Research Article

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On split twisted inner derivation triple systems with no restrictions on their 0-root spaces

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Abstract: The aim of this paper is to study the structure of arbitrary split twisted inner derivation triple systems. We obtain a sufficient condition for the decomposition of arbitrary twisted inner derivation triple system $\mathscr{T}$ which is of the form $\mathscr{T} = U + \sum_{[\theta]\in\Lambda^{\mathscr{T}}/\sim} I_{[\theta]}$ with U a subspace of $\mathscr{T}_0$ and any $I_{[\theta]}$ a well-described ideal of $\mathscr{T}$, satisfying $\{I_{[\theta]}, \mathscr{T}, I_{[\eta]}\} = \{I_{[\theta]}, I_{[\eta]}, \mathscr{T}\} = \{\mathscr{T}, I_{[\theta]}, I_{[\eta]}\} = \{I_{[\theta]}, \mathscr{T}, I_{[\eta]}\}' = \{I_{[\theta]}, I_{[\eta]}, \mathscr{T}\}' = \{\mathscr{T}, I_{[\theta]}, I_{[\eta]}\}' = 0$ if $[\theta] \neq [\eta]$. In particular, a necessary and sufficient condition for the simplicity of the triple system is given.

Keywords: split, twisted inner derivation triple system, root system, root space

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1 Introduction

Split algebras are very active in the research and applications of mathematics and physics. The split structure of an algebra has an important relationship with quantum theory and deformation. A special Lie algebra-split Lie algebra was first defined in [1,2], that is, a Lie algebra contains a split Cartan subalgebra. In 2008, Calderón used the techniques of connections of roots to study the decomposition and simplicity of split Lie algebras with symmetric roots on any dimension and any field, in particular, he obtained the necessary and sufficient conditions for the simplicity of split Lie algebras in [3]. Since then, many authors have started the research on the structure of different classes of split algebras. The decomposition and simplicity of split Leibniz algebras and one of the split Lie color algebras were determined in [4,5]. In [6,7], Cao studied the structure of split regular Hom-Lie color algebras and split regular Hom-Leibniz algebras.

Hopkins in 1985 introduced the definition of twisted inner derivation triple system in [8]. The twisted inner derivation triple systems are generalized Lie triple systems containing Lie triple systems and Jordan triple systems. It is one of the important fundamental topics in Lie theory to study the structure of split generalized Lie triple systems, and results of this project are important to the study of many subjects such as quantum theory, deformation theory, and conformal field theory. In [9], locally finite split Lie triple systems were introduced and studied. In [10–13], some applications of Lie triple systems were widely studied. In [14], the author used techniques of connections of roots to study the structure of arbitrary split Lie triple system with a coherent 0-root space. In [15], infinite dimensional simple split Lie triple systems were studied. The authors extended conclusions of splitting Lie triple systems with a coherent 0-root space to

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arbitrary Lie triple systems with no restrictions on their 0-root spaces in [16]. In [17,18], the structures of arbitrary split Leibniz triple systems and graded Leibniz triple systems were studied. In [19], a necessary and sufficient condition for the simplicity of a split Lie color triple system was determined. In [20], Calderón studied the structure of arbitrary split twisted inner derivation triple system with a coherent 0-root space, that is, those satisfying $\{\mathscr{T}_0, \mathscr{T}_0, \mathscr{T}\} = 0$ and $\{\mathscr{T}_0, \mathscr{T}_\theta, \mathscr{T}_0\} \neq 0$. In this paper, we need to study the structure of arbitrary split twisted inner derivation triple systems with no restrictions on their 0-root spaces.

In this paper, split twisted inner derivation triple system $\mathscr{T}$ are considered of arbitrary dimension and over an arbitrary base field $\mathbb{K}$. The structure of the paper is organized as follows. In Section 2, we give the preliminaries on split twisted inner derivation triple systems theory. In Section 3, we obtain a sufficient condition for the decomposition of arbitrary twisted inner derivation triple system $\mathscr{T}$, which is of the form $\mathscr{T} = U + \sum_{[\theta]\in\Lambda^{\mathscr{T}}/\sim} I_{[\theta]}$ with U a subspace of $\mathscr{T}_0$ and any $I_{[\theta]}$ a well-described ideal of $\mathscr{T}$, satisfying $\{I_{[\theta]}, \mathscr{T}, I_{[\eta]}\} = \{\mathscr{T}, I_{[\theta]}, I_{[\eta]}\} = \{I_{[\theta]}, I_{[\eta]}, \mathscr{T}\} = \{I_{[\theta]}, \mathscr{T}, I_{[\eta]}\}' = \{\mathscr{T}, I_{[\theta]}, I_{[\eta]}\}' = \{I_{[\theta]}, I_{[\eta]}, \mathscr{T}\}' = 0$ if $[\theta] \neq [\eta]$. In Section 4, we obtain a necessary and sufficient condition for the simplicity of the triple system.

2 Preliminaries

First we recall the definitions of Lie triple systems, Jordan triple systems, and twisted inner derivation triple systems. The following definition is well known from the theory of triple systems.

Definition 2.1. Let $\mathscr{T}$ be a triple system such that its triple product satisfies:
(1) $\{x, y, z\} = \varepsilon\{y, x, z\}$, with ε a fixed element in ± 1,
(2) $\{x, y, z\} + \{y, z, x\} + \{z, x, y\} = 0$,
(3) $\{x, y, \{a, b, c\}\} - \{a, b, \{x, y, c\}\} = \{\{x, y, a\}, b, c\} + \{a, \{x, y, b\}, c\}$,
for $x, y, z, a, b, c \in \mathscr{T}$. Then $\mathscr{T}$ is called Lie triple system if $\varepsilon = -1$, and an anti-Lie triple system if $\varepsilon = 1$.

Definition 2.2. A triple system $\mathscr{T}$ is called Jordan triple system if its triple product satisfies:
(1) $\{x, y, z\} = \{z, y, x\}$,
(2) $\{x, y, \{a, b, c\}\} - \{a, b, \{x, y, c\}\} = \{\{x, y, a\}, b, c\} - \{a, \{y, x, b\}, c\}$,
for $x, y, z, a, b, c \in T$.

Definition 2.3. [8] Let $(\mathscr{T}, \{\cdot,\cdot,\cdot\})$ be a triple system. We say that $\mathscr{T}$ is a twisted inner derivation triple system, if there exists a linear bijection, τ, of order one or two on $\mathscr{L} := \mathrm{span}_{\mathbb{K}}\{\mathscr{L}(x, y) : x, y \in \mathscr{T}\}$, where $\mathscr{L}(x, y)$ denotes the left multiplication operator in $\mathscr{T}$, $\mathscr{L}(x, y)(z) := \{x, y, z\}$, such that

$$\{x, y, \{a, b, c\}\} - \{a, b, \{x, y, c\}\} = \{\{x, y, a\}, b, c\} + \{a, \{x, y, b\}', c\} \tag{2.1}$$

and

$$\{x, y, \{a, b, c\}'\}' - \{a, b, \{x, y, c\}'\}' = \{\{x, y, a\}, b, c\}' + \{a, \{x, y, b\}', c\}' \tag{2.2}$$

for any $x, y, z, a, b, c \in \mathscr{T}$, where $\{a, b, c\}' := \tau[\mathscr{L}(a, b)](c)$.

When $\tau = Id$, $(\mathscr{T}, \{\cdot,\cdot,\cdot\})$ is called inner derivation triple system. Obviously, Lie triple system and anti-Lie triple system are inner derivation triple system and twisted inner derivation triple system. If $\tau\mathscr{L}(x, y) := -\mathscr{L}(x, y)$, then twisted inner derivation triple systems are Jordan triple systems, and the ternary algebras considered in [21]. Let us observe that (2.1) means that $\mathscr{L} := \mathrm{span}_{\mathbb{K}}\{\mathscr{L}(x, y) : x, y \in \mathscr{T}\}$ is a Lie algebra, the product being

$$[\mathscr{L}(x, y), \mathscr{L}(a, b)] = \mathscr{L}(\{x, y, a\}, b) + \mathscr{L}(a, \{x, y, b\}'), \tag{2.3}$$

and that equations (2.1), (2.2) give τ is a Lie algebra automorphism of $\mathscr{L}$.

Due to every $l \in \mathscr{L}$ is of the form $\Sigma\mathscr{L}(x_i, y_i)$, by equation (2.3) we have

$$[l, \mathscr{L}(a, b)] = \mathscr{L}(la, b) + \mathscr{L}(a, \tau(l)b), \tag{2.4}$$

with $a, b \in \mathscr{T}$ and $l \in \mathscr{L}$. Particularly, for any $x, y \in \mathscr{T}$

$$[\tau\mathscr{L}(x, y), \mathscr{L}(a, b)] = \mathscr{L}(\tau\mathscr{L}(x, y)a, b) + \mathscr{L}(a, \mathscr{L}(x, y)b) \tag{2.5}$$

or

$$\{x, y, \{a, b, c\}\}' - \{a, b, \{x, y, c\}'\} = \{\{x, y, a\}', b, c\} + \{a, \{x, y, b\}, c\}. \tag{2.6}$$

By acting τ on both sides of equation (2.5) we obtain

$$[\mathscr{L}(x, y), \tau\mathscr{L}(a, b)] = \tau\mathscr{L}(\tau\mathscr{L}(x, y)a, b) + \tau\mathscr{L}(a, \mathscr{L}(x, y)b) \tag{2.7}$$

or

$$\{x, y, \{a, b, c\}'\} - \{a, b, \{x, y, c\}\}' = \{\{x, y, a\}', b, c\}' + \{a, \{x, y, b\}, c\}'. \tag{2.8}$$

Identities (2.4)–(2.8) play a key role during the study of split twisted inner derivation triple system.

Definition 2.4. Let I be a linear subspace of a twisted inner derivation triple system $\mathscr{T}$. If $\{I, I, I\} + \{I, I, I\}' \subseteq I$, we say that I is a subsystem of $\mathscr{T}$. If $\{I, \mathscr{T}, \mathscr{T}\} + \{\mathscr{T}, I, \mathscr{T}\} + \{\mathscr{T}, \mathscr{T}, I\} + \{I, \mathscr{T}, \mathscr{T}\}' + \{\mathscr{T}, I, \mathscr{T}\}' \subseteq I$, we say that I is an ideal of $\mathscr{T}$.

Clearly, $\{\mathscr{T}, \mathscr{T}, I\} \subseteq I$ implies $\{\mathscr{T}, \mathscr{T}, I\}' \subseteq I$.

Definition 2.5. The annihilator of a twisted inner derivation triple system $\mathscr{T}$ is the set $\mathrm{Ann}(\mathscr{T}) = \{x \in \mathscr{T} : \{x, \mathscr{T}, \mathscr{T}\} + \{\mathscr{T}, x, \mathscr{T}\} + \{\mathscr{T}, \mathscr{T}, x\} = 0\}$.

Definition 2.6. [8] The standard embedding of a twisted inner derivation triple system $(\mathscr{T}, \{\cdot,\cdot,\cdot\})$ is the two-graded algebra $A = \mathscr{L} \oplus \mathscr{T}$, whose product is given by

$$xy = \mathscr{L}(x, y), \tag{2.9}$$

$$\mathscr{L}(x, y)z := -z\tau[\mathscr{L}(x, y)] := \{x, y, z\}, \tag{2.10}$$

$$l_1 l_2 := [l_1, l_2], \tag{2.11}$$

for any $x, y, z \in \mathscr{T}$, and $l_1, l_2 \in \mathscr{L}$.

By equation (2.10), we have

$$\{x, y, z\}' = \tau[\mathscr{L}(x, y)]z = -z\mathscr{L}(x, y) = -z(xy),$$

or

$$z(xy) = z\mathscr{L}(x, y) = -\{x, y, z\}'.$$

So the following hold for any $x, y, z \in \mathscr{T}$:

$$\begin{aligned} \mathscr{L}(x, y)z &= \{x, y, z\} = (xy)z, \\ \tau[\mathscr{L}(x, y)]z &= \{x, y, z\}' = -z(xy), \\ z\mathscr{L}(x, y) &= -\{x, y, z\}' = z(xy), \\ z\tau[\mathscr{L}(x, y)] &= -\{x, y, z\} = -(xy)z. \end{aligned}$$

Given an element x of a Lie algebra L, we denote by $ad(x)$ the adjoint mapping defined as $ad(x)(y) = [x, y]$ for any $y \in L$. The concept of a split Lie algebra and its related content can be seen in [3].

Definition 2.7. [20] Let $\mathscr{T}$ be a twisted inner derivation triple system, $A = \mathscr{L} \oplus \mathscr{T}$ be its standard embedding, and H be an $MASA$ of Lie algebra $\mathscr{L}$ satisfying $\tau(H) \subset H$. For a linear functional $\theta \in (H)^*$, we define the root space of $\mathscr{T}$ (with respect to H) associated with θ as the subspace $\mathscr{T}_\theta := \{t_\theta \in \mathscr{T} : ht_\theta = \theta(h)t_\theta \text{ for any } h \in H\}$. The elements $\theta \in (H)^*$ satisfying $\mathscr{T}_\theta \neq 0$ are called roots of $\mathscr{T}$ with respect to H and we denote $\Lambda^{\mathscr{T}} := \{\theta \in (H)^* \backslash \{0\} : \mathscr{T}_\theta \neq 0\}$.

Let us observe that $\mathscr{T}_0 := \{t_0 \in \mathscr{T} : ht_0 = 0 \text{ for any } h \in H\}$, and that for any $t_\theta \in \mathscr{T}_\theta$ and $h \in H$, $t_\theta h = -\tau(h)t_\theta = -\theta(\tau(h))t_\theta$.

Next, $\Lambda^{\mathscr{L}}$ stands for the set of all nonzero $\theta \in (H)^*$ such that $\mathscr{L}_\theta := \{e_\theta \in \mathscr{L} : [h, e_\theta] = \theta(h)e_\theta \text{ for any } h \in H\} \neq 0$.

Definition 2.8. [20] Let $\mathscr{T}$ be a twisted inner derivation triple system, $A = \mathscr{L} \oplus \mathscr{T}$ be its standard embedding, and let H be an *MASA* of Lie algebra $\mathscr{L}$ satisfying $\tau(H) \subset H$. We shall call that $\mathscr{T}$ is a split twisted inner derivation triple system with respect to H if

$$\mathscr{T} = \mathscr{T}_0 \oplus \left(\bigoplus_{\theta \in \Lambda^{\mathscr{T}}} \mathscr{T}_\theta \right).$$

We say that $\Lambda^{\mathscr{T}}$ is the root system of $\mathscr{T}$.

Lemma 2.9. *Let $\mathscr{T}$ be a split twisted inner derivation triple system, for any $\theta \in \Lambda^{\mathscr{L}}$, then $\theta\tau \in \Lambda^{\mathscr{L}}$ and $\mathscr{L}_{\theta\tau} = \tau(\mathscr{L}_\theta)$.*

Proof. Given any $0 \neq e_\theta \in \mathscr{L}_\theta$ and $h \in H$, one has $[h, \tau(e_\theta)] = \tau([\tau(h), e_\theta]) = \tau(\theta(\tau(h))e_\theta) = \theta(\tau(h))\tau(e_\theta) = (\theta\tau)(h)\tau(e_\theta)$, with $\tau(e_\theta) \neq 0$; therefore, $\theta\tau \in \Lambda^{\mathscr{L}}$ and $\tau(\mathscr{L}_\theta) \subset \mathscr{L}_{\theta\tau}$. Thus, for any $\theta \in \Lambda^{\mathscr{L}}$, one has $(\mathscr{L}_\theta) \subset \tau(\mathscr{L}_{\theta\tau})$, as $\theta\tau \in \Lambda^{\mathscr{L}}$, we have $\mathscr{L}_{\theta\tau} = \tau(\mathscr{L}_\theta)$. ☐

Lemma 2.10. *Let $\mathscr{T}$ be a split twisted inner derivation triple system with a standard embedding $A = \mathscr{L} \oplus \mathscr{T}$. H is an MASA of Lie algebra $\mathscr{L}$ satisfying $\tau(H) \subset H$. For any $\theta, \eta, \zeta \in \Lambda^{\mathscr{T}} \cup \{0\}$ and any $\delta, \varepsilon \in \Lambda^{\mathscr{L}} \cup \{0\}$, then the following statements hold*:

1. *If $\mathscr{T}_\theta\mathscr{T}_\eta \neq 0$, then $\mathscr{T}_\theta\mathscr{T}_\eta \subset \mathscr{L}_{\theta+\eta\tau}$, that is, $\theta + \eta\tau \in \Lambda^{\mathscr{L}} \cup \{0\}$.*
2. *If $\mathscr{L}_\delta\mathscr{T}_\theta \neq 0$, then $\mathscr{L}_\delta\mathscr{T}_\theta \subset \mathscr{T}_{\delta+\theta}$, that is, $\delta + \theta \in \Lambda^{\mathscr{T}} \cup \{0\}$.*
3. *If $\mathscr{T}_\theta\mathscr{L}_\delta \neq 0$, then $\mathscr{T}_\theta\mathscr{L}_\delta \subset \mathscr{T}_{\theta+\delta\tau}$, that is, $\theta + \delta\tau \in \Lambda^{\mathscr{T}} \cup \{0\}$.*
4. *If $[\mathscr{L}_\delta, \mathscr{L}_\varepsilon] \neq 0$, then $[\mathscr{L}_\delta, \mathscr{L}_\varepsilon] \subset \mathscr{L}_{\delta+\varepsilon}$, that is, $\delta + \varepsilon \in \Lambda^{\mathscr{L}} \cup \{0\}$.*
5. *If $\{\mathscr{T}_\theta, \mathscr{T}_\eta, \mathscr{T}_\zeta\} \neq 0$, then $\{\mathscr{T}_\theta, \mathscr{T}_\eta, \mathscr{T}_\zeta\} \subseteq \mathscr{T}_{\theta+\eta\tau+\zeta}$, that is, $\theta + \eta\tau + \zeta \in \Lambda^{\mathscr{T}} \cup \{0\}$.*
6. *If $\{\mathscr{T}_\theta, \mathscr{T}_\eta, \mathscr{T}_\zeta\}' \neq 0$, then $\{\mathscr{T}_\theta, \mathscr{T}_\eta, \mathscr{T}_\zeta\}' \subseteq \mathscr{T}_{\theta\tau+\eta+\zeta}$, that is, $\theta\tau + \eta + \zeta \in \Lambda^{\mathscr{T}} \cup \{0\}$.*

Proof.

(1) For any $t_\theta \in \mathscr{T}_\theta$, $t_\eta \in \mathscr{T}_\eta$, and $h \in H$, by equation (2.4), one has

$$[h, t_\theta t_\eta] = (ht_\theta)t_\eta + t_\theta(\tau(h)t_\eta) = (\theta + \eta\tau)(h)t_\theta t_\eta.$$

Therefore, $\mathscr{T}_\theta\mathscr{T}_\eta \subset \mathscr{L}_{\theta+\eta\tau}$.

(2) For any $e_\delta \in \mathscr{L}_\delta$, $t_\theta \in \mathscr{T}_\theta$, and $h \in H$, we can write

$$h = \sum_{i=1}^{m} x_i y_i$$

and

$$e_\delta = \sum_{j=1}^{n} z_j u_j$$

with $x_i, y_i, z_j, u_j \in \mathscr{T}$. By equations (2.1) and (2.4), one obtains

$$\begin{aligned}
h(e_\delta t_\theta) &= \sum_{i=1}^{m}\sum_{j=1}^{n}\{x_i, y_i, \{z_j, u_j, t_\theta\}\}\\
&= \sum_{i=1}^{m}\sum_{j=1}^{n}(\{\{x_i, y_i, z_j\}, u_j, t_\theta\} + \{z_j, \{x_i, y_i, u_j\}', t_\theta\} + \{z_j, u_j, \{x_i, y_i, t_\theta\}\})\\
&= \sum_{j=1}^{n}\{hz_j, u_j, t_\theta\} + \sum_{j=1}^{n}\{z_j, \tau(h)u_j, t_\theta\} + \sum_{j=1}^{n}\{z_j, u_j, \theta(h)t_\theta\}\\
&= \left(\sum_{j=1}^{n}(hz_j)u_j + \sum_{j=1}^{n}z_j(\tau(h)u_j)\right)t_\theta + \theta(h)e_\delta t_\theta\\
&= \left(\sum_{j=1}^{n}[h, z_j u_j]\right)t_\theta + \theta(h)e_\delta t_\theta\\
&= [h, e_\delta]t_\theta + \theta(h)e_\delta t_\theta\\
&= (\delta + \theta)(h)e_\delta t_\theta.
\end{aligned}$$

Therefore, $\mathscr{L}_\delta \mathscr{T}_\theta \subset \mathscr{T}_{\delta+\theta}$.

(3) Consequence of Lemma 2.9 and items (2).

(4) For any $e_\delta \in \mathscr{L}$, $e_\varepsilon \in \mathscr{L}$, and $h \in H$, by Jacobi identity of Lie algebra $\mathscr{L}$, one obtains $[h, [e_\delta, e_\varepsilon]] = [e_\delta, [h, e_\varepsilon]] + [[h, e_\delta], e_\varepsilon] = [e_\delta, \varepsilon(h)e_\varepsilon] + [\delta(h)e_\delta, e_\varepsilon] = (\delta + \varepsilon)(h)[e_\delta, e_\varepsilon]$. Therefore, $[\mathscr{L}_\delta, \mathscr{L}_\varepsilon] \subset \mathscr{L}_{\delta+\varepsilon}$.

(5) Consequence of items (1) and (2).

(6) Consequence of items (1) and (2), and Lemma 2.9. □

We noted that the facts $H \subset \mathscr{L} = TT$ and $\mathscr{T} = \mathscr{T}_0 \oplus (\oplus_{\theta\in\Lambda^{\mathscr{T}}}\mathscr{T}_\theta)$ imply

$$H = \mathscr{T}_0\mathscr{T}_0 + \sum_{\theta\in\Lambda^{\mathscr{T}}} \mathscr{T}_\theta \mathscr{T}_{-\theta\tau}. \tag{2.12}$$

Finally, as $\mathscr{T}_0\mathscr{T}_0 \subset \mathscr{L}_0 = H$, we have

$$\{\mathscr{T}_0, \mathscr{T}_0, \mathscr{T}_0\} = 0. \tag{2.13}$$

In [20], Calderón and Piulestán introduced and studied structures of arbitrary split twisted inner derivation triple system with a coherent 0-root space, that is, those satisfying $\{\mathscr{T}_0, \mathscr{T}_0, \mathscr{T}\} = 0$ and $\{\mathscr{T}_0, \mathscr{T}_\theta, \mathscr{T}_0\} = 0$ for any nonzero root and where $\mathscr{T}_0$ denotes the 0-root space and $\mathscr{T}_\theta$ the θ-root space. In this paper, we will study the structures of arbitrary inner derivation triple systems with no restrictions on their 0-root spaces.

Definition 2.11. Let $\Lambda^{\mathscr{T}}$ be a root system of a split twisted inner derivation triple system $\mathscr{T}$, if it satisfies that $\theta \in \Lambda^{\mathscr{T}}$ implies $-\theta, \theta\tau \in \Lambda^{\mathscr{T}}$, we say that $\Lambda^{\mathscr{T}}$ is symmetric.

A similar concept applies to the set $\Lambda^{\mathscr{L}}$ of nonzero roots of $\mathscr{L}$.

In the following, $\mathscr{T}$ denotes a split twisted inner derivation triple system with a symmetric root system $\Lambda^{\mathscr{T}}$, and $\mathscr{T} = \mathscr{T}_0 \oplus (\oplus_{\theta\in\Lambda^{\mathscr{T}}}\mathscr{T}_\theta)$ the corresponding root decomposition. Using the properties of connections of roots is an important method to study split twisted inner derivation triple systems. Next, we will give the definition of connections of roots of a split twisted inner derivation triple system $\mathscr{T}$.

Definition 2.12. Let θ and η be two nonzero roots, we shall say that θ and η are connected if there exists a family $\{\theta_1, \theta_2, \ldots, \theta_{2n}, \theta_{2n+1}\} \subset \Lambda^{\mathscr{T}} \cup \{0\}$ of roots of $\mathscr{T}$ such that

(1) $\{\theta_1, \theta_1 + \theta_2\tau + \theta_3, \theta_1 + \theta_2\tau + \theta_3 + \theta_4\tau + \theta_5, \ldots, \theta_1 + \theta_2\tau + \cdots + \theta_{2n}\tau + \theta_{2n+1}\} \subset \Lambda^{\mathscr{T}}$;

(2) $\{\theta_1 + \theta_2\tau, \theta_1 + \theta_2\tau + \theta_3 + \theta_4\tau, \ldots, \theta_1 + \theta_2\tau + \cdots + \theta_{2n}\tau\} \subset \Lambda^{\mathscr{L}}$;

(3) $\theta_1 = \theta$ and $\theta_1 + \theta_2\tau + \cdots + \theta_{2n}\tau + \theta_{2n+1} \in \{\pm\eta, \pm\eta\tau\}$.

We shall also say that $\{\theta_1, \theta_2, \ldots, \theta_{2n}, \theta_{2n+1}\}$ is a connection from θ to η.

We denote by

$$\Lambda_\theta^{\mathscr{T}} := \{\eta \in \Lambda^{\mathscr{T}} : \theta \text{ and } \eta \text{ are connected}\}.$$

Clearly, if $\eta \in \Lambda_\theta^{\mathscr{T}}$, then $-\eta, \pm\eta\tau \in \Lambda_\theta^{\mathscr{T}}$.

Definition 2.13. Let $\Omega^{\mathscr{T}}$ be a subset of a root system $\Lambda^{\mathscr{T}}$, if it is symmetric, and given $\theta, \eta, \zeta \in \Omega^{\mathscr{T}} \cup \{0\}$ such that $\theta + \eta\tau \in \Lambda^{\mathscr{L}}$ and $\theta + \eta\tau + \zeta \in \Lambda^{\mathscr{T}}$, then $\theta + \eta\tau + \zeta \in \Omega^{\mathscr{T}}$, and we say that $\Omega^{\mathscr{T}}$ is a root subsystem.

Definition 2.14. Let $\Omega^{\mathscr{T}}$ be a root subsystem of $\Lambda^{\mathscr{T}}$. We define

$$\mathscr{T}_{0,\Omega^{\mathscr{T}}} := \operatorname{span}_{\mathbb{K}}\{\{\mathscr{T}_\theta, \mathscr{T}_\eta, \mathscr{T}_\zeta\} : \theta + \eta\tau + \zeta = 0; \quad \{\mathscr{T}_\theta, \mathscr{T}_\eta, \mathscr{T}_\zeta\}' : \theta\tau + \eta + \zeta = 0; \text{ where } \theta, \eta, \zeta \in \Omega^{\mathscr{T}} \cup \{0\}\} \subset \mathscr{T}_0,$$

and $V_{\Omega^{\mathscr{T}}} := \oplus_{\theta\in\Omega^{\mathscr{T}}}\mathscr{T}_\theta$. Taking into account the fact that $\{\mathscr{T}_0, \mathscr{T}_0, \mathscr{T}_0\} = 0$, it is straightforward to verify that

$$\mathscr{T}_{\Omega^{\mathscr{T}}} := \mathscr{T}_{0,\Omega^{\mathscr{T}}} \oplus V_{\Omega^{\mathscr{T}}}$$

is a subsystem of $\mathscr{T}$. We will say that $\mathscr{T}_{\Omega^{\mathscr{T}}}$ is a twisted inner derivation triple subsystem associated with the root subsystem $\Omega^{\mathscr{T}}$.

Proposition 2.15. *If $\Lambda^{\mathscr{L}}$ is symmetric, then the relation $\sim$ in $\Lambda^{\mathscr{T}}$, defined by $\theta \sim \eta$ if and only if $\eta \in \Lambda_\theta^{\mathscr{T}}$, is of equivalence.*

The proof process is similar to Proposition 4.2 in [20].

Proposition 2.16. *Let θ be a nonzero root and suppose $\Lambda^{\mathscr{L}}$ is symmetric. Then $\Lambda_\theta^{\mathscr{T}}$ is a root subsystem.*

The proof process is similar to Lemma 4.5 in [20].

3 Decompositions

In this section, $\mathscr{T}$ denotes a split twisted inner derivation triple system, we will state a series of previous results in order to show that for a fixed $\theta_0 \in \Lambda^{\mathscr{T}}$, the twisted inner derivation triple-subsystem $\mathscr{T}_{\Lambda_{\theta_0}^{\mathscr{T}}}$ associated with the root subsystem $\Lambda_{\theta_0}^{\mathscr{T}}$ is an ideal of $\mathscr{T}$.

Lemma 3.1. *The following assertions hold*:
(1) *Suppose $\theta, \eta \in \Lambda^{\mathscr{T}}$, $\mathscr{T}_\theta\mathscr{T}_\eta \neq 0$, then $\theta \sim \eta$.*
(2) *Suppose $\theta, \eta \in \Lambda^{\mathscr{T}}$, $\theta \in \Lambda^{\mathscr{L}}$, $\mathscr{L}_\theta\mathscr{T}_\eta \neq 0$, then $\theta \sim \eta$.*
(3) *Suppose $\theta, \eta \in \Lambda^{\mathscr{T}}$, $\theta \in \Lambda^{\mathscr{L}}$, $\mathscr{T}_\eta\mathscr{L}_\theta \neq 0$, then $\theta \sim \eta$.*
(4) *Suppose $\theta, \eta \in \Lambda^{\mathscr{T}}$, $\theta, \eta \in \Lambda^{\mathscr{L}}$, $[\mathscr{L}_\theta, \mathscr{L}_\eta] \neq 0$, then $\theta \sim \eta$.*
(5) *Suppose $\theta, \bar\eta \in \Lambda^{\mathscr{T}}$ such that θ is not connected with $\bar\eta$, then $\mathscr{T}_\theta\mathscr{T}_{\bar\eta} = 0$, $\mathscr{L}_\theta\mathscr{T}_{\bar\eta} = 0$ and $\mathscr{T}_{\bar\eta}\mathscr{L}_\theta = 0$ if furthermore $\theta \in \Lambda^{\mathscr{L}}$. Suppose $\theta, \bar\eta \in \Lambda^{\mathscr{T}}$ such that θ is not connected with $\bar\eta$, $[\mathscr{L}_\theta, \mathscr{L}_{\bar\eta}] = 0$ if furthermore $\theta, \bar\eta \in \Lambda^{\mathscr{L}}$.*

Proof.
(1) Suppose $\mathscr{T}_\theta\mathscr{T}_\eta \neq 0$, by Lemma 2.10 (1), we obtain $\theta + \eta\tau \in \Lambda^{\mathscr{L}} \cup \{0\}$. If $\theta + \eta\tau = 0$, then $\eta = -\theta\tau$ and so $\theta \sim \eta$. Suppose $\theta + \eta\tau \neq 0$, by $\theta + \eta\tau \in \Lambda^{\mathscr{L}}$, we find $\{\theta, \eta, -\theta\}$ is a connection from θ to η, so $\theta \sim \eta$.
(2) Suppose $\mathscr{L}_\theta\mathscr{T}_\eta \neq 0$, by Lemma 2.10 (2), we obtain $\theta + \eta \in \Lambda^{\mathscr{T}} \cup \{0\}$. If $\theta + \eta = 0$, then $\theta \sim \eta$. Suppose $\theta + \eta \neq 0$, by $\theta + \eta \in \Lambda^{\mathscr{T}}$, we find $\{\theta, 0, -\theta - \eta\}$ is a connection from θ to η, so $\theta \sim \eta$.

(3) Suppose $\mathcal{T}_\eta\mathcal{L}_\theta \neq 0$, by Lemma 2.10 (3), we obtain $\eta + \theta\tau \in \Lambda^{\mathcal{T}} \cup \{0\}$. If $\eta + \theta\tau = 0$, then $\theta \sim \eta$. Suppose $\eta + \theta\tau \neq 0$, by $\eta + \theta\tau \in \Lambda^{\mathcal{T}}$, we find $\{\theta, 0, -\eta\tau - \theta\}$ is a connection from θ to η, so $\theta \sim \eta$.
(4) Suppose $[\mathcal{L}_\theta, \mathcal{L}_\eta] \neq 0$, by Lemma 2.10 (4), we obtain $\theta + \eta \in \Lambda^{\mathcal{L}} \cup \{0\}$. If $\theta + \eta = 0$, then $\theta \sim \eta$. Suppose $\theta + \eta \neq 0$, by $\theta + \eta \in \Lambda^{\mathcal{L}}$, we find $\{\theta, \eta\tau, -\theta\}$ is a connection from θ to η, so $\theta \sim \eta$.
(5) It is a consequence of items (1), (2), (3), and (4). □

Lemma 3.2. *If $\theta, \bar{\eta} \in \Lambda^{\mathcal{T}}$ are not connected, then*

$$\bar{\eta}(\mathcal{T}_\theta\mathcal{T}_{-\theta\tau}) = \bar{\eta}(\tau(\mathcal{T}_\theta\mathcal{T}_{-\theta\tau})) = 0.$$

Proof. If $\mathcal{T}_\theta\mathcal{T}_{-\theta\tau} = 0$ it is clear. One suppose that $\mathcal{T}_\theta\mathcal{T}_{-\theta\tau} \neq 0$ and $\bar{\eta}(\mathcal{T}_\theta\mathcal{T}_{-\theta\tau}) \neq 0$, one obtains $[\mathcal{T}_\theta\mathcal{T}_{-\theta\tau}, \mathcal{L}_{\bar{\eta}}] = \bar{\eta}(\mathcal{T}_\theta\mathcal{T}_{-\theta\tau})\mathcal{L}_{\bar{\eta}} \neq 0$. But see equation (2.4), one obtains

$$\left[\mathcal{T}_\theta\mathcal{T}_{-\theta\tau}, \mathcal{L}_{\bar{\eta}}\right] = -\left[\mathcal{L}_{\bar{\eta}}, \mathcal{T}_\theta\mathcal{T}_{-\theta\tau}\right] \subset \left(\mathcal{L}_{\bar{\eta}}\mathcal{T}_\theta\right)\mathcal{T}_{-\theta\tau} + \mathcal{T}_\theta\Big(\tau\Big(\mathcal{L}_{\bar{\eta}}\Big)\mathcal{T}_{-\theta\tau}\Big).$$

By Lemmas 2.9 and 3.1 (5), one obtains $\mathcal{L}_{\bar{\eta}}\mathcal{T}_\theta = \tau(\mathcal{L}_{\bar{\eta}})\mathcal{T}_{-\theta\tau} = 0$, that is, $[\mathcal{T}_\theta\mathcal{T}_{-\theta\tau}, \mathcal{L}_{\bar{\eta}}] = 0$, a contradiction. Finally, one suppose $\mathcal{T}_\theta\mathcal{T}_{-\theta\tau} \neq 0$ and $\bar{\eta}(\tau(\mathcal{T}_\theta\mathcal{T}_{-\theta\tau})) \neq 0$, one obtains $[\tau(\mathcal{T}_\theta\mathcal{T}_{-\theta\tau}), \mathcal{L}_{\bar{\eta}}] = \bar{\eta}(\tau(\mathcal{T}_\theta\mathcal{T}_{-\theta\tau}))\mathcal{L}_{\bar{\eta}} \neq 0$. But see equation (2.7), one obtains

$$\left[\tau(\mathcal{T}_\theta\mathcal{T}_{-\theta\tau}), \mathcal{L}_{\bar{\eta}}\right] = -\left[\mathcal{L}_{\bar{\eta}}, \tau(\mathcal{T}_\theta\mathcal{T}_{-\theta\tau})\right] \subset \tau\Big(\Big(\Big(\tau\mathcal{L}_{\bar{\eta}}\Big)\mathcal{T}_\theta\Big)\mathcal{T}_{-\theta\tau}\Big) + \tau\Big(\mathcal{T}_\theta\Big(\mathcal{L}_{\bar{\eta}}\mathcal{T}_{-\theta\tau}\Big)\Big).$$

By Lemmas 2.9 and 3.1 (5), one obtains $(\tau\mathcal{L}_{\bar{\eta}})\mathcal{T}_\theta = \mathcal{L}_{\bar{\eta}}\mathcal{T}_{-\theta\tau} = 0$, that is, $[\tau(\mathcal{T}_\theta\mathcal{T}_{-\theta\tau}), \mathcal{L}_{\bar{\eta}}] = 0$, a contradiction. □

Lemma 3.3. *Fix $\theta_0 \in \Lambda^{\mathcal{T}}$ and suppose $\Lambda^{\mathcal{L}}$ is symmetric. For $\theta \in \Lambda^{\mathcal{T}}_{\theta_0}$ and $\eta, \zeta \in \Lambda^{\mathcal{T}} \cup \{0\}$, then the following assertions hold*:
(1) *If $\{\mathcal{T}_\theta, \mathcal{T}_\eta, \mathcal{T}_\zeta\} \neq 0$ (resp., $\{\mathcal{T}_\theta, \mathcal{T}_\eta, \mathcal{T}_\zeta\}' \neq 0$), then $\eta, \zeta, \theta + \eta\tau + \zeta \in \Lambda^{\mathcal{T}}_{\theta_0} \cup \{0\}$ (resp., $\eta, \zeta, \theta\tau + \eta + \zeta \in \Lambda^{\mathcal{T}}_{\theta_0} \cup \{0\}$).*
(2) *If $\{\mathcal{T}_\eta, \mathcal{T}_\theta, \mathcal{T}_\zeta\} \neq 0$ (resp., $\{\mathcal{T}_\eta, \mathcal{T}_\theta, \mathcal{T}_\zeta\}' \neq 0$), then $\eta, \zeta, \eta + \theta\tau + \zeta \in \Lambda^{\mathcal{T}}_{\theta_0} \cup \{0\}$ (resp., $\eta, \zeta, \eta\tau + \theta + \zeta \in \Lambda^{\mathcal{T}}_{\theta_0} \cup \{0\}$).*
(3) If $\{\mathcal{T}_\eta, \mathcal{T}_\zeta, \mathcal{T}_\theta\} \neq 0$ (*resp.*, $\{\mathcal{T}_\eta, \mathcal{T}_\zeta, \mathcal{T}_\theta\}' \neq 0$), *then* $\eta, \zeta, \eta + \zeta\tau + \theta \in \Lambda^{\mathcal{T}}_{\theta_0} \cup \{0\}$ (*resp.* $\eta, \zeta, \eta\tau + \zeta + \theta \in \Lambda^{\mathcal{T}}_{\theta_0} \cup \{0\}$).

Proof. (1) It is easy to see that $\mathcal{T}_\theta\mathcal{T}_\eta \neq 0$ (resp., $\tau(\mathcal{T}_\theta\mathcal{T}_\eta) \neq 0$), for $\theta \in \Lambda^{\mathcal{T}}_{\theta_0}$ and $\eta \in \Lambda^{\mathcal{T}} \cup \{0\}$. By Lemma 3.1 (1), one obtains $\theta \sim \eta$ in the case $\eta \neq 0$. From here, $\eta \in \Lambda^{\mathcal{T}}_{\theta_0} \cup \{0\}$. In order to complete the proof, we will show $\zeta, \theta + \eta\tau + \zeta \in \Lambda^{\mathcal{T}}_{\theta_0} \cup \{0\}$ (resp., $\zeta, \theta\tau + \eta + \zeta \in \Lambda^{\mathcal{T}}_{\theta_0} \cup \{0\}$). We distinguish two cases.

Case 1. Suppose $\theta + \eta\tau + \zeta = 0$ (resp., $\theta\tau + \eta + \zeta = 0$). It is clear that $\theta + \eta\tau + \zeta \in \Lambda^{\mathcal{T}}_{\theta_0} \cup \{0\}$ (resp., $\theta\tau + \eta + \zeta \in \Lambda^{\mathcal{T}}_{\theta_0} \cup \{0\}$). If we have $\zeta \neq 0$, as $\theta + \eta\tau = -\zeta$ (resp., $\theta\tau + \eta = -\zeta$), $\{\theta, 0, \eta\tau\}$ would be a connection from θ to ζ and we conclude $\zeta \in \Lambda^{\mathcal{T}}_{\theta_0} \cup \{0\}$.

Case 2. Suppose $\theta + \eta\tau + \zeta \neq 0$ (resp., $\theta\tau + \eta + \zeta \neq 0$). We treat separately two cases.

Suppose $\theta + \eta\tau \neq 0$ (resp., $\theta\tau + \eta \neq 0$). By Lemma 2.10 (1), we have $\theta + \eta\tau \in \Lambda^{\mathcal{L}}$ and so $\{\theta, \eta, \zeta\}$ (resp., $\{\theta, \eta, \zeta\tau\}$) is a connection from θ to $\theta + \eta\tau + \zeta$ (resp., from θ to $\theta\tau + \eta + \zeta$). Hence, $\theta + \eta\tau + \zeta \in \Lambda^{\mathcal{T}}_{\theta_0}$ (resp., $\theta\tau + \eta + \zeta \in \Lambda^{\mathcal{T}}_{\theta_0}$). In the case $\zeta \neq 0$, we have $\{\theta, \eta, -\theta - \eta\tau - \zeta\}$ (resp., $\{\theta, \eta, -\theta - \eta\tau - \zeta\tau\}$) is a connection from θ to ζ. So $\zeta \in \Lambda^{\mathcal{T}}_{\theta_0}$. Hence, $\zeta \in \Lambda^{\mathcal{T}}_{\theta_0} \cup \{0\}$.

Suppose $\theta + \eta\tau = 0$ (resp., $\theta\tau + \eta = 0$). Then necessarily $\zeta \in \Lambda^{\mathcal{T}}_{\theta_0} \cup \{0\}$. Indeed, if $\zeta \neq 0$ and θ is not connected with ζ, by Lemma 3.2, $\{\mathcal{T}_\theta, \mathcal{T}_\eta, \mathcal{T}_\zeta\} = \zeta(\mathcal{T}_\theta\mathcal{T}_\eta)\mathcal{T}_\zeta = 0$ (resp., $\{\mathcal{T}_\theta, \mathcal{T}_\eta, \mathcal{T}_\zeta\}' = \zeta(\tau(\mathcal{T}_\theta\mathcal{T}_\eta))\mathcal{T}_\zeta = 0$), a contradiction. We also have $\theta + \eta\tau + \zeta = \zeta \in \Lambda^{\mathcal{T}}_{\theta_0} \cup \{0\}$ (resp., $\theta\tau + \eta + \zeta = \zeta \in \Lambda^{\mathcal{T}}_{\theta_0} \cup \{0\}$).

Item (2) can be proved similar to item (1); and item (3) is a consequence of items (1) and (2). □

Lemma 3.4. *Fix $\theta_0 \in \Lambda^{\mathscr{T}}$ and suppose $\Lambda^{\mathscr{L}}$ is symmetric. For $\theta, \eta, \zeta \in \Lambda^{\mathscr{T}}_{\theta_0} \cup \{0\}$ with $\theta + \eta\tau + \zeta = 0$ (resp., $\theta\tau + \eta + \zeta = 0$) and $\delta, \varepsilon \in \Lambda^{\mathscr{T}} \cup \{0\}$, then the following assertions hold*:

(1) *If $\{\{\mathscr{T}_\theta, \mathscr{T}_\eta, \mathscr{T}_\zeta\}, \mathscr{T}_\delta, \mathscr{T}_\varepsilon\} \neq 0$ (resp., $\{\{\mathscr{T}_\theta, \mathscr{T}_\eta, \mathscr{T}_\zeta\}', \mathscr{T}_\delta, \mathscr{T}_\varepsilon\}' \neq 0$), then δ, ε, $\delta\tau + \varepsilon \in \Lambda^{\mathscr{T}}_{\theta_0} \cup \{0\}$ (resp., $\delta, \varepsilon, \delta + \varepsilon \in \Lambda^{\mathscr{T}}_{\theta_0} \cup \{0\}$).*

(2) *If $\{\{\mathscr{T}_\theta, \mathscr{T}_\eta, \mathscr{T}_\zeta\}, \mathscr{T}_\delta, \mathscr{T}_\varepsilon\}' \neq 0$ (resp., $\{\{\mathscr{T}_\theta, \mathscr{T}_\eta, \mathscr{T}_\zeta\}', \mathscr{T}_\delta, \mathscr{T}_\varepsilon\} \neq 0$), then δ, ε, $\delta + \varepsilon \in \Lambda^{\mathscr{T}}_{\theta_0} \cup \{0\}$ (resp., $\delta, \varepsilon, \delta\tau + \varepsilon \in \Lambda^{\mathscr{T}}_{\theta_0} \cup \{0\}$).*

(3) *If $\{\mathscr{T}_\delta, \{\mathscr{T}_\theta, \mathscr{T}_\eta, \mathscr{T}_\zeta\}, \mathscr{T}_\varepsilon\} \neq 0$ (resp., $\{\mathscr{T}_\delta, \{\mathscr{T}_\theta, \mathscr{T}_\eta, \mathscr{T}_\zeta\}', \mathscr{T}_\varepsilon\}' \neq 0$), then δ, ε, $\delta + \varepsilon \in \Lambda^{\mathscr{T}}_{\theta_0} \cup \{0\}$ (resp., $\delta, \varepsilon, \delta\tau + \varepsilon \in \Lambda^{\mathscr{T}}_{\theta_0} \cup \{0\}$).*

(4) *If $\{\mathscr{T}_\delta, \{\mathscr{T}_\theta, \mathscr{T}_\eta, \mathscr{T}_\zeta\}, \mathscr{T}_\varepsilon\}' \neq 0$ (resp., $\{\mathscr{T}_\delta, \{\mathscr{T}_\theta, \mathscr{T}_\eta, \mathscr{T}_\zeta\}', \mathscr{T}_\varepsilon\} \neq 0$), then δ, ε, $\delta\tau + \varepsilon \in \Lambda^{\mathscr{T}}_{\theta_0} \cup \{0\}$ (resp., $\delta, \varepsilon, \delta + \varepsilon \in \Lambda^{\mathscr{T}}_{\theta_0} \cup \{0\}$).*

(5) *If $\{\mathscr{T}_\delta, \mathscr{T}_\varepsilon, \{\mathscr{T}_\theta, \mathscr{T}_\eta, \mathscr{T}_\zeta\}\} \neq 0$ (resp., $\{\mathscr{T}_\delta, \mathscr{T}_\varepsilon, \{\mathscr{T}_\theta, \mathscr{T}_\eta, \mathscr{T}_\zeta\}'\}' \neq 0$), then δ, ε, $\delta + \varepsilon\tau \in \Lambda^{\mathscr{T}}_{\theta_0} \cup \{0\}$ (resp., $\delta, \varepsilon, \delta\tau + \varepsilon \in \Lambda^{\mathscr{T}}_{\theta_0} \cup \{0\}$).*

(6) *If $\{\mathscr{T}_\delta, \mathscr{T}_\varepsilon, \{\mathscr{T}_\theta, \mathscr{T}_\eta, \mathscr{T}_\zeta\}\}' \neq 0$ (resp., $\{\mathscr{T}_\delta, \mathscr{T}_\varepsilon, \{\mathscr{T}_\theta, \mathscr{T}_\eta, \mathscr{T}_\zeta\}'\} \neq 0$), then δ, ε, $\delta\tau + \varepsilon \in \Lambda^{\mathscr{T}}_{\theta_0} \cup \{0\}$ (resp., $\delta, \varepsilon, \delta + \varepsilon\tau \in \Lambda^{\mathscr{T}}_{\theta_0} \cup \{0\}$).*

Proof. (1) Suppose that at least two distinct elements in $\{\theta, \eta, \zeta\}$ are nonzero, since $\theta + \eta\tau + \zeta = 0$, $\{\mathscr{T}_0, \mathscr{T}_0, \mathscr{T}_0\} = 0$, and $\{\mathscr{T}_\theta, \mathscr{T}_{-\theta\tau}, \mathscr{T}_0\} = 0$. Taking into account $\{\mathscr{T}_\theta, \mathscr{T}_\eta, \mathscr{T}_\zeta\} \neq 0$, $\theta + \eta\tau \neq 0$, and $\zeta \neq 0$. Since

$$0 \neq \{\{\mathscr{T}_\theta, \mathscr{T}_\eta, \mathscr{T}_\zeta\}, \mathscr{T}_\delta, \mathscr{T}_\varepsilon\} \subset \{\mathscr{T}_\theta, \mathscr{T}_\eta, \{\mathscr{T}_\zeta, \mathscr{T}_\delta, \mathscr{T}_\varepsilon\}\} + \{\mathscr{T}_\zeta, \{\mathscr{T}_\theta, \mathscr{T}_\eta, \mathscr{T}_\delta\}', \mathscr{T}_\varepsilon\} + \{\mathscr{T}_\zeta, \mathscr{T}_\delta, \{\mathscr{T}_\theta, \mathscr{T}_\eta, \mathscr{T}_\varepsilon\}\},$$

any of the aforementioned three summands is nonzero. Suppose

$$\{\mathscr{T}_\theta, \mathscr{T}_\eta, \{\mathscr{T}_\zeta, \mathscr{T}_\delta, \mathscr{T}_\varepsilon\}\} \neq 0,$$

as $\zeta \neq 0$ and $\{\mathscr{T}_\zeta, \mathscr{T}_\delta, \mathscr{T}_\varepsilon\} \neq 0$, Lemma 3.3 (1) shows δ, ε, and $\zeta + \delta\tau + \varepsilon$ are connected with ζ in the case of being nonzero roots and so $\delta, \varepsilon, \zeta + \delta\tau + \varepsilon \in \Lambda^{\mathscr{T}}_{\theta_0} \cup \{0\}$. If $\zeta + \delta\tau + \varepsilon = 0$, then $\zeta + \delta\tau = -\varepsilon \in \Lambda^{\mathscr{T}}_{\theta_0}$. If $\zeta + \delta\tau + \varepsilon \neq 0$, taking into account $0 \neq \{\mathscr{T}_\theta, \mathscr{T}_\eta, \{\mathscr{T}_\zeta, \mathscr{T}_\delta, \mathscr{T}_\varepsilon\}\} \subset \{\mathscr{T}_\theta, \mathscr{T}_\eta, \mathscr{T}_{\zeta+\delta\tau+\varepsilon}\}$, Lemma 3.3 (3) gives us that $\theta + \eta\tau + \zeta + \delta\tau + \varepsilon = \delta\tau + \varepsilon \in \Lambda^{\mathscr{T}}_{\theta_0}$. Therefore, $\delta, \varepsilon, \delta\tau + \varepsilon \in \Lambda^{\mathscr{T}}_{\theta_0} \cup \{0\}$.

By Lemma 3.3, we argue similarly if either $\{\mathscr{T}_\zeta, \{\mathscr{T}_\theta, \mathscr{T}_\eta, \mathscr{T}_\delta\}', \mathscr{T}_\varepsilon\} \neq 0$ or $\{\mathscr{T}_\zeta, \mathscr{T}_\delta, \{\mathscr{T}_\theta, \mathscr{T}_\eta, \mathscr{T}_\varepsilon\} \neq 0$ to obtain δ, ε, $\delta\tau + \varepsilon \in \Lambda^{\mathscr{T}}_{\theta_0} \cup \{0\}$. Similarly, if one suppose $\{\{\mathscr{T}_\theta, \mathscr{T}_\eta, \mathscr{T}_\zeta\}', \mathscr{T}_\delta, \mathscr{T}_\varepsilon\}' \neq 0$, one obtains δ, ε, $\delta + \varepsilon \in \Lambda^{\mathscr{T}}_{\theta_0} \cup \{0\}$.

Items (2), (3), (4), (5), and (6) can be proved as item (1). □

Definition 3.5. A split twisted inner derivation triple system $\mathscr{T}$ is said to be simple, if $\{\mathscr{T}, \mathscr{T}, \mathscr{T}\} \neq 0$ and its only ideals are $\{0\}$ and $\mathscr{T}$.

Theorem 3.6. *Suppose $\Lambda^{\mathscr{L}}$ is symmetric, the following assertions hold*:

(1) *For any $\theta_0 \in \Lambda^{\mathscr{T}}$, the split twisted inner derivation triple subsystem*

$$\mathscr{T}_{\Lambda^{\mathscr{T}}_{\theta_0}} = \mathscr{T}_{0,\Lambda^{\mathscr{T}}_{\theta_0}} \oplus V_{\Lambda^{\mathscr{T}}_{\theta_0}}$$

of $\mathscr{T}$ associated with the root subsystem $\Lambda^{\mathscr{T}}_{\theta_0}$ is an ideal of $\mathscr{T}$.

(2) *If $\mathscr{T}$ is simple, then there exists a connection from θ to η for any $\theta, \eta \in \Lambda^{\mathscr{T}}$.*

Proof. (1) Recall that

$$\mathscr{T}_{0,\Lambda^{\mathscr{T}}_{\theta_0}} := \mathrm{span}_{\mathbb{K}}\Big\{\{\mathscr{T}_\theta, \mathscr{T}_\eta, \mathscr{T}_\zeta\} : \theta + \eta\tau + \zeta = 0; \quad \{\mathscr{T}_\theta, \mathscr{T}_\eta, \mathscr{T}_\zeta\}' : \theta\tau + \eta + \zeta = 0; \\ \text{where } \theta, \eta, \zeta \in \Lambda^{\mathscr{T}}_{\theta_0} \cup \{0\}\Big\} \subset \mathscr{T}_0,$$

and $V_{\Lambda^{\mathcal{T}}_{\theta_0}} := \oplus_{\zeta\in\Lambda^{\mathcal{T}}_{\theta_0}}\mathcal{T}_\zeta$. Obviously,

$$\left\{\mathcal{T}_{\Lambda^{\mathcal{T}}_{\theta_0}}, \mathcal{T}, \mathcal{T}\right\} = \left\{\mathcal{T}_{0,\Lambda^{\mathcal{T}}_{\theta_0}}, \mathcal{T}, \mathcal{T}\right\} + \left\{V_{\Lambda^{\mathcal{T}}_{\theta_0}}, \mathcal{T}, \mathcal{T}\right\}.$$

First of all, we will prove that $\{\mathcal{T}_{0,\Lambda^{\mathcal{T}}_{\theta_0}}, \mathcal{T}, \mathcal{T}\} \subset \mathcal{T}_{\Lambda^{\mathcal{T}}_{\theta_0}}$. Find that

$$\begin{aligned}\left\{\mathcal{T}_{0,\Lambda^{\mathcal{T}}_{\theta_0}}, \mathcal{T}, \mathcal{T}\right\} &= \{\mathcal{T}_{0,\Lambda^{\mathcal{T}}_{\theta_0}}, \mathcal{T}_0 \oplus (\bigoplus_{\theta\in\Lambda^{\mathcal{T}}}\mathcal{T}_\theta), \mathcal{T}_0 \oplus (\oplus_{\theta\in\Lambda^{\mathcal{T}}}\mathcal{T}_\theta)\}\\ &= \left\{\mathcal{T}_{0,\Lambda^{\mathcal{T}}_{\theta_0}}, \mathcal{T}_0, \mathcal{T}_0\right\} + \{\mathcal{T}_{0,\Lambda^{\mathcal{T}}_{\theta_0}}, \mathcal{T}_0, \bigoplus_{\theta\in\Lambda^{\mathcal{T}}}\mathcal{T}_\theta\}\\ &\quad + \{\mathcal{T}_{0,\Lambda^{\mathcal{T}}_{\theta_0}}, \bigoplus_{\theta\in\Lambda^{\mathcal{T}}}\mathcal{T}_\theta, \mathcal{T}_0\} + \{\mathcal{T}_{0,\Lambda^{\mathcal{T}}_{\theta_0}}, \bigoplus_{\theta\in\Lambda^{\mathcal{T}}}\mathcal{T}_\theta, \bigoplus_{\eta\in\Lambda^{\mathcal{T}}}\mathcal{T}_\eta\}.\end{aligned}$$

One obtains that $\{\mathcal{T}_{0,\Lambda^{\mathcal{T}}_{\theta_0}}, \mathcal{T}_0, \mathcal{T}_0\} \subset \{\mathcal{T}_0, \mathcal{T}_0, \mathcal{T}_0\} = 0$. Taking into account $\{\mathcal{T}_{0,\Lambda^{\mathcal{T}}_{\theta_0}}, \mathcal{T}_0, \mathcal{T}_\theta\}$, for $\theta \in \Lambda^{\mathcal{T}}$, if $\theta \in \Lambda^{\mathcal{T}}_{\theta_0}$, by Lemma 2.10 (5), one obtains $\{\mathcal{T}_{0,\Lambda^{\mathcal{T}}_{\theta_0}}, \mathcal{T}_0, \mathcal{T}_\theta\} \subset V_{\Lambda^{\mathcal{T}}_{\theta_0}}$. If $\theta \notin \Lambda^{\mathcal{T}}_{\theta_0}$, by Lemma 3.4 (1), one obtains $\{\mathcal{T}_{0,\Lambda^{\mathcal{T}}_{\theta_0}}, \mathcal{T}_0, \mathcal{T}_\theta\} = 0$. Hence, $\{\mathcal{T}_{0,\Lambda^{\mathcal{T}}_{\theta_0}}, \mathcal{T}_0, \oplus_{\theta\in\Lambda^{\mathcal{T}}}\mathcal{T}_\theta\} \subset \mathcal{T}_{\Lambda^{\mathcal{T}}_{\theta_0}}$. Similarly, one obtains that $\{\mathcal{T}_{0,\Lambda^{\mathcal{T}}_{\theta_0}}, \oplus_{\theta\in\Lambda^{\mathcal{T}}}\mathcal{T}_\theta, \mathcal{T}_0\} \subset \mathcal{T}_{\Lambda^{\mathcal{T}}_{\theta_0}}$. Next, we will consider that $\{\mathcal{T}_{0,\Lambda^{\mathcal{T}}_{\theta_0}}, \mathcal{T}_\theta, \mathcal{T}_\eta\}$, where $\theta, \eta \in \Lambda^{\mathcal{T}}$. We discuss five cases.

(i) If $\theta \in \Lambda^{\mathcal{T}}_{\theta_0}$, $\eta \in \Lambda^{\mathcal{T}}_{\theta_0}$, $\theta\tau + \eta = 0$, this means that $\{\mathcal{T}_{0,\Lambda^{\mathcal{T}}_{\theta_0}}, \mathcal{T}_\theta, \mathcal{T}_\eta\} \subset \mathcal{T}_{0,\Lambda^{\mathcal{T}}_{\theta_0}}$.

(ii) If $\theta \in \Lambda^{\mathcal{T}}_{\theta_0}$, $\eta \in \Lambda^{\mathcal{T}}_{\theta_0}$, $\theta\tau + \eta \neq 0$, by $\Lambda^{\mathcal{T}}_{\theta_0}$ is a root subsystem, this means that $\{\mathcal{T}_{0,\Lambda^{\mathcal{T}}_{\theta_0}}, \mathcal{T}_\theta, \mathcal{T}_\eta\} \subset V_{\Lambda^{\mathcal{T}}_{\theta_0}}$.

(iii) If $\theta \in \Lambda^{\mathcal{T}}_{\theta_0}$, $\eta \notin \Lambda^{\mathcal{T}}_{\theta_0}$, satisfy Lemma 3.4 (1), this means that $\{\mathcal{T}_{0,\Lambda^{\mathcal{T}}_{\theta_0}}, \mathcal{T}_\theta, \mathcal{T}_\eta\} = 0$.

(iv) If $\eta \in \Lambda^{\mathcal{T}}_{\theta_0}$, $\theta \notin \Lambda^{\mathcal{T}}_{\theta_0}$, satisfy Lemma 3.4 (1), this means that $\{\mathcal{T}_{0,\Lambda^{\mathcal{T}}_{\theta_0}}, \mathcal{T}_\theta, \mathcal{T}_\eta\} = 0$.

(v) If $\eta \notin \Lambda^{\mathcal{T}}_{\theta_0}$, $\theta \notin \Lambda^{\mathcal{T}}_{\theta_0}$, satisfy Lemma 3.4 (1), this means that $\{\mathcal{T}_{0,\Lambda^{\mathcal{T}}_{\theta_0}}, \mathcal{T}_\theta, \mathcal{T}_\eta\} = 0$.

So, one has $\{\mathcal{T}_{0,\Lambda^{\mathcal{T}}_{\theta_0}}, \oplus_{\theta\in\Lambda^{\mathcal{T}}}\mathcal{T}_\theta, \oplus_{\eta\in\Lambda^{\mathcal{T}}}\mathcal{T}_\eta\} \subset \mathcal{T}_{\Lambda^{\mathcal{T}}_{\theta_0}}$. Therefore, $\{\mathcal{T}_{0,\Lambda^{\mathcal{T}}_{\theta_0}}, \mathcal{T}, \mathcal{T}\} \subset \mathcal{T}_{\Lambda^{\mathcal{T}}_{\theta_0}}$.

Next, we will prove that $\{V_{\Lambda^{\mathcal{T}}_{\theta_0}}, \mathcal{T}, \mathcal{T}\} \subset \mathcal{T}_{\Lambda^{\mathcal{T}}_{\theta_0}}$. Note that

$$\begin{aligned}\left\{V_{\Lambda^{\mathcal{T}}_{\theta_0}}, \mathcal{T}, \mathcal{T}\right\} &= \{\bigoplus_{\zeta\in\Lambda^{\mathcal{T}}_{\theta_0}}\mathcal{T}_\zeta, \mathcal{T}_0 \oplus (\bigoplus_{\theta\in\Lambda^{\mathcal{T}}}\mathcal{T}_\theta), \mathcal{T}_0 \oplus (\bigoplus_{\theta\in\Lambda^{\mathcal{T}}}\mathcal{T}_\theta)\}\\ &= \{\bigoplus_{\zeta\in\Lambda^{\mathcal{T}}_{\theta_0}}\mathcal{T}_\zeta, \mathcal{T}_0, \mathcal{T}_0\} + \{\oplus_{\zeta\in\Lambda^{\mathcal{T}}_{\theta_0}}\mathcal{T}_\zeta, \mathcal{T}_0, \bigoplus_{\theta\in\Lambda^{\mathcal{T}}}\mathcal{T}_\theta\}\\ &\quad + \{\oplus_{\zeta\in\Lambda^{\mathcal{T}}_{\theta_0}}\mathcal{T}_\zeta, \bigoplus_{\theta\in\Lambda^{\mathcal{T}}}\mathcal{T}_\theta, \mathcal{T}_0\} + \{\oplus_{\zeta\in\Lambda^{\mathcal{T}}_{\theta_0}}\mathcal{T}_\zeta, \bigoplus_{\theta\in\Lambda^{\mathcal{T}}}\mathcal{T}_\theta, \bigoplus_{\eta\in\Lambda^{\mathcal{T}}}\mathcal{T}_\eta\}.\end{aligned}$$

One obtains that $\{\oplus_{\zeta\in\Lambda^{\mathcal{T}}_{\theta_0}}\mathcal{T}_\zeta, \mathcal{T}_0, \mathcal{T}_0\} \subset V_{\Lambda^{\mathcal{T}}_{\theta_0}}$. Next, we will consider $\{\mathcal{T}_\zeta, \mathcal{T}_0, \mathcal{T}_\theta\}$, for $\zeta \in \Lambda^{\mathcal{T}}_{\theta_0}$, $\theta \in \Lambda^{\mathcal{T}}$. We discuss the following three cases.

(i) If $\zeta \in \Lambda^{\mathcal{T}}_{\theta_0}$, $\theta \notin \Lambda^{\mathcal{T}}_{\theta_0}$, satisfying Lemma 3.3 (1), this means that $\{\mathcal{T}_\zeta, \mathcal{T}_0, \mathcal{T}_\theta\} = 0$.

(ii) If $\zeta \in \Lambda^{\mathcal{T}}_{\theta_0}$, $\theta \in \Lambda^{\mathcal{T}}_{\theta_0}$, $\zeta + \theta \neq 0$, by $\Lambda^{\mathcal{T}}_{\theta_0}$ is a root subsystem, this means that $\{\mathcal{T}_\zeta, \mathcal{T}_0, \mathcal{T}_\theta\} \subset V_{\Lambda^{\mathcal{T}}_{\theta_0}}$.

(iii) If $\zeta \in \Lambda^{\mathcal{T}}_{\theta_0}$, $\theta \in \Lambda^{\mathcal{T}}_{\theta_0}$, $\zeta + \theta = 0$, it is clear that $\{\mathcal{T}_\zeta, \mathcal{T}_0, \mathcal{T}_\theta\} \subset \mathcal{T}_{0,\Lambda^{\mathcal{T}}_{\theta_0}}$. Hence, $\{\oplus_{\zeta\in\Lambda^{\mathcal{T}}_{\theta_0}}\mathcal{T}_\zeta, \mathcal{T}_0, \oplus_{\theta\in\Lambda^{\mathcal{T}}}\mathcal{T}_\theta\} \subset \mathcal{T}_{\Lambda^{\mathcal{T}}_{\theta_0}}$. Similarly, one obtains that $\{\oplus_{\zeta\in\Lambda^{\mathcal{T}}_{\theta_0}}\mathcal{T}_\zeta, \oplus_{\theta\in\Lambda^{\mathcal{T}}}\mathcal{T}_\theta, \mathcal{T}_0\} \subset \mathcal{T}_{\Lambda^{\mathcal{T}}_{\theta_0}}$. At last, we will consider $\{\mathcal{T}_\zeta, \mathcal{T}_\theta, \mathcal{T}_\eta\}$, where $\zeta \in \Lambda^{\mathcal{T}}_{\theta_0}$, $\theta \in \Lambda^{\mathcal{T}}$ and $\eta \in \Lambda^{\mathcal{T}}$. We discuss the following five cases.

(i) If $\zeta \in \Lambda^{\mathcal{T}}_{\theta_0}$, $\theta \in \Lambda^{\mathcal{T}}_{\theta_0}$, $\eta \in \Lambda^{\mathcal{T}}_{\theta_0}$, $\zeta + \theta\tau + \eta = 0$, this means that $\{\mathcal{T}_\zeta, \mathcal{T}_\theta, \mathcal{T}_\eta\} \subset \mathcal{T}_{0,\Lambda^{\mathcal{T}}_{\theta_0}}$.

(ii) If $\zeta \in \Lambda^{\mathcal{T}}_{\theta_0}$, $\theta \in \Lambda^{\mathcal{T}}_{\theta_0}$, $\eta \in \Lambda^{\mathcal{T}}_{\theta_0}$, $\zeta + \theta\tau + \eta \neq 0$, this means that $\{\mathcal{T}_\zeta, \mathcal{T}_\theta, \mathcal{T}_\eta\} \subset V_{\Lambda^{\mathcal{T}}_{\theta_0}}$.

(iii) If $\zeta \in \Lambda^{\mathcal{T}}_{\theta_0}$, $\theta \in \Lambda^{\mathcal{T}}_{\theta_0}$, $\eta \notin \Lambda^{\mathcal{T}}_{\theta_0}$, satisfying Lemma 3.3 (1), this means that $\{\mathcal{T}_\zeta, \mathcal{T}_\theta, \mathcal{T}_\eta\} = 0$.

(iv) If $\zeta \in \Lambda^{\mathcal{T}}_{\theta_0}$, $\theta \notin \Lambda^{\mathcal{T}}_{\theta_0}$, $\eta \in \Lambda^{\mathcal{T}}_{\theta_0}$, satisfying Lemma 3.3 (1), this means that $\{\mathcal{T}_\zeta, \mathcal{T}_\theta, \mathcal{T}_\eta\} = 0$.

(v) If $\zeta \in \Lambda^{\mathcal{T}}_{\theta_0}$, $\theta \notin \Lambda^{\mathcal{T}}_{\theta_0}$, $\eta \notin \Lambda^{\mathcal{T}}_{\theta_0}$, satisfying Lemma 3.3 (1), this means that $\{\mathcal{T}_\zeta, \mathcal{T}_\theta, \mathcal{T}_\eta\} = 0$.

So, one has $\{\oplus_{\zeta\in\Lambda^{\mathcal{T}}_{\theta_0}}\mathcal{T}_\zeta, \oplus_{\theta\in\Lambda^{\mathcal{T}}}\mathcal{T}_\theta, \oplus_{\eta\in\Lambda^{\mathcal{T}}}\mathcal{T}_\eta\} \subset \mathcal{T}_{\Lambda^{\mathcal{T}}_{\theta_0}}$. Therefore, $\{V_{\Lambda^{\mathcal{T}}_{\theta_0}}, \mathcal{T}, \mathcal{T}\} \subset \mathcal{T}_{\Lambda^{\mathcal{T}}_{\theta_0}}$. Then $\{\mathcal{T}_{\Lambda^{\mathcal{T}}_{\theta_0}}, \mathcal{T}, \mathcal{T}\} \subset \mathcal{T}_{\Lambda^{\mathcal{T}}_{\theta_0}}$.

Similarly, one has

$$\begin{aligned}\left\{\mathscr{T}_{0,\Lambda^{\mathscr{T}}_{\theta_0}}, \mathscr{T}, \mathscr{T}\right\}' &= \{\mathscr{T}_{0,\Lambda^{\mathscr{T}}_{\theta_0}}, \mathscr{T}_0 \oplus (\bigoplus_{\theta\in\Lambda^{\mathscr{T}}}\mathscr{T}_\theta), \mathscr{T}_0 \oplus (\bigoplus_{\theta\in\Lambda^{\mathscr{T}}}\mathscr{T}_\theta)\}' \\ &= \left\{\mathscr{T}_{0,\Lambda^{\mathscr{T}}_{\theta_0}}, \mathscr{T}_0, \mathscr{T}_0\right\}' + \{\mathscr{T}_{0,\Lambda^{\mathscr{T}}_{\theta_0}}, \mathscr{T}_0, \bigoplus_{\theta\in\Lambda^{\mathscr{T}}}\mathscr{T}_\theta\}' \\ &\quad + \{\mathscr{T}_{0,\Lambda^{\mathscr{T}}_{\theta_0}}, \bigoplus_{\theta\in\Lambda^{\mathscr{T}}}\mathscr{T}_\theta, \mathscr{T}_0\}' + \{\mathscr{T}_{0,\Lambda^{\mathscr{T}}_{\theta_0}}, \bigoplus_{\theta\in\Lambda^{\mathscr{T}}}\mathscr{T}_\theta, \bigoplus_{\eta\in\Lambda^{\mathscr{T}}}\mathscr{T}_\eta\}' \subset \mathscr{T}_{\Lambda^{\mathscr{T}}_{\theta_0}},\end{aligned}$$

and

$$\begin{aligned}\left\{V_{\Lambda^{\mathscr{T}}_{\theta_0}}, \mathscr{T}, \mathscr{T}\right\}' &= \{\bigoplus_{\zeta\in\Lambda^{\mathscr{T}}_{\theta_0}}\mathscr{T}_\zeta, \mathscr{T}_0 \oplus (\bigoplus_{\theta\in\Lambda^{\mathscr{T}}}\mathscr{T}_\theta), \mathscr{T}_0 \oplus (\bigoplus_{\theta\in\Lambda^{\mathscr{T}}}\mathscr{T}_\theta)\}' \\ &= \{\bigoplus_{\zeta\in\Lambda^{\mathscr{T}}_{\theta_0}}\mathscr{T}_\zeta, \mathscr{T}_0, \mathscr{T}_0\}' + \{\bigoplus_{\zeta\in\Lambda^{\mathscr{T}}_{\theta_0}}\mathscr{T}_\zeta, \mathscr{T}_0, \bigoplus_{\theta\in\Lambda^{\mathscr{T}}}\mathscr{T}_\theta\}' \\ &\quad + \{\bigoplus_{\zeta\in\Lambda^{\mathscr{T}}_{\theta_0}}\mathscr{T}_\zeta, \bigoplus_{\theta\in\Lambda^{\mathscr{T}}}\mathscr{T}_\theta, \mathscr{T}_0\}' + \{\bigoplus_{\zeta\in\Lambda^{\mathscr{T}}_{\theta_0}}\mathscr{T}_\zeta, \bigoplus_{\theta\in\Lambda^{\mathscr{T}}}\mathscr{T}_\theta, \bigoplus_{\eta\in\Lambda^{\mathscr{T}}}\mathscr{T}_\eta\}' \subset \mathscr{T}_{\Lambda^{\mathscr{T}}_{\theta_0}}.\end{aligned}$$

Then

$$\left\{\mathscr{T}_{\Lambda^{\mathscr{T}}_{\theta_0}}, \mathscr{T}, \mathscr{T}\right\}' = \left\{\mathscr{T}_{0,\Lambda^{\mathscr{T}}_{\theta_0}}, \mathscr{T}, \mathscr{T}\right\}' + \left\{V_{\Lambda^{\mathscr{T}}_{\theta_0}}, \mathscr{T}, \mathscr{T}\right\}' \subset \mathscr{T}_{\Lambda^{\mathscr{T}}_{\theta_0}}.$$

By the same argument, Lemmas 3.3 and 3.4 show $\{\mathscr{T}, \mathscr{T}_{\Lambda^{\mathscr{T}}_{\theta_0}}, \mathscr{T}\} + \{\mathscr{T}, \mathscr{T}_{\Lambda^{\mathscr{T}}_{\theta_0}}, \mathscr{T}\}' + \{\mathscr{T}, \mathscr{T}, \mathscr{T}_{\Lambda^{\mathscr{T}}_{\theta_0}}\} \subset \mathscr{T}_{\Lambda^{\mathscr{T}}_{\theta_0}}$.

(2) The simplicity of $\mathscr{T}$ implies $\mathscr{T}_{\Lambda^{\mathscr{T}}_{\theta_0}} = \mathscr{T}$. Hence, $\Lambda^{\mathscr{T}}_{\theta_0} = \Lambda^{\mathscr{T}}$. □

Theorem 3.7. *Suppose $\Lambda^{\mathscr{L}}$ is symmetric. Then for a vector space complement U of*

$$\mathrm{span}_{\mathbb{K}}\{\{\mathscr{T}_\theta, \mathscr{T}_\eta, \mathscr{T}_\zeta\} : \theta + \eta\tau + \zeta = 0; \quad \{\mathscr{T}_\theta, \mathscr{T}_\eta, \mathscr{T}_\zeta\}' : \theta\tau + \eta + \zeta = 0; \quad \text{where } \theta, \eta, \zeta \in \Lambda^{\mathscr{T}} \cup \{0\}\}$$

in $\mathscr{T}_0$, we have

$$\mathscr{T} = U + \sum_{[\theta]\in\Lambda^{\mathscr{T}}/\sim} I_{[\theta]},$$

where any $I_{[\theta]}$ is one of the ideals $\mathscr{T}_{\Lambda^{\mathscr{T}}_{\theta_0}}$ of $\mathscr{T}$ described in Theorem 3.6. *Moreover*, $\{I_{[\theta]}, \mathscr{T}, I_{[\eta]}\} = \{I_{[\theta]}, I_{[\eta]}, \mathscr{T}\} = \{\mathscr{T}, I_{[\theta]}, I_{[\eta]}\} = \{I_{[\theta]}, \mathscr{T}, I_{[\eta]}\}' = \{I_{[\theta]}, I_{[\eta]}, \mathscr{T}\}' = \{\mathscr{T}, I_{[\theta]}, I_{[\eta]}\}' = 0$ *if* $[\theta] \neq [\eta]$.

Proof. Let us denote $\xi_0 := \mathrm{span}_{\mathbb{K}}\{\{\mathscr{T}_\theta, \mathscr{T}_\eta, \mathscr{T}_\zeta\} : \theta + \eta\tau + \zeta = 0;\ \{\mathscr{T}_\theta, \mathscr{T}_\eta, \mathscr{T}_\zeta\}' : \theta\tau + \eta + \zeta = 0;$ where $\theta, \eta, \zeta \in \Lambda^{\mathscr{T}} \cup \{0\}\}$ in $\mathscr{T}_0$. By Proposition 2.15, we can consider the quotient set $\Lambda^{\mathscr{T}}/\sim := \{[\theta] : \theta \in \Lambda^{\mathscr{T}}\}$. By denoting $I_{[\theta]} := \mathscr{T}_{\Lambda^{\mathscr{T}}_\theta}$, $\mathscr{T}_{0,[\theta]} := \mathscr{T}_{0,\Lambda^{\mathscr{T}}_\theta}$ and $V_{[\theta]} := V_{\Lambda^{\mathscr{T}}_\theta}$, one obtains $I_{[\theta]} := \mathscr{T}_{0,[\theta]} \oplus V_{[\theta]}$. From

$$\mathscr{T} = \mathscr{T}_0 \oplus (\bigoplus_{\theta\in\Lambda^{\mathscr{T}}}\mathscr{T}_\theta) = (U + \xi_0) \oplus (\bigoplus_{\theta\in\Lambda^{\mathscr{T}}}\mathscr{T}_\theta),$$

it follows $\oplus_{\theta\in\Lambda^{\mathscr{T}}}\mathscr{T}_\theta = \oplus_{[\theta]\in\Lambda^{\mathscr{T}}/\sim}V_{[\theta]}$, $\xi_0 = \sum_{[\theta]\in\Lambda^{\mathscr{T}}/\sim}\mathscr{T}_{0,[\theta]}$, which implies

$$\mathscr{T} = (U + \xi_0) \oplus (\bigoplus_{\theta\in\Lambda^{\mathscr{T}}}\mathscr{T}_\theta) = U + \sum_{[\theta]\in\Lambda^{\mathscr{T}}/\sim} I_{[\theta]},$$

where each $I_{[\theta]}$ is an ideal of $\mathscr{T}$ by Theorem 3.6.

It is sufficient to show that $\{I_{[\theta]}, \mathscr{T}, I_{[\eta]}\} = 0$ if $[\theta] \neq [\eta]$. Note that,

$$\begin{aligned}\{I_{[\theta]}, \mathscr{T}, I_{[\eta]}\} &= \left\{\mathscr{T}_{0,[\theta]} \oplus V_{[\theta]}, \mathscr{T}_0 \oplus \left(\oplus_{\zeta\in\Lambda^{\mathscr{T}}}\mathscr{T}_\zeta\right), \mathscr{T}_{0,[\eta]} \oplus V_{[\eta]}\right\} \\ &= \{\mathscr{T}_{0,[\theta]}, \mathscr{T}_0, \mathscr{T}_{0,[\eta]}\} + \{\mathscr{T}_{0,[\theta]}, \mathscr{T}_0, V_{[\eta]}\} + \left\{\mathscr{T}_{0,[\theta]}, \oplus_{\zeta\in\Lambda^{\mathscr{T}}}\mathscr{T}_\zeta, \mathscr{T}_{0,[\eta]}\right\} \\ &\quad + \left\{\mathscr{T}_{0,[\theta]}, \oplus_{\zeta\in\Lambda^{\mathscr{T}}}\mathscr{T}_\zeta, V_{[\eta]}\right\} + \{V_{[\theta]}, \mathscr{T}_0, \mathscr{T}_{0,[\eta]}\} + \{V_{[\theta]}, \mathscr{T}_0, V_{[\eta]}\} \\ &\quad + \left\{V_{[\theta]}, \oplus_{\zeta\in\Lambda^{\mathscr{T}}}\mathscr{T}_\zeta, \mathscr{T}_{0,[\eta]}\right\} + \left\{V_{[\theta]}, \oplus_{\zeta\in\Lambda^{\mathscr{T}}}\mathscr{T}_\zeta, V_{[\eta]}\right\}.\end{aligned}$$

Here, it is clear that $\{\mathscr{T}_{0,[\theta]}, \mathscr{T}_0, \mathscr{T}_{0,[\eta]}\} \subset \{\mathscr{T}_0, \mathscr{T}_0, \mathscr{T}_0\} = 0$. If $[\theta] \neq [\eta]$, by Lemmas 3.3 and 3.4, it is easy to see $\{\mathscr{T}_{0,[\theta]}, \mathscr{T}_0, V_{[\eta]}\} = 0$, $\{\mathscr{T}_{0,[\theta]}, \oplus_{\zeta\in\Lambda^{\mathscr{T}}}\mathscr{T}_\zeta, V_{[\eta]}\} = 0$, $\{V_{[\theta]}, \mathscr{T}_0, \mathscr{T}_{0,[\eta]}\} = 0$, $\{V_{[\theta]}, \mathscr{T}_0, V_{[\eta]}\} = 0$, $\{V_{[\theta]}, \oplus_{\zeta\in\Lambda^{\mathscr{T}}}\mathscr{T}_\zeta, \mathscr{T}_{0,[\eta]}\} = 0$, $\{V_{[\theta]}, \oplus_{\zeta\in\Lambda^{\mathscr{T}}}\mathscr{T}_\zeta, V_{[\eta]}\} = 0$.

Next, we will prove $\{\mathscr{T}_{0,[\theta]}, \oplus_{\zeta\in\Lambda^{\mathscr{T}}}\mathscr{T}_\zeta, \mathscr{T}_{0,[\eta]}\} = 0$. In fact, for $\{\mathscr{T}_{\theta_1}, \mathscr{T}_{\theta_2}, \mathscr{T}_{\theta_3}\} \in \mathscr{T}_{0,[\theta]}$ with $\theta_1, \theta_2, \theta_3 \in \Lambda_\theta^{\mathscr{T}} \cup \{0\}$, $\theta_1 + \theta_2\tau + \theta_3 = 0$, and for $\{\mathscr{T}_{\eta_1}, \mathscr{T}_{\eta_2}, \mathscr{T}_{\eta_3}\} \in \mathscr{T}_{0,[\eta]}$ with $\eta_1, \eta_2, \eta_3 \in \Lambda_\eta^{\mathscr{T}} \cup \{0\}$, $\eta_1 + \eta_2\tau + \eta_3 = 0$, by the definition of twisted inner derivation triple system, one obtains

$$\begin{aligned}&\Big\{\big\{\mathscr{T}_{\theta_1}, \mathscr{T}_{\theta_2}, \mathscr{T}_{\theta_3}\big\}, \oplus_{\zeta\in\Lambda^{\mathscr{T}}}\mathscr{T}_\zeta, \big\{\mathscr{T}_{\eta_1}, \mathscr{T}_{\eta_2}, \mathscr{T}_{\eta_3}\big\}\Big\}\\ &\quad\subset \Big\{\Big\{\big\{\mathscr{T}_{\theta_1}, \mathscr{T}_{\theta_2}, \mathscr{T}_{\theta_3}\big\}, \oplus_{\zeta\in\Lambda^{\mathscr{T}}}\mathscr{T}_\zeta, \mathscr{T}_{\eta_1}\Big\}, \mathscr{T}_{\eta_2}, \mathscr{T}_{\eta_3}\Big\} + \Big\{\mathscr{T}_{\eta_1}, \Big\{\big\{\mathscr{T}_{\theta_1}, \mathscr{T}_{\theta_2}, \mathscr{T}_{\theta_3}\big\}, \oplus_{\zeta\in\Lambda^{\mathscr{T}}}\mathscr{T}_\zeta, \mathscr{T}_{\eta_2}\Big\}', \mathscr{T}_{\eta_3}\Big\}\\ &\qquad + \Big\{\mathscr{T}_{\eta_1}, \mathscr{T}_{\eta_2}, \Big\{\big\{\mathscr{T}_{\theta_1}, \mathscr{T}_{\theta_2}, \mathscr{T}_{\theta_3}\big\}, \oplus_{\zeta\in\Lambda^{\mathscr{T}}}\mathscr{T}_\zeta, \mathscr{T}_{\eta_3}\Big\}\Big\}.\end{aligned}$$

By Lemma 3.4, it is easy to see that

$$\begin{aligned}&\Big\{\Big\{\big\{\mathscr{T}_{\theta_1}, \mathscr{T}_{\theta_2}, \mathscr{T}_{\theta_3}\big\}, \oplus_{\zeta\in\Lambda^{\mathscr{T}}}\mathscr{T}_\zeta, \mathscr{T}_{\eta_1}\Big\}, \mathscr{T}_{\eta_2}, \mathscr{T}_{\eta_3}\Big\} = 0,\\ &\Big\{\mathscr{T}_{\eta_1}, \Big\{\big\{\mathscr{T}_{\theta_1}, \mathscr{T}_{\theta_2}, \mathscr{T}_{\theta_3}\big\}, \oplus_{\zeta\in\Lambda^{\mathscr{T}}}\mathscr{T}_\zeta, \mathscr{T}_{\eta_2}\Big\}', \mathscr{T}_{\eta_3}\Big\} = 0,\\ &\Big\{\mathscr{T}_{\eta_1}, \mathscr{T}_{\eta_2}, \Big\{\big\{\mathscr{T}_{\theta_1}, \mathscr{T}_{\theta_2}, \mathscr{T}_{\theta_3}\big\}, \oplus_{\zeta\in\Lambda^{\mathscr{T}}}\mathscr{T}_\zeta, \mathscr{T}_{\eta_3}\Big\}\Big\} = 0,\end{aligned}$$

for $\theta_1, \theta_2, \theta_3 \in \Lambda_\theta^{\mathscr{T}} \cup \{0\}$, $\theta_1 + \theta_2\tau + \theta_3 = 0$, $\eta_1, \eta_2, \eta_3 \in \Lambda_\eta^{\mathscr{T}} \cup \{0\}$, $\eta_1 + \eta_2\tau + \eta_3 = 0$, $[\theta] \neq [\eta]$. In fact, if for $\{\mathscr{T}_{\theta_1}, \mathscr{T}_{\theta_2}, \mathscr{T}_{\theta_3}\}' \in \mathscr{T}_{0,[\theta]}$ with $\theta_1, \theta_2, \theta_3 \in \Lambda_\theta^{\mathscr{T}} \cup \{0\}$, and $\theta_1\tau + \theta_2 + \theta_3 = 0$, or for $\{\mathscr{T}_{\eta_1}, \mathscr{T}_{\eta_2}, \mathscr{T}_{\eta_3}\}' \in \mathscr{T}_{0,[\eta]}$ with $\eta_1, \eta_2, \eta_3 \in \Lambda_\eta^{\mathscr{T}} \cup \{0\}$, and $\eta_1\tau + \eta_2 + \eta_3 = 0$, one will obtain $\{\mathscr{T}_{0,[\theta]}, \oplus_{\zeta\in\Lambda^{\mathscr{T}}}\mathscr{T}_\zeta, \mathscr{T}_{0,[\eta]}\} = 0$.

Finally, by Lemmas 3.3 and 3.4, we also obtain $\{\mathscr{T}, I_{[\theta]}, I_{[\eta]}\} = \{I_{[\theta]}, I_{[\eta]}, \mathscr{T}\} = \{I_{[\theta]}, \mathscr{T}, I_{[\eta]}\}' = \{\mathscr{T}, I_{[\theta]}, I_{[\eta]}\}' = \{I_{[\theta]}, I_{[\eta]}, \mathscr{T}\}' = 0$. □

Corollary 3.8. *Suppose* $\Lambda^{\mathscr{L}}$ *is symmetric. If* $\mathrm{Ann}(\mathscr{T}) = 0$, *and* $\{\mathscr{T}, \mathscr{T}, \mathscr{T}\} = \mathscr{T}$, *then* $\mathscr{T}$ *is the direct sum of the ideals given in* Theorem 3.7,

$$\mathscr{T} = \bigoplus_{[\theta]\in\Lambda^{\mathscr{T}}/\sim} I_{[\theta]}.$$

Proof. From $\{\mathscr{T}, \mathscr{T}, \mathscr{T}\} = \mathscr{T}$ and Theorem 3.7, we have

$$\left\{U + \sum_{[\theta]\in\Lambda^{\mathscr{T}}/\sim} I_{[\theta]}, U + \sum_{[\theta]\in\Lambda^{\mathscr{T}}/\sim} I_{[\theta]}, U + \sum_{[\theta]\in\Lambda^{\mathscr{T}}/\sim} I_{[\theta]}\right\} = U + \sum_{[\theta]\in\Lambda^{\mathscr{T}}/\sim} I_{[\theta]}.$$

Taking into account $U \subset \mathscr{T}_0$, $\{\mathscr{T}_0, \mathscr{T}_0, \mathscr{T}_0\} = 0$, Lemma 3.3, and the fact that $\{I_{[\theta]}, \mathscr{T}, I_{[\eta]}\} = 0$ if $[\theta] \neq [\eta]$ (see Theorem 3.7) give us that $U = 0$. That is,

$$\mathscr{T} = \sum_{[\theta]\in\Lambda^{\mathscr{T}}/\sim} I_{[\theta]}.$$

To finish, it is sufficient to show the direct character of the sum. For $x \in I_{[\theta]} \cap \sum_{[\eta]\in\Lambda^{\mathscr{T}}/\sim, \eta\neq\theta} I_{[\eta]}$, using again the equation $\{I_{[\theta]}, \mathscr{T}, I_{[\eta]}\} = 0$ for $[\theta] \neq [\eta]$, we obtain

$$\{x, \mathscr{T}, I_{[\theta]}\} = \left\{x, \mathscr{T}, \sum_{\substack{[\eta]\in\Lambda^{\mathscr{T}}/\sim\\ \eta\neq\theta}} I_{[\eta]}\right\} = 0.$$

So $\{x, \mathscr{T}, \mathscr{T}\} = \{x, \mathscr{T}, I_{[\theta]} + \sum_{[\eta]\in\Lambda^{\mathscr{T}}/\sim, \eta\neq\theta} I_{[\eta]}\} = \{x, \mathscr{T}, I_{[\theta]}\} + \{x, \mathscr{T}, \sum_{[\eta]\in\Lambda^{\mathscr{T}}/\sim, \eta\neq\theta} I_{[\eta]}\} = 0 + 0 = 0$. A same argument shows $\{\mathscr{T}, x, \mathscr{T}\} = 0$ and $\{\mathscr{T}, \mathscr{T}, x\} = 0$. That is, $x \in \mathrm{Ann}(\mathscr{T}) = 0$. Thus $x = 0$, as desired. □

4 The simple components

In this section, we study if any of the components in the decomposition given in Corollary 3.8 is simple. Under certain conditions we give an affirmative answer. From now on char($\mathbb{K}$) = 0.

Lemma 4.1. *Let $\mathscr{T} = \mathscr{T}_0 \oplus (\oplus_{\theta\in\Lambda^{\mathscr{T}}}\mathscr{T}_\theta)$ be a split twisted inner derivation triple system. If I is an ideal of $\mathscr{T}$, then $I = (I \cap \mathscr{T}_0) \oplus (\oplus_{\theta\in\Lambda^{\mathscr{T}}}(I \cap \mathscr{T}_\theta))$.*

Proof. We can see that $\mathscr{T} = \mathscr{T}_0 \oplus (\oplus_{\theta\in\Lambda^{\mathscr{T}}}\mathscr{T}_\theta)$ as a weight module with respect to the split Lie algebra $\mathscr{L}$ with MASA H. The character of ideal of I and the fact $\mathscr{L} = \mathscr{T}\mathscr{T}$ give us that I is a submodule of $\mathscr{T}$. It is well-known that a submodule of a weight module is again a weight module. From here, I is a weight module with respect to $\mathscr{L}$ (and H) and so $I = (I \cap \mathscr{T}_0) \oplus (\oplus_{\theta\in\Lambda^{\mathscr{T}}}(I \cap \mathscr{T}_\theta))$. □

Definition 4.2. We say that a split twisted inner derivation triple system T is root-multiplicative if $\theta, \eta, \zeta \in \Lambda^T \cup \{0\}$ are such that $\theta + \eta\tau \in \Lambda^{\mathscr{L}}$, and $\theta + \eta\tau + \zeta \in \Lambda^T$, then $\{T_\theta, T_\eta, T_\zeta\} \neq 0$.

Lemma 4.3. *Let $\mathscr{T}$ be a root-multiplicative split twisted inner derivation triple system with* Ann($\mathscr{T}$) = 0. *If for any $\theta \in \Lambda^{\mathscr{T}}$, we have* $\dim\mathscr{L}_\theta \leq 1$. *Then there is not any nonzero ideal of $\mathscr{T}$ contained in $\mathscr{T}_0$.*

Proof. Suppose there exists a nonzero ideal I of $\mathscr{T}$ such that $I \subset \mathscr{T}_0$. Taking into account $\{\mathscr{T}_0, \mathscr{T}_0, \mathscr{T}_0\} = 0$, $\{I, \mathscr{T}_0, \mathscr{T}_0\} = 0$, $\{\mathscr{T}_0, I, \mathscr{T}_0\} = 0$. Given $\theta \in \Lambda^{\mathscr{T}}$, as $\{I, \mathscr{T}_0, \mathscr{T}_\theta\} \subset \mathscr{T}_\theta \cap \mathscr{T}_0$, $\{\mathscr{T}_0, I, \mathscr{T}_\theta\} \subset \mathscr{T}_\theta \cap \mathscr{T}_0$, $\{\mathscr{T}_\theta, I, \mathscr{T}_0\} \subset \mathscr{T}_\theta \cap \mathscr{T}_0$ and $\{I, \mathscr{T}_\theta, \mathscr{T}_0\} \subset \mathscr{T}_{\theta\tau} \cap \mathscr{T}_0$, $\{I, \mathscr{T}_0, \mathscr{T}_\theta\} = \{\mathscr{T}_0, I, \mathscr{T}_\theta\} = \{\mathscr{T}_\theta, I, \mathscr{T}_0\} = \{I, \mathscr{T}_\theta, \mathscr{T}_0\} = 0$. Given also $\eta \in \Lambda^{\mathscr{T}}$, if $\theta + \eta \neq 0$, then $\{\mathscr{T}_\theta, I, \mathscr{T}_\eta\} \subset \mathscr{T}_{\theta+\eta} \cap \mathscr{T}_0 = 0$. If $\theta\tau + \eta \neq 0$, $\{I, \mathscr{T}_\theta, \mathscr{T}_\eta\} \subset \mathscr{T}_{\theta\tau+\eta} \cap \mathscr{T}_0 = 0$. In addition, $\{\mathscr{T}, \mathscr{T}, I\} = 0$.

As Ann($\mathscr{T}$) = 0, we have either $\{I, \mathscr{T}_\theta, \mathscr{T}_{-\theta\tau}\} \neq 0$ or $\{\mathscr{T}_\theta, I, \mathscr{T}_{-\theta}\} \neq 0$ for some $\theta \in \Lambda^{\mathscr{T}}$. In the case $\{I, \mathscr{T}_\theta, \mathscr{T}_{-\theta\tau}\} \neq 0$, there exist $t_\theta \in \mathscr{T}_\theta$, $t_{-\theta\tau} \in \mathscr{T}_{-\theta\tau}$ and $t_0 \in I$ such that $\{t_0, t_\theta, t_{-\theta\tau}\} \neq 0$. Hence, $0 \neq t_0 t_\theta \in \mathscr{L}_{\theta\tau}$ and so necessarily $\dim\mathscr{L}_{\theta\tau} = 1$. The root-multiplicativity of $\mathscr{T}$ (consider the roots $0, \theta\tau, 0 \in \Lambda^{\mathscr{T}} \cup \{0\}$), and the fact that $\dim\mathscr{L}_{\theta\tau} = 1$ give us the existence of $0 \neq t'_0 \in \mathscr{T}_0$ such that $0 \neq \{t_0, t_\theta, t'_0\} \in \mathscr{T}_{\theta\tau}$. As $t_0 \in I$, we conclude $0 \neq t'_{\theta\tau} \coloneqq \{t_0, t_\theta, t'_0\} \in I \subset \mathscr{T}_0 \cap \mathscr{T}_{\theta\tau}$, a contradiction. If $\{\mathscr{T}_\theta, I, \mathscr{T}_{-\theta}\} \neq 0$, we similarly obtain a contradiction. Similarly, if $\{I, \mathscr{T}, \mathscr{T}\}' \neq 0$, $\{\mathscr{T}, I, \mathscr{T}\}' \neq 0$, we can obtain a contradiction. So I is not contained in $\mathscr{T}_0$. □

Theorem 4.4. *Let $\mathscr{T}$ be a root-multiplicative, with* Ann($\mathscr{T}$) = 0 *and satisfying $\mathscr{T} = \{\mathscr{T}, \mathscr{T}, \mathscr{T}\}$. If $\Lambda^{\mathscr{L}}$ is symmetric and for any $\theta \in \Lambda^{\mathscr{T}}$, we have* $\dim\mathscr{T}_\theta = 1$, $\dim\mathscr{L}_\theta \leq 1$, *then $\mathscr{T}$ is simple if and only if it has all its nonzero roots connected.*

Proof. If $\mathscr{T}$ is simple, satisfy Theorem 3.6, it has all its nonzero roots connected. Let us prove the converse: Consider I a nonzero ideal of $\mathscr{T}$. By Lemmas 4.1 and 4.3, $I = (I \cap \mathscr{T}_0) \oplus (\oplus_{\theta\in\Lambda^{\mathscr{T}}}(I \cap \mathscr{T}_\theta))$ with $I \cap \mathscr{T}_{\theta_0} \neq 0$ for some $\theta_0 \in \Lambda^{\mathscr{T}}$. Taking into account $\dim\mathscr{T}_{\theta_0} = 1$, we have $\mathscr{T}_{\theta_0} \subset I$. The fact $\{\mathscr{T}_0, \mathscr{T}_{\theta_0}, \mathscr{T}_0\} \neq 0$ gives us $\mathscr{T}_{\theta_0\tau} \subset I$. Given $\eta_0 \in \Lambda^{\mathscr{T}}$ with $\eta_0 \notin \{\pm\theta_0, \pm\theta_0\tau\}$, as θ_0 and η_0 are connected, the root-multiplicativity of $\mathscr{T}$ and the assumption $\dim\mathscr{T}_\theta = 1$ for any $\theta \in \Lambda^{\mathscr{T}}$ give us a connection $\{\theta_1, \dots, \theta_{2r+1}\}$ from θ_0 to η_0 such that $\theta_1 = \theta_0$, $\theta_1 + \theta_2\tau + \theta_3, \dots, \theta_1 + \theta_2\tau + \dots + \theta_{2r}\tau + \theta_{2r+1} \in \Lambda^{\mathscr{T}}$, $\theta_1 + \theta_2\tau, \cdots, \theta_1 + \theta_2\tau + \dots + \theta_{2r}\tau \in \Lambda^{\mathscr{L}}$, and $\theta_1 + \theta_2\tau + \dots + \theta_{2r}\tau + \theta_{2r+1} \in p\eta_0\nu$, where $p \in \pm 1$, $\nu \in \{Id|_H, \tau\}$, and $\mathscr{T}_{\theta_1} = \mathscr{T}_{\theta_0}$, $\{\mathscr{T}_{\theta_1}, \mathscr{T}_{\theta_2}, \mathscr{T}_{\theta_3}\} = \mathscr{T}_{\theta_1+\theta_2\tau+\theta_3}$, $\{\{\mathscr{T}_{\theta_1}, \mathscr{T}_{\theta_2}, \mathscr{T}_{\theta_3}\}, \mathscr{T}_{\theta_4}, \mathscr{T}_{\theta_5}\} = \mathscr{T}_{\theta_1+\theta_2\tau+\theta_3+\theta_4\tau+\theta_5}, \cdots,$

$$\left\{\left\{\cdots\left\{\left\{\mathscr{T}_{\theta_1}, \mathscr{T}_{\theta_2}, \mathscr{T}_{\theta_3}\right\}, \mathscr{T}_{\theta_4}, \mathscr{T}_{\theta_5}\right\}, \dots\right\}, \mathscr{T}_{\theta_{2r+1}}\right\} \in \mathscr{T}_{p\eta_0\nu}.$$

If $\mathscr{T}_{\eta_0} \subset I$, we have $\{\mathscr{T}_0, \mathscr{T}_{\eta_0}, \mathscr{T}_0\} \subset I$, that is, $\mathscr{T}_{\eta_0\tau} \subset I$. If $\mathscr{T}_{-\eta_0\tau} \subset I$, we have $\{\mathscr{T}_0, \mathscr{T}_{-\eta_0\tau}, \mathscr{T}_0\} \subset I$, that is, $\mathscr{T}_{-\eta_0} \subset I$. From here, either

$$\mathscr{T}_{\eta_0} \subset I \text{ or } \mathscr{T}_{-\eta_0\tau} \subset I, \tag{4.14}$$

for any $\eta_0 \subset \Lambda^{\mathscr{T}}$, and so $\{\mathscr{T}_{\eta_0}, \mathscr{T}_{-\eta_0\tau}, \mathscr{T}\} \subset I$.

Observe that as a consequence of $\mathcal{T} = \{\mathcal{T}, \mathcal{T}, \mathcal{T}\}$, we have

$$\mathcal{T}_0 = \sum_{\substack{\theta+\eta\tau+\zeta=0\\ \theta,\eta,\zeta\in\Lambda^{\mathcal{T}}\cup\{0\}}} \{\mathcal{T}_\theta, \mathcal{T}_\eta, \mathcal{T}_\zeta\} + \sum_{\substack{\theta\tau+\eta+\zeta=0\\ \theta,\eta,\zeta\in\Lambda^{\mathcal{T}}\cup\{0\}}} \{\mathcal{T}_\theta, \mathcal{T}_\eta, \mathcal{T}_\zeta\}'. \tag{4.15}$$

In order to show $\mathcal{T}_0 \subset I$, we carry out the following steps.

First, let us study the products $\{\mathcal{T}_\theta, \mathcal{T}_\eta, \mathcal{T}_\zeta\}$ with $\theta, \eta, \zeta \in \Lambda^{\mathcal{T}} \cup \{0\}$, $\theta + \eta\tau + \zeta = 0$. Taking into account $\{\mathcal{T}_0, \mathcal{T}_0, \mathcal{T}_0\} = 0$, and the fact $\theta + \eta\tau + \zeta = 0$ with $\theta, \eta, \zeta \in \Lambda^{\mathcal{T}} \cup \{0\}$, we can suppose $\zeta \neq 0$ and either $\theta \neq 0$ or $\eta \neq 0$. Suppose $\theta \neq 0$ and $\eta = 0$ (resp. $\theta = 0$ and $\eta \neq 0$), then $\theta = -\zeta$ (resp., $\eta\tau = -\zeta$) and by equation (4.14), $\{\mathcal{T}_\theta, \mathcal{T}_\eta, \mathcal{T}_\zeta\} = \{\mathcal{T}_{-\zeta}, \mathcal{T}_0, \mathcal{T}_\zeta\} \subset I$ (resp., $\{\mathcal{T}_\theta, \mathcal{T}_\eta, \mathcal{T}_\zeta\} = \{\mathcal{T}_0, \mathcal{T}_{-\zeta\tau}, \mathcal{T}_\zeta\} \subset I$). If the three elements in $\{\theta, \eta, \zeta\}$ are nonzero, in case some $\mathcal{T}_\varepsilon \subset I$, $\varepsilon \in \{\theta, \eta, \zeta\}$, then clearly $\{\mathcal{T}_\theta, \mathcal{T}_\eta, \mathcal{T}_\zeta\} \subset I$. Finally, consider the case in which any of the $\mathcal{T}_\varepsilon$ does not belong to I. If $\{\mathcal{T}_\theta, \mathcal{T}_\eta, \mathcal{T}_\zeta\} = 0$, then $\{\mathcal{T}_\theta, \mathcal{T}_\eta, \mathcal{T}_\zeta\} \subset I$. If $\{\mathcal{T}_\theta, \mathcal{T}_\eta, \mathcal{T}_\zeta\} \neq 0$, necessarily $\theta + \eta\tau \neq 0$ and so $\theta + \eta\tau \in \Lambda^{\mathcal{L}}$. From here, we have by root-multiplicativity $\{\mathcal{T}_\theta, \mathcal{T}_\eta, \mathcal{T}_{-\eta\tau}\} = \mathcal{T}_\theta$. Equation (4.14) gives us $\mathcal{T}_\eta \subset I$ or $\mathcal{T}_{-\eta\tau} \subset I$, then $\mathcal{T}_\theta \subset I$ and so

$$\{\mathcal{T}_\theta, \mathcal{T}_\eta, \mathcal{T}_\zeta\} \subset I. \tag{4.16}$$

Second, let us study the products $\{\mathcal{T}_\theta, \mathcal{T}_\eta, \mathcal{T}_\zeta\}'$ with $\theta, \eta, \zeta \in \Lambda^{\mathcal{T}} \cup \{0\}$, $\theta\tau + \eta + \zeta = 0$. Similarly, we can obtain

$$\{\mathcal{T}_\theta, \mathcal{T}_\eta, \mathcal{T}_\zeta\}' \subset I. \tag{4.17}$$

Therefore, equations (4.15), (4.16), and (4.17) imply

$$\mathcal{T}_0 \subset I. \tag{4.18}$$

Fix now any $\theta_0 \in \Lambda^{\mathcal{T}}$. By equation (4.14) either $\mathcal{T}_{\theta_0} \subset I$ or $\mathcal{T}_{-\theta_0\tau} \subset I$. That is, either $\mathcal{T}_{\theta_0} \subset I$ or $\mathcal{T}_{-\theta_0} \subset I$. Indeed, if $\mathcal{T}_{-\theta_0\tau} \subset I$, $\{\mathcal{T}_0, \mathcal{T}_{-\theta_0\tau}, \mathcal{T}_0\} \neq 0$, one obtains $\mathcal{T}_{-\theta_0} \subset I$. Write $\mathcal{T}_{\lambda\theta_0} \subset I$ with $\lambda \in \pm 1$, then we can show $\mathcal{T}_{-\lambda\theta_0} \subset I$. Indeed, since $\theta_0 \neq 0$, there exists $h_0 \in H$ such that $\theta_0(h_0) \neq 0$ and so we have

$$t_{-\lambda\theta_0} = -\lambda\theta_0(h_0)^{-1}\Big(h_0\Big(t_{-\lambda\theta_0}\Big)\Big), \tag{4.19}$$

for any $t_{-\lambda\theta_0} \in \mathcal{T}_{-\lambda\theta_0}$.

As

$$H = \mathcal{T}_0\mathcal{T}_0 + \sum_{\theta\in\Lambda^{\mathcal{T}}} \mathcal{T}_\theta\mathcal{T}_{-\theta\tau},$$

one suppose that either $h_0 = t_0t_0'$ with $t_0, t_0' \in \mathcal{T}_0$ or $h_0 = t_\theta t_{-\theta\tau}$ with $t_\theta \in \mathcal{T}_\theta$, $t_{-\theta\tau} \in \mathcal{T}_{-\theta\tau}$. In the first case, by equations (4.18) and (4.19), we obtain

$$t_{-\lambda\theta_0} = -\lambda\theta_0(h_0)^{-1}\Big\{t_0, t_0', t_{-\lambda\theta_0}\Big\} \in I, \tag{4.20}$$

for any $t_{-\lambda\theta_0} \in \mathcal{T}_{-\lambda\theta_0}$.

In the second case, by equations (4.14) and (4.19), we obtain

$$t_{-\lambda\theta_0} = -\lambda\theta_0(h_0)^{-1}\Big\{t_\theta, t_{-\theta\tau}, t_{-\lambda\theta_0}\Big\} \in I, \tag{4.21}$$

for any $t_{-\lambda\theta_0} \in \mathcal{T}_{-\lambda\theta_0}$.

Since $\dim\mathcal{T}_{-\lambda\theta_0} = 1$, we conclude $\mathcal{T}_{-\lambda\theta_0} \subset I$. So $\mathcal{T}_{\pm\theta_0} \subset I$ for any $\theta_0 \in \Lambda^{\mathcal{T}}$. From here, and taking into account equation (4.18), we conclude $I = \mathcal{T}$ and so $\mathcal{T}$ is simple. □

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