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Research Article

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# New properties for the Ramanujan *R*-function

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**Abstract:** In the article, we establish some monotonicity and convexity (concavity) properties for certain combinations of polynomials and the Ramanujan $R$-function by use of the monotone form of L'Hôpital's rule and present serval new asymptotically sharp bounds for the Ramanujan $R$-function that improve some previously known results.

**Keywords:** Ramanujan $R$-function, Gaussian hypergeometric function, Riemann zeta function, generalized elliptic integral, monotonicity, convexity

**MSC 2020:** 33C05, 33B15, 26A51, 26D20

## 1 Introduction

Let $\mathrm{Re}(a) > 0$ and $\mathrm{Re}(b) > 0$. Then, the classical gamma function $\Gamma(\cdot)$ [1–4], psi function $\psi(\cdot)$ [5–8], beta function $B(\cdot,\cdot)$ [9,10], and Euler-Mascheroni constant $\gamma$ [11–15] are defined by

$$\Gamma(a) = \int_0^\infty t^{a-1}e^{-t}\mathrm{d}t, \quad \psi(a) = \frac{\Gamma'(a)}{\Gamma(a)}, \quad B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \tag{1.1}$$

and

$$\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \log n\right) = 0.57721566\ldots,$$

respectively, and the Ramanujan $R$-function $R(\cdot)$ [16–18] is defined by

$$R(a) = -2\gamma - \psi(a) - \psi(1-a), \quad a \in (0,1), \tag{1.2}$$

and it is the special case of the two parameter function

$$R(a,b) = -2\gamma - \psi(a) - \psi(b), \tag{1.3}$$

and sometimes $R(a,b)$ is also said to be the Ramanujan constant although it is a function of $a$ and $b$.

For $a \in (0,1)$, we denote

$$B(a) = B(a, 1-a) = \Gamma(a)\Gamma(1-a) = \frac{\pi}{\sin(\pi a)}. \tag{1.4}$$

By the symmetry, in what follows, we assume that $a \in (0, 1/2]$ in (1.2) and (1.4).

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For $a, b, c \in \mathbb{R}$ with $c \neq 0, -1, -2, \ldots$, the Gaussian hypergeometric function [19–30] is defined by

$$F(a, b; c; x) = {}_2F_1(a, b; c; x) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n} \frac{x^n}{n!}, \quad x \in (-1, 1), \tag{1.5}$$

where $(x)_n$ denotes the shifted factorial function defined by $(x)_n = x(x+1)\cdots(x+n-1)$ for $n \geq 1$ and $(x)_0 = 1$ for $x \neq 0$. It is well known that $F(a, b; c; x)$ has wide applications in many branches of mathematics and physics. Many elementary and special functions in mathematical physics are the particular or limiting cases of the Gaussian hypergeometric function. $F(a, b; c; x)$ is said to be zero-balanced if $c = a + b$. The asymptotic properties of $F(a, b; a + b; x)$ are related to $B(a, b)$ and $R(a, b)$ as $x \to 1$ [31–36]. For example, $F(a, b; a + b; x)$ satisfies the Ramanujan asymptotic identity [37]:

$$B(a, b)F(a, b;a + b;x) + \log(1 - x) = R(a, b) + O((1 - x)\log(1 - x)) \tag{1.6}$$

when $x \to 1$.

Let $a, r \in (0, 1)$. Then, the generalized elliptic integral $\mathcal{K}_a(r)$ [38–52] of the first kind is defined by

$$\mathcal{K}_a(r) = \frac{\pi}{2}F(a, 1 - a;1;r^2), \quad \mathcal{K}_a(0^+) = \frac{\pi}{2}, \quad \mathcal{K}_a(1^-) = \infty. \tag{1.7}$$

For the sake of simplicity and convenience, in what follows, we denote $r' = \sqrt{1 - r^2}$ for $r \in [0, 1]$ and $\mathcal{K}'(r) = \mathcal{K}_a(r')$. By the symmetry, we only need to discuss the case of $a \in (0, 1/2]$. From (1.6), we obtain the asymptotic identity:

$$B(a)\mathcal{K}_a(r) - \pi \log\frac{e^{R(a)/2}}{r'} = O((r')^2 \log(r')) \tag{1.8}$$

when $r \to 1$.

Recently, the properties and bounds involving $\mathcal{K}_a(r)$ and $R(a)$ have attracted the attention of many researchers. Wang et al. [53] proved that the double inequality

$$\frac{\pi}{R(a) + [B(a) - R(a)]r^2} < \frac{\mathcal{K}_a(r)}{\log(e^{R(a)/2}/r')} < \frac{\pi}{B(a)[1 - a(1 - a) + a(1 - a)r^2]} \tag{1.9}$$

holds for all $a \in (0, 1/2]$ and $r \in (0, 1)$.

In [54], the authors established the following two-sided inequality:

$$\frac{\sin(\pi a)}{1 - b + br^2}\left(\log\frac{e^{R(a)/2}}{r'} - C\right) \leq \mathcal{K}_a(r) \leq \frac{\sin(\pi a)}{1 - b + br^2}\left(\log\frac{e^{R(a)/2}}{r'} - Cr'^2\right),$$

for all $a \in (0, 1/2]$ and $r \in (0, 1)$, where $C = [R(a) - (1 - b)B(a)]/2$ and $b = a(1 - a)$.

Qiu et al. [17] presented the power series expansions of the function $R(a)$ at $a = 0$ and $a = 1/2$, and provided the monotonicity and convexity properties of certain combinations defined in terms of polynomials and the difference between the Ramanujan $R$-function $R(a)$ and the beta function $B(a)$, and proved that

$$2(2a_{2n} + \alpha_{2n-1})x^{2n+1} \leq B(x) - R(x) + 2\sum_{k=1}^{2n} a_k x^k \leq (2a_{2n} + \alpha_{2n-1})x^{2n} \tag{1.10}$$

and

$$(2a_{2n+1} + \alpha_{2n})x^{2n+1} \leq B(x) - R(x) + 2\sum_{k=1}^{2n+1} a_k x^k \leq 2(2a_{2n+1} + \alpha_{2n})x^{2(n+1)} \tag{1.11}$$

for $n \in \mathbb{N}$ and $x \in (0, 1/2]$, where

$$a_n = \{(-1)^n + [1 + (-1)^{n+1}]2^{-n-1}\}\zeta(n + 1),$$

$$\alpha_n = 2^{n+1}\left(\pi - \log 16 + 2\sum_{k=1}^{n} 2^{-k} a_k\right)$$

and

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s} \quad (\mathrm{Re}(s) > 1) \tag{1.12}$$

is the Riemann zeta function [55,56].

Huang et al. [54] established several monotonicity, concavity, and convexity properties for certain combinations defined in terms of $R(a)$ and $b = a(1-a)$, and proved that

$$(7\log 2 - 3)b \le b(2-b)R(a) + b - 1, \tag{1.13}$$

$$A_2 b \le 1 - b^2 - b(1-b)R(a) < 2b, \tag{1.14}$$

$$\log 2 + D_1(1-2a)^2 - D_2(1-2a)^4 - D_3(1-2a)^5 \le bR(a) \le \log 2 + (1-\log 2)(1-2a), \tag{1.15}$$

where $D_1 = \lambda(3) - \log 2$, $D_2 = \lambda(3) - \lambda(5)$, $D_3 = \lambda(5) - 1$, $A_2 = 15/4 - \log 8$ and

$$\lambda(n+1) = \sum_{k=0}^{\infty} \frac{1}{(2k+1)^{n+1}}. \tag{1.16}$$

From [31], one has

$$\lambda(n+1) = (1 - 2^{-n-1})\zeta(n+1). \tag{1.17}$$

We also denote

$$\beta(n) = \sum_{k=0}^{\infty} (-1)^k \frac{1}{(2k+1)} \tag{1.18}$$

in the full article.

The Ramanujan $R$-function often appears in the studies of the properties of the generalized Grötzsch ring function $\mu_a(r)$ [57,58], the generalized Hersch-Pfluger distortion function $\varphi_{\mathcal{K}}^{a}(r)$ [59–61], and the generalized $\eta_{\mathcal{K}}$-distortion function $\eta_{\mathcal{K}}^{a}(t)$ [62]. Therefore, the function $R(a)$ is indispensable for studying the properties of $\mathcal{K}_a(r)$ and many other special functions [63].

The main purpose of this article is to establish new asymptotically sharp lower and upper bounds for the Ramanujan $R$-function $R(a)$ and to improve inequalities (1.14) and (1.15).

## 2 Lemmas

To prove our main results, we need several technical lemmas, which we present in this section.

**Lemma 2.1.** (See [54, Lemma 2.1]) *The following two statements are true*:

(1) *The function $\lambda(n)$ defined by* (1.17) *is strictly decreasing with respect to $n \in \mathbb{N}\backslash\{1\}$ with $\lambda(2) = \pi^2/8$ and* $\lim_{n\to\infty}\lambda(n) = 1$, *and the function $\beta(n)$ defined by* (1.18) *is strictly increasing with respect to $n \in \mathbb{N}$ with* $\beta(1) = \pi/4$ *and* $\lim_{n\to\infty}\beta(n) = 1$.

(2) *The following identities*

$$\begin{aligned} R(a) &= \frac{1}{a} + 2\sum_{n=1}^{\infty}\zeta(2n+1)a^{2n} \\ &= \log 16 + 4\sum_{n=1}^{\infty}\lambda(2n+1)(1-2a)^{2n} \\ &= \log 16 + 4\sum_{n=1}^{\infty}(1-2^{-2n-1})\zeta(2n+1)(1-2a)^{2n} \\ &= \frac{1}{b} - 4(1-\log 2) + 4\sum_{n=1}^{\infty}[\lambda(2n+1)-1](1-2a)^{2n} \end{aligned} \tag{2.1}$$

*and*

$$B(a) = \frac{1}{a} + 2\sum_{n=1}^{\infty}(1 - 2^{1-2n})\zeta(2n)a^{2n-1} = 4\sum_{n=1}^{\infty}\beta(2n+1)(1-2a)^{2n} \tag{2.2}$$

*hold for all* $a \in (0, 1/2]$.

**Lemma 2.2.** *The function* $f(s) = -2\zeta(2s+3) + \zeta(2s+5) + 9\zeta(2s+7)/8$ *is strictly increasing from* $[1, \infty)$ *onto* $[f(1), 1/8)$, *where* $f(1) = -2\zeta(5) + \zeta(7) + 9\zeta(9)/8 = 0.0617\ldots$.

**Proof.** It follows from (1.12) that

$$\begin{aligned} f(s) &= -2\zeta(2s+3) + \zeta(2s+5) + \frac{9}{8}\zeta(2s+7) \\ &= \sum_{k=1}^{\infty}\left[-2k^{-(2s+3)} + k^{-(2s+5)} + \frac{9}{8}k^{-(2s+7)}\right] \\ &= \sum_{k=1}^{\infty}\left(-2k^{-3} + k^{-5} + \frac{9}{8}k^{-7}\right)k^{-2s} \\ &= \sum_{k=1}^{\infty}\frac{-16k^4 + 8k^2 + 9}{8k^7}k^{-2s}. \end{aligned}$$

Differentiating $f(s)$ leads to

$$f'(s) = \sum_{k=1}^{\infty}(-2\log k)\frac{-16k^4 + 8k^2 + 9}{8k^7}k^{-2s} = \sum_{k=2}^{\infty}2\log k\frac{16k^4 - 8k^2 - 9}{8k^7}k^{-2s}.$$

Therefore, $f'(s) > 0$ for all $s > 0$ and $f$ is strictly increasing due to $16k^4 - 8k^2 - 9 > 0$ and $\log k > 0$ for $k \geq 2$. Note that $f(1) = -2\zeta(5) + \zeta(7) + 9\zeta(9)/8 = 0.06175\ldots$ and $f(\infty) = 1/8$. □

The following monotone form of L'Hôpita's rule can be found in the literature [[32], Theorem 1.25].

**Lemma 2.3.** *Let* $-\infty < a < b < \infty$, $f, g : [a, b] \to \mathbb{R}$ *be continuous on* $[a, b]$ *and be differentiable on* $(a, b)$, *and* $g'(x) \neq 0$ *on* $(a, b)$. *If* $f'(x)/g'(x)$ *is increasing (decreasing) on* $(a, b)$, *then so are the functions*

$$[f(x) - f(a)]/[g(x) - g(a)], \quad [f(x) - f(b)]/[g(x) - g(b)].$$

*If* $f'(x)/g'(x)$ *is strictly monotone, then the monotonicity in the conclusion is also strict.*

# 3 Main results

Throughout this paper, we always assume that $a \in (0, 1/2]$ and $b = a(1-a)$. Let

$$\begin{cases} A_1(2) = \log 2 - \lambda(3), \quad A_1(2k) = \lambda(2k-1) - \lambda(2k+1) \quad (k \geq 2), \\ B_1(1) = 1, \quad B_1(2) = 0, \\ B_1(2k-1) = -2\zeta(2k-1), \quad B_1(2k) = 2\zeta(2k-1) \quad (k \geq 2). \end{cases} \tag{3.1}$$

**Theorem 3.1.** *Let* $n \in \mathbb{N}$ *and* $f_1(a) = -bR(a) = -a(1-a)R(a)$. *Then one has*

(1) $f_1$ *has the power series formula*

$$f_1(a) = -\log 2 + \sum_{n=1}^{\infty}A_1(2n)(1-2a)^{2n} = -1 + \sum_{n=1}^{\infty}B_1(n)a^n. \tag{3.2}$$

(2) *$f_1$ is increasing, concave, and $(-1)^k f_1^{(k)}(a) > 0$ on $(0, 1/2)$ for $k \geq 3$.*

(3) *Let*

$$F_{1,n}(a) = \frac{f_1(a) - P_{1,n}(a)}{(1-2a)^{2(n+1)}}, \tag{3.3}$$

*where $P_{1,n}(a) = -\log 2 + \sum_{k=1}^{n} A_1(2k)(1-2a)^{2k}$ and $A_1(2k)(k \geq 1)$ is defined by (3.1). Then, $F_{1,n}$ is strictly decreasing and convex from $(0, 1/2)$ onto $(A_1(2n+2), \lambda(2n+1) - 1)$. In particular, the double inequality*

$$0 \leq -bR(a) - P_{1,n}(a) - A_1(2n+2)(1-2a)^{2n+2} \leq [\lambda(2n+1) - 1 - A_1(2n+2)](1-2a)^{2n+3} \tag{3.4}$$

*holds for all $n \in \mathbb{N}$ and $a \in (0, 1/2]$, with equality in each instance if and only if $a = 1/2$.*

(4) *Let*

$$G_{1,n}(a) = \frac{f_1(a) - R_{1,n}(a)}{a^{n+1}}, \tag{3.5}$$

*where $R_{1,n}(a) = -1 + \sum_{k=1}^{n} B_1(k)a^k$ and $B_1(k)(k \geq 1)$ is defined by (3.1). Then, $G_{1,2n}(G_{1,(2n+1)})$ is strictly increasing (decreasing) and concave (convex) on $(0, 1/2]$ with $G_{1,n}(0^+) = B_1(n+1)$. In particular, the inequalities*

$$\begin{aligned} 2(G_{1,2n}(1/2) + 2\zeta(2n+1))a^{2n+2} &\leq -bR(a) - R_{1,2n}(a) + 2\zeta(2n+1)a^{2n+1} \\ &\leq (G_{1,2n}(1/2) + 2\zeta(2n+1))a^{2n+1} \end{aligned} \tag{3.6}$$

*and*

$$\begin{aligned} (G_{1,(2n+1)}(1/2) - 2\zeta(2n+1))a^{2n+2} &\leq -bR(a) - R_{1,(2n+1)}(a) - 2\zeta(2n+1)a^{2n+2} \\ &\leq 2(G_{1,(2n+1)}(1/2) - 2\zeta(2n+1))a^{2n+3} \end{aligned} \tag{3.7}$$

*hold for all $n \in \mathbb{N}$ and $a \in (0, 1/2]$, with equality in each instance if and only if $a = 1/2$.*

**Proof.** (1) It follows from Lemma 2.1(2) and $b = a(1-a) = [1-(1-2a)^2]/4$ that

$$\begin{aligned} f_1(a) = -a(1-a)R(a) &= -\frac{1-(1-2a)^2}{4}\left[\log 16 + 4\sum_{n=1}^{\infty}\lambda(2n+1)(1-2a)^{2n}\right] \\ &= -\log 2 + \sum_{n=1}^{\infty} A_1(2n)(1-2a)^{2n}. \end{aligned} \tag{3.8}$$

On the other hand, from Lemma 2.1(2), $f_1(a)$ can be expressed by

$$f_1(a) = -a(1-a)R(a) = -a(1-a)\left(\frac{1}{a} + 2\sum_{n=1}^{\infty}\zeta(2n+1)a^{2n}\right) = -1 + \sum_{n=1}^{\infty} B_1(n)a^n. \tag{3.9}$$

(2) From Lemma 2.1(1), we know that $A_1(2n) > 0$ for all $n \geq 2$. Differentiating $f_1$ gives

$$f_1^{(k)}(a) = \sum_{n=[(k+1)/2]}^{\infty} (-1)^k \frac{2^k(2n)!}{(2n-k)!} A_1(2n)(1-2a)^{2n-k}, \tag{3.10}$$

where and in what follows $[\cdot]$ denotes the greatest integral function. Therefore, $(-1)^k f_1^{(k)}(a) \geq 0$ for all $a \in (0, 1/2]$ and $k = 3, 4, \ldots$ and $f_1''$ is decreasing on $(0, 1/2]$ due to $f_1'''(a) \leq 0$. It follows from (3.9) that

$$f_1''(a) = \sum_{n=2}^{\infty} n(n-1)B_1(n)a^{n-2}$$

with $f_1''(0^+) = 0$. Therefore, $f_1''(a) < 0$ for $a \in (0, 1/2]$, so that $f_1'(a)$ is strictly decreasing on $(0, 1/2]$ with $f_1'(1/2) = 0$, which leads to the desired monotonicity and concavity of $f_1$.

(3) Differentiating

$$F_{1,n}(a) = \frac{f_1(a) - P_{1,n}(a)}{(1-2a)^{2(n+1)}} = \sum_{k=n+1}^{\infty} A_1(2k)(1-2a)^{2(k-n-1)} = \sum_{k=0}^{\infty} A_1(2(k+n+1))(1-2a)^{2k}$$

gives

$$F_{1,n}^{(l)}(a) = \sum_{k=[(l+1)/2]}^{\infty} (-1)^l \frac{2^k(2k)!}{(2k-l)!} A_1(2(k+n+1))(1-2a)^{2k-l}, \tag{3.11}$$

which implies that

$$F_{1,n}'(a) = -4\sum_{k=1}^{\infty} kA_1(2(k+n+1))(1-2a)^{2k-1}$$

and

$$F_{1,n}''(a) = 8\sum_{k=1}^{\infty} k(2k-1)A_1(2(k+n+1))(1-2a)^{2k-2}.$$

Therefore, the monotonicity and concavity properties of $F_{1,n}$ follow easily from $A_1(2n) > 0$ for all $n \geq 2$. Clearly, $F_{1,n}(0^+) = \lambda(2n+1) - 1$, $F_{1,n}(1/2^-) = A_1(2n+2)$, the double inequality follows from the monotonicity and convexity properties of $F_{1,n}(a)$.

(4) Let $G_{1,n}(a) = h_1(a)/h_2(a)$, where $h_2(a) = a^{n+1}$ and

$$h_1(a) = f_1(a) - R_{1,n}(a). \tag{3.12}$$

Then $h_1^{(m)}(0^+) = h_2^{(m)}(0^+) = 0$ for all $m \in \mathbb{N} \cup \{0\}$ with $0 \leq m \leq n$ and

$$\frac{h_1^{(n+1)}(a)}{h_2^{(n+1)}(a)} = \frac{f_1^{(n+1)}(a)}{(n+1)!}. \tag{3.13}$$

From Lemma 2.3, we know that the function $G_{1,2n}(G_{1,(2n+1)})$ has the same monotonicity with the function $f^{(2n+1)}(f^{(2n+2)})$. It follows from $(-1)^n f_1^{(n)}(a) \geq 0$ for $a \in (0, 1/2]$ and $n \geq 3$ that $G_{1,2n}(G_{1,(2n+1)})$ is increasing (decreasing) on $(0, 1/2]$. Differentiating $G_{1,n}$ gives

$$(G_{1,n}(a))' = \left(\frac{h_1(a)}{h_2(a)}\right)' = \frac{a(f_1(a) - R_{1,n}(a))' - (n+1)(f_1(a) - R_{1,n}(a))}{a^{n+2}}.$$

Let

$$h_3(a) = a(f_1(a) - R_{1,n}(a))' - (n+1)(f_1(a) - R_{1,n}(a)),$$

and $h_4(a) = a^{n+2}$. Then from (3.9), we know that $h_3(a)$ can be rewritten as follows:

$$h_3(a) = \sum_{k=n+1}^{\infty} (k-n-1)B_1(k)a^k,$$

which leads to $h_3^{(m)}(0^+) = h_4^{(m)}(0^+) = 0$ for all $m \in \mathbb{N} \cup \{0\}$ with $0 \leq m \leq n$ and

$$\frac{h_3^{(n+1)}(a)}{h_4^{(n+1)}(a)} = \frac{f_1^{(n+2)}(a)}{(n+2)!}. \tag{3.14}$$

From Lemma 2.3, we know that the function $(G_{1,n})'$ has the same monotonicity with the function $f^{(n+2)}$. Therefore, the desired monotonicity of $(G_{1,2n})'$ and $(G_{1,(2n+1)})'$ is obtained, the concavity (convexity) property of $G_{1,2n}(G_{1,(2n+1)})$ follows from part (2), and the double inequality (3.6) (3.7) and its equality case follow from the monotonicity and concavity (convexity) properties of $G_{1,2n}(G_{1,(2n+1)})$. □

A function $f$ is said to be strictly completely monotonic on an interval $I \subset \mathbb{R}$ if $(-1)^n f^{(n)}(x) > 0$ for all $x \in I$ and $n = 0, 1, 2, 3 \ldots$. If $(-1)^n f^{(n)}(x) \geq 0$ for all $x \in I$ and $n = 0, 1, 2, 3 \ldots$, then $f$ is called completely monotonic on $I$. The completely monotonic functions have important applications in areas of numerical analysis [64,65], probability theory [66], physics [67], and so on.

**Corollary 3.2.** *For each $n \in \mathbb{N}$, the function $F_{1,n}$ is completely monotonic on* $(0, 1/2]$.

**Proof.** It follows from (3.11) that $(-1)^l F_{1,n}^{(l)}(a) \geq 0$ for all $a \in (0, 1/2]$, and $l = 0, 1, 2, 3 \ldots$. Therefore, $F_{1,n}$ is completely monotonic on $(0, 1/2]$. □

Let

$$\begin{cases} A_2(2) = [9\lambda(3) - 10\log 2]/4, \quad A_2(4) = [9\lambda(5) - 10\lambda(3) + \log 2]/4, \\ A_2(2k) = [9\lambda(2k+1) - 10\lambda(2k-1) + \lambda(2k-3)]/4 \qquad (k \geq 3), \\ B_2(1) = -1, \quad B_2(2) = -2, \quad B_2(3) = 4\zeta(3) + 1, \quad B_2(4) = -2\zeta(3), \\ B_2(2k+1) = -4(\zeta(2k-1) - \zeta(2k+1)) \qquad (k \geq 2), \\ B_2(2k+2) = 2(\zeta(2k-1) - \zeta(2k+1)) \qquad (k \geq 2). \end{cases} \tag{3.15}$$

**Theorem 3.3.** *Let $f_2(a) = b(2+b)R(a)$. Then, the following statements are true*:

(1) *$f_2$ has the power series formula*:

$$f_2(a) = \frac{9\log 2}{4} + \sum_{n=1}^{\infty} A_2(2n)(1-2a)^{2n} = 2 + \sum_{n=1}^{\infty} B_2(n)a^n. \tag{3.16}$$

(2) *The function $f_2$ is strictly decreasing on $(0, 1/2]$, and $f_2$ is neither convex nor concave on $(0, 1/2]$ and $(-1)^k f_2^{(k)}(a) > 0$ for $(0, 1/2]$ and $k \geq 5$.*

(3) *Let*

$$F_{2,n}(a) = \frac{f_2(a) - P_{2,n}(a)}{(1-2a)^{2(n+1)}}, \tag{3.17}$$

*where $P_{2,n}(a) = (9\log 2)/4 + \sum_{k=1}^{n} A_2(2k)(1-2a)^{2k}$ and $A_2(2k)(k \geq 1)$ is given in (3.15). Then, $F_{2,n}$ is strictly decreasing and convex from $(0, 1/2)$ onto $(A_2(n+1), F_{2,n}(0^+))$. In particular, the double inequality*

$$0 \leq b(2+b)R(a) - P_{2,n}(a) - A_2(n+1)(1-2a)^{2(n+1)} \leq [F_{2,n}(0^+) - A_2(n+1)](1-2a)^{2(n+1)+1} \tag{3.18}$$

*holds for all $n \in \mathbb{N}$ and $a \in (0, 1/2]$, with equality in each instance if and only if $a = 1/2$.*

(4) *Let*

$$G_{2,n}(a) = \frac{f_2(a) - R_{2,n}(a)}{a^{n+1}}, \tag{3.19}$$

*where $R_{2,n}(a) = 2 + \sum_{k=1}^{n} B_2(k)a^k$ and $B_2(k)(k \geq 1)$ is defined by (3.15). Then, for $n \geq 2$, $G_{2,2n}(G_{2,(2n-1)})$ is strictly increasing (decreasing) and concave (convex) on $(0, 1/2]$ with $G_{2,n}(0^+) = B_2(n+1)$. In particular, inequalities*

$$\begin{aligned} 2(G_{2,2n}(1/2) - B_2(2n+1))a^{2n+2} &\leq b(2+b)R(a) - R_{2,2n}(a) - B_2(2n+1)a^{2n+1} \\ &\leq (G_{2,2n}(1/2) - B_2(2n+1))a^{2n+1} \end{aligned} \tag{3.20}$$

*and*

$$\begin{aligned} (G_{2,(2n-1)}(1/2) - B_2(2n))a^{2n+2} &\leq b(2+b)R(a) - R_{2,2n}(a) - B_2(2n)a^{2n+2} \\ &\leq 2(G_{2,(2n-1)}(1/2) - B_2(2n))a^{2n+3} \end{aligned} \tag{3.21}$$

*hold for all $n \in \mathbb{N}$ and $a \in (0, 1/2]$, with equality in each instance if and only if $a = 1/2$.*

**Proof.** (1) It follows from Lemma 2.1(2) and $b = a(1-a) = [1-(1-2a)^2]/4$ that

$$\begin{aligned} f_2(a) = b(2+b)R(a) &= \frac{9-10(1-2a)^2+(1-2a)^4}{16}\left[\log 16 + 4\sum_{n=1}^{\infty}\lambda(2n+1)(1-2a)^{2n}\right] \\ &= \frac{9\log 2}{4} + \sum_{n=1}^{\infty} A_2(2n)(1-2a)^{2n}, \end{aligned} \tag{3.22}$$

where $A_2(2n)(n \geq 1)$ is defined by (3.15). Again use Lemma 2.1(2), we obtain

$$\begin{aligned} f_2(a) = b(2+b)R(a) &= b(2+b)\left(\frac{1}{a} + 2\sum_{n=1}^{\infty}\zeta(2n+1)a^{2n}\right) \\ &= 2 - a - 2a^2 + (4\zeta(3)+1)a^3 - 2\zeta(3)a^4 + \sum_{n=2}^{\infty}[-4\zeta(2n-1) + 4\zeta(2n+1)]a^{2n+1} \\ &\quad + [2\zeta(2n-1) - 2\zeta(2n+1)]a^{2n+2} = 2 + \sum_{n=1}^{\infty} B_2(n)a^n. \end{aligned} \tag{3.23}$$

(2) It follows from (1.16) that $A_2(2n) = [9\lambda(2n+1) - 10\lambda(2n-1) + \lambda(2n-3)]/4$ and

$$A_2(2n) = \frac{1}{4}\sum_{k=1}^{\infty}\frac{9-10(2k+1)^2+(2k+1)^4}{(2k+1)^{2n+1}} = 4\sum_{k=2}^{n}\frac{(k^2+2k)(k^2-1)}{(2k+1)^{2n+1}} > 0$$

for $n > 2$. Differentiating $f_2$ gives

$$f_2^{(k)}(a) = \sum_{n=[(k+1)/2]}^{\infty}(-1)^k\frac{2^k(2n)!}{(2n-k)!}A_2(2n)(1-2a)^{2n-k}, \tag{3.24}$$

$$f_2^{(k)}(a) = \sum_{n=[(k+1)/2]}^{\infty}\frac{(n)!}{(n-k)!}B_2(n)a^{n-k}. \tag{3.25}$$

From (3.24), we know that $(-1)^k f_2^{(k)}(a) \geq 0$ for $a \in (0, 1/2]$ and $k = 5, 6, \ldots$; and $f_2^{(5)}(a)$ is increasing on $(0, 1/2]$. By the series formula of $f_2(a)$, we obtain $f_2^{(5)}(1/2) = 0$; therefore, $f_2^{(5)}(a) \leq 0$ for $a \in (0, 1/2]$, so that $f_2^{(4)}$ is strictly decreasing on $(0, 1/2]$.

It follows from (3.25) that $f_2^{(4)}(0^+) = -48\zeta(3) < 0$, so $f_2^{(4)}(a) < 0$ for $a \in (0, 1/2]$. Hence, $f_2'''(a)$ are strictly decreasing on $(0, 1/2]$ with $f_2'''(1/2) = 0$, which implies that $f_2'''(a) > 0$ and $f_2''$ are strictly increasing on $(0, 1/2]$.

By (3.24) and (3.25), we obtain $f_2''(0^+) = -4$ and $f_2''(1/2) = -20\log 2 + 63\zeta(3)/4 = 5.070\ldots > 0$. Therefore, there exists $a^* \in (0, 1/2]$ such that $f_2''(a^*) = 0$, and $f_2'$ is decreasing on $(0, a^*]$ and $f_2'$ is increasing on on $(a^*, 1/2]$. Since $f_2'(0^+) = -1$ and $f_2'(1/2) = 0$, so $f_2'(a) < 0$ and $f_2$ is strictly decreasing on $(0, 1/2]$ with $f_2(0^+) = 2$ and $f_2(1/2) = 9\log 2/4$.

(3) Differentiating

$$F_{2,n}(a) = \frac{f_2(a) - P_{2,n}(a)}{(1-2a)^{2(n+1)}} = \sum_{k=n+1}^{\infty} A_2(2k)(1-2a)^{2(k-n-1)} = \sum_{k=0}^{\infty} A_2(2(k+n+1))(1-2a)^{2k}$$

leads to

$$F_{2,n}^{(l)}(a) = \sum_{k=[(l+1/2)]}^{\infty}(-1)^l\frac{2^k(2k)!}{(2k-l)!}A_2(2(k+n+1))(1-2a)^{2k-l}, \tag{3.26}$$

which implies that

$$F_{2,n}'(a) = -4\sum_{k=1}^{\infty} kA_2(2(k+n+1))(1-2a)^{2k-1}$$

and

$$F''_{2,n}(a) = 8\sum_{k=1}^{\infty} k(2k-1)A_2(2(k+n+1))(1-2a)^{2k-2}.$$

Therefore, the monotonicity and concavity properties of $F_{2,n}$ follow from $A_2(2n) > 0$ for all $n > 2$, and the desired inequalities follow from the monotonicity and convexity properties of $F_{2,n}$ together with $F_{2,n}((1/2)^-) = A_2(2n+2)$.

(4) Let $G_{2,n}(a) = h_5(a)/h_6(a)$, where $h_6(a) = a^{n+1}$ and

$$h_5(a) = f_2(a) - R_{2n}(a). \tag{3.27}$$

Then it is easy to verify that $h_5^{(m)}(0^+) = h_6^{(m)}(0^+) = 0$ for all $m \in \mathbb{N} \cup \{0\}$ with $0 \le m \le n$. Simple computations lead to

$$\frac{h_5^{(n+1)}(a)}{h_6^{(n+1)}(a)} = \frac{f_2^{(n+1)}(a)}{(n+1)!}. \tag{3.28}$$

From Lemma 2.3, we know that $G_{2,2n}(G_{2,(2n-1)})$ has the same monotonicity with the function $f_2^{(2n+1)}(f_2^{(2n)})$. Therefore, $G_{2,2n}(G_{2,(2n-1)})$ is increasing (decreasing) on $(0,1/2]$ due to $(-1)^n f_2^{(n)}(a) \ge 0$ for $a \in (0,1/2]$ and $n \ge 5$.

Differentiating $G_{2,n}$ gives

$$(G_{2,n}(a))' = \left(\frac{h_5(a)}{h_6(a)}\right)' = \frac{a(f_2(a) - R_{2,n}(a))' - (n+1)(f_2(a) - R_{2,n}(a))}{a^{n+2}}.$$

Let $h_8(a) = a^{n+2}$ and

$$h_7(a) = a(f_2(a) - R_{2,n}(a))' - (n+1)(f_2(a) - R_{2,n}(a)).$$

Then from (3.23), we know that $h_7(a)$ can be rewritten as follows:

$$h_7(a) = \sum_{k=n+1}^{\infty} (k-n-1)B_2(k)a^k.$$

Therefore, $h_7^{(m)}(0^+) = h_8^{(m)}(0^+) = 0$ for all $m \in \mathbb{N} \cup \{0\}$ with $0 \le m \le n$ and

$$\frac{h_7^{(n+1)}(a)}{h_8^{(n+1)}(a)} = \frac{f_2^{(n+2)}(a)}{(n+2)!}, \tag{3.29}$$

which shows that $G'_{2,n}$ has the same monotonicity with the function $f_2^{(n+2)}$ by Lemma 2.3. Therefore, the desired monotonicity of $G'_{2,2n}$ and $G'_{2,(2n-1)}$, the concavity (convexity) of $G_{2,2n}(G_{2,(2n-1)})$, are obtained, and the inequalities (3.20) and (3.21) follow easily from the monotonicity and concavity (convexity) properties of $G_{2,2n}(G_{2,(2n+1)})$. □

**Corollary 3.4.** *The function $F_{2,n}(n > 2)$ is completely monotonic on* $(0,1/2]$.

**Proof.** From (3.26), we clearly see that $A_2(2n) > 0$ for all $n > 2$. Therefore, $(-1)^l F_{2,n}^{(l)}(a) \ge 0$ for all $a \in (0,1/2]$ and $l = 0, 1, 2, 3 \ldots$, and $F_{2,n}$ is completely monotonic on $(0,1/2]$. □

**Corollary 3.5.** *$G_{2,1}$ is increasing and concave, and $G_{2,2}$ is decreasing and concave on* $(0,1/2]$.

**Proof.** It follows from (3.28) and Lemma 2.3 that $G_{2,2}(G_{2,1})$ has the same monotonicity with the function $f_2'''(f_2'')$. Since $f_2'''(f_2'')$ is strictly decreasing (increasing) on $(0,1/2]$, we can obtain the desired monotonicity of $G_{2,2}$ and $G_{2,1}$. It follows from (3.29) and Lemma 2.3 that $G'_{2,2}(G'_{2,1})$ has the same monotonicity with the function $f_2^{(4)}(f_2''')$. Since $f_2'''$ and $f_2^{(4)}$ is strictly decreasing on $(0,1/2]$, the desired monotonicity of $G_{2,2}'(a)$ and $G_{2,1}'(a)$, and the concavity (convexity) of $G_{2,2}(a)$ and $G_{2,1}(a)$ are obtained. □

Let

$$\begin{cases} A_3(0) = 7\log 2 - 4 = 0.8520\ldots, \\ A_3(2) = 7\lambda(3) + \log 2 - 4 = 4.055745\ldots, \\ A_3(2k) = 7\lambda(2k+1) + \lambda(2k-1) - 4 \quad (k \geq 2), \\ B_3(1) = 0, \quad B_3(2) = 4\zeta(3) - 1, \\ B_3(2k-1) = -1 - 2\zeta(2k-1) \quad (k \geq 2), \\ B_3(2k) = -1 + 2\zeta(2k-1) + 4\zeta(2k+1) \quad (k \geq 2). \end{cases} \tag{3.30}$$

**Theorem 3.6.** *Let $f_3(a) = (2-b)R(a) - 1/b$. Then the following statements are true*:
(1) *$f_3$ has the power series formula*:

$$f_3(a) = 7\log 2 - 4 + \sum_{n=1}^{\infty} A_3(2n)(1-2a)^{2n} = \frac{1}{a} - 2 + \sum_{n=1}^{\infty} B_3(n)a^n. \tag{3.31}$$

(2) *$f_3(a)$ is completely monotonic on* $(0,1/2]$.
(3) *Let*

$$F_{3,n}(a) = \frac{f_3(a) - P_{3,n}(a)}{(1-2a)^{2(n+1)}}, \tag{3.32}$$

*where $P_{3,n}(a) = \sum_{k=0}^{n} A_3(2k)(1-2a)^{2k}$ and $A_3(k)(k \geq 0)$ is given in* (3.30). *Then, $F_{3,n}$ is strictly decreasing and convex from $(0,1/2]$ onto $(A_3(n+1), \infty)$. In particular, the inequality*

$$A_3(n+1)(1-2a)^{2(n+1)} < (2-b)R - 1/b - P_{3,n}(a) \tag{3.33}$$

*holds for all $a \in (0,1/2]$ and $n \in \mathbb{N}$, with equality in each instance if and only if $a = 1/2$.*
(4) *Let*

$$G_{3,n}(a) = \frac{f_3(a) - R_{3,2n}(a)}{a^{2n+1}}, \tag{3.34}$$

*where $R_{3,n}(a) = 1/a - 2 + \sum_{k=1}^{n} B_3(k)a^k$ and $B_3(k)(k \geq 1)$ is defined in* (3.30). *Then, $G_{3,n}$ is strictly increasing from $(0,1/2]$ onto $(B_3(2n-1), G_{3,n}(1/2)]$. In particular, the inequality*

$$B_3(2n-1)a^{2n-1} < (2-b)R(a) - 1/b - R_{3,n}(a) \leq G_{3,n}\left(\frac{1}{2}\right)a^{2n-1}, \tag{3.35}$$

*holds for all $a \in (0,1/2]$ and $n \in \mathbb{N}$, with equality in each instance if and only if $a = 1/2$.*
*Let*

$$H_{3,n}(a) = \frac{f_3(a) - R_{3,(2n-1)}(a)}{a^{2n}}, \tag{3.36}$$

*where $R_{3,n}(a) = 1/a - 2 + \sum_{k=1}^{n} B_3(k)a^k$ and $B_3(k)(k \geq 1)$ is given in* (3.30). *Then, $H_{3,n}$ is convex on $(0,1/2]$ and there exists $a_0 \in (0,1/2]$ such that $H_{3,n}$ is strictly decreasing on $(0,a_0]$ and strictly increasing on $(a_0,1/2]$. In particular, the inequality*

$$(2-b)R(a) - 1/b - R_{3,(2n-1)}(a) < \min\left\{B_3(2n)a^{2n-1}, H_{3,n}\left(\frac{1}{2}\right)a^{2n-1}\right\} \tag{3.37}$$

*holds for $a \in (0,1/2]$ and $n \in \mathbb{N}$.*

**Proof.** (1) It follows from Lemma 2.1(2) that

$$\begin{aligned} f_3(a) = (2-b)R - \frac{1}{b} &= \frac{7+(1-2a)^2}{4}R - \frac{4}{1-(1-2a)^2} \\ &= [7+(1-2a)^2]\left[\log 2 + \sum_{k=1}^{\infty}\lambda(2k+1)(1-2a)^{2k}\right] - 4\sum_{k=0}^{\infty}(1-2a)^{2k} \\ &= 7\log 2 - 4 + \sum_{n=1}^{\infty} A_3(2n)(1-2a)^{2n}, \end{aligned} \tag{3.38}$$

where $A_3(2n)(n \geq 1)$ is given by (3.30), which leads to

$$\begin{aligned} f_3(a) &= (2-b)R - \frac{1}{b} \\ &= [2 - a(1-a)]\left[\frac{1}{a} + 2\sum_{n=1}^{\infty}\zeta(2n+1)a^{2n}\right] - \frac{1}{a(1-a)} \\ &= \frac{2}{a} - (1-a) + 2(2-a+a^2)\sum_{n=1}^{\infty}\zeta(2n+1)a^{2n} - \left(\frac{1}{a} + \sum_{n=0}^{\infty}a^n\right) \\ &= \frac{1}{a} - 2 + (4\zeta(3)-1)a^2 + \sum_{n=2}^{\infty}[(-1-2\zeta(2n-1))a^{2n-1} + (-1+2\zeta(2n-1)+4\zeta(2n+1))a^{2n}] \\ &= \frac{1}{a} - 2 + \sum_{n=1}^{\infty}B_3(n)a^n, \end{aligned} \tag{3.39}$$

where $B_3(n)(n \geq 2)$ is defined by (3.30).

(2) From Lemma 2.1(1), we know that $7\log 2 - 4 = 0.8520\ldots$, $7\lambda(3) + \log 2 - 4 = 4.055745\ldots$ and $A_3(n) > 4$ for all $n \in \mathbb{N}$. Simple computations lead to

$$f_3^{(k)}(a) = \sum_{n=[(k+1)/2]}^{\infty} (-1)^k \frac{2^k(2n)!}{(2n-k)!} A_3(2n)(1-2a)^{2n-k}, \tag{3.40}$$

therefore, $(-1)^k f_3^{(k)}(a) \geq 0$ for all $a \in (0, 1/2]$ and $k = 0, 1, 2, \ldots$, so $f_3$ is completely monotonic on $(0, 1/2]$.

(3) Differentiating

$$F_{3,n}(a) = \frac{f_3(a) - P_{3,n}(a)}{(1-2a)^{2(n+1)}} = \sum_{k=n+1}^{\infty} A_3(2k)(1-2a)^{2(k-n-1)} = \sum_{k=0}^{\infty} A_3(2(k+n+1))(1-2a)^{2k}$$

gives

$$F_{3,n}^{(l)}(a) = \sum_{k=[(l+1)/2]}^{\infty} (-1)^l \frac{2^k(2k)!}{(2k-l)!} A_3(2(k+n+1))(1-2a)^{2k-l}, \tag{3.41}$$

which implies

$$F'_{3,n}(a) = -4\sum_{k=1}^{\infty} kA_3(k+n+1)(1-2a)^{2k-1}$$

and

$$F''_{3,n}(a) = 8\sum_{k=1}^{\infty} k(2k-1)A_3(k+n+1)(1-2a)^{2k-2}.$$

From Lemma 2.1(1), we know that $7\lambda(3) + \log 2 - 4 = 4.055745\ldots$ and $A_3(2n) > 4$ for all $n \in \mathbb{N}$. Therefore, we obtain the desired monotonicity and concavity of $F_{3,n}$, and inequality (3.33) follows from the monotonicity of $F_{3,n}$ and $F_{3,n}(1/2) = A_3(n+1)$.

(4) Differentiating

$$\begin{aligned} G_{3,n}(a) &= \frac{f_3(a) - R_{3,(2n)}(a)}{a^{2n+1}} \\ &= \sum_{k=2n+1}^{\infty} B_3(k)a^{k-(2n+1)} \\ &= \sum_{k=0}^{\infty} [B_3(2k+2n+1)a^{2k} + B_3(2k+2n+2)a^{2k+1}] \end{aligned} \tag{3.42}$$

gives

$$\begin{aligned} G'_{3,n}(a) &= \sum_{k=0}^{\infty} [(2k+1)B_3(2k+2n+2)a^{2k} + (2k+2)B_3(2k+2n+3)a^{2k+1}] \\ &= \sum_{k=0}^{\infty} [(2k+1)B_3(2k+2n+2) + (2k+2)B_3(2k+2n+3)a]a^{2k}. \end{aligned}$$

Note that $\zeta(2n+1)$ is decreasing, $0 < \zeta(2n+1) < 2$, $B_3(2k+2n+2) > 5$, and $B_3(2k+2n+3) > -5$. Let $M_1(a) = (2k+1)B_3(2k+2n+2) + (2k+2)B_3(2k+2n+3)a$. Then, $M_1$ is strictly decreasing on $(0, 1/2]$, $M_1(a) \geq M_1(1/2) > 5(2k+1) - 5(k+1) = 5k \geq 0$ for $n \in \mathbb{N}$, $k \geq 0$ and $a \in (0, 1/2]$. Therefore, $G'_{3,n}(a) > 0$ and $G_{3,n}$ is strictly increasing on $(0, 1/2]$. Clearly, $G_{3,n}(0^+) = B_3(2n-1)$.

From (3.39), we obtain

$$\begin{aligned} H_{3,n}(a) &= \frac{f_3(a) - R_{3,(2n-1)}(a)}{a^{2n}} \\ &= \sum_{k=n}^{\infty} [B_3(2k)a^{2(k-n)} + B_3(2k+1)a^{2(k-n)+1}] \\ &= \sum_{k=0}^{\infty} [B_3(2k+2n)a^{2k} + B_3(2k+2n+1)a^{2k+1}]. \end{aligned} \tag{3.43}$$

Differentiating $H_{3,n}$ yields

$$\begin{aligned} H'_{3,n}(a) &= B_3(2n+1) + \sum_{k=1}^{\infty} [2kB_3(2k+2n)a^{2k-1} + (2k+1)B_3(2k+2n+1)a^{2k}] \\ &= B_3(2n+1) + \sum_{k=1}^{\infty} [2kB_3(2k+2n) + (2k+1)B_3(2k+2n+1)a]a^{2k-1} \end{aligned}$$

and

$$\begin{aligned} H''_{3,n}(a) &= \sum_{k=1}^{\infty} [2k(2k-1)B_3(2k+2n)a^{2k-2} + 2k(2k+1)B_3(2k+2n+1)a^{2k-1}] \\ &= \sum_{k=1}^{\infty} [2k(2k-1)B_3(2k+2n) + 2k(2k+1)B_3(2k+2n+1)a]a^{2k-2}. \end{aligned}$$

Note that $\zeta(2n+1)$ is decreasing, $0 < \zeta(2n+1) < 2$, $B_3(2k+2n) > 5$ and $B_3(2k+2n+1) > -5$. Let $M_2(a) = 2kB_3(2k+2n) + (2k+1)B_3(2k+2n+1)a$. Then, $M_2(a)$ is strictly decreasing on $(0, 1/2]$ and $M_2(a) \geq M_2(1/2) > 10k - 5(2k+1)/2 = 5k - 5/2 > 0$ for $k \geq 1$ and $a \in (0, 1/2]$. Let $M_3(a) = 2k(2k-1)B_3(2k+2n) + 2k(2k+1)B_3(2k+2n+1)a$. Then, $M_3(a)$ is strictly decreasing on $(0, 1/2]$.

If $k = 1$ and $a \in (0, 1/2]$, then we obtain

$$2B_3(2+2n) + 6B_3(2n+3)a \geq 2B_3(2+2n) + 3B_3(2n+3) = -5 + 4\zeta(2n+1) + 2\zeta(2n+3) > 1.$$

If $k \geq 2$ and $a \in (0, 1/2]$, then one has

$$M_3(a) \geq M_3(1/2) > 10k(2k-1) - 5k(2k+1) = 5k(2k-3) > 0.$$

Therefore, $M_3(a) > 0$ for $k \geq 1$ and $a \in (0, 1/2]$, $H''_{3,n}(a) > 0$, $H'_{3,n}$ is strictly increasing on $(0, 1/2]$, and $H_{3,n}$ is convex on $(0, 1/2]$. Clearly, $H'_{3,n}(0^+) = B_3(2n+1) < 0$ and

$$\begin{aligned} H'_{3,n}\left(\frac{1}{2}\right) &= B_3(2n+1) + \sum_{k=1}^{\infty}\left[2kB_3(2k+2n) + \frac{(2k+1)B_3(2k+2n+1)}{2}\right]\left(\frac{1}{2}\right)^{2k-1} \\ &> B_3(2n+1) + \sum_{k=1}^{2}\left[2kB_3(2k+2n) + \frac{(2k+1)B_3(2k+2n+1)}{2}\right]\left(\frac{1}{2}\right)^{2k-1} \\ &= -\frac{57}{16} + \frac{7\zeta(2n+3)}{2} + \frac{11\zeta(2n+5)}{8} > \frac{21}{11}, \end{aligned}$$

which leads to the conclusion that there exists $a_0 \in (0, 1/2]$ such that $H_{3,n}$ is decreasing on $(0, a_0]$ and increasing on $[a_0, 1/2]$. □

Let

$$\begin{cases} A_4(0) = 4\log 2 - 15/4 = -0.9774\ldots, \\ A_4(2) = 4\lambda(3) - 17/4 = -0.0426\ldots, \\ A_4(2k) = 4(\lambda(2k+1) - 1) \quad (k \geq 2), \\ B_4(1) = 0, \quad B_4(2) = 2\zeta(3) - 2, \\ B_4(2k-1) = -1, \quad B_4(2k) = 2\zeta(2k+1) - 1 \quad (k \geq 2). \end{cases} \tag{3.44}$$

**Theorem 3.7.** *Let $f_4(a) = R(a) + b - 1/b$. Then the following statements are true*:

(1) *$f_4$ has the power series formula*

$$f_4(a) = 4\log 2 - \frac{15}{4} + \sum_{n=1}^{\infty} A_4(2n)(1-2a)^{2n} = -1 + \sum_{k=1}^{\infty} B_4(k)a^k. \tag{3.45}$$

(2) *The function $f_4(a)$ is strictly increasing on* $(0, 1/2]$, and *$f_4$ is neither convex nor concave on* $(0, 1/2]$ *and $(-1)^k f_4^{(k)}(a) > 0$ for $k \geq 3$ and $a \in (0, 1/2]$.*

(3) *Let*

$$F_{4,n}(a) = \frac{f_4(a) - P_{4,n}(a)}{(1-2a)^{2(n+1)}}, \tag{3.46}$$

*where $P_{4,n}(a) = \sum_{k=0}^{n} A_4(2k)(1-2a)^{2k}$ and $A_4(2k) (k \geq 0)$ is defined in* (3.44). *Then $F_{4,n}$ is strictly decreasing and convex from* $(0, 1/2]$ *onto* $[A_4(2n+2), F_{4,n}(0^+))$, *and the inequality*

$$0 \leq R(a) + b - \frac{1}{b} - P_{4,n}(a) - A_4(2n+2)(1-2a)^{2(n+1)} \leq (F_{4,n}(0^+) - A_4(2n+2))(1-2a)^{2n+3} \tag{3.47}$$

*holds for all $n \in \mathbb{N}$ and $a \in$ $(0, 1/2]$, *with equality in each instance if and only if* $a = 1/2$.

(4) *Let*

$$G_{4,n}(a) = \frac{f_4(a) - R_{4,n}(a)}{a^{n+1}}, \tag{3.48}$$

*where $R_{4,n}(a) = -1 + \sum_{k=1}^{n} B_4(k)a^k$ and $B_4(k) (k \geq 1)$ is given by* (3.44). *Then $G_{2,2n}(G_{2,(2n-1)})$ is strictly increasing* (*decreasing*) *and concave* (*convex*) *on* $(0, 1/2]$ *for $n \geq 2$ with $G_{4n}(0^+) = B_4(n+1)$. In particular, the inequalities*

$$\begin{aligned} 2(G_{4,2n}(1/2) - B_4(2n+1))a^{2n+2} &\leq R(a) + b - \frac{1}{b} - R_{4,2n}(a) - B_4(2n+1)a^{2n+1} \\ &\leq (G_{4,2n}(1/2) - B_4(2n+1))a^{2n+1} \end{aligned} \tag{3.49}$$

*and*

$$(G_{4,(2n-1)}(1/2) - B_4(2n))a^{2n+2} \le R(a) + b - \frac{1}{b} - R_{4,2n}(a) - B_4(2n)a^{2n+2} \\ \le 2(G_{4,(2n-1)}(1/2) - B_4(2n))a^{2n+3} \tag{3.50}$$

*hold for all* $n \in \mathbb{N}$ *and* $a \in (0, 1/2]$, *with equality in each instance if and only if* $a = 1/2$.

**Proof.** (1) It follows from Lemma 2.1(2) that

$$\begin{aligned} f_4(a) = R(a) + b - \frac{1}{b} &= -\frac{1}{b} + b + \left[\frac{1}{b} - 4(1 - \log 2) + 4\sum_{n=1}^{\infty}[\lambda(2n+1) - 1](1-2a)^{2n}\right] \\ &= \frac{1}{4} - \frac{(1-2a)^2}{4} - 4(1 - \log 2) + 4\sum_{n=1}^{\infty}[\lambda(2n+1) - 1](1-2a)^{2n} \\ &= 4\log 2 - \frac{15}{4} + \sum_{n=1}^{\infty} A_4(2n)(1-2a)^{2n}, \end{aligned} \tag{3.51}$$

where $A_4(n)(n \ge 1)$ is given in (3.44). Therefore,

$$\begin{aligned} f_4(a) = R(a) + b - \frac{1}{b} &= \left[\frac{1}{a} + \sum_{n=1}^{\infty} 2\zeta(2n+1)a^{2n}\right] + a(1-a) - \frac{1}{a(1-a)} \\ &= \frac{1}{a} + \sum_{n=1}^{\infty} 2\zeta(2n+1)a^{2n} - \left(\frac{1}{a} + 1 + a + a^2 + \sum_{n=3}^{\infty} a^n\right) + a - a^2 \\ &= -1 - (2 - 2\zeta(3))a^2 - \sum_{n=2}^{\infty}[a^{2n-1} + (1 - 2\zeta(2n+1))a^{2n}] \\ &= -1 + \sum_{k=1}^{\infty} B_4(k)a^k. \end{aligned} \tag{3.52}$$

(2) It follows from Lemma 2.1(1) that $A_4(2n) = 4(\lambda(2k+1) - 1) > 0$ for all $n \ge 2$ and $n \in \mathbb{N}$. Elaborated computations lead to

$$f_4^{(k)}(a) = \sum_{n=[(k+1)/2]}^{\infty} (-1)^k \frac{2^k(2n)!}{(2n-k)!} A_4(2n)(1-2a)^{2n-k} \tag{3.53}$$

and

$$f_4^{(k)}(a) = \sum_{n=[(k+1)/2]}^{\infty} \frac{(n)!}{(n-k)!} B_4(n)a^{n-k}, \tag{3.54}$$

which imply that $(-1)^k f_4^{(k)}(a) \ge 0$ for all $a \in (0, 1/2]$ and $k = 3, 4, \ldots$. Therefore, $f_4'''(a) \le 0$ and $f_4''(a)$ are decreasing on $(0, 1/2]$. Note that $f_4''(a)$ can be written as follows:

$$f_4''(a) = \sum_{n=2}^{\infty} n(n-1)B_4(n)a^{n-2}.$$

It is easy to check that $f_4''(0^+) = 4\zeta(3) - 4 > 0$ and $f_4''(1/2) = -34 + 28\zeta(3) = -0.341\ldots$. Therefore, there exists $a_1 \in (0, 1/2]$ such that $f_4''(a) \ge 0$ on $(0, a_1]$ and $f_4''(a) < 0$ on $(a_1, 1/2]$, and $f'(a)$ is increasing on $(0, a_1]$ and decreasing on $[a_1, 1/2]$. Note that

$$f_4'(a) = \sum_{n=1}^{\infty} nB_4(n)a^{n-1}, \quad f_4'(0^+) = f_4'(1/2) = 0,$$

and $f_4'(a) \ge 0$ on $(0, 1/2]$. Therefore, we obtain the desired monotonicity and concavity of $f_4$.

(3) Differentiating

$$F_{4,n}(a) = \frac{f_4(a) - P_{4,n}(a)}{(1-2a)^{2(n+1)}} = \sum_{k=n+1}^{\infty} A_4(2k)(1-2a)^{2(k-n-1)} = \sum_{k=0}^{\infty} A_4(2(k+n+1))(1-2a)^{2k} \tag{3.55}$$

yields

$$F_{4,n}^{(l)}(a) = \sum_{k=[(l+1)/2]}^{\infty} (-1)^l \frac{2^k(2k)!}{(2k-l)!} A_4(2(k+n+1))(1-2a)^{2k-l}, \tag{3.56}$$

which leads to

$$F_{4,n}'(a) = -2\sum_{k=1}^{\infty} 2kA_4(2(k+n+1))(1-2a)^{2k-1}$$

and

$$F_{4,n}''(a) = 4\sum_{k=1}^{\infty} 2k(2k-1)A_4(2(k+n+1))(1-2a)^{2k-2}.$$

It follows from Lemma 2.1(1) that $A_4(2n) > 0$ for all $n \geq 2$ and $n \in \mathbb{N}$, $F_{4,n}'(a) < 0$, $F_{4,n}''(a) > 0$. Therefore, we obtain the desired monotonicity and concavity of $F_{4,n}$, and the desired inequality follows from the monotonicity of $F_{4,n}$ and $F_{4,n}(0^+) = 1 - P_{4,n}(0)$, $F_{4,n}(1/2^+) = A_4(2n+2)$.

(4) Let $G_{4,n}(a) = h_9(a)/h_{10}(a)$, where $h_{10}(a) = a^{n+1}$ and

$$h_9(a) = f_4(a) - R_{4,n}(a). \tag{3.57}$$

Then $h_9^{(m)}(0^+) = h_{10}^{(m)}(0^+) = 0$ for all $m \in N \cup \{0\}$ with $0 \leq m \leq n$ and

$$\frac{h_9^{(n+1)}(a)}{h_{10}^{(n+1)}(a)} = \frac{f_4^{(n+1)}(a)}{(n+1)!}. \tag{3.58}$$

From Lemma 2.3, we know that $G_{4,2n}(G_{4,(2n-1)})$ have the same monotonicity with the function $f_4^{(2n+1)}(f_4^{(2n)})$. Therefore, $G_{4,2n}(G_{4,(2n-1)})$ is increasing (decreasing) on $(0, 1/2]$ follows from the monotonicity of $f_4^{(n)}$.

Differentiating $G_{4,n}$ gives

$$(G_{4,n}(a))' = \left(\frac{h_9(a)}{h_{10}(a)}\right)' = \frac{a(f_4(a) - R_{4n}(a))' - (n+1)(f_4(a) - R_{4,n}(a))}{a^{n+2}}.$$

Let $h_{12}(a) = a^{n+2}$ and

$$h_{11}(a) = a[f_2(a) - R_{2,n}(a)]' - (n+1)[f_2(a) - R_{2,n}(a)].$$

Then from (3.52), we clearly see that $h_{11}(a)$ can be rewritten as follows:

$$h_{11}(a) = \sum_{k=n+1}^{\infty} (k-n-1)B_4(k)a^k.$$

Therefore, $h_{11}^{(m)}(0^+) = h_{12}^{(m)}(0^+) = 0$ for all $m \in \mathbb{N} \cup \{0\}$ with $0 \leq m \leq n$. Note that

$$\frac{h_{11}^{(n+1)}(a)}{h_{12}^{(n+1)}(a)} = \frac{f_4^{(n+2)}(a)}{(n+2)!}. \tag{3.59}$$

It follows from Lemma 2.3 that $G_{4,n}'$ has the same monotonicity with the function $f_4^{(n+2)}$. Therefore, we get the desired monotonicity of $G_{4,2n}'$ and $G_{4,(2n-1)}'$, and the desired concavity (convexity) of $G_{4,2n}'$ $(G_{4,(2n-1)}')$.

The double inequality (3.49) (3.50) follows easily from the monotonicity and concavity (convexity) of $G_{4,2n}(G_{4,(2n+1)})$. □

Let $n = 1$. Then Theorems 3.1, 3.3, 3.6 and 3.7 lead to Corollary 3.8 immediately.

**Corollary 3.8.** *The inequalities*

$$\log 2 - A_1(2)(1-2a)^2 - A_1(4)(1-2a)^4 - (\lambda(5)-1)(1-2a)^5 \le bR(a) \le \log 2 - A_1(2)(1-2a)^2 - A_1(4)(1-2a)^4,$$

$$1 - a + (4\log 2 - 2)a^3 \le bR(a) \le 1 - a + 2\zeta(3)a^3 - 4(1 - 2\log 2 + \zeta(3))a^4, \tag{3.60}$$

$$\begin{aligned} &1 - a + 2\zeta(3)a^3 - 2\zeta(3)a^4 + 4[12 + 8\log 2 - \zeta(3)]a^5 \\ &\le\ bR(a) \le 1 - a + 2\zeta(3)a^3 + 4[12 + 8\log 2 - \zeta(3)]a^4, \end{aligned} \tag{3.61}$$

$$\begin{aligned} &\frac{9\log 2}{4} + A_2(2)[(1-2a)^2 + (1-2a)^4] \le b(2+b)R(a) \\ &\le\ \frac{9\log 2}{4} + A_2(2)[(1-2a)^2 + (1-2a)^4] + [(8 - 9\log 2)/4 - 2A_2(2)](1-2a)^5, \end{aligned}$$

$$\begin{aligned} &2 - a - 2a^2 + (4\zeta(3) + 1)a^3 + 2[18\log 2 - 4\zeta(3) - 9]a^4 \\ &\le b(2+b)R(a) \le 2 - a - 2a^2 + 2(9\log 2 - 4)a^3, \end{aligned} \tag{3.62}$$

$$2 - a - 2a^2 + (9\log 2 - 6)a^4 \le b(2+b)R(a) \le 2 - a - 2a^2 - 2a^4 + 2(9\log 2 - 4)a^5, \tag{3.63}$$

$$7\log 2 - 4 + (7\lambda(3) + \log 2 - 4)[(1-2a)^2 + (1-2a)^4] < (2-b)R - 1/b,$$

$$\frac{1}{a} - 2 < (2-b)R - 1/b \le \frac{1}{a} - 2 + [56\log 2 - 8\zeta(3) - 30]a, \tag{3.64}$$

$$(2-b)R - \frac{1}{b} < \frac{1}{a} + 2 + \min\{(4\zeta(3) - 1)a, (28\log 2 - 16)a\}, \tag{3.65}$$

$$\begin{aligned} &A_4(0) + A_4(2)(1-2a)^2 + A_4(4)(1-2a)^4 \le R(a) + b - \frac{1}{b} \\ &\le\ A_4(0) + A_4(2)(1-2a)^2 + A_4(4)(1-2a)^4 + (F_{4,1}(0^+) - A_4(4))(1-2a)^5, \\ &-1 + B_4(2)a^2 - a^3 + 2(G_{4,2}(1/2) + 1)a^4 \le R(a) + b - \frac{1}{b} \le -1 + B_4(2)a^2 + G_{4,2}(1/2)a^3 \end{aligned} \tag{3.66}$$

*and*

$$-1 + B_4(2)a^2 + G_{4,1}(1/2)a^4 \le R(a) + b - \frac{1}{b} \le -1 + B_4(2)a^2 + B_4(2)a^4 + 2(G_{4,1}(1/2) - B_4(2))a^5 \tag{3.67}$$

*hold for all* $a \in (0, 1/2]$, *with equality in each instance if and only if* $a = 1/2$, *where*

$$A_1(2) = \log 2 - \lambda(3) = -0.358653\ldots, \quad A_1(4) = \lambda(3) - \lambda(5) = 0.047276\ldots,$$

$$A_2(2) = (9\lambda(3) - 10\log 2)/4 = 0.633682\ldots,$$

$$A_4(0) = 4\log 2 - \frac{15}{4} = -0.9774\ldots, \quad A_4(2) = 4\lambda(3) - \frac{17}{4} = -0.0426\ldots,$$

$$A_4(4) = 4(\lambda(3) - 1) = 0.2072\ldots, \quad F_{4,1}(0^+) = -1 - A_4(0) - A_4(2) = -2.02\ldots.$$

$$G_{4,1}(1/2) = 16\log 2 - 11 = -3.90964512\ldots$$

$$G_{4,2}(1/2) = 32\log 2 - 4\zeta(3) - 18 = -0.0264894\ldots,$$

*and*

$$B_4(2) = 2\zeta(3) - 2 = 0.1035996\ldots.$$

**Remark 3.9.** It is not very difficult to prove that the upper and lower bounds for $bR(a)$ in (3.60) are better than that in (3.61), the upper and lower bounds for $b(2+b)R(a)$ in (3.62) are better than that in (3.63), the upper bound for $(2-b)R(a) - 1/b$ in (3.65) is better than that in (3.64), and the upper and lower bounds for $R(a) + b - 1/b$ in (3.66) are better than that in (3.67). We leave the details of the proof to the interested readers.

# 4 Conclusion

In the article, we have established some monotonicity and convexity (concavity) properties for certain combinations of polynomials and the Ramanujan $R$-function, $R(a) = -2\gamma - \psi(a) - \psi(1-a)$ by use of the L'hôpital's monotone rule, and presented serval novel bounds for the Ramanujan $R$-function, which are the refinements and improvements of the results given in [54]. The proposed method and technology may lead to a lot of follow-up work in the future.

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