Research Article

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Connectivity with respect to α-discrete closure operators

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Abstract: We discuss certain closure operators that generalize the Alexandroff topologies. Such a closure operator is defined for every ordinal $\alpha > 0$ in such a way that the closure of a set A is given by closures of certain α-indexed sequences formed by points of A. It is shown that connectivity with respect to such a closure operator can be viewed as a special type of path connectivity. This makes it possible to apply the operators in solving problems based on employing a convenient connectivity such as problems of digital image processing. One such application is presented providing a digital analogue of the Jordan curve theorem.

Keywords: closure operator, ordinal (number), ordinal-indexed sequence, connectivity, digital Jordan curve

MSC 2020: 54A05, 54D05

1 Introduction

One of the basic problems of digital topology is to equip the digital plane $\mathbb{Z}^2$ with a convenient connectivity, i.e., a connectivity behaving analogously to the connectivity in the Euclidean plane $\mathbb{R}^2$. In particular, such a connectivity is required to satisfy a digital analogue of the Jordan curve theorem because digital Jordan curves represent borders of objects in digital pictures. In this note, we study certain closure operators (more general than the Kuratowski ones) and show that they provide a convenient connectivity for the digital plane.

By a *closure operator* u on a set X, we mean a map $u : \exp X \to \exp X$ (where $\exp X$ denotes the power set of X), which is

(i) grounded (i.e., $u\emptyset = \emptyset$),
(ii) extensive (i.e., $A \subseteq X \Rightarrow A \subseteq uA$), and
(iii) monotonic (i.e., $A \subseteq B \subseteq X \Rightarrow uA \subseteq uB$).

The pair (X, u) is then called a *closure space*.

These closure operators were studied in the pioneering paper [1] by Čech published as early as in 1936.

Recall that a closure operator u on X which is

(iv) additive (i.e., $u(A \cup B) = uA \cup uB$ whenever $A, B \subseteq X$) and
(v) idempotent (i.e., $uuA = uA$ whenever $A \subseteq X$)

is called a *Kuratowski closure operator* or a *topology*, and the pair (X, u) is called a *topological space*.

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In the literature, closure operators are usually understood to be idempotent and not necessarily grounded, while the ones introduced above are named *preclosure operators*. But, in this note, we follow the terminology of categorical topology where closure operators in the above sense are commonly used.

Given a cardinal $m > 1$, a closure operator u on a set X and the closure space (X, u) are called an S_m*-closure operator* and an S_m*-closure space* (briefly, an S_m*-space*), respectively, if the following condition is satisfied:

$$A \subseteq X \Rightarrow uA = \bigcup\{uB; B \subseteq A, \text{ card } B < m\}.$$

The closure operators usually employed in algebra are the idempotent ones. The so-called *algebraic closure operators* are then simply the idempotent $S_{\aleph_0}$-closure operators (cf. [2]).

In the categorical topology, an important role is played by the closure operators on categories studied, e.g., in [3]. They generalize the above closure operators by considering them on subobject lattices in a given category (instead of on power sets, i.e., the subobject lattices in the category *Set*).

In [4], (additive) S_2-closure operators and S_2-spaces are called *quasi-discrete*. S_2-topologies (S_2-topological spaces) are usually called *Alexandroff topologies* (*Alexandroff spaces*) [5]. Alexandroff topologies play an important role in general topology and many other branches of mathematics, particularly in general and topological algebra (Scott topologies, monoid actions, commutative algebra) and computer science (Khalimsky topology), cf. [6–11]. Of course, every S_2-closure operator is additive, and every S_α-closure operator is an S_β-closure operator whenever $\alpha < \beta$. Since any closure operator on a set X is obviously an S_α-closure operator for each cardinal α with $\alpha >$ card X, there exists a least cardinal α such that u is an S_α-closure operator. Such a cardinal is then an important invariant of the closure operator u. Evidently, if $\alpha \le \aleph_0$, then every additive S_α-closure operator is an S_2-closure operator.

We will use some known topological concepts (see, e.g., [12]) naturally extended to closure spaces. Given a closure space (X, u), a subset $A \subseteq X$ is called *closed* if $uA = A$, and *open* if $X - A$ is closed. A closure space (X, u) is said to be a *subspace* of a closure space (Y, v) if $uA = vA \cap X$ for each subset $A \subseteq X$. We will speak briefly about a subspace X of (Y, v). A closure space (X, u) is said to be *connected* if $\emptyset$ and X are the only subsets of X to be both closed and open. A subset $X \subseteq Y$ is connected in a closure space (Y, v) if the subspace X of (Y, v) is connected. A maximal connected subset of a closure space is called a *component* of this space. All the basic properties of connected subsets and components in topological spaces are also preserved in closure spaces. In particular, if $\{A_i;\ i \in I\}$ is a family of connected subsets with a non-empty intersection, then $\bigcup_{i \in I} A_i$ is connected, too.

In the paper, we will work with α-indexed sequences of points of a set X, where α is an ordinal, i.e., with sequences of the form $(x_i|\ i < \alpha) \subseteq X^\alpha$. We will write $(x_i|\ i \le \alpha)$ instead of $(x_i|\ i < \alpha + 1)$.

2 α-discrete closure operators

In the sequel, $\alpha > 0$ will be an ordinal.

Definition 2.1. Let (X, u) be a closure space. A sequence $(x_i|\ i \le \alpha) \subseteq X^{\alpha+1}$ is called a *u-connected element* if $x_j \in u\{x_i;\ i < j\}$ for each j, $0 < j \le \alpha$. The elements x_0 and x_α are called the *end points* of the u-connected element $(x_i|\ i \le \alpha)$.

Definition 2.2. A closure operator u on a set X is called α*-discrete* if the following condition is fulfilled:

For any $A \subseteq X$ and any $x \in uA$, there exists a u-connected element $(x_i|\ i \le \alpha)$ such that $(x_i|\ i < \alpha) \subseteq A^\alpha$ and $x_\alpha = x$.

Thus, every α-discrete closure operator is an $S_{\langle\alpha\rangle}$-space, where $\langle\alpha\rangle$ denotes the least of all cardinals that are greater than α. The 1-discrete closure operators coincide with the S_2-closure operators so that idempotent 1-discrete closure operators are simply Alexandroff topologies, i.e., topologies with completely additive closures.

Example 2.3. Let u be the closure operator on the set $X = \{a, b, c\}$ given by $u\{a\} = \{a, b\}$, $u\{b\} = \{b\}$, $u\{c\} = \{c\}$, $u\{a, b\} = u\{a, c\} = X$, and $u\{b, c\} = \{b, c\}$. Then, u is a 2-discrete closure operator on X (which is not 1-discrete). This results from the fact that the sequences (a, a, a), (b, b, b), (c, c, c), (a, a, b), and (a, b, c) are u-connected elements and, for every $A \subseteq X$ and $x \in uA$, there exists $(x_i|\ i \le 2) \in \{(a, a, a), (b, b, b), (c, c, c), (a, a, b), (a, b, c)\}$ such that $(x_i|\ i < 2) \in A^2$ and $x_2 = x$.

For a closure operator u on a set X, we denote by u_α the closure operator on X given by $u_\alpha A = \{x \in X;$ there is a u-connected element $(x_i|\ i \le \alpha)$ such that $x = x_{i_0}$ for some $i_0,\ 0 < i_0 \le \alpha$, and $x_i \in A$ for all $i < i_0\}$.

Proposition 2.4. *A closure operator u on a set X is α-discrete if and only if $u = u_\alpha$.*

Proof. Let u be a closure operator on X. Let u be α-discrete and let $A \subseteq X$ be a subset. It is evident that $uA \subseteq u_\alpha A$. To show the converse inclusion, let $x \in u_\alpha A$. Then, there is a u-connected element $(x_i|\ i \le \alpha)$ such that $x = x_{i_0}$ for some $i_0,\ 0 < i_0 \le \alpha$, and $x_i \in A$ for all $i < i_0$. Thus, $x \in u\{x_i;\ i < i_0\} \subseteq uA$. Therefore, $u = u_\alpha$.

Conversely, let $u = u_\alpha$. Let $A \subseteq X$ and $x \in uA$. Then, there exists a u-connected element $(x_i|\ i \le \alpha)$ such that $x = x_{i_0}$ for some $i_0,\ 0 < i_0 \le \alpha$, and $x_i \in A$ for all $i < i_0$. Put $x_i' = x_i$ for all $i < i_0$, $x_i' = x_0$ for all $i, i_0 \le i < \alpha$, and $x_\alpha' = x_{i_0}$. Then, $(x_i'|\ i \le \alpha)$ is a u-connected element such that $(x_i'|\ i < \alpha) \subseteq A^\alpha$ and $x_\alpha = x$. Therefore, u is α-discrete. □

Example 2.5. As usual, we denote by ω the least infinite ordinal. Let u be the closure operator on the set (ordinal) $\omega + 1$ given by $uA = \{\alpha; \alpha$ is an ordinal such that $\min A \le \alpha \le \omega\}$. Let $x \in uA$ be a point. Then, the points $y \in A$ with $y < x$ form an increasing sequence $(y_i|\ i < \alpha)$, where $\alpha \le \omega$, $y_0 = \min A$, and $x_j \in u\{x_i;\quad i < j\}$ for each j, $\quad 0 < j < \alpha$. Put $y_i' = y_i$ for all $i < \alpha$ and $y_i' = x$ for all $i, \alpha \le i \le \omega$. Then, $(y_i'|\ i \le \omega)$ is a u-connected element such that $x = y_\alpha'$ and $y_i' \in A$ for all $i < \alpha$. Therefore, $u = u_\omega$. By Proposition 2.4, u is an ω-discrete closure operator (which is not α-discrete for any finite ordinal α).

Proposition 2.6. *Let u be an α-discrete closure operator on a set X. Then:*
(1) *The union of a system of closed subsets of (X, u) is a closed subset of (X, u).*
(2) *u is idempotent if and only if (X, u) is an Alexandroff space.*

Proof. (1) Let $\{A_j;\ j \in J\}$ be a system of closed subsets of (X, u) and let $x \in u\bigcup_{j\in J} A_j$. Then, there exists a u-connected element $(x_i|\ i \le \alpha)$ such that $x = x_\alpha$ and $x_i \in \bigcup_{j\in J} A_j$ for all $i < \alpha$. In particular, we have $x_0 \in \bigcup_{j\in J} A_j$, and so there exists $j_0 \in J$ such that $x_0 \in A_{j_0}$. Suppose that $\{x_i;\ i < \alpha\}$ is not a subset of A_{j_0}. Then, there is the smallest ordinal $i_1 < \alpha$ such that $x_{i_1} \notin A_{j_0}$. Consequently, $0 < i_1$ and $x_i \in A_{j_0}$ for all $i < i_1$. Thus, we have $x_{i_1} \in u\{x_i;\ i < i_1\} \subseteq uA_{j_0} = A_{j_0}$, which is a contradiction. Therefore, $\{x_i;\ i < \alpha\} \subseteq A_{j_0}$, and hence, $x \in u\{x_i;\ i < \alpha\} \subseteq uA_{j_0} \subseteq \bigcup_{j\in J} uA_j$. We have shown that $u\bigcup_{j\in J} A_j \subseteq \bigcup_{j\in J} uA_j$. As the converse inclusion is obvious, the proof is complete.

(2) Let u be idempotent, let $A \subseteq X$ and $x \in uA$. Then, there is a u-connected element $(x_i|\ i \le \alpha)$ such that $(x_i|\ i < \alpha) \subseteq A^\alpha$ and $x_\alpha = x$. Thus, we have $x_j \in u\{x_i;\ i < j\}$ for all j, $0 < j \le \alpha$, and clearly, $x_0 \in u\{x_0\}$. Let j be an ordinal, $0 < j \le \alpha$, such that $x_i \in u\{x_0\}$ for all $i < j$. Then, $x_j \in u\{x_i;\ i < j\} \subseteq uu\{x_0\} = u\{x_0\}$. Therefore, by transfinite induction, $x_j \in u\{x_0\}$ for every $j \le \alpha$. Consequently, $x = x_\alpha \in u\{x_0\}$. Hence, u is an Alexandroff topology. The converse implication is obvious. □

Definition 2.7. Let u be an α-discrete closure operator on a set X and $x, y \in X$. A finite sequence of u-connected elements $(p_j)_{j=1}^k = ((x_i^j\,|i \le \alpha))_{j=1}^k$ (k a positive integer) is said to be an *α-path* connecting x and y if

- x is an end point of p_1 with the other end point of p_1 coinciding with an end point of p_2,
- an end point of p_j coincides with an end point of p_{j-1} and the other end point of p_j coincides with an end point of p_{j+1} for $j = 2, 3, \dots, k-1$, and
- y is an end point of p_k and the other end point of p_k coincides with an end point of p_{k-1}.

Thus, every u-connected element $(x_i| \, i < \alpha)$ is an α-path connecting x_0 and x_α (and also an α-path connecting x_α and x_0). Clearly, if $(p_j)_{j=1}^k$ is an α-path connecting x and y, then $(p_{k-j+1})_{j=1}^k$ is an α-path connecting y and x. It is also evident that any α-path (connecting a pair of points) $((x_i^j| \, i \le \alpha))_{j=1}^k$ is a connected set, namely, the set $\bigcup_{j=1}^k \{x_i^j; \, i \le \alpha\}$. If $(p_j)_{j=1}^k$ is an α-path connecting x and y and $(q_j)_{j=1}^l$ is an α-path connecting y and z, then the α-path $(r_j)_{j=1}^{k+l}$, where $r_j = p_j$ whenever $1 \le j \le k$ and $r_j = q_{j-k}$ whenever $k < j \le k + l$ is an α-path connecting x and z.

Theorem 2.8. *Let u be an α-discrete closure operator on a set X and $A \subseteq X$ a subset. Then, A is connected in (X, u) if and only if any two points of A can be joined by an α-path contained in A.*

Proof. If $A = \emptyset$, then the statement is trivial. Let $A \neq \emptyset$. In (X, u), if any two points of A can be connected by an α-path, then A is clearly connected (because, choosing a point $x \in A$, we have $\bigcup_{y \in A}\{P_y, \; P_y$ is an α-path contained in A connecting x and $y\} = A$, i.e., A is the union of connected sets with a non-empty intersection). Conversely, let A be connected and suppose that there are points $x, y \in A$ that cannot be connected by an α-path contained in A. Let B be the set of all points of A that can be connected with x by an α-path contained in A. Let $z \in uB \cap A$ be a point. Then, there is a u-connected element $(x_i| \, i \le \alpha)$ such that $(x_i| \, i < \alpha) \subseteq B^\alpha$ and $x_\alpha = z$. Hence, $(x_i| \, i \le \alpha)$ is a u-connected element contained in A, thus an α-path connecting the points $x_0 \in B$ and $z \in A$. Since x and x_0 can also be connected by an α-path contained in A, so can x and z. Therefore, $z \in B$, i.e., $uB \cap A = B$. Consequently, B is closed in the subspace A of (X, u). Next, let $z \in u(A - B) \cap A$ be a point. Then, there is a u-connected element $(x_i| \, i \le \alpha)$ such that $(x_i| \, i < \alpha) \subseteq (A - B)^\alpha$ and $x_\alpha = z$. Suppose that $z \in B$. Then, x can be connected with z by an α-path contained in A. Further, z can be connected with x_0 by an α-path – the u-connected element $(x_i| \, i \le \alpha)$ – contained in A. Consequently, x and x_0 can be connected by an α-path contained in A, which is a contradiction with $x_0 \notin B$. Thus, $z \notin B$, i.e., $u(A - B) \cap A = A - B$ implying that $A - B$ is closed in the subspace A of (X, u). Hence, A is the union of the non-empty disjoint sets B and $A - B$ closed in the subspace A of (x, u). But this is a contradiction because A is connected. Therefore, any two points of A can be connected by an α-path contained in A. □

It is well known [13] that closure operators that are more general than the Kuratowski ones have useful applications in computer science. By Theorem 2.8, connectivity with respect to an α-discrete closure operator is a certain type of path connectivity. This enables us to apply these closure operators in digital image processing because they provide connectivity structures suitable for studying the geometric and topological properties of digital images (cf. [14]). There are two well-known 1-discrete topologies on $\mathbb{Z}^2$ employed in digital image processing, *Marcus topology* [15] and *Khalimsky topology* [16]. In [17], a 2-discrete closure operator on $\mathbb{Z}^2$ is used to formulate and prove a digital analogue of the Jordan curve theorem – see the following example.

Example 2.9. Let u be a closure operator on $\mathbb{Z}^2$ and G be a simple undirected graph (without loops) with the vertex set $\mathbb{Z}^2$. A circuit in G is said to be a *simple closed curve* in G with respect to u if it is a minimal (with respect to set inclusion of vertex sets) circuit in G that is a connected subset of $(\mathbb{Z}^2, u)$. A simple closed curve in G with respect to u is called a *Jordan curve* (with respect to u) if it separates the space $(\mathbb{Z}^2, u)$ into exactly two components, i.e., if the subspace $\mathbb{Z}^2 - J$ of $(\mathbb{Z}^2, u)$ has exactly two components.

Let $z = (x, y) \in \mathbb{Z}^2$ be a point. We put

$$H(z) = \{(x + i, y); \quad i \in \{-1, 0, 1\}\},$$
$$V(z) = \{(x, y + i); \quad i \in \{-1, 0, 1\}\},$$
$$D(z) = \{(x + i, y + i); \quad i \in \{-1, 0, 1\}\},$$
$$D'(z) = \{(x + i, y - i); \quad i \in \{-1, 0, 1\}\}.$$

Next, we put

$$A(z) = H(z) \cup V(z) \cup D(z) \cup D'(z).$$

In the literature, the points of $H(z) \cup V(z)$ and $A(z)$ different from z are said to be *4-adjacent* and *8-adjacent* to z, respectively. Here, for all $z \in \mathbb{Z}^2$, each of the sets $H(z)$, $V(z)$, $D(z)$, and $D'(z)$ will be called a *basic segment*. Note that basic segments may be regarded as digital (three-element) line segments (where $H(z)$ is oriented horizontally, $V(z)$ is oriented vertically, and $D(z)$ and $D'(z)$ are oriented diagonally in $\mathbb{Z}^2$).

For every point $z \in \mathbb{Z}^2$, we put

$$w\{z\} = \begin{cases} H(z) & \text{if } z = (4k+2, y) \text{ where } k \in \mathbb{Z} \text{ and } y \neq 4l+2 \text{ for every } l \in \mathbb{Z}, \\ V(z) & \text{if } z = (x, 4l+2) \text{ where } l \in \mathbb{Z} \text{ and } x \neq 4k+2 \text{ for every } k \in \mathbb{Z}, \\ A(z) & \text{if } z = (4k+2, 4l+2), \quad k, l \in \mathbb{Z}, \\ \{z\} & \text{otherwise} \end{cases}$$

and, for every two-element subset $\{z, t\} \subseteq \mathbb{Z}^2$, we put $w\{z, t\} = w\{z\} \cup w\{t\} \cup \langle z, t\rangle$ where

$$\langle z, t\rangle = \begin{cases} H(z) & \text{if } z = (4k+2+i, l) \text{ and } t = (4k+2, l), \quad k, l \in \mathbb{Z}, \quad i \in \{-1, 1\}, \\ V(z) & \text{if } z = (k, 4l+2+i) \text{ and } t = (k, 4l+2), \quad k, l \in \mathbb{Z}, \quad i \in \{-1, 1\}, \\ D(z) & \text{if } z = (4k+2+i, 4l+2+i) \text{ and } t = (4k+2, 4l+2), \quad k, l \in \mathbb{Z}, \quad i \in \{-1, 1\}, \\ D'(z) & \text{if } z = (4k+2+i, 4l+2-i) \text{ and } t = (4k+2, 4l+2), \quad k, l \in \mathbb{Z}, \quad i \in \{-1, 1\}, \\ \{z, t\} & \text{otherwise.} \end{cases}$$

It was pointed out by one of the referees that $\langle z, t\rangle = \{z, t, z + 2(t - z)\}$ if one of the following three conditions is satisfied

(1) $z = (4k+2, 4l+2)$ and $t \in w(z) = A(z)$,
(2) $z = (4k+2, 4l)$ and $t \in w(z) = H(z)$,
(3) $z = (4k, 4l+2)$ and $t \in w(z) = V(z)$

or one of the three conditions obtained by interchanging z and t in (1)–(3) is satisfied while $\langle z, t\rangle = \{z, t\}$ otherwise.

Now, putting $wA = \bigcup\{wB;\ B \subseteq A, \text{ card } B < 3\}$ for every subset $A \subseteq X$ (card$A > 2$), we obtain an S_3-closure operator on $\mathbb{Z}^2$. The closure operator w is demonstrated in Figure 1. For any point $z \in \mathbb{Z}^2$, a point $u \in \mathbb{Z}^2$, $u \neq z$, belongs to $w\{z\}$ if and only if there is an edge from z to u in the directed graph demonstrated in the left part of Figure 1. If $\{z, t\} \subseteq \mathbb{Z}^2$ is a two-element subset, then a point $u \in \mathbb{Z}^2$ with $u \notin w\{z\} \cup w\{t\}$ belongs to $w\{z, t\}$ if and only if, in the directed graph demonstrated in the right part of Figure 1, z and t are the end points of a dotted line segment containing no other point of $\mathbb{Z}^2$ (the dotted line segments are not edges of the graph), there is an edge from z or t to u, and the points z, t, and u lie on a line (with z or t lying between the other two points so that the set $\{z, t, u\}$ is a basic segment with $t \in w\{z\}$ or $z \in w\{t\}$ – cf, the directed graph in the left part of the figure).

Clearly, any sequence $((x_i, y_i)|\ i \leq 2) \in (\mathbb{Z}^2)^3$ satisfying one of the following nine conditions is a w-connected element:

(1) $x_0 = x_1 = x_2$ and $y_0 = y_1 = y_2$,
(2) $x_0 = x_1 = x_2$ and there is $k \in \mathbb{Z}$ such that $y_i = 4k + i$ for all $i < 3$,

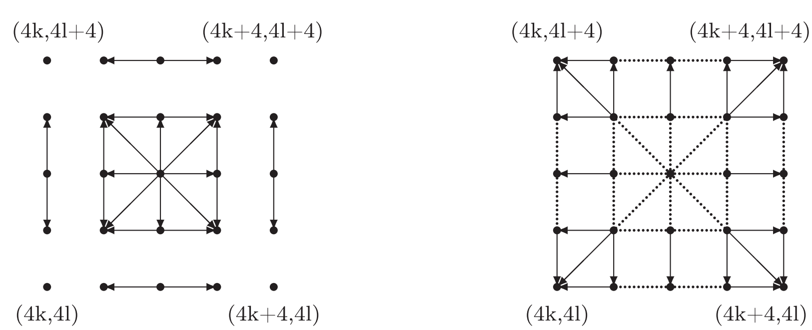

Figure 1: Closure operator w.

(3) $x_0 = x_1 = x_2$ and there is $k \in \mathbb{Z}$ such that $y_i = 4k - i$ for all $i < 3$,
(4) $y_0 = y_1 = y_2$ and there is $k \in \mathbb{Z}$ such that $x_i = 4k + i$ for all $i < 3$,
(5) $y_0 = y_1 = y_2$ and there is $k \in \mathbb{Z}$ such that $x_i = 4k - i$ for all $i < 3$,
(6) there is $k \in \mathbb{Z}$ such that $x_i = 4k + i$ for all $i < 3$ and there is $l \in \mathbb{Z}$ such that $y_i = 4l + i$ for all $i < 3$,
(7) there is $k \in \mathbb{Z}$ such that $x_i = 4k + i$ for all $i < 3$ and there is $l \in \mathbb{Z}$ such that $y_i = 4l - i$ for all $i < 3$,
(7) there is $k \in \mathbb{Z}$ such that $x_i = 4k - i$ for all $i < 3$ and there is $l \in \mathbb{Z}$ such that $y_i = 4l + i$ for all $i < 3$, and
(9) there is $k \in \mathbb{Z}$ such that $x_i = 4k - i$ for all $i < 3$ and there is $l \in \mathbb{Z}$ such that $y_i = 4l - i$ for all $i < 3$.

It may easily be seen that, for any subset $A \subseteq \mathbb{Z}^2$, $z \in wA$ if and only if there exists a sequence $((x_i, y_i)|\ i \le 2) \in (\mathbb{Z}^2)^3$ satisfying one of the above nine conditions such that $z = (x_{i_0}, y_{i_0})$ for some i_0, $0 < i_0 \le 2$, and $(x_i, y_i) \in A$ for all $i < 2$. Thus, $w = w_2$ so that w is a 2-discrete closure operator on $\mathbb{Z}^2$. The w-connected elements $((x_i, y_i)|\ i \le 2) \in (\mathbb{Z}^2)^3$ satisfying (an arbitrary single) one of the conditions (2)–(9) are demonstrated in Figure 2 by directed line segments with the starting point (x_0, y_0) and the end point (x_2, y_2) (so that (x_1, y_1) is the midpoint of the segment).

We denote by H the graph with the vertex set $\mathbb{Z}^2$ such that, for all $z, t \in \mathbb{Z}^2$, z and t are adjacent in H if and only if they are different and one of the following two conditions is satisfied:
(1) $z \in w\{t\}$ or $t \in w\{z\}$,
(2) there is a point $u \in \mathbb{Z}^2$, $z \ne u \ne t$, such that either $z \in w\{u\}$ and $t \in w\{u, z\}$ or $t \in w\{u\}$ and $z \in w\{u, t\}$.

A graphical representation of (a section of) the graph H is given by Figure 2 when forgetting all edge directions.

The following results may be proved by applying Theorem 2.8 (in [17)], they were obtained by the help of Lemma 3.3 proved there, which is nothing but a straightforward consequence of Theorem 2.8):
(1) The closure space $(\mathbb{Z}^2, w)$ is connected.
(2) Every circuit C in the graph H that turns only at some of the points $(4k, 4l)$, $k, l \in \mathbb{Z}$, is a Jordan curve in H with respect to the closure operator w and has the property that its union with any of the two components of the subspace $\mathbb{Z}^2 - C$ of $(\mathbb{Z}^2, w)$ is connected.

Jordan curves in the graph H with respect to the closure operator w will be briefly called Jordan curves in the closure space $(\mathbb{Z}^2, w)$. The possible turning points of the Jordan curves in $(\mathbb{Z}^2, w)$ determined in statement (2) are the points $(4k, 4l)$, $k, l \in \mathbb{Z}$, where the curves may turn at the acute angle $\frac{\pi}{4}$. This is an advantage over the Jordan curves with respect to the Khalimsky topology because they may never turn at the acute angle $\frac{\pi}{4}$ – cf. [16].

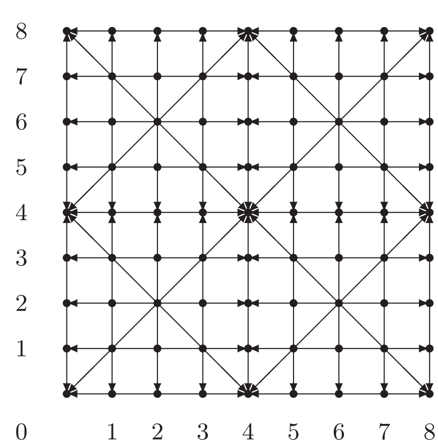

Figure 2: w-connected elements satisfying (an arbitrary but) one of the conditions (2)–(9).

3 Conclusion

We have introduced, for every ordinal $\alpha > 0$, closure operators that generalize the quasi-discrete closure operators [4]. These closure operators, called α-discrete, are studied. In particular, it is shown that the connectivity with respect to α-discrete closure operators is a certain type of path connectivity. This enables us to apply the operators to solving various problems of a discrete nature based on connectivity. One such application in digital topology is discussed. We have shown that there is a 2-discrete closure operator on the digital plane $\mathbb{Z}^2$ allowing for a digital analogue of the Jordan curve theorem, thus providing a convenient structure on $\mathbb{Z}^2$ for the study of digital images.

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