,x)=J_nu^0(x)\ \text{ in }\mathbb{R}^3.\end{cases}$$

- By Cauchy-Lipschitz theorem, we obtain a unique solution $u_n\in C^1(\mathbb{R}^+,L^2(\mathbb{R}^3))$ of $(S_n)$. Moreover, $J_nu_n=u_n$ and

$$\|u_n(t)\|_{L^2}^2+2\int_0^t\|\nabla u_n\|_{L^2}^2+2\alpha\int_0^t\|\left(e^{\beta|u_n|^2}-1\right)|u_n|^2\|_{L^1}\leq\|u^0\|_{L^2}^2. \tag{3.5}$$

- By inequality (3.5), we obtain $(u_n)$ is bounded in $L^2_{\mathrm{loc}}(\mathbb{R}^+, H^1(\mathbb{R}^3))$:

$$\int_0^T \left(\|u_n\|_{L^2}^2 + \|\nabla u_n\|_{L^2}^2\right) \leq \|u^0\|_{L^2}^2 T + \frac{\|u^0\|_{L^2}^2}{2} = M_T, \quad \forall T \geq 0. \tag{3.6}$$

- By using inequality (3.5) and the fact that

$$\left(e^{\beta|u_n|^2} - 1\right)|u_n|^2 = \sum_{k=1}^{\infty} \frac{\beta^k}{k!}|u_n|^{2k+2}, \tag{3.7}$$

  we obtain for all $k \in \mathbb{N}$, $(u_n)$ is bounded in $L^{2k+2}(\mathbb{R}^+ \times \mathbb{R}^3)$ and

$$\int_0^\infty \int_{\mathbb{R}^3} |u_n|^{2k+2} \leq \frac{k!2\alpha}{\beta^k}. \tag{3.8}$$

- By applying Proposition 3.1 and Remark 2.8, we obtain

$$\text{the sequence } (u_n) \text{ is equicontinuous in } C_b(\mathbb{R}^+, H^{-1}(\mathbb{R}^3)). \tag{3.9}$$

- Let $(T_q)_q \in (0,\infty)^{\mathbb{N}}$ such that $T_q < T_{q+1}$ and $T_q \to \infty$ as $q \to \infty$. Let $(\theta_q)_{q\in\mathbb{N}}$ be a sequence in $C_0^\infty(\mathbb{R}^3)$ such that: for all $q \in \mathbb{N}$,

$$\begin{cases} \theta_q(x) = 1, & \forall x \in B\left(0, q+1+\frac{1}{4}\right), \\ \theta_q(x) = 0, & \forall x \in B(0, q+2)^c, \\ 0 \leq \theta_q \leq 1. \end{cases}$$

Using (3.5)–(3.9) and classical argument by combining Ascoli's theorem and the Cantor diagonal process, we obtain a nondecreasing $\varphi : \mathbb{N} \to \mathbb{N}$ and $u \in L^\infty(\mathbb{R}^+, L^2(\mathbb{R}^3)) \cap C(\mathbb{R}^+, H^{-3}(\mathbb{R}^3))$ such that for all $q \in \mathbb{N}$, we have

$$\lim_{n\to\infty} \|\theta_q(u_{\varphi(n)} - u)\|_{L^\infty([0,T_q],H^{-4})} = 0. \tag{3.10}$$

- Combining inequalities (3.7)–(3.10) and applying Lemma 2.10, we obtain

$$u_{\varphi(n)} \to u \text{ strongly in } L^p_{\mathrm{loc}}(\mathbb{R}^+, L^p_{\mathrm{loc}}(\mathbb{R}^3)), \quad \forall p \in [2, \infty). \tag{3.11}$$

- The sequence $(u_n)$ is bounded in the Hilbert space $L^2(\mathbb{R}^+, \dot{H}^1(\mathbb{R}^3))$, then

$$u_{\varphi(n)} \to u \text{ weakly in } L^2(\mathbb{R}^+, \dot{H}^1(\mathbb{R}^3)).$$

Particularly $u \in L^2(\mathbb{R}^+, \dot{H}^1(\mathbb{R}^3))$ and

$$\int_0^t \|\nabla u\|_{L^2}^2 \leq \liminf_{n\to\infty} \int_0^t \|\nabla u_{\varphi(n)}\|_{L^2}^2, \quad \forall t \geq 0. \tag{3.12}$$

- Prove that for all $T > 0$ and $k \in \mathbb{N}$,

$$\int_0^T \int_{\mathbb{R}^3} |u|^{2k+2} \leq \liminf_{n\to\infty} \int_0^T \int_{\mathbb{R}^3} |u_{\varphi(n)}|^{2k+2}. \tag{3.13}$$

For this take $m \in \mathbb{N}$ and put $\Omega_m = ]0, T[ \times B_m$. Applying Lemma 2.10 to the sequence $(\mathbf{1}_{\Omega_m} u_{\varphi(n)})$ with $p_0 = 2k + 4$ and $p = 2k + 2$, with taking into account (3.8), we obtain

$$\int_0^T \int_{B_m} |u|^{2k+2} \leq \liminf_{n\to\infty} \int_0^T \int_{B_m} |u_{\varphi(n)}|^{2k+2}.$$

Then

$$\int_0^T\int_{B_m}|u|^{2k+2}\leq\liminf_{n\to\infty}\int_0^T\int_{\mathbb{R}^3}|u_{\varphi(n)}|^{2k+2}.$$

By using monotonic convergence theorem, we obtain the desired result.

- Prove that for all $T>0$,

$$\int_0^T\int_{\mathbb{R}^3}\Big(e^{\beta|u|^2}-1\Big)|u|^2\leq\liminf_{n\to\infty}\int_0^T\int_{\mathbb{R}^3}\Big(e^{\beta|u_{\varphi(n)}|^2}-1\Big)|u_{\varphi(n)}|^2. \tag{3.14}$$

For this, take $m\in\mathbb{N}$ and by (3.13), we obtain

$$\int_0^T\int_{\mathbb{R}^3}P_m(\beta|u|^2)|u|^2\leq\liminf_{n\to\infty}\int_0^T\int_{\mathbb{R}^3}P_m(\beta|u_{\varphi(n)}|^2)|u_{\varphi(n)}|^2.$$

Then

$$\int_0^T\int_{\mathbb{R}^3}P_m(\beta|u|^2)|u|^2\leq\liminf_{n\to\infty}\int_0^T\int_{\mathbb{R}^3}\Big(e^{\beta|u_{\varphi(n)}|^2}-1\Big)|u_{\varphi(n)}|^2.$$

Monotonic convergence theorem gives the desired result.

- Combining the aforementioned inequalities, we obtain for all $t\geq0$,

$$\|u(t)\|_{L^2}^2+2\int_0^t\|\nabla u(z)\|_{L^2}^2dz+2\alpha\int_0^t\|\Big(e^{\beta|u(z)|^2}-1\Big)|u(z)|^2\|_{L^1}dz\leq\|u^0\|_{L^2}^2. \tag{3.15}$$

It remains to show that $u$ is a solution of the system $(S)$. To do this, we must verify that

$$\lim_{n\to\infty}J_{\varphi(n)}(u^i_{\varphi(n)}u^j_{\varphi(n)})=u^iu^j\text{ in }\mathcal{D}'([0,\infty)\times\mathbb{R}^3), \tag{3.16}$$

$$\lim_{n\to\infty}J_{\varphi(n)}[\Big(e^{\beta|u_{\varphi(n)}|^2}-1\Big)u_{\varphi(n)}]=\Big(e^{\beta|u|^2}-1\Big)u\text{ in }\mathcal{D}'([0,\infty)\times\mathbb{R}^3). \tag{3.17}$$

- Let $\Phi$ in $C_0^\infty(\mathbb{R}^+\times\mathbb{R}^3)$. There is an integer $q\in\mathbb{N}$ such that

$$\mathrm{supp}(\Phi)\subset[0,T_q)\times B(0,q).$$

We start by proving (3.16). For this, we write

$$\Phi u^i_{\varphi(n)}u^j_{\varphi(n)}-\Phi u^iu^j=\theta_q\Phi u^i_{\varphi(n)}u^j_{\varphi(n)}-\theta_q\Phi u^iu^j=\theta_q(u^i_{\varphi(n)}-u^i)\Phi u^j_{\varphi(n)}+\theta_q(u^j_{\varphi(n)}-u^j)\Phi u^i.$$

Using Lemma 2.6, we obtain, for $\varepsilon_0\in(0,1)$,

$$\begin{aligned}
&\|\theta_q(u^i_{\varphi(n)}-u^i)\Phi u^j_{\varphi(n)}\|_{L^2(\mathbb{R}^+,H^{-\varepsilon_0-\frac12})}\\
&\leq C\|\theta_q(u^i_{\varphi(n)}-u^i)\|_{L^\infty\left([0,T_q],H^{-\frac{\varepsilon_0}{2}}\right)}\|\Phi u^j_{\varphi(n)}\|_{L^2\left([0,T_q],H^{1-\frac{\varepsilon_0}{2}}\right)}\\
&\leq C\|\theta_q(u^i_{\varphi(n)}-u^i)\|_{L^\infty\left([0,T_q],H^{-\frac{\varepsilon_0}{2}}\right)}\|\Phi u^j_{\varphi(n)}\|_{L^2([0,T_q],H^1)}\\
&\leq C'\|\theta_q(u^i_{\varphi(n)}-u^i)\|_{L^\infty\left([0,T_q];\,H^{-\frac{\varepsilon_0}{2}}\right)}\|u^j_{\varphi(n)}\|_{L^2([0,T_q];\,H^1)}\\
&\leq C'M_{T_q}\|\theta_q(u^i_{\varphi(n)}-u^i)\|_{L^\infty\left([0,T_q];\,H^{-\frac{\varepsilon_0}{2}}\right)},
\end{aligned}$$

where $M_{T_q}$ is given by (3.6). Then, equation (3.10) implies

$$\lim_{n\to\infty} \|\theta_q(u^i_{\varphi(n)} - u^i)\Phi u^j_{\varphi(n)}\|_{L^2(\mathbb{R}^+,H^{-\varepsilon_0-\frac{1}{2}})} = 0.$$

In a similar way, we show that

$$\lim_{n\to\infty} \theta_q(u^j_{\varphi(n)} - u^j)\Phi u^i = 0 \text{ in } L^2(\mathbb{R}^+, H^{-\varepsilon_0-\frac{1}{2}}(\mathbb{R}^3)).$$

Then, we obtain

$$\lim_{n\to\infty} \Phi u^i_{\varphi(n)} u^j_{\varphi(n)} = \Phi u^i u^j \text{ in } L^2(\mathbb{R}^+, H^{-\varepsilon_0-\frac{1}{2}}(\mathbb{R}^3)).$$

Using the fact that $(u_n)$ is bounded in $L^\infty(\mathbb{R}^+, L^2(\mathbb{R}^3)) \cap L^2(\mathbb{R}^+, \dot{H}^1(\mathbb{R}^3))$, we obtain

$$\begin{aligned}\|(Id - J_{\varphi(n)})u^i(t)u^j(t)\|_{H^{-\varepsilon_0-\frac{1}{2}}} &\leq \|(Id - J_{\varphi(n)})u^i(t)u^j(t)\|_{\dot{H}^{-\varepsilon_0-\frac{1}{2}}}\\ &\leq (\varphi(n))^{-\varepsilon_0}\|u^i(t)u^j(t)\|_{\dot{H}^{-\frac{1}{2}}}\\ &\leq C(\varphi(n))^{-\varepsilon_0}\|u(t)\|_{L^2}\|u(t)\|_{\dot{H}^1}.\end{aligned}$$

Then, we obtain

$$\|(Id - J_{\varphi(n)})u^i(t)u^j(t)\|_{L^2\left(\mathbb{R}^+,H^{-\varepsilon_0-\frac{1}{2}}\right)} \leq C(\varphi(n))^{-\varepsilon_0}\|u\|_{L^\infty(\mathbb{R}^+,L^2)}\|u\|_{L^2(\mathbb{R}^+,\dot{H}^1)},$$

which implies (3.16).

- Now, we want to prove (3.17). Let $\varepsilon > 0$ and $R_0 > 0$ such that

$$\frac{1}{R_0} < \frac{\varepsilon}{4\|u^0\|^2_{L^2}\|\Phi\|_{L^\infty(\mathbb{R}^+\times\mathbb{R}^3)} + 4}. \tag{3.18}$$

Let $m_0 \in \mathbb{N}$ such that

$$\left(e^{\beta R_0^2} - 1 - P_{m_0}(\beta R_0^2)\right)R_0 = \sum_{k=m_0+1}^{\infty} \frac{\beta^k}{k!}R_0^{2k+1} < \frac{\varepsilon}{4\alpha\|\Phi\|_{L^1(\mathbb{R}^+\times\mathbb{R}^3)} + 4}. \tag{3.19}$$

We have

$$\int_0^\infty\int_{\mathbb{R}^3} J_{\varphi(n)}\left[\left(e^{\beta|u_{\varphi(n)}|^2} - 1\right)u_{\varphi(n)}\right]\Phi - \int_0^\infty\int_{\mathbb{R}^3}\left(e^{\beta|u|^2} - 1\right)u\Phi = \left(\sum_{i=1}^3 L^i_n\right) + S,$$

with

$$\begin{aligned}L^1_n &= \int_0^\infty\int_{\mathbb{R}^3}\left[P_{m_0}(\beta|u_{\varphi(n)}|^2)u_{\varphi(n)} - P_{m_0}(\beta|u|^2)u\right]\Phi,\\ L^2_n &= \int_0^\infty\int_{\mathbb{R}^3}(Id - J_{\varphi(n)})\left[\left(e^{\beta|u_{\varphi(n)}|^2} - 1\right)u_{\varphi(n)}\right]\Phi,\\ L^3_n &= \int_0^\infty\int_{\mathbb{R}^3}\left[\left(e^{\beta|u_{\varphi(n)}|^2} - 1 - P_{m_0}(\beta|u_{\varphi(n)}|^2)\right)u_{\varphi(n)}\right]\Phi,\\ S &= \int_0^\infty\int_{\mathbb{R}^3}\left[\left(e^{\beta|u|^2} - 1 - P_{m_0}(\beta|u|^2)\right)u\right]\Phi.\end{aligned}$$

– Estimate of term $L_n^1$: Using Lemma 2.4, we obtain

$$
\begin{aligned}
|L_n^1| &\le C_{m_0,\beta}\int_0^{T_q}\int_{B(0,q)}\Big[P_{m_0}(\beta|u_{\varphi(n)}|^2)+P_{m_0}(\beta|u|^2)\Big]\theta_q|u_{\varphi(n)}-u||\Phi|\\
&\le C_{m_0,\beta}\|\Phi\|_{L^\infty(\mathbb{R}^+\times\mathbb{R}^3)}\int_0^{T_q}\int_{B(0,q)}\Big[P_{m_0}(\beta|u_{\varphi(n)}|^2)+P_{m_0}(\beta|u|^2)\Big]\theta_q|u_{\varphi(n)}-u|.
\end{aligned}
$$

Applying Cauchy-Schwarz and using Lemma 2.3, we obtain

$$
\begin{aligned}
|L_n^1| &\le C_{m_0,\beta}\|\Phi\|_{L^\infty(\mathbb{R}^+\times\mathbb{R}^3)}(\|\Big[P_{m_0}(\beta|u_{\varphi(n)}|^2)\|_{L^2([0,T_q]\times B(0,q))}+\|P_{m_0}(\beta|u|^2)\Big]\|_{L^2([0,T_q]\times B(0,q))})\|\theta_q(u_{\varphi(n)}-u)\|_{L^2([0,T_q]\times B(0,q))}\\
&\le c_{m_0,\beta}C_{m_0,\beta}\|\Phi\|_{L^\infty(\mathbb{R}^+\times\mathbb{R}^3)}(\|\Big(e^{\beta|u_{\varphi(n)}|^2}-1\Big)|u_{\varphi(n)}|^2\|^{1/2}_{L^1([0,T_q],L^1(\mathbb{R}^3))}+\|\Big(e^{\beta|u|^2}-1\Big)|u|^2\|^{1/2}_{L^1([0,T_q],L^1(\mathbb{R}^3))})\|\theta_q(u_{\varphi(n)}\\
&\quad -u)\|_{L^2([0,T_q]\times B(0,q))}.
\end{aligned}
$$

Energy inequalities (3.5)–(3.15) give

$$
|L_n^1| \le c_{m_0,\beta}C_{m_0,\beta}\|\Phi\|_{L^\infty(\mathbb{R}^+\times\mathbb{R}^3)}\frac{\sqrt{2}\|u^0\|_{L^2}}{\sqrt{\alpha}}\|\theta_q(u_{\varphi(n)}-u)\|_{L^2([0,T_q],L^2)}.
$$

Then, the convergence result (3.10) gives an integer $n_1\in\mathbb{N}$ such that: for all $n\ge n_1$, we have

$$
|L_n^1| < \varepsilon/4. \tag{3.20}
$$

– Estimate of term $L_n^2$: we have

$$
\begin{aligned}
|L_n^2| &= |C\int_0^{T_q}\int_{\mathbb{R}^3}(Id-J_{\varphi(n)})\Big[\Big(e^{\beta|u_{\varphi(n)}|^2}-1\Big)u_{\varphi(n)}\Big]\Phi|\\
&= |C\int_0^{T_q}\Big((Id-J_{\varphi(n)})\Big[\Big(e^{\beta|u_{\varphi(n)}|^2}-1\Big)u_{\varphi(n)}\Big]|\Phi\Big)_{L^2}|\\
&\le C\int_0^{T_q}\|(Id-J_{\varphi(n)})\Big[\Big(e^{\beta|u_{\varphi(n)}|^2}-1\Big)u_{\varphi(n)}\Big]\|_{H^{-4}}\|\Phi(t)\|_{H^4}\\
&\le \frac{C}{\varphi(n)}\int_0^{T_q}\|\Big[\Big(e^{\beta|u_{\varphi(n)}|^2}-1\Big)u_{\varphi(n)}\Big]\|_{H^{-3}}\|\Phi\|_{L^\infty(\mathbb{R}^+,H^4)}\\
&\le \frac{C}{\varphi(n)}\|\Big[\Big(e^{\beta|u_{\varphi(n)}|^2}-1\Big)u_{\varphi(n)}\Big]\|_{L^1([0,T_q],H^{-3})}\|\Phi\|_{L^\infty(\mathbb{R}^+,H^4)}.
\end{aligned}
$$

By using Proposition 2.1(3.2) with the choice ($R=1$, $s=3$, $T=T_q$), we obtain

$$
|L_n^2| \le \frac{C}{\varphi(n)}\Big(M_{\beta,1}T_q+\frac{1}{2\alpha}\Big)\sigma_{3,3}\|u^0\|^2_{L^2}\|\Phi\|_{L^\infty(\mathbb{R}^+,H^4)},
$$

which implies that there is an integer $n_2\in\mathbb{N}$ such that

$$
|L_n^2| < \varepsilon/4,\ \ \forall n\ge n_2. \tag{3.21}
$$

– Estimate of term $L_n^3$: Put the following set

$$
Y_{n,R_0} = \{(t,x)\in[0,T_q]\times B(0,q);\ |u_{\varphi(n)}(t,x)|\le R_0\}.
$$

We have

$$
\begin{aligned}
|L_n^3| &\le \int\limits_{Y_{n,R_0}} \Big[\Big(e^{\beta|u_{\varphi(n)}|^2} - 1 - P_{m_0}(\beta|u_{\varphi(n)}|^2)\Big)|u_{\varphi(n)}|\Big]|\Phi| + \int\limits_{Y_{n,R_0}^c} \Big[\Big(e^{\beta|u_{\varphi(n)}|^2} - 1 - P_{m_0}(\beta|u_{\varphi(n)}|^2)\Big)|u_{\varphi(n)}|\Big]|\Phi| \\
&\le \Big[\Big(e^{\beta R_0^2} - 1 - P_{m_0}(\beta R_0^2)\Big)R_0\Big]\|\Phi\|_{L^1(\mathbb{R}^+\times\mathbb{R}^3)} + \frac{1}{R_0}\int\limits_{Y_{n,R_0}^c} \Big[\Big(e^{\beta|u_{\varphi(n)}|^2} - 1 - P_{m_0}(\beta|u_{\varphi(n)}|^2)\Big)|u_{\varphi(n)}|^2\Big]\|\Phi\|_{L^\infty(\mathbb{R}^+\times\mathbb{R}^3)} \\
&\le \Big[\Big(e^{\beta R_0^2} - 1 - P_{m_0}(\beta R_0^2)\Big)R_0\Big]\|\Phi\|_{L^1(\mathbb{R}^+\times\mathbb{R}^3)} + \frac{1}{R_0}\int\limits_{\mathbb{R}^+\times\mathbb{R}^3} \Big[\Big(e^{\beta|u_{\varphi(n)}|^2} - 1\Big)|u_{\varphi(n)}|^2\Big]\|\Phi\|_{L^\infty(\mathbb{R}^+\times\mathbb{R}^3)} \\
&\le \Big[\Big(e^{\beta R_0^2} - 1 - P_{m_0}(\beta R_0^2)\Big)R_0\Big]\|\Phi\|_{L^1(\mathbb{R}^+\times\mathbb{R}^3)} + \frac{1}{R_0}\|u^0\|_{L^2}^2\|\Phi\|_{L^\infty(\mathbb{R}^+\times\mathbb{R}^3)},
\end{aligned}
$$

Using the choices (3.18)–(3.19), we obtain

$$|L_n^3| < \varepsilon/4, \quad \forall n \in \mathbb{N}. \tag{3.22}$$

– Estimate of term $S$: Put the following subset of $\mathbb{R}^4$,

$$Z_{R_0} = \{(t,x) \in [0,T_q] \times B(0,q);\ |u(t,x)| \le R_0\}.$$

We have

$$
\begin{aligned}
|S| &\le \int\limits_{Z_{R_0}} \Big[\Big(e^{\beta|u|^2} - 1 - P_{m_0}(\beta|u|^2)\Big)|u|\Big]|\Phi| + \int\limits_{Z_{R_0}^c} \Big[\Big(e^{\beta|u|^2} - 1 - P_{m_0}(\beta|u|^2)\Big)|u|\Big]|\Phi| \\
&\le \Big(e^{\beta R_0^2} - 1 - P_{m_0}(\beta R_0^2)\Big)R_0\|\Phi\|_{L^1(\mathbb{R}^+\times\mathbb{R}^3)} + \frac{1}{R_0}\left(\int\limits_{Z_{R_0}^c} \Big(e^{\beta|u|^2} - 1 - P_{m_0}(\beta|u|^2)\Big)|u|^2\right)\|\Phi\|_{L^\infty(\mathbb{R}^+\times\mathbb{R}^3)} \\
&\le \Big(e^{\beta R_0^2} - 1 - P_{m_0}(\beta R_0^2)\Big)R_0\|\Phi\|_{L^1(\mathbb{R}^+\times\mathbb{R}^3)} + \frac{1}{R_0}\left(\int\limits_{\mathbb{R}^+\times\mathbb{R}^3} \Big(e^{\beta|u|^2} - 1\Big)|u|^2\right)\|\Phi\|_{L^\infty(\mathbb{R}^+\times\mathbb{R}^3)} \\
&\le \Big(e^{\beta R_0^2} - 1 - P_{m_0}(\beta R_0^2)\Big)R_0\|\Phi\|_{L^1(\mathbb{R}^+\times\mathbb{R}^3)} + \frac{1}{2\alpha R_0}\|u^0\|_{L^2}^2\|\Phi\|_{L^\infty(\mathbb{R}^+\times\mathbb{R}^3)}.
\end{aligned}
$$

By the choices (3.18)–(3.19), we obtain

$$|S| < \varepsilon/4. \tag{3.23}$$

Combining (3.20), (3.21), (3.22), and (3.23), we obtain

$$\left|\int_0^\infty\int_{\mathbb{R}^3} J_{\varphi(n)}\Big[\Big(e^{\beta|u_{\varphi(n)}|^2} - 1\Big)u_{\varphi(n)}\Big]\Phi - \int_0^\infty\int_{\mathbb{R}^3}\Big(e^{\beta|u|^2} - 1\Big)u\Phi\right| < \varepsilon, \quad \forall n \ge \max(n_1, n_2),$$

which implies (3.17), and the proof of Theorem 1.3 is finished.

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