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Research Article

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Existence results for neutral evolution equations with nonlocal conditions and delay via fractional operator

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Abstract: In this paper, we study the existence of solutions for the neutral evolution equations with nonlocal conditions and delay in α-norm, which are more general than in many previous publications. We assume that the linear part generates an analytic semigroup and transforms them into suitable integral equations. By using the Kuratowski measure of noncompactness and fixed-point theory, some existence theorems are established without the assumption of compactness on the associated semigroup. Particularly, our results cover the cases where the nonlinear term F takes values in different spaces such as X_α. An example of neutral partial differential system is also given to illustrate the feasibility of our abstract results.

Keywords: neutral evolution equations, nonlocal conditions, nonlocal conditions, delay, measure of noncompactness, α-norm

MSC 2020: 34K30, 35D35, 35K55, 47J35

1 Introduction

In this paper, we are concerned with the existence of solutions for the following neutral evolution equations with nonlocal conditions and delay

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}[u(t) + G(t, u(t), u_t)] + Au(t) = F(t, u(t), u_t), \quad t \in [0, a], \\ u(t) = g(u)(t) + \phi(t), \quad t \in [-h, 0], \end{cases} \tag{1.1}$$

where $u(\cdot)$ takes value in a subspace $D(A^\alpha)$ of Banach space X, which will be defined later, $A : D(A) \subset X \to X$ is a linear operator and $-A$ generates an analytic semigroup $T(t)(t \geq 0)$, α, h and a be three constants such that $0 < \alpha < 1$ and $0 < h, a < +\infty$. For every $t \in [0, a]$, F, G are given functions satisfying some assumptions, $\phi \in C([-h, 0], D(A^\alpha))$ is *a priori* given history, while the function $g : C([-h, a], D(A^\alpha)) \to C([-r, 0], D(A^\alpha))$ implicitly defines a complementary history, chosen by the system itself. Following the standard notation (see [1]), if $u \in C([-h, a], D(A^\alpha))$ and $t \in [0, a]$, we denote by u_t the function $u_t : [-h, 0] \to D(A^\alpha)$, defined by $u_t(\theta) coloneu(t + \theta)$ for $\theta \in [-h, 0]$.

Many complex processes in nature and technology are described by functional differential equations, which are dominant nowadays because the functional components in equations allow one to consider after-effect or prehistory influence. Delay differential equations are one of the important types of functional

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differential equations, in which the response of the system depends not only on the current state of the system but also on the history of the system. For more details on this topic, see, for example, the books of Hale and Verduyn Lunel [1] and Wu [2], and the papers of Chen et al. [3], Chen [4], Dong and Li [5], Fu [6], Fu and Ezzinibi [7], Li [8], Travis and Webb [9,10], Vrabie [11,12] and Wang et al. [13].

The theory of neutral partial differential equations with a delay has an extensive physical background and realistic mathematical model; hence, it has been considerably developed and the numerous properties of their solutions have been studied, see [6,8,14–19] and the references therein. The problem concerning neutral partial differential equations with nonlocal conditions and delay is an important area of investigation in recent years. Especially, the existence of solutions of neutral evolution equations with nonlocal conditions and delay have been considered by several authors, see [7,20].

In problem (1.1), when $\alpha = 0$ in interpolation space X_α, Dong et al. [21] studied the neutral partial functional differential equations with nonlocal conditions

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}[u(t) + G(t, u(t), u_t)] + Au(t) = F(t, u(t), u_t), \quad t \in [0, a], \\ u_0 = \phi + g(u). \end{cases} \tag{1.2}$$

With the aid of Hausdorff's measure of noncompactness and fixed-point theory, the authors established some existence results of mild solutions for the problem (1.2). However, their results cannot be applied to equations with terms involving spatial derivatives.

The existence and regularity problem on the compact interval $[0, a]$, in the very simplest case when $r = 0$, i.e., when the delay is absent, was studied by Chang and Liu [22]. In this case, C_0^α identifies with X_α, $F(t, u, u_0)$ identifies with a function F from $[0, a] \times X_\alpha \to X$, and $G(t, u, u_0)$ identifies with a function G from $[0, a] \times X_\alpha \to X$, and so the previous paper [22] has considered the problem:

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}[u(t) + G(t, u(t))] + Au(t) = F(t, u(t)), \quad t \in [0, a], \\ u(0) + g(u) = u_0 \in X_\alpha, \end{cases} \tag{1.3}$$

where the operator $-A : D(A) \subset X \to X$ generates an analytic and compact semigroup. The authors of [22] showed existence results of solutions to neutral evolution equations with nonlocal conditions (1.3) in the α-norm under the assumptions that the linear part of equations generates a compact analytic semigroup and the nonlinear part satisfies some Lipschitz conditions with respect to the α-norm. In their work, a key assumption is that the associated semigroup is compact.

In 2013, Fu [6] investigated the existence of mild solutions for the following abstract neutral evolution equation with an infinite delay

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}[u(t) + G(t, u_t)] + Au(t) = F(t, u_t), \quad t \in [0, a], \\ u_0 = \phi \in \mathcal{B}_\alpha, \end{cases} \tag{1.4}$$

where $u(\cdot)$ takes values in a subspace of Banach space X. Under the assumption that the linear part of equations generates a compact analytic semigroup, the author obtained the existence of mild solutions to the problem (1.4) by using fractional power theory and α-norm. The results in [6] can be applied to equations with terms involving spatial derivatives. However, to the best of the authors' knowledge, in all of the existing articles, such as [6], the neutral evolution equations with nonlocal conditions via fractional operator have been studied under the hypothesis that the corresponding linear partial differential operator generates a compact semigroup, and the existence of mild solutions for neutral evolution equations with nonlocal conditions via fractional operator with noncompact semigroup has not been investigated yet.

Motivated by all the aforementioned aspects, in this article, we apply the theory of fractional power operator, α-norm, Kuratowski measure of noncompactness and corresponding fixed-point theory to obtain the existence and uniqueness of mild solutions for neutral evolution equations with nonlocal conditions and delay via fractional operator (1.1) without the assumptions of compactness on the associated semigroup. Particularly, our results cover the cases where the nonlinear term F takes values in different spaces such as X_α and X_μ.

The rest of this paper is organized as follows. In Section 2, we recall some preliminary results on the fractional powers of unbounded linear operators generating analytic semigroups, and the results of Kuratowski's measure of noncompactness, which will be used in the proof of our main results. Section 3 states and proves the existence and uniqueness of mild solutions for the problems (1.1) in the α-norm by utilizing the fixed-point theorem and Kuratowski's measure of noncompactness. An example of the neutral partial differential system is also given in Section 4 to illustrate the feasibility of our abstract results.

2 Preliminaries

In this section, we introduce some notations, definitions and preliminary facts that are used throughout this paper.

Assume that X is a real Banach space with norm $\|\cdot\|$, $A : D(A) \subset X \to X$ is a densely defined closed linear operator and $-A$ generates an analytic semigroup $T(t)(t \geq 0)$. Then there exists a constant $M \geq 1$ such that $\|T(t)\| \leq M$ for $t \geq 0$. Without loss of generality, we suppose that $0 \in \rho(A)$; otherwise instead of A, we take $A - \lambda I$, where λ is chosen such that $0 \in \rho(A - \lambda I)$ where $\rho(A)$ is the resolvent set of A. Then it is possible to define the fractional power A^α for $0 < \alpha < 1$, as a closed linear operator on its domain $D(A^\alpha)$. Furthermore, the subspace $D(A^\alpha)$ is dense in X and

$$\|x\|_\alpha = \|A^\alpha x\|, \quad x \in D(A^\alpha)$$

defines a norm on $D(A^\alpha)$. Hereafter, we denote by X_α the Banach space $D(A^\alpha)$ normed with $\|\cdot\|_\alpha$. In addition, we have the following properties.

Lemma 2.1. [23] *Let $0 < \alpha < 1$. Then*

(i) *$T(t) : X \to X_\alpha$ for each $t > 0$;*
(ii) *$A^\alpha T(t)x = T(t)A^\alpha x$, for each $x \in X_\alpha$ and $t \geq 0$;*
(iii) *For every $t > 0$, the linear operator $A^\alpha T(t)$ is bounded and $\|A^\alpha T(t)\| < \frac{M_\alpha}{t^\alpha}$, where M_α is a positive real constant;*
(iv) *For $0 \leq \alpha \leq 1$, one has $\|A^{-\alpha}\| \leq N_\alpha$, where C_α is a positive real constant;*
(v) *For $0 < \alpha < \beta \leq 1$, we obtain $X_\beta \hookrightarrow X_\alpha$.*

Set

$$C_0^\alpha := C([-r, 0], X_\alpha) = \{u : [-h, 0] \to X_\alpha \text{ is continuous}\},$$
$$C_a^\alpha := C([-r, a], X_\alpha) = \{u : [-h, a] \to X_\alpha \text{ is continuous}\}.$$

It is easy to see that C_0^α and C_a^α are Banach spaces with norms

$$\|u\|_{C_0^\alpha} = \sup_{-h\leq t\leq 0} \|u(t)\|_\alpha, \quad \|u\|_{C_a^\alpha} = \sup_{-h\leq t\leq a} \|u(t)\|_\alpha,$$

respectively. For any $R > 0$, let $D_R = \{u \in X_\alpha : \|u\|_\alpha \leq R\}$, $D_R(C_0^\alpha) = \{u \in C_0^\alpha : \|u\|_{C_0^\alpha} \leq R\}$, $D_R(C_a^\alpha) = \{u \in C_a^\alpha : \|u\|_{C_a^\alpha} \leq R\}$.

Moreover, we introduce the Kuratowski measure of noncompactness $\mu(\cdot)$ defined by

$$\mu(D) := \inf\left\{\delta > 0 : D = \bigcup_{i=1}^{n} D_i \text{ and } \operatorname{diam}(D_i) \leq \delta \text{ for } i = 1, 2, \cdots, n\right\}$$

for a bounded set D in the Banach space X. In this article, we denote by $\mu(\cdot)$, $\mu_\alpha(\cdot)$ and $\mu_{C_a^\alpha}(\cdot)$ the Kuratowski measure of noncompactness on the bounded set of X, X_α and C_a^α, respectively. For any $D \subset C_a^\alpha$, $s \in [-h, a]$ and $t \in [0, a]$, set

$$D(s) = \{u(s) : u \in D\} \subset X, \quad D_t = \{u_t : u \in D\}.$$

Lemma 2.2. [24] *Let X be a Banach space and $D \subset C_0^\alpha$ be a bounded and equicontinuous set. Then $\mu(D(t))$ is continuous on $[-h, a]$, and $\mu_{C_0^\alpha}(D) = \max_{t\in[-h,a]}\mu(D(t))$.*

Lemma 2.3. [25] *Let X be a Banach space and $D = \{u_n\}_{n=1}^\infty \subset C([0, a], X)$ be a bounded and countable set. Then $\mu(D(t))$ is Lebesgue integral on $[0, a]$, and*

$$\mu\left(\left\{\int_0^a u_n(t)\mathrm{d}t : n \in \mathbb{N}\right\}\right) \leq 2\int_0^a \mu(D(t))\mathrm{d}t.$$

Lemma 2.4. [26] *Let X be a Banach space, and let $D \subset X$ be bounded. Then there exists a countable subset $D^* \subset D$, such that $\mu(D) \leq 2\mu(D^*)$.*

3 Existence results

This section is devoted to investigate the existence of mild solutions for problem (1.1) in the subspace X_α of Banach space X. By comparison with the abstract Cauchy problem

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}u(t) + Au(t) = h(t), & t \in [0, a], \\ u(0) = \tilde{u}, \end{cases} \tag{3.1}$$

whose properties are well known [23], we can obtain the definition of mild solution for the problem (1.1) in X_α.

Definition 3.1. A function $u \in C_a^\alpha$ is said to be a mild solution to problem (1.1) in X_α if it satisfies

(i) For each $t \in [-h, 0]$, $u(t) = g(u)(t) + \phi(t)$;

(ii) For each $t \in [0, a]$ the function $s \mapsto AT(t - s)G(s, u(s), u_s)$ is integrable on $[0, t]$, and the following integral equation is satisfied:

$$\begin{aligned} u(t) = {} & T(t)[g(u)(0) + \phi(0) + G(0, g(u)(0) + \phi(0), g(u) + \phi)] - G(t, u(t), u_t) \\ & + \int_0^t AT(t - s)G(s, u(s), u_s)\mathrm{d}s + \int_0^t T(t - s)F(s, u(s), u_s)\mathrm{d}s. \end{aligned} \tag{3.2}$$

Definition 3.2. A function $F : [0, a] \times X_\alpha \times C_0^\alpha \to X$ is said to be Carathéodory continuous if

(i) For all $(u, v) \in X_\alpha \times C_0^\alpha \to X$, $F(\cdot, u, v) : [0, a] \to X$ is measurable,

(ii) For a.e. $t \in [0, a]$, $F(t, \cdot, \cdot) : X_\alpha \times C_0^\alpha \to X$ is continuous.

In what follows, we make the following hypotheses on the data of problem (1.1).

(P_1) The function $F : [0, a] \times X_\alpha \times C_0^\alpha \to X$ is Carathéodory continuous, and there exist constants $q \in [0, 1 - \alpha)$, $\gamma > 0$ and function $\varphi_R \in L^{\frac{1}{q}}([0, a], \mathbb{R}^+)$ such that for some positive constant R,

$$\|F(t, u, v)\| \leq \varphi_R(t), \quad \liminf_{R\to+\infty} \frac{\|\varphi_R\|_{L^{\frac{1}{q}}([0,a])}}{R} := \gamma < +\infty,$$

for any $u \in D_R$, $v \in D_R(C_0^\alpha)$ and a.e. $t \in [0, a]$.

(P_2) There exists a constant $\beta \in (0, 1)$ with $\alpha + \beta \leq 1$, such that the function $G : [0, a] \times X_\alpha \times C_0^\alpha \to X_{\alpha+\beta}$ satisfies Lipschitz condition, i.e., there exists a constant $L_G > 0$ such that

$$\|A^\beta G(t, u_2, v_2) - A^\beta G(t, u_1, v_1)\|_\alpha \leq L_G(|t_2 - t_1| + \|u_2 - u_1\|_\alpha + \|v_2 - v_1\|_{C_0^\alpha}),$$

where $t_1, t_2 \in [0, a]$, $u_1, u_2 \in X_\alpha$ and $v_1, v_2 \in C_0^\alpha$.

(P_3) There exists a positive constant L_g such that

$$\|g(u) - g(v)\|_{C_0^\alpha} \leq L_g\|u - v\|_{C_a^\alpha}, \quad u, v \in C_a^\alpha.$$

(P_4) $\mathcal{H} + M_\alpha\Big(\frac{1-q}{1-\alpha-q}\Big)^{1-q} a^{1-\alpha-q}\gamma < 1$, where $\mathcal{H} = M(L_g + 2N_\beta L_G) + 2N_\beta L_G + \frac{2M_{1-\beta}L_G a^\beta}{\beta}$.

(P_5) There exists a function $\eta \in L^1([0, a], \mathbb{R}^+)$ with $\|\eta\|_{L^1} < \frac{1-\mathcal{H}}{8}$ such that for any bounded and countable set $D \subset C_a^\alpha$,

$$\mu_\alpha(T(s)F(t, D(t), D_t)) \leq \eta(t)\Big(\mu_\alpha(D(t)) + \sup_{-h\leq\tau\leq 0} \mu_\alpha(D(t+\tau))\Big), \quad \text{a.e. } t, s \in [0, a].$$

Theorem 3.3. *Assume that conditions* (P_1)–(P_5) *are satisfied. Then for every* $\phi \in C_0^\alpha$, *problem* (1.1) *exists at least one mild solution in* C_a^α.

Proof. For some positive constant R, consider an operator $\mathcal{F} : D_R(C_a^\alpha) \to C_a^\alpha$ defined by

$$(\mathcal{F}u)(t) = \begin{cases} T(t)[g(u)(0) + \phi(0) + G(0, g(u)(0) + \phi(0), g(u) + \phi)] \\ - G(t, u(t), u_t) + \int_0^t AT(t-s)G(s, u(s), u_s)\mathrm{d}s \\ + \int_0^t T(t-s)F(s, u(s), u_s)\mathrm{d}s, \quad t \in [0, a], \\ g(u)(t) + \phi(t), \quad t \in [-h, 0]. \end{cases} \tag{3.3}$$

From hypothesis (P_2) and Lemma 2.1, one has

$$\begin{aligned} \|AT(t-s)G(s, u(s), u_s)\|_\alpha &\leq \|A^{1-\beta}T(t-s)\| \cdot \|G(s, u(s), u_s)\|_{\alpha+\beta} \\ &\leq \frac{M_{1-\beta}}{(t-s)^{1-\beta}}\Big[L_G\Big(\|u(s)\|_\alpha + \|u_s\|_{C_0^\alpha} + a\Big) + \|G(0, \theta, \theta)\|_{\alpha+\beta}\Big] \\ &\leq \frac{M_{1-\beta}}{(t-s)^{1-\beta}}\Big[L_G\Big(2\|u\|_{C_a^\alpha} + a\Big) + \|G(0, \theta, \theta)\|_{\alpha+\beta}\Big]. \end{aligned}$$

Then from Bochner theorem, it follows that $AT(t-s)G(s, u(s), u_s)$ is integrable on $[0, t)$ for every $t \in (0, a]$ and $u \in D_R(C_a^\alpha)$. Therefore, $\mathcal{F}$ is well defined and has values in C_a^α. In accordance with Definition 3.2, it is easy to see that the mild solution of problem (1.1) is equivalent to the fixed-point of the operator $\mathcal{F}$ defined by (3.3). In the following, we will show that operator $\mathcal{F}$ has a fixed-point by applying the famous Sadovskii's fixed-point theorem, which can be found in [27].

First, we prove that there exists a positive constant R such that the operator $\mathcal{F}$ defined by (3.2) maps the bounded set $D_R(C_a^\alpha)$ into itself. If this is not true, there would exist $u^r \in D_r(C_a^\alpha)$ and $t^r \in [-h, a]$ such that $\|(\mathcal{F}u^r)(t^r)\|_\alpha > r$ for each $r > 0$. For $t^r \in [-h, 0]$, by (3.3) and the hypothesis (P_3), one has

$$\begin{aligned} r &< \|(\mathcal{F}u^r)(t^r)\|_\alpha \leq \|g(u^r)(t^r) + \phi(t^r)\|_\alpha \leq \|g(u^r)\|_{C_0^\alpha} + \|\phi\|_{C_0^\alpha} \leq +L_g\|u^r\|_{C_a^\alpha} + \|g(\theta)\|_{C_0^\alpha} + \|\phi\|_{C_0^\alpha} \\ &\leq L_g r + \|g(\theta)\|_{C_0^\alpha} + \|\phi\|_{C_0^\alpha}, \end{aligned} \tag{3.4}$$

and for $t^r \in [0, a]$, by (3.3), Lemma 2.1, the conditions (P_1)–(P_3) and Hölder inequality, we obtain that

$$\begin{aligned} r &< \|(\mathcal{F}u^r)(t^r)\|_\alpha \\ &\leq M(\|g(u^r)(0)\|_\alpha + \|\phi(0)\|_\alpha + \|G(0, u^r(0), (u^r)_0)\|_\alpha) + \|G\Big(t^r, u^r(t^r), (u^r)_{t^r}\Big)\|_\alpha \\ &\quad + \int_0^{t^r} \|AT(t^r - s)G(s, u^r(s), (u^r)_s)\|_\alpha \mathrm{d}s + \int_0^{t^r} \|T(t^r - s)F(s, u^r(s), (u^r)_s)\|_\alpha \mathrm{d}s \end{aligned} \tag{3.5}$$

$$
\begin{aligned}
&\le M\Big(\|g(u^r)\|_{C_0^\alpha}+\|\phi\|_{C_0^\alpha}+\|A^{-\beta}\|\cdot\|G(0,u^r(0),(u^r)_0)\|_{\alpha+\beta}\Big)+\|A^{-\beta}\|\cdot\|G\big(t^r,u^r(t^r),(u^r)_{t^r}\big)\|_{\alpha+\beta}\\
&\quad+\int_0^{t^r}\|A^{1-\beta}T(t^r-s)\|\cdot\|G(s,u^r(s),(u^r)_s)\|_{\alpha+\beta}\mathrm{d}s+\int_0^{t^r}\|A^\alpha T(t^r-s)F(s,u^r(s),(u^r)_s)\|\mathrm{d}s\\
&\le M\Big(L_g\|u^r\|_{C_a^\alpha}+\|g(\theta)\|_{C_0^\alpha}+\|\phi\|_{C_0^\alpha}+N_\beta\|G(0,u^r(0),(u^r)_0)\|_{\alpha+\beta}\Big)+N_\beta\|G\big(t^r,u^r(t^r),(u^r)_{t^r}\big)\|_{\alpha+\beta}\\
&\quad+\int_0^{t^r}\frac{M_{1-\beta}}{(t^r-s)^{1-\beta}}\|G(s,u^r(s),(u^r)_s)\|_{\alpha+\beta}\mathrm{d}s+\int_0^{t^r}\frac{M_\alpha}{(t^r-s)^\alpha}\|F(s,u^r(s),(u^r)_s)\|\mathrm{d}s\\
&\le M\Big(L_gr+\|g(\theta)\|_{C_0^\alpha}+\|\phi\|_{C_0^\alpha}\Big)+MN_\beta\Big[L_G\Big(\|u^r(0)\|_\alpha+\|(u^r)_0\|_{C_0^\alpha}\Big)+\|G(0,\theta,\theta)\|_{\alpha+\beta}\Big]\\
&\quad+N_\beta\Big[L_G\Big(t^r+\|u^r(t^r)\|_\alpha+\|(u^r)_{t^r}\|_{C_0^\alpha}\Big)+\|G(0,\theta,\theta)\|_{\alpha+\beta}\Big]\\
&\quad+\int_0^{t^r}\frac{M^{1-\beta}}{(t^r-s)^{1-\beta}}\Big[L_G\Big(s+\|u^r(s)\|_\alpha+\|(u^r)_s\|_{C_0^\alpha}\Big)+\|G(0,\theta,\theta)\|_{\alpha+\beta}\Big]\mathrm{d}s\\
&\quad+M_\alpha\left(\int_0^{t_r}(t_r-s)^{\frac{-\alpha}{1-q}}\mathrm{d}s\right)^{1-q}\left(\int_0^{t_r}\varphi_r^{\frac{1}{q}}(s)\mathrm{d}s\right)^q\\
&\le M\Big(\|g(\theta)\|_{C_0^\alpha}+\|\phi\|_{C_0^\alpha}+C_\beta\|G(0,\theta,\theta)\|_{\alpha+\beta}\Big)+M(L_g+2C_\beta L_G)r+N_\beta(L_Ga+\|G(0,\theta,\theta)\|_{\alpha+\beta})+2N_\beta L_Gr\\
&\quad+\frac{M_{1-\beta}a^\beta}{\beta}(L_Ga+\|G(0,\theta,\theta)\|_{\alpha+\beta})+\frac{2M_{1-\beta}L_Ga^\beta}{\beta}r+M_\alpha\left(\frac{1-q}{1-\alpha-q}\right)^{1-q}a^{1-\alpha-q}\|\varphi_r\|_{L^{\frac{1}{q}}([0,a])}=:H_r.
\end{aligned}
$$

Combining with (3.4) and (3.5), we obtain that

$$
r<\max\Big\{L_gr+\|g(\theta)\|_{C_0^\alpha}+\|\phi\|_{C_0^\alpha},H_r\Big\}.
$$

Therefore, by the fact $M\ge 1$, one gets

$$
r<H_r. \tag{3.6}
$$

Dividing both side of the inequality (3.6) by r and taking the lower limit as $r\to+\infty$, combined with the assumption (P_4), we obtain that

$$
1\le M(L_g+2N_\beta L_G)+2N_\beta L_G+\frac{2M_{1-\beta}L_Ga^\beta}{\beta}+M_\alpha\left(\frac{1-q}{1-\alpha-q}\right)^{1-q}a^{1-\alpha-q}\gamma<1,
$$

which is a contradiction. Hence, we have proved that $\mathcal{F}:D_R(C_a^\alpha)\to D_R(C_a^\alpha)$.

Second, we prove that the operator $\mathscr{F}:D_R(C_a^\alpha)\to D_R(C_a^\alpha)$ is condensing. For this purpose, we first decompose $\mathscr{F}$ as $\mathscr{F}=\mathscr{F}_1+\mathscr{F}_2$, where

$$
(\mathscr{F}_1u)(t)=\begin{cases}T(t)[g(u)(0)+\phi(0)+G(0,g(u)(0)+\phi(0),g(u)+\phi)]\\ -G(t,u(t),u_t)+\displaystyle\int_0^t AT(t-s)G(s,u(s),u_s)\mathrm{d}s, \quad t\in[0,a],\\ g(u)(t)+\phi(t),\quad t\in[-h,0],\end{cases}
$$

$$
(\mathscr{F}_2u)(t)=\begin{cases}\displaystyle\int_0^t T(t-s)F(s,u(s),u_s)\mathrm{d}s,\quad t\in[0,a],\\ 0,\quad t\in[-h,0].\end{cases}
$$

Now, we show that $\mathscr{F}_1:D_R(C_a^\alpha)\to D_R(C_a^\alpha)$ is Lipschitzian with Lipschitz constant $\mathcal{H}$. In fact, take u and v in $D_R(C_a^\alpha)$. For $t\in[-h,0]$, by the formulation of the operator $\mathscr{F}_1$ and the hypothesis (P_3), we obtain that for $t\in[0,a]$,

$$
\|(\mathscr{F}_1u)(t)-(\mathscr{F}_1v)(t)\|_\alpha=\|g(u)(t)-g(v)(t)\|_\alpha\le\|g(u)-g(v)\|_{C_0^\alpha}\le L_g\|u-v\|_{C_a^\alpha},\quad\forall u,v\in D_R(C_a^\alpha). \tag{3.7}
$$

On the basis of the definition of the operator $\mathscr{F}_1$, Lemma 2.1 and the hypotheses (P_2) and (P_3), one obtains that

$$\begin{aligned}
&\|(\mathscr{F}_1u)(t)-(\mathscr{F}_1v)(t)\|_\alpha\\
&\le M(\|g(u)(0)-g(v)(0)\|_\alpha+\|G(0,u(0),u_0)-G(0,v(0),v_0)\|_\alpha)+\|G(t,u(t),u_t)-G(t,v(t),v_t)\|_\alpha\\
&\quad+\int_0^t\|AT(t-s)[G(s,u(s),u_s)-G(s,v(s),v_s)]\|_\alpha\mathrm{d}s\\
&\le M\Big(\|g(u)-g(v)\|_{C_0^\alpha}+\|A^{-\beta}\|\cdot\|G(0,u(0),u_0)-G(0,v(0),v_0)\|_{\alpha+\beta}\Big)+\|A^{-\beta}\|\cdot\|G(t,u(t),u_t)\\
&\quad-G(t,v(t),v_t)\|_{\alpha+\beta}+\int_0^t\|A^{1-\beta}T(t-s)\|\cdot\|G(s,u(s),u_s)-G(s,v(s),v_s)\|_{\alpha+\beta}\mathrm{d}s\\
&\le M\Big[L_g\|u-v\|_{C_a^\alpha}+N_\beta L_G\Big(\|u(0)-v(0)\|_\alpha+\|u_0-v_0\|_{C_0^\alpha}\Big)\Big]+N_\beta L_G\Big(\|u(t)-v(t)\|_\alpha+\|u_t-v_t\|_{C_0^\alpha}\Big)\\
&\quad+\int_0^t\frac{M_{1-\beta}}{(t-s)^{1-\beta}}L_G\Big(\|u(s)-v(s)\|_\alpha+\|u_s-v_s\|_{C_0^\alpha}\Big)\mathrm{d}s\\
&\le\left\{M(L_g+2N_\beta L_G)+2N_\beta L_G+2L_G\int_0^t\frac{M_{1-\beta}}{(t-s)^{1-\beta}}\mathrm{d}s\right\}\|u-v\|_{C_a^\alpha}\\
&\le\left\{M(L_g+2N_\beta L_G)+2N_\beta L_G+\frac{2L_GM_{1-\beta}a^\beta}{\beta}\right\}\|u-v\|_{C_a^\alpha}\quad=:\mathcal{H}\|u-v\|_{C_a^\alpha},\quad\forall u,v\in D_R(C_a^\alpha).
\end{aligned}\tag{3.8}$$

Notice that $M\ge1$ yields that

$$L_g\le\mathcal{H}.\tag{3.9}$$

Thus, from (3.7), (3.8) and (3.9), it follows that

$$\|(Q_1u)(t)-(Q_1v)(t)\|_\alpha\le\mathcal{H}\|u-v\|_{C_a^\alpha},\quad t\in[-h,a].$$

Taking supremum over t, we obtain that

$$\|Q_1u-Q_1v\|_{C_a^\alpha}\le\mathcal{H}\|u-v\|_{C_a^\alpha}.\tag{3.10}$$

This means that $\mathscr{F}_1$ is Lipschitzian with Lipschitz constant $\mathcal{H}$.

Subsequently, we prove that the operator $\mathscr{F}_2:D_R(C_a^\alpha)\to D_R(C_a^\alpha)$ is continuous. To this end, letting the sequence $\{u_n\}_{n=1}^\infty\subset D_R(C_a^\alpha)$ such that $\lim_{n\to+\infty}u_n=u$ in $D_R(C_a^\alpha)$. Then

$$\lim_{n\to\infty}u_n(t)=u(t),\ \ t\in[-h,a],\qquad\lim_{n\to\infty}(u_n)_t=u_t,\ \ t\in[0,a].$$

Combining with this and the condition (P_1), one obtain that

$$\lim_{n\to\infty}F(s,u_n(s),(u_n)_s)=F(s,u(s),u_s),\qquad\text{a.e. } s\in[0,a].\tag{3.11}$$

From the hypothesis (P_1), we obtain that for a.e. $s\in[0,t]$, $t\in[0,a]$,

$$(t-s)^{-\alpha}\|F(s,u_n(s),(u_n)_s)-F(s,u(s),u_s)\|\le2(t-s)^{-\alpha}\varphi_R(s).\tag{3.12}$$

Using the fact the function $s\mapsto2(t-s)^{-\alpha}\varphi_R(s)$ is Lebesgue integrable for a.e. $s\in[0,t]$, $t\in[0,a]$, by (3.11), (3.12) and Lebesgue dominated convergence theorem, we obtain that

$$\begin{aligned}
\|(\mathscr{F}_2u_n)(t)-(\mathscr{F}_2u)(t)\|_\alpha&\le\int_0^t\|A^\alpha T(t-s)\|\cdot\|F(s,u_n(s),(u_n)_s)-F(s,u(s),u_s)\|\mathrm{d}s\\
&\le M_\alpha\int_0^t(t-s)^{-\alpha}\|F(s,u_n(s),(u_n)_s)-F(s,u(s),u_s)\|\mathrm{d}s\\
&\to0\quad\text{as}\quad n\to\infty.
\end{aligned}$$

Hence,

$$\|\mathscr{F}_2 u_n - \mathscr{F}_2 u\|_{C_a^\alpha} \to 0 \quad \text{as } n \to \infty,$$

which means that the operator $\mathscr{F}_2 : D_R(C_a^\alpha) \to D_R(C_a^\alpha)$ is continuous.

Below we demonstrate that $\{\mathscr{F}_2 u : u \in D_R(C_a^\alpha)\}$ is a family of equi-continuous functions. For any $u \in D_R(C_a^\alpha)$ and $0 \le t' < t'' \le a$, by the formulation of the operator $\mathscr{F}_2$, we obtain that

$$\begin{aligned}
&\|(\mathscr{F}_2 u)(t'') - (\mathscr{F}_2 u)(t')\|_\alpha \\
&\le \int_{t'}^{t''} \|T(t''-s)F(s,u(s),u_s)\mathrm{d}s\|_\alpha \mathrm{d}s + \int_0^{t'} \|[T(t''-s) - T(t'-s)]F(s,u(s),u_s)\|_\alpha \mathrm{d}s \\
&:= J_1 + J_2.
\end{aligned} \tag{3.13}$$

Therefore, we only need to check J_i tend to 0 independently of $u \in D_R(C_a^\alpha)$ when $t'' - t' \to 0$ for $i = 1, 2$. For J_1, taking (3.13), the assumption (P_1), Lemma 2.1 and Hölder inequality into account, we obtain

$$\begin{aligned}
J_1 &\le \int_{t'}^{t''} \|A^\alpha T(t''-s)F(s,u(s),u_s)\|\mathrm{d}s \\
&\le \int_{t'}^{t''} \frac{M_\alpha}{(t''-s)^\alpha}\varphi_R(s)\mathrm{d}s \\
&\le M_\alpha \left(\frac{1-q}{1-\alpha-q}\right)^{1-q} (t''-t')^{1-\alpha-q} \|\varphi_R\|_{L^{\frac{1}{q}}([0,a])} \\
&\to 0 \quad \text{as} \quad t''-t' \to 0.
\end{aligned}$$

For $t' = 0$, $0 < t'' \le a$, it is easy to see that $J_2 = 0$. For $t' > 0$ and $0 < \varepsilon < t'$ small enough, by (3.13), the condition (P_1), Lemma 2.1, Hölder inequality and the equi-continuity of $T(t)$, we know that

$$\begin{aligned}
J_2 &\le \int_0^\varepsilon \left\| \left[T\left(t''-t'+\frac{s}{2}\right) - T\left(\frac{s}{2}\right)\right] A^\alpha T\left(\frac{s}{2}\right) F\left(t'-s, u(t'-s), u_{t'-s}\right) \right\| \mathrm{d}s \\
&\quad + \int_\varepsilon^{t'} \left\| \left[T\left(t''-t'+\frac{s}{2}\right) - T\left(\frac{s}{2}\right)\right] A^\alpha T\left(\frac{s}{2}\right) F\left(t'-s, u(t'-s), u_{t'-s}\right) \right\| \mathrm{d}s \\
&\le 2M \int_0^\varepsilon \left\| A^\alpha T\left(\frac{s}{2}\right) F\left(t'-s, u(t'-s), u_{t'-s}\right) \right\| \mathrm{d}s \\
&\quad + \sup_{s\in[\varepsilon,t']} \left\| T\left(t''-t'+\frac{s}{2}\right) - T\left(\frac{s}{2}\right) \right\| \int_\varepsilon^{t'} \left\| A^\alpha T\left(\frac{s}{2}\right) F\left(t'-s, u(t'-s), u_{t'-s}\right) \right\| \mathrm{d}s \\
&\le 2MM_\alpha \left(\frac{1-q}{1-\alpha-q}\right)^{1-q} \left(\frac{\varepsilon}{2}\right)^{1-\alpha-q} \|\varphi_R\|_{L^{\frac{1}{q}}([0,a])} + \sup_{s\in[\varepsilon,t']} \left\| T\left(t''-t'+\frac{s}{2}\right) - T\left(\frac{s}{2}\right) \right\| \\
&\quad \times M_\alpha \left(\frac{1-q}{1-\alpha-q}\right)^{1-q} \left[\left(\frac{t'}{2}\right)^{\frac{1-\alpha-q}{1-q}} - \left(\frac{\varepsilon}{2}\right)^{\frac{1-\alpha-q}{1-q}}\right]^{1-q} \|\varphi_R\|_{L^{\frac{1}{q}}([0,a])} \\
&\le 2MM_\alpha \left(\frac{1-q}{1-\alpha-q}\right)^{1-q} \left(\frac{\varepsilon}{2}\right)^{1-\alpha-q} \|\varphi_R\|_{L^{\frac{1}{q}}([0,a])} \\
&\quad + \sup_{s\in[\varepsilon,t']} \left\| T\left(t''-t'+\frac{s}{2}\right) - T\left(\frac{s}{2}\right) \right\| M_\alpha \left(\frac{1-q}{1-\alpha-q}\right)^{1-q} \left(\frac{t'-\varepsilon}{2}\right)^{1-\alpha-q} \|\varphi_R\|_{L^{\frac{1}{q}}([0,a])} \\
&\to 0 \quad \text{as } t''-t' \to 0 \quad \text{and} \quad \varepsilon \to 0.
\end{aligned}$$

As a result, $\|(\mathscr{F}_2u)(t'') - (\mathscr{F}_2u)(t')\|_\alpha \to 0$ independently of $u \in D_R(C_a^\alpha)$ as $t'' - t' \to 0$, which means that $\mathscr{F}_2$ maps $D_R(C_a^\alpha)$ into a family of equi-continuous functions. Here, we consider only the case $0 \le t' < t'' \le a$, since the other cases $-h \le t' < t'' \le 0$ and $-h \le t' \le 0 < t'' \le a$ are very simple.

For any bounded set $D \subset D_R(C_a^\alpha)$, by Lemma 2.4, there exists a countable subset $D^* = \{u_n\}_{n=1}^{\infty}$, such that

$$\mu_{C_a^\alpha}(\mathscr{F}_2(D)) \le 2\mu_{C_a^\alpha}(\mathscr{F}_2(D^*)). \tag{3.14}$$

Since $\mathscr{F}_2(D^*) \subset \mathscr{F}_2(D_R(C_a^\alpha))$ is equi-continuous, by Lemma 2.2 and the definition of the operator $\mathscr{F}_2$, we arrive at

$$\mu_{C_a^\alpha}(\mathscr{F}_2(D^*)) = \max_{t\in[-h,a]} \mu_\alpha((\mathscr{F}_2D^*)(t)) = \max_{t\in[0,a]} \mu_\alpha((\mathscr{F}_2D^*)(t)). \tag{3.15}$$

From the formulation of the operator $\mathscr{F}_2$, using the condition (P_5), we obtain that for any $t \in [0, a]$

$$\begin{aligned}\mu_\alpha((\mathscr{F}_2D^*)(t)) &\le 2\int_0^t \mu_\alpha(\{T(t-s)f(s, u_n(s), (u_n)_s)\})\mathrm{d}s \\ &\le 2\int_0^t \eta(s)(\mu_\alpha(D^*(s)) + \sup_{-h\le\tau\le 0} \mu_\alpha(D^*(s+\tau)))\mathrm{d}s \\ &\le 4\mu_{C_a^\alpha}(D)\int_0^t \eta(s)\mathrm{d}s \le 4\|\eta\|_{L^1}\mu_{C_a^\alpha}(D).\end{aligned} \tag{3.16}$$

By (3.13)–(3.15), one has

$$\mu_{C_a^\alpha}(\mathscr{F}_2(D)) \le 8\|\eta\|_{L^1}\mu_{C_a^\alpha}(D). \tag{3.17}$$

Therefore, by (3.10), (3.17), the properties of the Kuratowski measure of noncompactness and the condition (P_5), we arrive at

$$\mu_{C_a^\alpha}(\mathscr{F}(D)) \le \mu_{C_a^\alpha}(\mathscr{F}_1(D)) + \mu_{C_a^\alpha}(\mathscr{F}_2(D)) \le \big(\mathcal{H} + 8\|\eta\|_{L^1}\big)\mu_{C_a^\alpha}(D) < \mu_{C_a^\alpha}(D),$$

which means that $\mathscr{F} : D_R(C_a^\alpha) \to D_R(C_a^\alpha)$ is a condensing operator. According to the famous Sadovskii's fixed-point theorem, we know that the operator $\mathscr{F}$ has at least one fixed-point $u \in D_R(C_a^\alpha)$, and this fixed-point is just the mild solution of problem (1.1) in C_a^α. □

In the second half of this section, we discuss the uniqueness of mild solutions to problem (1.1) in α-norm. To do this, we need the following hypothesis:

(P_6) There exist positive constants L_F^1 and L_F^2 such that

$$\|F(t, u_1, v_1) - F(t, u_2, v_2)\| \le L_F^1\|u_1 - u_2\|_\alpha + L_F^2\|v_1 - v_2\|_{C_0^\alpha},$$

where $t \in [0, a]$, $u_1, u_2 \in X_\alpha$ and $v_1, v_2 \in C_0^\alpha$.

Theorem 3.4. *Assume that conditions (P_2), (P_3) and (P_6) are satisfied. Then for every $\phi \in C_0^\alpha$, the problem* (1.1) *has a unique mild solution in C_a^α if*

$$\mathcal{H} + \frac{a^{1-\alpha}M_\alpha(L_F^1 + L_F^2)}{1-\alpha} < 1, \tag{3.18}$$

where $\mathcal{H} = M(L_g + 2N_\beta L_G) + 2N_\beta L_G + \frac{2L_G M_{1-\beta}a^\beta}{\beta}$.

Proof. From the proof of Theorem 3.3, we can see that there exists a positive constant R such that the operator $\mathscr{F}$ defined by (3.3) is well defined and has values in C_a^α. In accordance with Definition 3.2, it is easy to see that the mild solution of problem (1.1) is equivalent to the fixed-point of the operator $\mathscr{F}$ defined by (3.3).

In the following, we will prove the operator $\mathscr{F}$ has a unique fixed-point in C_a^α. For any $u, v \in D_R(C_a^\alpha)$, by the proof of Theorem 3.3, one has

$$\|\mathscr{F}_1 u - \mathscr{F}_1 v\|_{C_a^\alpha} \le \mathcal{H}\|u - v\|_{C_a^\alpha}. \tag{3.19}$$

Furthermore, by the formulation of the operator $\mathscr{F}_2$, Lemma 2.1 and the hypothesis (P_7), we obtain that

$$\begin{aligned}\|(\mathscr{F}_2 u)(t) - (\mathscr{F}_2 v)(t)\|_\alpha &\le \int_0^t \|A^\alpha T(t-s)\|\cdot\|F(s, u(s), u_s) - F(s, v(s), v_s)\|\mathrm{d}s \\ &\le M_\alpha \int_0^t (t-s)^{-\alpha}\Big(L_F^1\|u(s) - v(s)\|_\alpha + L_F^2\|u_s - v_s\|_{C_0^\alpha}\Big)\mathrm{d}s \\ &\le M_\alpha(L_F^1 + L_F^2)\|u - v\|_{C_a^\alpha}\int_0^t (t-s)^{-\alpha}\mathrm{d}s \\ &\le \frac{a^{1-\alpha}M_\alpha(L_F^1 + L_F^2)}{1-\alpha}\|u - v\|_{C_a^\alpha}, \quad t \in [0, a].\end{aligned}$$

Therefore,

$$\|\mathscr{F}_2 u - \mathscr{F}_2 v\|_{C_a^\alpha} \le \frac{a^{1-\alpha}M_\alpha(L_F^1 + L_F^2)}{1-\alpha}\|u - v\|_{C_a^\alpha}. \tag{3.20}$$

Combining with (3.19) and (3.20), we arrive at

$$\|\mathscr{F}u - \mathscr{F}v\|_{C_a^\alpha} \le \|\mathscr{F}_1 u - \mathscr{F}_1 v\|_{C_a^\alpha} + \|\mathscr{F}_2 u - \mathscr{F}_2 v\|_{C_a^\alpha} \le \left\{\mathcal{H} + \frac{a^{1-\alpha}M_\alpha(L_F^1 + L_F^2)}{1-\alpha}\right\}\|u - v\|_{C_a^\alpha}.$$

From the aforementioned inequality and (3.18), we obtain that

$$\|\mathscr{F}u - \mathscr{F}v\|_{C_a^\alpha} < \|u - v\|_{C_a^\alpha}.$$

This illustrates $\mathscr{F} : D_R(C_a^\alpha) \to C_a^\alpha$ a contractive mapping. By using Banach contraction mapping principle, we know that the operator $\mathscr{F}$ has a unique fixed-point u^* in C_a^α, and this fixed-point is the unique mild solution of problem (1.1) in C_a^α. □

4 An example

By using the abstract results obtained in Section 3, we can solve the following neutral partial differential system

$$\begin{cases}\dfrac{\partial}{\partial t}\left[w(t,x) + \displaystyle\int_{t-h}^{t}\int_0^\pi b(s-t, y, x)(w(s,y) + \frac{\partial}{\partial y}w(s,y))\mathrm{d}y\mathrm{d}s\right] - \dfrac{\partial^2}{\partial x^2}w(t,x) \\ \quad = l\left(w(t,x) + \dfrac{\partial}{\partial y}w(t,x)\right) + \displaystyle\int_{t-h}^{t} b_0(s-t)\left(w(s,x) + \frac{\partial}{\partial y}w(s,x)\right)\mathrm{d}s, \quad (t,x) \in [0,a]\times[0,\pi], \\ w(t,0) = w(t,\pi) = 0, \quad t \in [0,a], \\ w(s,x) = \displaystyle\sum_{i=1}^{p}\beta_i\left(w(t_i+s,x) + \frac{\partial}{\partial x}w(t_i+s,x)\right) + \phi(x,s), \quad (s,x) \in [-h,0]\times[0,\pi],\end{cases} \tag{4.1}$$

where p is a positive integer, $0 < t_i < a$, β_i, $i = 1, 2, \dots, p$, are fixed numbers, the functions b and b_0 will be described later.

Let $X = L^2([0,\pi],\mathbb{R})$ be a Banach space with L^2-norm $\|\cdot\|_2$. Define an operator A on X by

$$Af = -f''$$

with the domain

$$D(A) = \{f \in X | f', f'' \in X,\ f(0) = f(\pi) = 0\}.$$

Then $-A$ generates a strong continuous semigroup $T(t)(t \geq 0)$, which is analytic. Furthermore, $-A$ has a discrete spectrum, the eigenvalues are $-n^2$, $n \in \mathbb{N}$, with the corresponding normalized eigenvectors $e_n(x) = \sqrt{\frac{2}{\pi}}\sin(nx)$. Then, the following properties hold:

(a) If $f \in D(A)$, then

$$Af = \sum_{n=1}^{\infty} n^2 \langle f, e_n\rangle e_n.$$

(b) For every $f \in X$,

$$T(t)f = \sum_{n=1}^{\infty} e^{-n^2 t}\langle f, e_n\rangle e_n, \quad t \geq 0, \quad A^{-\frac{1}{2}}f = \sum_{n=1}^{\infty}\frac{1}{n}\langle f, e_n\rangle e_n.$$

In particular, $\|T(t)\| \leq 1$, $\|A^{-\frac{1}{2}}\| = 1$.

(c) The operator $A^{\frac{1}{2}}$ is given by

$$A^{\frac{1}{2}}f = \sum_{n=1}^{\infty} n\langle f, e_n\rangle e_n$$

on the space $D(A^{\frac{1}{2}}) = \{f(\cdot) \in X, \sum_{n=1}^{\infty} n\langle f, e_n\rangle e_n \in X\}$.

(d) For every $f \in X$,

$$A^{\frac{1}{2}}T(t)f = \sum_{n=1}^{\infty} n e^{-n^2 t}\langle f, e_n\rangle e_n, \quad t \geq 0.$$

In particular, $\|A^{\frac{1}{2}}T(t)\| \leq \Gamma\left(\frac{1}{2}\right)t^{-\frac{1}{2}}, \quad t > 0.$

The following lemma is also needed in order to prove our main result of this section.

Lemma 4.1. [10] *If $u \in X_{\frac{1}{2}}$, then u is absolutely continuous with $u' \in X$* and $\|u'\|_2 = \|A^{\frac{1}{2}}u\|_2 = \|u\|_{\frac{1}{2}}$.

For solving the neutral partial differential system (4.1), the following assumptions are needed.

(i) The functions $b(\theta, y, x)$, $\frac{\partial}{\partial x}b(\theta, y, x)$ are measurable, $b(\theta, y, 0) = b(\theta, y, \pi)$ for all (θ, y) and

$$\tilde{c} = \left\{\int_0^{\pi}\int_{-h}^{0}\int_0^{\pi}\left(\frac{\partial^2 b(\theta, y, x)}{\partial x^2}\right)^2 \mathrm{d}y\mathrm{d}\theta\mathrm{d}x\right\}^{\frac{1}{2}} < \infty.$$

(ii) The function $b_0 : \mathbb{R} \to \mathbb{R}$ is continuous and $c := \left(\int_{-h}^{0}(b_0(\theta))^2\mathrm{d}\theta\right)^{\frac{1}{2}} < \infty.$

Now define the abstract functions

$$u(t)(x) = w(t, x), \quad t \in [-h, a],$$

$$(t, \phi)(x) = \int_{-h}^{0}\int_0^{\pi} b(\theta, y, x)(\phi(\theta)(y) + \phi(\theta)'(y))\mathrm{d}y\mathrm{d}\theta, \quad (t, \phi) \in [0, a] \times C_0^{\frac{1}{2}},$$

$$F(t,\varphi,\phi)(x)=l(\varphi(x)+\varphi'(x))+\int_{-h}^{0}b_0(\theta)(\phi(\theta)(x)+\phi(\theta)'(x))\mathrm{d}\theta,\quad (t,\varphi,\phi)\in[0,a]\times X_{\frac{1}{2}}\times C_0^{\frac{1}{2}},$$

$$g(u)(s)(x)=\sum_{i=1}^{p}\beta_i(u(t_i+s)(x)+u(t_i+s)'(x)),\quad s\in[-h,0],\ u\in C_a^{\frac{1}{2}}.$$

Then system (4.1) can be rewritten as the abstract from (1.1). Here, we will verify that G, g and F satisfy the condition (P_2), (P_3) and (P_6), respectively.

For any $t_1,t_2\in[0,a]$, $\phi_1,\phi_2\in C_0^{\frac{1}{2}}$, by the definition of G and assumption (i), we see that $G(\cdot,\cdot)\in D(A)$ and

$$\begin{aligned}\|AG(t_2,\phi_2)-AG(t_1,\phi_1)\|_2^2&=\int_0^{\pi}\int_{-h}^{0}\int_0^{\pi}\left[\frac{\partial^2 b(\theta,y,x)}{\partial x^2}[(\phi_2(\theta)(y)-\phi_1(\theta)(y))+(\phi_2(\theta)'(y)-\phi_1(\theta)'(y))]\right]^2\mathrm{d}y\mathrm{d}\theta\mathrm{d}x\\&=\pi\int_0^{\pi}\int_{-h}^{0}\int_0^{\pi}\left(\frac{\partial^2 b(\theta,y,x)}{\partial x^2}\right)^2\mathrm{d}y\mathrm{d}\theta\mathrm{d}x\int_{-h}^{0}\int_0^{\pi}[(\phi_2(\theta)(y)-\phi_1(\theta)(y))\\&\quad+(\phi_2(\theta)'(y)-\phi_1(\theta)'(y))]^2\mathrm{d}y\mathrm{d}\theta\\&\le\pi(\tilde{c})^2\int_{-h}^{0}(\|\phi_2(\theta)-\phi_1(\theta)\|_2+\|\phi_2(\theta)'-\phi_1(\theta)'\|_2)^2\mathrm{d}\theta\\&=\pi(\tilde{c})^2\int_{-h}^{0}(\|A^{-\frac{1}{2}}\|\|\phi_2(\theta)-\phi_1(\theta)\|_{\frac{1}{2}}+\|\phi_2(\theta)-\phi_1(\theta)\|_{\frac{1}{2}})^2\mathrm{d}\theta\\&\le(2\tilde{c}\sqrt{\pi h}\|\phi_2-\phi_1\|_{C_0^{\frac{1}{2}}})^2.\end{aligned}$$

For any $t\in[0,a]$, $\varphi_1,\varphi_2\in X_{\frac{1}{2}}$, $\phi_1,\phi_2\in C_0^{\frac{1}{2}}$, by the definition of F and assumption (ii), we obtain that $F(\cdot,\cdot,\cdot)\in X$ and

$$\begin{aligned}&\|F(t,\varphi_2,\phi_2)-F(t_1,\varphi_1\phi_1)\|_2^2\\&=\int_0^{\pi}\left[l(\varphi_2(x)-\varphi_2(x))+l(\varphi_2'(x)-\varphi_2'(x))+\int_{-h}^{0}b_0(\theta)[(\phi_2(\theta)(x)-\phi_1(\theta)(x))+(\phi_2(\theta)'(x)-\phi_1(\theta)'(x))]\mathrm{d}\theta\right]^2\mathrm{d}x\\&\le\left[l\|\varphi_2-\varphi_1\|_2+l\|\varphi_2'-\varphi_1'\|_2+\left(\int_{-h}^{0}b_0^2(\theta)\mathrm{d}\theta\right)^{\frac{1}{2}}\left(\int_{-h}^{0}(\|\phi_2(\theta)-\phi_1(\theta)\|_2+\|\phi_2(\theta)'-\phi_1(\theta)'\|_2)^2\mathrm{d}\theta\right)^{\frac{1}{2}}\right]^2\\&\le\left[2l\|\varphi_2-\varphi_1\|_{\frac{1}{2}}+2\sqrt{ch}\|\phi_2-\phi_1\|_{C_0^{\frac{1}{2}}}\right]^2.\end{aligned}$$

For any $s\in[-h,0]$, $u_1,u_2\in C_a^{\frac{1}{2}}$, by the definition of g, we see that $g(\cdot)\in C_0^{\frac{1}{2}}$ and

$$\|g(u_2)(s)-g(u_2)(s)\|_2^2\le\left(2\sum_{i=1}^{p}\beta_i\|u_2-u_1\|_{C_2^{\frac{1}{2}}}\right)^2.$$

Suppose further that

$$2\sum_{i=1}^{p}\beta_i+2\tilde{c}\sqrt{\pi h}(3+2a^{\frac{1}{2}}\Gamma(1/2))+4a^{\frac{1}{2}}(l+\sqrt{ch}\Gamma(1/2))<1.$$

Then system (4.1) exists a unique mild solution follows from Theorem 3.2.

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