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Research Article

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An $\mathcal{H}$-tensor-based criteria for testing the positive definiteness of multivariate homogeneous forms

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Abstract: A positive definite homogeneous multivariate form plays an important role in the field of optimization, and positive definiteness of the form can be identified by a special structured tensor. In this paper, based on the equivalence between the form and the corresponding tensor, and the links of the positive definiteness of a tensor with $\mathcal{H}$-tensor, we propose an $\mathcal{H}$-tensor-based criterion for identifying the positive definiteness of multivariate homogeneous forms. Some numerical examples are provided to illustrate the efficiency and validity of our results.

Keywords: homogeneous multivariate form, positive definiteness, $\mathcal{H}$-tensors, diagonally dominant tensors, irreducible, non-zero element chain

MSC: 15A69, 15A18, 65F15, 65H17

1 Introduction

Let $R(C)$ be the real(complex) field, $N = \{1, 2, \ldots, n\}$. An mth-order n-dimensional real(complex) tensor $\mathcal{A} = (a_{j_1 j_2 \cdots j_m})$ consists of n^m real(complex) entrise:

$$a_{j_1 j_2 \cdots j_m} \in R(C),$$

where $j_i = 1, 2, \ldots, n,\ i = 1, 2, \ldots, m$ [1–6]. Obviously, a matrix is a 2nd-order tensor. Moreover, tensor $\mathcal{A} = (a_{j_1 j_2 \cdots j_m})$ is called symmetric [7] if

$$a_{j_1 j_2 \cdots j_m} = a_{\pi(j_1 j_2 \cdots j_m)}, \quad \forall \pi \in \Pi_m,$$

where Π_m is the permutation group of m indices. If $a_{j_1 j_2 \cdots j_m} \geq 0$, then tensor $\mathcal{A}$ is called a nonnegative tensor. Tensor $\mathcal{I} = (\delta_{j_1 j_2 \cdots j_m})$ is called the unit tensor [8], where

$$\delta_{j_1 j_2 \cdots j_m} = \begin{cases} 1, & \text{if} \quad j_1 = \cdots = j_m, \\ 0, & \text{otherwise.} \end{cases}$$

Denote an m-th degree multivariate homogeneous form of n variables $f(x)$ as follows:

$$f(x) = \sum_{j_1, j_2, \ldots, j_m \in N} a_{j_1 j_2 \cdots j_m} x_{j_1} x_{j_2} \cdots x_{j_m}, \tag{1}$$

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where $x \in R^n$. While m is even, $f(t)$ is labeled positive definite if

$$f(t) > 0, \quad \text{for any } t \in R^n, t \neq 0.$$

Function $f(x)$ in (1) can be expressed as the tensor product of an mth-order n-dimensional symmetric tensor $\mathcal{A}$ and x^m defined as follows:

$$f(x) \equiv \mathcal{A}x^m = \sum_{j_1, j_2, \ldots, j_m \in N} a_{j_1 j_2 \cdots j_m} x_{j_1} x_{j_2} \cdots x_{j_m}, \tag{2}$$

where $x \in R^n$ and x^m is an mth-order n-dimensional rank-one tensor with entries $x_{j_1} \cdots x_{j_m}$ [5]. The symmetric tensor $\mathcal{A}$ is positive definite if $f(x)$ is positive definite [9].

The positive definiteness of tensor has received much attention of researchers' in recent decade [10–12]. By the Sturm theorem, the positive definiteness of a multivariate homogeneous form can be identified for $n \leq 3$ [13]. Ni et al. [9] presented an eigenvalue method for checking positive definiteness of a symmetric tensor. While, all the eigenvalues should be calculated in this method, and it is not practical when tensor order or dimension is large.

Recently, Li et al. [14] provided an $\mathcal{H}$-tensor-based method for identifying the positive definiteness of an even-order symmetric tensor. It is well known that an even-order symmetric $\mathcal{H}$-tensor with positive diagonal entries is positive definite. Hence, we can check the positive definiteness of a tensor via the aid of $\mathcal{H}$-tensor. Subsequently, with the help of generalized diagonal dominance, miscellaneous criterions for $\mathcal{H}$-tensors and $\mathcal{M}$-tensors are provided [15–20], which depends on the entries of tensors and is efficient to determine $\mathcal{H}$-tensor ($\mathcal{M}$-tensor).

Theorem 1. [20] *Let $\mathcal{A} = (a_{j_1 \cdots j_m}) \in C^{[m,n]}$. If*

$$|a_{jj\cdots j}|r_j > \sum_{j_2 j_3 \cdots j_m \in N_0^{m-1}} |a_{jj_2\cdots j_m}| + \sum_{\substack{j_2 j_3 \cdots j_m \in N_2^{m-1} \\ \delta_{jj_2\cdots j_m}=0}} \max_{t \in \{j_2, j_3, \ldots, j_m\}} \{r_t\} |a_{jj_2\cdots j_m}|$$
$$+ \sum_{j_2 j_3 \cdots j_m \in N_3^{m-1}} \max_{t \in \{j_2, j_3, \ldots, j_m\}} \{l_t\} |a_{jj_2\cdots j_m}|, \quad \forall j \in N_2$$

and

$$|a_{ii\cdots i}| \neq \sum_{\substack{i_2 i_3 \cdots i_m \in N_0^{m-1} \\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}|, \quad \forall i \in N_1 \neq \varnothing \quad (or\ N_1 = \varnothing),$$

then $\mathcal{A}$ is an $\mathcal{H}$-tensor.

In this paper, we continue to present new criteria for $\mathcal{H}$-tensors. These new results improve the corresponding conclusions [20–22]. As an application, some sufficient conditions of the positive definiteness for an even-order real symmetric tensor are given. Several numerical experiments demonstrate its efficiency.

Several specially definitions extended from matrices are as follows.

Definition 1. [17] Let $\mathcal{A} = (a_{j_1 j_2 \cdots j_m})$ be a complex tensor of order m dimension n. $\mathcal{A}$ is an $\mathcal{H}$-tensor if there exists a positive vector $x = (x_1, x_2, \ldots, x_n)^T \in R^n$, such that

$$|a_{jj\cdots j}| x_j^{m-1} > \sum_{\substack{j_2, \ldots, j_m \in N \\ \delta_{jj_2\cdots j_m}=0}} |a_{jj_2\cdots j_m}| x_{j_2} \cdots x_{j_m}, \quad \forall j \in N.$$

Definition 2. [8] A complex tensor $\mathcal{A} = (a_{j_1 j_2 \cdots j_m})$ of order m dimension n is called reducible if there is a nonempty subset $I \subset N$, such that

$$a_{j_1 j_2 \cdots j_m} = 0, \quad \forall j_1 \in I, \quad \forall j_2, \ldots, j_m \notin I.$$

Otherwise, $\mathcal{A}$ is called irreducible.

Definition 3. [18] Let $\mathcal{A} = (a_{j_1 j_2 \cdots j_m})$ be a complex tensor of order m dimension n. For $i, j \in N(i \neq j)$, if there are indices $k_1, k_2, \ldots, k_r$, such that

$$\sum_{\substack{j_2, \ldots, j_m \in N \\ \delta_{k_s j_2 \ldots j_m}=0, \quad k_{s+1} \in \{j_2, \ldots, j_m\}}} |a_{k_s j_2 \cdots j_m}| \neq 0, \quad s = 0, 1, \ldots, r.$$

where $k_0 = j$, $k_{r+1} = i$, and then there exists a nonzero elements chain from j to i.

The remainder of this paper is as follows. In Section 2, we gives some criteria for the identification of $\mathcal{H}$-tensors. In Section 3, some new conditions for the identification of positive definite tensors are presented. Some numerical experiments are also provided to show the effectiveness of new methods.

2 Criteria for $\mathcal{H}$-tensors

In this section, three new criteria for identifying $\mathcal{H}$-tensors are presented. First, for the convenience of description, some notation and lemmas are given. For a tensor $\mathcal{A} = (a_{j_1 j_2 \cdots j_m})$ of order m dimension n, denote

$$\begin{aligned}
S^{m-1} &= \{j_2 j_3 \cdots j_m : j_i \in S, i = 2, 3, \cdots, m\}, \quad \varnothing \neq S \subset N;\\
N^{m-1} \backslash S^{m-1} &= \{j_2 j_3 \cdots j_m : j_2 j_3 \cdots j_m \in N^{m-1} \quad \text{and} \quad j_2 j_3 \cdots j_m \notin S^{m-1}\};\\
N_0^{m-1} &= N^{m-1} \backslash (N_2^{m-1} \cup N_3^{m-1});\\
\Lambda_j(\mathcal{A}) &= \sum_{\substack{j_2, \ldots, j_m \in N \\ \delta_{jj_2 \ldots j_m}=0}} |a_{jj_2 \cdots j_m}| = \sum_{j_2, \ldots, j_m \in N} |a_{jj_2 \cdots j_m}| - |a_{jj \cdots j}|;\\
N_1 &= \{j \in N : 0 < |a_{jj \cdots j}| = \Lambda_j(\mathcal{A})\};\\
N_2 &= \{j \in N : 0 < |a_{jj \cdots j}| < \Lambda_j(\mathcal{A})\};\\
N_3 &= \{j \in N : |a_{jj \cdots j}| > \Lambda_j(\mathcal{A})\};\\
w_j &= \frac{\Lambda_j(\mathcal{A})}{\Lambda_j(\mathcal{A}) + |a_{jj \cdots j}|}, \quad w = \max\{w_j\}, \quad \forall j \in N_2;\\
r &= \max_{j \in N_3} \left(\frac{\sum_{j_2 \cdots j_m \in N_0^{m-1}} |a_{jj_2 \cdots j_m}| + \sum_{j_2 \cdots j_m \in N_2^{m-1}} |a_{jj_2 \cdots j_m}|}{|a_{jj \cdots j}| - \sum_{\substack{j_2 \cdots j_m \in N_3^{m-1} \\ \delta_{jj_2 \ldots j_m}=0}} |a_{jj_2 \cdots j_m}|} \right);\\
P_{j,r}(\mathcal{A}) &= \sum_{j_2 \cdots j_m \in N_0^{m-1}} |a_{jj_2 \cdots j_m}| + \sum_{j_2 \cdots j_m \in N_2^{m-1}} |a_{jj_2 \cdots j_m}| + r \sum_{\substack{j_2 \cdots j_m \in N_3^{m-1} \\ \delta_{jj_2 \ldots j_m}=0}} |a_{jj_2 \cdots j_m}|, \quad j \in N_3;\\
r_1 &= \max_{j \in N_3} \left(\frac{\sum_{j_2 \cdots j_m \in N_0^{m-1}} |a_{jj_2 \cdots j_m}| + \sum_{j_2 \cdots j_m \in N_2^{m-1}} |a_{jj_2 \cdots j_m}|}{|a_{jj \cdots j}| - \sum_{\substack{j_2 \cdots j_m \in N_3^{m-1} \\ \delta_{jj_2 \ldots j_m}=0}} \max_{t \in \{j_2, j_3, \ldots, j_m\}} \frac{P_{t,r}(\mathcal{A})}{|a_{tt \cdots t}|} |a_{jj_2 \cdots j_m}|} \right);\\
\mu &= \max\{w_j, r_1\}, \quad j \in N_2;\\
P_{j,r_1}(\mathcal{A}) &= \sum_{j_2 \cdots j_m \in N_0^{m-1}} |a_{jj_2 \cdots j_m}| + \sum_{j_2 \cdots j_m \in N_2^{m-1}} |a_{jj_2 \cdots j_m}| + r_1 \sum_{\substack{j_2 \cdots j_m \in N_3^{m-1} \\ \delta_{jj_2 \ldots j_m}=0}} \max_{t \in \{j_2, j_3, \ldots, j_m\}} \frac{P_{t,r}(\mathcal{A})}{|a_{tt \cdots t}|} |a_{jj_2 \cdots j_m}|, \quad j \in N_3;\\
h &= \max_{j \in N_3} \left(\frac{\mu \sum_{j_2 \cdots j_m \in N_0^{m-1}} |a_{jj_2 \cdots j_m}| + \sum_{j_2 \cdots j_m \in N_2^{m-1}} |a_{jj_2 \cdots j_m}| w}{P_{j,r_1}(\mathcal{A}) - \sum_{\substack{j_2 \cdots j_m \in N_3^{m-1} \\ \delta_{jj_2 \ldots j_m}=0}} \max_{t \in \{j_2, j_3, \ldots, j_m\}} \frac{P_{t,r_1}(\mathcal{A})}{|a_{tt \cdots t}|} |a_{jj_2 \cdots j_m}|} \right).
\end{aligned}$$

Throughout this paper, we assume that $N_1 \neq \emptyset$ and $N_2 \neq \emptyset$. Meanwhile, we also assume that tensor $\mathcal{A}$ satisfies: $a_{jj\cdots j} \neq 0$, $\Lambda_j(\mathcal{A}) \neq 0$, $\forall j \in N$.

Lemma 1. [17] *If tensor $\mathcal{A} = (a_{j_1 j_2 \cdots j_m})$ is strictly diagonally dominant, then $\mathcal{A}$ is an $\mathcal{H}$-tensor.*

Lemma 2. [14] *Let $\mathcal{A} = (a_{j_1 j_2 \cdots j_m})$ be an mth-order n-dimensional complex tensor. If there is a positive diagonal matrix X such that $\mathcal{A}X^{m-1}$ is an $\mathcal{H}$-tensor, then $\mathcal{A}$ is an $\mathcal{H}$-tensor.*

Lemma 3. [14] *Let $\mathcal{A} = (a_{j_1 j_2 \cdots j_m})$ be an mth-order n-dimensional complex tensor. If $\mathcal{A}$ is irreducible,*

$$|a_{jj\cdots j}| \geq \Lambda_j(\mathcal{A}), \quad \forall j \in N,$$

and $N_3(\mathcal{A}) \neq \emptyset$, then $\mathcal{A}$ is an $\mathcal{H}$-tensor.

Lemma 4. [18] *Let $\mathcal{A} = ((a_{j_1 j_2 \cdots j_m}))$ be an mth-order n-dimensional complex tensor. If*
(i) $|a_{jj\cdots j}| \geq \Lambda_j(\mathcal{A}), \quad \forall j \in N$;
(ii) $N_3 = \{j \in N : |a_{jj\cdots j}| > \Lambda_j(\mathcal{A})\} \neq \emptyset$;
(iii) *For $\forall j \notin N_1$, there exists a nonzero elements chain from j to i such that $i \in N_1$,*

then $\mathcal{A}$ is an $\mathcal{H}$-tensor.

Next, we give some new criteria for $\mathcal{H}$-tensors.

Theorem 2. *Let $\mathcal{A} = (a_{j_1 j_2 \cdots j_m})$ be an mth-order n-dimensional complex tensor. If $\mathcal{A}$ satisfies*

$$|a_{jj\cdots j}| > \frac{\mu}{w_j} \sum_{j_2 j_3 \cdots j_m \in N_0^{m-1}} |a_{jj_2\cdots j_m}| + \sum_{\substack{j_2 j_3 \cdots j_m \in N_2^{m-1} \\ \delta_{jj_2\cdots j_m}=0}} |a_{jj_2\cdots j_m}| \frac{w}{w_j} + \frac{h}{w_j} \sum_{j_2 j_3 \cdots j_m \in N_3^{m-1}} \max_{t \in \{j_2, j_3, \ldots, j_m\}} \frac{P_{t,r_1}(\mathcal{A})}{|a_{tt\cdots t}|} |a_{jj_2\cdots j_m}|, \ \forall j \in N_2, \tag{3}$$

and there exists $j_2 j_3 \cdots j_m \in N^{m-1} \backslash N_1^{m-1}$ for any $j \in N_1$ such that $a_{jj_2 j_3 \cdots j_m} \neq 0$, then $\mathcal{A}$ is an $\mathcal{H}$-tensor.

Proof. By $0 \leq r_1 \leq r < 1$, according to the definition of r, $P_{i,r}(\mathcal{A})$, r_1 and $P_{i,r_1}(\mathcal{A})$, for any $i \in N_3$,

$$r_1 |a_{ii\cdots i}| \geq \sum_{i_2 \cdots i_m \in N_0^{m-1}} |a_{ii_2\cdots i_m}| + \sum_{i_2 \cdots i_m \in N_2^{m-1}} |a_{ii_2\cdots i_m}| + r_1 \sum_{\substack{i_2 \ldots i_m \in N_3^{m-1} \\ \delta_{ii_2\ldots i_m}=0}} \max_{j \in \{i_2, i_3, \ldots, i_m\}} \frac{P_{j,r}(\mathcal{A})}{|a_{jj\cdots j}|} |a_{ii_2\cdots i_m}|,$$

that is, $P_{i,r_1}(\mathcal{A}) \leq r_1 |a_{ii\cdots i}| (\forall i \in N_3)$, so

$$0 < \frac{P_{i,r_1}(\mathcal{A})}{|a_{ii\cdots i}|} \leq r_1 \leq r < 1, \quad \forall i \in N_3.$$

By the definitions of w_i and μ, for any $i \in N_3$,

$$\begin{aligned} &\frac{\mu \sum_{i_2 \cdots i_m \in N_0^{m-1}} |a_{ii_2\cdots i_m}| + \sum_{i_2 \cdots i_m \in N_2^{m-1}} |a_{ii_2\cdots i_m}| w}{P_{i,r_1}(\mathcal{A}) - \sum_{\substack{i_2 \ldots i_m \in N_3^{m-1} \\ \delta_{ii_2\ldots i_m}=0}} \max_{j \in \{i_2, i_3, \ldots, i_m\}} \frac{P_{j,r_1}(\mathcal{A})}{|a_{jj\cdots j}|} |a_{ii_2\cdots i_m}|} \\ &\leq \frac{\mu \left(P_{i,r_1}(\mathcal{A}) - r_1 \sum_{\substack{i_2 \cdots i_m \in N_3^{m-1} \\ \delta_{ii_2\ldots i_m}=0}} \max_{j \in \{i_2, i_3, \ldots, i_m\}} \frac{P_{j,r}(\mathcal{A})}{|a_{jj\cdots j}|} |a_{ii_2\cdots i_m}| \right)}{P_{i,r_1}(\mathcal{A}) - \sum_{\substack{i_2 \ldots i_m \in N_3^{m-1} \\ \delta_{ii_2\ldots i_m}=0}} \max_{j \in \{i_2, i_3, \ldots, i_m\}} \frac{P_{j,r_1}(\mathcal{A})}{|a_{jj\cdots j}|} |a_{ii_2\cdots i_m}|} \\ &< 1. \end{aligned}$$

For $\forall i \in N_3$, from the definition of h, we have $0 < h < 1$, and

$$hP_{i,r_1}(\mathcal{A}) \geq \mu \sum_{i_2 \cdots i_m \in N_0^{m-1}} |a_{ii_2\cdots i_m}| + \sum_{i_2 \cdots i_m \in N_2^{m-1}} |a_{ii_2\cdots i_m}| w + h \sum_{\substack{i_2 \ldots i_m \in N_3^{m-1} \\ \delta_{ii_2\ldots i_m}=0}} \max_{j \in \{i_2, i_3, \ldots, i_m\}} \frac{P_{j,r_1}(\mathcal{A})}{|a_{jj\cdots j}|} |a_{ii_2\cdots i_m}|. \tag{4}$$

By inequality (3), for all $i \in N_2$, we have

$$w_i|a_{ii\cdots i}| > \mu \sum_{i_2 i_3\cdots i_m \in N_0^{m-1}} |a_{ii_2\cdots i_m}| + \sum_{\substack{i_2 i_3\cdots i_m \in N_2^{m-1}\\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}|w + h \sum_{i_2 i_3\cdots i_m \in N_3^{m-1}} \max_{j\in\{i_2,i_3,\ldots,i_m\}} \frac{P_{j,r_1}(\mathcal{A})}{|a_{jj\cdots j}|}|a_{ii_2\cdots i_m}|.$$

Let

$$\begin{aligned} R_j \equiv \frac{1}{\sum_{j_2 j_3\cdots j_m\in N_3^{m-1}}|a_{jj_2\cdots j_m}|}\Bigg[w_j|a_{jj\cdots j}| - \Bigg(\mu \sum_{j_2 j_3\cdots j_m\in N_0^{m-1}}|a_{jj_2\cdots j_m}| + \sum_{\substack{j_2 j_3\cdots j_m\in N_2^{m-1}\\ \delta_{jj_2\cdots j_m}=0}}|a_{jj_2\cdots j_m}|w \\ + h\sum_{j_2 j_3\cdots j_m\in N_3^{m-1}} \max_{t\in\{j_2,j_3,\ldots,j_m\}}\frac{P_{t,r_1}(\mathcal{A})}{|a_{tt\cdots t}|}|a_{jj_2\cdots j_m}|\Bigg)\Bigg], \quad \forall j\in N. \end{aligned} \tag{5}$$

If $\sum_{j_2 j_3\cdots j_m\in N_3^{m-1}}|a_{jj_2\cdots j_m}| = 0$, we denote $R_j = +\infty$. By Equality (5), we have

$$R_j > 0(\forall j\in N_2), \quad 0 < \frac{hP_{t,r_1}(\mathcal{A})}{|a_{tt\cdots t}|} < 1(\forall t\in N_3).$$

Hence, there exists a positive number ε such that

$$0<\varepsilon<\min_{j\in N_2} R_j \le +\infty, \quad \max_{t\in N_3}\left\{\frac{hP_{t,r_1}(\mathcal{A})}{|a_{tt\cdots t}|}+\varepsilon\right\}<1.$$

Let the matrix $D = \operatorname{diag}(d_1, d_2, \ldots, d_n)$, denote $\mathcal{B} = \mathcal{A}D^{m-1} = (b_{i_1 i_2\cdots i_m})$, where

$$d_i = \begin{cases} (\mu)^{\frac{1}{m-1}}, & i\in N_1,\\ (w_i)^{\frac{1}{m-1}}, & i\in N_2,\\ \left(\varepsilon + \frac{hP_{i,r_1}(\mathcal{A})}{|a_{ii\cdots i}|}\right)^{\frac{1}{m-1}}, & i\in N_3. \end{cases}$$

For $\forall j\in N_1$, there exists $j_2 j_3\cdots j_m \in N^{m-1}\backslash N_1^{m-1}$ such that $a_{jj_2 j_3\cdots j_m}\ne 0$, and for $\forall t\in N_3$, let $\varepsilon>0$ satisfy $0<\varepsilon+\frac{hP_{t,r_1}(\mathcal{A})}{|a_{tt\cdots t}|}<\mu<1$, we obtain

$$\begin{aligned} \Lambda_j(\mathcal{B}) &= \mu \sum_{\substack{j_2\cdots j_m\in N_0^{m-1}\\ \delta_{jj_2\cdots j_m}=0}}|a_{jj_2\cdots j_m}| + \sum_{j_2\cdots j_m\in N_2^{m-1}}|a_{jj_2\cdots j_m}|w + \sum_{j_2\cdots j_m\in N_3^{m-1}}|a_{jj_2\cdots j_m}|\left(\varepsilon+\frac{hP_{j_2,r_1}(\mathcal{A})}{|a_{j_2 j_2\cdots j_2}|}\right)^{\frac{1}{m-1}}\cdots \\ &\quad \left(\varepsilon+\frac{hP_{j_m,r_1}(\mathcal{A})}{|a_{j_m j_m\cdots j_m}|}\right)^{\frac{1}{m-1}} \\ &\le \mu \sum_{\substack{j_2\cdots j_m\in N_0^{m-1}\\ \delta_{jj_2\cdots j_m}=0}}|a_{jj_2\cdots j_m}| + \sum_{j_2\cdots j_m\in N_2^{m-1}}|a_{jj_2\cdots j_m}|w + \sum_{j_2\cdots j_m\in N_3^{m-1}}|a_{jj_2\cdots j_m}|\left(\varepsilon+\max_{t\in\{j_2,j_3,\ldots,j_m\}}\frac{hP_{t,r_1}(\mathcal{A})}{|a_{tt\cdots t}|}\right) \\ &< \mu\left(\sum_{\substack{j_2\cdots j_m\in N_0^{m-1}\\ \delta_{jj_2\cdots j_m}=0}}|a_{jj_2\cdots j_m}| + \sum_{j_2\cdots j_m\in N_2^{m-1}}|a_{jj_2\cdots j_m}| + \sum_{j_2\cdots j_m\in N_3^{m-1}}|a_{jj_2\cdots j_m}|\right) = \mu\Lambda_j(\mathcal{A}) = \mu|a_{jj\cdots j}| = |b_{jj\cdots j}|. \end{aligned}$$

For $\forall j \in N_2$, by equality (5), we have

$$\begin{aligned}
\Lambda_j(\mathcal{B}) &= \mu \sum_{j_2\cdots j_m \in N_0^{m-1}} |a_{jj_2\cdots j_m}| + \sum_{\substack{j_2\cdots j_m \in N_2^{m-1}\\ \delta_{jj_2\cdots j_m}=0}} |a_{jj_2\cdots j_m}|w + \sum_{j_2\cdots j_m \in N_3^{m-1}} |a_{jj_2\cdots j_m}|\left(\varepsilon + \frac{hP_{j_2,r_1}(\mathcal{A})}{|a_{j_2j_2\cdots j_2}|}\right)^{\frac{1}{m-1}} \cdots \\
&\quad \left(\varepsilon + \frac{hP_{j_m,r_1}(\mathcal{A})}{|a_{j_mj_m\cdots j_m}|}\right)^{\frac{1}{m-1}} \\
&\le \mu \sum_{j_2\cdots j_m \in N_0^{m-1}} |a_{jj_2\cdots j_m}| + \sum_{\substack{j_2\cdots j_m \in N_2^{m-1}\\ \delta_{jj_2\cdots j_m}=0}} |a_{jj_2\cdots j_m}|w + \sum_{j_2\cdots j_m \in N_3^{m-1}} |a_{jj_2\cdots j_m}|\left(\varepsilon + \max_{t\in\{j_2,j_3,\ldots,j_m\}} \frac{hP_{t,r_1}(\mathcal{A})}{|a_{tt\cdots t}|}\right) \\
&= \varepsilon \sum_{j_2j_3\cdots j_m \in N_3^{m-1}} |a_{jj_2\cdots j_m}| + \mu \sum_{j_2j_3\cdots j_m \in N_0^{m-1}} |a_{jj_2\cdots j_m}| + \sum_{\substack{j_2j_3\cdots j_m \in N_2^{m-1}\\ \delta_{jj_2\cdots j_m}=0}} |a_{jj_2\cdots j_m}|w \\
&\quad + h \sum_{j_2j_3\cdots j_m \in N_3^{m-1}} \max_{t\in\{j_2,j_3,\ldots,j_m\}} \frac{P_{t,r_1}(\mathcal{A})}{|a_{tt\cdots t}|}|a_{jj_2\cdots j_m}| \\
&< R_j \sum_{j_2j_3\cdots j_m \in N_3^{m-1}} |a_{jj_2\cdots j_m}| + \mu \sum_{j_2j_3\cdots j_m \in N_0^{m-1}} |a_{jj_2\cdots j_m}| + \sum_{\substack{j_2j_3\cdots j_m \in N_2^{m-1}\\ \delta_{jj_2\cdots j_m}=0}} |a_{jj_2\cdots j_m}|w \\
&\quad + h \sum_{j_2j_3\cdots j_m \in N_3^{m-1}} \max_{t\in\{j_2,j_3,\ldots,j_m\}} \frac{P_{t,r_1}(\mathcal{A})}{|a_{tt\cdots t}|}|a_{jj_2\cdots j_m}| \\
&= w_j|a_{jj\ldots j}| = |b_{jj\ldots j}|.
\end{aligned}$$

Finally, for $j \in N_3$, by inequality (4), we have

$$\begin{aligned}
\Lambda_j(\mathcal{B}) &= \mu \sum_{j_2\cdots j_m \in N_0^{m-1}} |a_{jj_2\cdots j_m}| + \sum_{j_2\cdots j_m \in N_2^{m-1}} |a_{jj_2\cdots j_m}|w \\
&\quad + \sum_{\substack{j_2\cdots j_m \in N_3^{m-1}\\ \delta_{jj_2\cdots j_m}=0}} |a_{jj_2\cdots j_m}|\left(\varepsilon + \frac{hP_{j_2,r_1}(\mathcal{A})}{|a_{j_2j_2\cdots j_2}|}\right)^{\frac{1}{m-1}} \cdots \left(\varepsilon + \frac{hP_{j_m,r_1}(\mathcal{A})}{|a_{j_mj_m\cdots j_m}|}\right)^{\frac{1}{m-1}} \\
&\le \mu \sum_{j_2\cdots j_m \in N_0^{m-1}} |a_{jj_2\cdots j_m}| + \sum_{j_2\cdots j_m \in N_2^{m-1}} |a_{jj_2\cdots j_m}|w + \sum_{\substack{j_2\cdots j_m \in N_3^{m-1}\\ \delta_{jj_2\cdots j_m}=0}} |a_{jj_2\cdots j_m}|\left(\varepsilon + \max_{t\in\{j_2,j_3,\ldots,j_m\}} \frac{hP_{t,r_1}(\mathcal{A})}{|a_{tt\cdots t}|}\right) \\
&= \varepsilon \sum_{\substack{j_2j_3\cdots j_m \in N_3^{m-1}\\ \delta_{jj_2\cdots j_m}=0}} |a_{jj_2\cdots j_m}| + \mu \sum_{i_2i_3\cdots j_m \in N_0^{m-1}} |a_{jj_2\cdots j_m}| + \sum_{j_2j_3\cdots j_m \in N_2^{m-1}} |a_{jj_2\cdots j_m}|w \\
&\quad + h \sum_{\substack{j_2j_3\cdots j_m \in N_3^{m-1}\\ \delta_{jj_2\cdots j_m}=0}} \max_{t\in\{j_2,j_3,\ldots,j_m\}} \frac{P_{t,r_1}(\mathcal{A})}{|a_{tt\cdots t}|}|a_{jj_2\cdots j_m}| \\
&\le \varepsilon \sum_{\substack{j_2j_3\cdots j_m \in N_3^{m-1}\\ \delta_{jj_2\cdots j_m}=0}} |a_{jj_2\cdots j_m}| + hP_{j,r_1}(\mathcal{A}) < \varepsilon|a_{jj\ldots j}| + hP_{j,r_1}(\mathcal{A}) = |b_{jj\ldots j}|.
\end{aligned}$$

Therefore, $|b_{jj\ldots j}| > \Lambda_j(\mathcal{B})(\forall i \in N)$. So $\mathcal{B}$ is an $\mathcal{H}$-tensor by Lemma 1. From Lemma 2, $\mathcal{A}$ is an $\mathcal{H}$-tensor. □

Remark 1. It is hard to theoretically give the comparison between our result and the known ones in [20]. The following numerical example illustrates that our proposed criteria is more effective to theirs in some cases. Moreover, we provide an example that satisfies our conditions in Theorem 2 but not those in Theorem 1 of [20].

Let

$$K(\mathcal{A})=\left\{j\in N_2:|a_{jj\cdots j}|>\frac{\mu}{w_j}\sum_{j_2j_3\cdots j_m\in N_0^{m-1}}|a_{jj_2\cdots j_m}|+\sum_{\substack{j_2j_3\cdots j_m\in N_2^{m-1}\\ \delta_{jj_2\cdots j_m}=0}}|a_{jj_2\cdots j_m}|\frac{w}{w_j}\right.$$
$$\left.+\frac{h}{w_j}\sum_{j_2j_3\cdots j_m\in N_3^{m-1}}\max_{t\in\{j_2,j_3,\ldots,j_m\}}\frac{P_{t,r_1}(\mathcal{A})}{|a_{tt\cdots t}|}|a_{jj_2\cdots j_m}|\right\}.$$

Theorem 3. *Let $\mathcal{A}=(a_{j_1j_2\cdots j_m})$ be an mth-order n-dimensional complex tensor. If $\mathcal{A}$ is irreducible,*

$$\begin{aligned}|a_{jj\cdots j}|\geq&\frac{\mu}{w_j}\sum_{j_2j_3\cdots j_m\in N_0^{m-1}}|a_{jj_2\cdots j_m}|+\sum_{\substack{j_2j_3\cdots j_m\in N_2^{m-1}\\ \delta_{jj_2\cdots j_m}=0}}|a_{jj_2\cdots j_m}|\frac{w}{w_j}\\&+\frac{h}{w_j}\sum_{j_2j_3\cdots j_m\in N_3^{m-1}}\max_{t\in\{j_2,j_3,\ldots,j_m\}}\frac{P_{t,r_1}(\mathcal{A})}{|a_{tt\cdots t}|}|a_{jj_2\cdots j_m}|,\quad\forall j\in N_2,\end{aligned}\tag{6}$$

and $K(\mathcal{A})\neq\varnothing$, then $\mathcal{A}$ is an $\mathcal{H}$-tensor.

Proof. Since $\mathcal{A}$ is irreducible, so

$$\sum_{i_2i_3\cdots i_m\in N^{m-1}\setminus N_3^{m-1}}|a_{ii_2\cdots i_m}|>0,\quad\forall i\in N_3.$$

By inequality (6), we obtain

$$|a_{ii\cdots i}|w_i\geq\mu\sum_{i_2i_3\cdots i_m\in N_0^{m-1}}|a_{ii_2\cdots i_m}|+\sum_{\substack{i_2i_3\cdots i_m\in N_2^{m-1}\\ \delta_{ii_2\cdots i_m}=0}}|a_{ii_2\cdots i_m}|w+h\sum_{i_2i_3\cdots i_m\in N_3^{m-1}}\max_{j\in\{i_2,i_3,\ldots,i_m\}}\frac{P_{j,r_1}(\mathcal{A})}{|a_{jj\cdots j}|}|a_{ii_2\cdots i_m}|,\quad\forall i\in N_2.\tag{7}$$

and at least one strict inequality in (7) holds.

Let the matrix $D=\operatorname{diag}(d_1,d_2,\ldots,d_n)$, denote $\mathcal{B}=\mathcal{A}D^{m-1}=(b_{i_1i_2\cdots i_m})$, where

$$d_i=\begin{cases}(\mu)^{\frac{1}{m-1}}, & i\in N_1,\\ (w_i)^{\frac{1}{m-1}}, & i\in N_2,\\ \left(\dfrac{hP_{i,r_1}(\mathcal{A})}{|a_{ii\cdots i}|}\right)^{\frac{1}{m-1}}, & i\in N_3.\end{cases}$$

For $\forall i\in N_1$, by $\mu>\frac{hP_{i,r_1}(\mathcal{A})}{|a_{ii\cdots i}|}(\forall i\in N_3)$, we have

$$\begin{aligned}\Lambda_i(\mathcal{B})=&\mu\sum_{\substack{i_2\cdots i_m\in N_0^{m-1}\\ \delta_{ii_2\cdots i_m}=0}}|a_{ii_2\cdots i_m}|+\sum_{i_2\cdots i_m\in N_2^{m-1}}|a_{ii_2\cdots i_m}|w+\sum_{i_2\cdots i_m\in N_3^{m-1}}|a_{ii_2\cdots i_m}|\left(\frac{hP_{i_2,r_1}(\mathcal{A})}{|a_{i_2i_2\cdots i_2}|}\right)^{\frac{1}{m-1}}\cdots\left(\frac{hP_{i_m,r_1}(\mathcal{A})}{|a_{i_mi_m\cdots i_m}|}\right)^{\frac{1}{m-1}}\\ \leq&\mu\sum_{\substack{i_2\cdots i_m\in N_0^{m-1}\\ \delta_{ii_2\cdots i_m}=0}}|a_{ii_2\cdots i_m}|+\sum_{i_2\cdots i_m\in N_2^{m-1}}|a_{ii_2\cdots i_m}|w+\sum_{i_2\cdots i_m\in N_3^{m-1}}\max_{j\in\{i_2,i_3,\ldots,i_m\}}\frac{hP_{j,r_1}(\mathcal{A})}{|a_{jj\cdots j}|}|a_{ii_2\cdots i_m}|\\ <&\mu\left(\sum_{\substack{i_2\cdots i_m\in N_0^{m-1}\\ \delta_{ii_2\cdots i_m}=0}}|a_{ii_2\cdots i_m}|+\sum_{i_2\cdots i_m\in N_2^{m-1}}|a_{ii_2\cdots i_m}|+\sum_{i_2\cdots i_m\in N_3^{m-1}}|a_{ii_2\cdots i_m}|\right)=\mu|a_{ii\cdots i}|=|b_{ii\cdots i}|.\end{aligned}$$

For $\forall i \in N_2$, by inequality (7), we obtain

$$\begin{aligned}\Lambda_i(\mathcal{B}) &= \mu \sum_{i_2\cdots i_m\in N_0^{m-1}} |a_{ii_2\cdots i_m}| + \sum_{\substack{i_2\cdots i_m\in N_2^{m-1}\\ \delta_{ii_2\ldots i_m}=0}} |a_{ii_2\cdots i_m}|w + \sum_{i_2\ldots i_m\in N_3^{m-1}} |a_{ii_2\cdots i_m}|\left(\frac{hP_{i_2,r_1}(\mathcal{A})}{|a_{i_2i_2\cdots i_2}|}\right)^{\frac{1}{m-1}}\cdots\left(\frac{hP_{i_m,r_1}(\mathcal{A})}{|a_{i_mi_m\cdots i_m}|}\right)^{\frac{1}{m-1}}\\
&\le \mu \sum_{i_2\cdots i_m\in N_0^{m-1}} |a_{ii_2\cdots i_m}| + \sum_{\substack{i_2\cdots i_m\in N_2^{m-1}\\ \delta_{ii_2\ldots i_m}=0}} |a_{ii_2\cdots i_m}|w + \sum_{i_2\ldots i_m\in N_3^{m-1}} \max_{j\in\{i_2,i_3,\ldots,i_m\}} \frac{hP_{j,r_1}(\mathcal{A})}{|a_{jj\ldots j}|}|a_{ii_2\cdots i_m}|\\
&\le w_i|a_{ii\ldots i}| = |b_{ii\ldots i}|.\end{aligned}$$

Next, we consider $i \in N_3$, by inequality (4), we obtain

$$\begin{aligned}\Lambda_i(\mathcal{B}) &= \mu \sum_{i_2\cdots i_m\in N_0^{m-1}} |a_{ii_2\cdots i_m}| + \sum_{i_2\cdots i_m\in N_2^{m-1}} |a_{ii_2\cdots i_m}|w + \sum_{\substack{i_2\ldots i_m\in N_3^{m-1}\\ \delta_{ii_2\ldots i_m}=0}} |a_{ii_2\cdots i_m}|\left(\frac{hP_{i_2,r_1}(\mathcal{A})}{|a_{i_2i_2\cdots i_2}|}\right)^{\frac{1}{m-1}}\cdots\left(\frac{hP_{i_m,r_1}(\mathcal{A})}{|a_{i_mi_m\cdots i_m}|}\right)^{\frac{1}{m-1}}\\
&\le \mu \sum_{i_2\cdots i_m\in N_0^{m-1}} |a_{ii_2\cdots i_m}| + \sum_{i_2\cdots i_m\in N_2^{m-1}} |a_{ii_2\cdots i_m}|w + \sum_{\substack{i_2\ldots i_m\in N_3^{m-1}\\ \delta_{ii_2\ldots i_m}=0}} \max_{j\in\{i_2,i_3,\ldots,i_m\}} \frac{hP_{j,r_1}(\mathcal{A})}{|a_{jj\ldots j}|}|a_{ii_2\cdots i_m}|\\
&\le hP_{i,r_1}(\mathcal{A}) = |b_{ii\ldots i}|.\end{aligned}$$

Therefore, $|b_{ii\ldots i}| \ge \Lambda_i(\mathcal{B})$ for all $i \in N$, and at least one strict inequality in (6) holds, that is, there exists an $i_0 \in N_2$ such that $|b_{i_0i_0\cdots i_0}| > \Lambda_{i_0}(\mathcal{B})$.

Notice that $\mathcal{A}$ is irreducible and so is $\mathcal{B}$. Hence, we obtain that $\mathcal{B}$ is an $\mathcal{H}$-tensor by Lemma 3 and $\mathcal{A}$ is an $\mathcal{H}$-tensor by Lemma 2. □

Theorem 4. *Let $\mathcal{A} = (a_{j_1j_2\cdots j_m})$ be an mth-order n-dimensional complex tensor. If $\mathcal{A}$ satisfies*

$$|a_{jj\cdots j}| \ge \frac{\mu}{w_j}\sum_{j_2j_3\cdots j_m\in N_0^{m-1}}|a_{jj_2\cdots j_m}| + \sum_{\substack{j_2j_3\cdots j_m\in N_2^{m-1}\\ \delta_{jj_2\cdots j_m}=0}}|a_{jj_2\cdots j_m}|\frac{w}{w_j} + \frac{h}{w_j}\sum_{j_2j_3\cdots j_m\in N_3^{m-1}}\max_{t\in\{j_2,j_3,\ldots,j_m\}}\frac{P_{t,r_1}(\mathcal{A})}{|a_{tt\cdots t}|}|a_{jj_2\cdots j_m}|,\ \forall j \in N_2 \tag{8}$$

and if any $j \in N\backslash K(\mathcal{A}) \ne \emptyset$, there is a nonzero elements chain from j to i such that $i \in K(\mathcal{A}) \ne \emptyset$, then $\mathcal{A}$ is an $\mathcal{H}$-tensor.

Proof. Let matrix $D = \mathrm{diag}(d_1, d_2, \ldots, d_n)$, denote $\mathcal{B} = \mathcal{A}D^{m-1} = (b_{j_1j_2\cdots j_m})$, where

$$d_j = \begin{cases}(\mu)^{\frac{1}{m-1}}, & j \in N_1,\\ (w_j)^{\frac{1}{m-1}}, & j \in N_2,\\ \left(\dfrac{hP_{j,r_1}(\mathcal{A})}{|a_{jj\cdots j}|}\right)^{\frac{1}{m-1}}, & j \in N_3.\end{cases}$$

By a similar proof to that of Theorem 3, we obtain that $|b_{jj\ldots j}| \ge \Lambda_j(\mathcal{B})(\forall j \in N)$, and there exists at least an $i_0 \in N_2$ such that $|b_{i_0i_0\cdots i_0}| > \Lambda_{i_0}(\mathcal{B})$.

On the other hand, if $|b_{jj\cdots j}| = \Lambda_j(\mathcal{B})$, then $j \in N\backslash K(\mathcal{A})$, so there is a nonzero elements chain of $\mathcal{A}$ from j to i satisfying $i \in K(\mathcal{A})$. Hence, there is a nonzero elements chain of $\mathcal{B}$ from j to i satisfying $|b_{ii\ldots i}| > \Lambda_i(\mathcal{B})$.

Therefore, we obtain that $\mathcal{B}$ is an $\mathcal{H}$-tensor by Lemma 4 and $\mathcal{A}$ is an $\mathcal{H}$-tensor by Lemma 2. □

Below, we present a numerical example to show that the criteria for $\mathcal{H}$-tensor is obtained by only using the conditions of Theorem 2 but not the ones in [20].

Example 1. Given a 3rd-order three-dimensional tensor $\mathcal{A} = (a_{ijk})$ as follows:

$$\mathcal{A} = [A(1,:,:), A(2,:,:), A(3,:,:)],$$

$$A(1,:,:) = \begin{pmatrix} 12 & 1 & 0 \\ 1 & 6 & 0 \\ 1 & 1 & 15 \end{pmatrix}, \quad A(2,:,:) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{pmatrix}, \quad A(3,:,:) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 16 \end{pmatrix}.$$

Obviously,

$$|a_{111}| = 12, \quad \Lambda_1(\mathcal{A}) = 25, \quad |a_{222}| = 4, \quad \Lambda_2(\mathcal{A}) = 8, \quad |a_{333}| = 16, \quad \Lambda_3(\mathcal{A}) = 2,$$

so $N_1 = \varnothing$, $N_2 = \{1, 2\}$, $N_3 = \{3\}$. By calculations, we have

$$w_1 = \frac{25}{25+12} = \frac{25}{37}, \quad w_2 = \frac{8}{8+4} = \frac{2}{3}, \quad w = \frac{25}{37}, \quad r = \frac{0+2}{16-0} = \frac{1}{8},$$

$$P_{3,r}(\mathcal{A}) = 0 + 2 + \frac{1}{8} \times 0 = 2, \quad r_1 = \frac{0+2}{16 - \frac{1}{8} \times 0} = \frac{1}{8}, \quad \mu = \frac{25}{37},$$

$$P_{3,r_1}(\mathcal{A}) = 0 + 2 + \frac{1}{8} \times \frac{1}{8} \times 0 = 2, \quad h = \frac{\frac{25}{37} \times 0 + \frac{25}{37} \times 2}{2 - \frac{1}{8} \times 0} = \frac{25}{37}.$$

When $i = 1$, we obtain

$$\begin{aligned} &\mu \sum_{i_2 i_3 \in N_0^2} |a_{1 i_2 i_3}| + \sum_{\substack{i_2 i_3 \in N_2^2 \\ \delta_{1 i_2 i_3} = 0}} |a_{1 i_2 i_3}| w + h \sum_{i_2 i_3 \in N_3^2} \max_{j \in \{i_2, i_3\}} \frac{P_{j,r_1}(\mathcal{A})}{|a_{jjj}|} |a_{1 i_2 i_3}| \\ &= \frac{25}{37} \times 2 + \frac{25}{37} \times 8 + \frac{25}{37} \times \frac{1}{8} \times 15 = \frac{2375}{296} < \frac{300}{37} = 12 \times \frac{25}{37} = |a_{111}| \times w_1. \end{aligned}$$

When $i = 2$, we obtain

$$\begin{aligned} &\mu \sum_{i_2 i_3 \in N_0^2} |a_{2 i_2 i_3}| + \sum_{\substack{i_2 i_3 \in N_2^2 \\ \delta_{2 i_2 i_3} = 0}} |a_{2 i_2 i_3}| w + h \sum_{i_2 i_3 \in N_3^2} \max_{j \in \{i_2, i_3\}} \frac{P_{j,r_1}(\mathcal{A})}{|a_{jjj}|} |a_{2 i_2 i_3}| \\ &= \frac{25}{37} \times 0 + \frac{25}{37} \times 2 + \frac{25}{37} \times \frac{1}{8} \times 6 = \frac{275}{148} \\ &< \frac{100}{37} = 4 \times \frac{25}{37} = |a_{111}| \times w_1. \end{aligned}$$

Hence, the conditions of Theorem 2 are satisfied, then $\mathcal{A}$ is an $\mathcal{H}$-tensor. However,

$$p_3 = \frac{26}{25}, \quad s_3 = \frac{13}{200}, \quad M_3 = 1, \quad F_3 = \frac{26}{25}, \quad t_3 = \frac{13}{200}.$$

$$\sum_{i_2 i_3 \in N_0^2} |a_{1 i_2 i_3}| + \sum_{\substack{i_2 i_3 \in N_2^2 \\ \delta_{1 i_2 i_3} = 0}} \max_{j \in \{i_2, i_3\}} \{r_j\} |a_{1 i_2 i_3}| + \sum_{i_2 i_3 \in N_3^2} \max_{j \in \{i_2, i_3\}} \{t_j\} |a_{1 i_2 i_3}| = 2 + \frac{13}{25} \times 8 + \frac{13}{200} \times 15 = \frac{1427}{200} \frac{156}{25} = |a_{111}| r_1.$$

So, $\mathcal{A}$ does not satisfy the conditions of Theorem 1.

3 An application

In this section, we provide some $\mathcal{H}$-tensor-based criteria for identifying the positive definite tensor. First, we recall the following lemma.

Lemma 5. [14] *Let $\mathcal{A} = (a_{j_1 j_2 \cdots j_m})$ be an mth-order n-dimensional even-order real symmetric tensor with $a_{jj \cdots j} > 0$ for $\forall j \in N$. If $\mathcal{A}$ is an $\mathcal{H}$-tensor, then $\mathcal{A}$ is positive definite.*

By Theorems 2–4 and Lemma 5, we can prove easily the following result.

Theorem 5. *Let $\mathcal{A} = (a_{j_1 j_2 \cdots j_m})$ be an mth-order n-dimensional even-order real symmetric tensor $a_{jj\cdots j} > 0$ for $\forall j \in N$. If one of the following conditions is satisfied*:
(*i*) *All the conditions of* Theorem 2;
(*ii*) *All the conditions of* Theorem 3;
(*iii*) *All the conditions of* Theorem 4,

then $\mathcal{A}$ is a positive definite tensor.

Example 2. Consider an 4th-degree homogeneous polynomial:

$$f(x) = \mathcal{A}x^4 = 24x_1^4 + 36x_2^4 + 25x_3^4 + 10x_4^4 - 8x_2^3x_4 + 12x_1^2x_2x_3 - 12x_2x_3^2x_4 - 24x_1x_2x_3x_4.$$

We can obtain an 4th-order four-dimensional real symmetric tensor $\mathcal{A} = (a_{j_1 j_2 j_3 j_4})$ with entries

$$\begin{aligned}
&a_{1111} = 24, \quad a_{2222} = 36, \quad a_{3333} = 25, \quad a_{4444} = 10,\\
&a_{2224} = a_{2242} = a_{2422} = a_{4222} = -2,\\
&a_{1123} = a_{1132} = a_{1213} = a_{1312} = a_{1231} = a_{1321} = 1,\\
&a_{2113} = a_{2131} = a_{2311} = a_{3112} = a_{3121} = a_{3211} = 1,\\
&a_{2334} = a_{2343} = a_{2433} = a_{4233} = a_{4323} = a_{4332} = -1,\\
&a_{3234} = a_{3243} = a_{3324} = a_{3342} = a_{3423} = a_{3432} = -1,\\
&a_{1234} = a_{1243} = a_{1324} = a_{1342} = a_{1423} = a_{1432} = -1,\\
&a_{2134} = a_{2143} = a_{2314} = a_{2341} = a_{2413} = a_{2431} = -1,\\
&a_{3124} = a_{3142} = a_{3214} = a_{3241} = a_{3412} = a_{3421} = -1,\\
&a_{4123} = a_{4132} = a_{4213} = a_{4231} = a_{4312} = a_{4321} = -1,
\end{aligned}$$

and other $a_{i_1i_2i_3i_4} = 0$. By calculations, we have

$$a_{4444} = 10 < 11 = \Lambda_4(\mathcal{A})$$

and

$$a_{1111}(a_{4444} - \Lambda_4(\mathcal{A}) + |a_{4111}|) = -24 < 0 = \Lambda_4(\mathcal{A})|a_{4111}|.$$

Therefore, tensor $\mathcal{A}$ is not strictly diagonally dominate as defined in [21], or quasidoubly strictly diagonally dominant as defined in [22]. However, all the conditions of Theorem 2 can be satisfied.

$$\Lambda_1(\mathcal{A}) = 12, \quad \Lambda_2(\mathcal{A}) = 18, \quad \Lambda_3(\mathcal{A}) = 15, \quad \Lambda_4(\mathcal{A}) = 11,$$

so $N_1 = \varnothing$, $N_2 = \{4\}$, $N_3 = \{1, 2, 3\}$. By calculations, we have

$$\begin{aligned}
&w_4 = \frac{11}{21} = w, \quad r = \frac{6}{11}, \quad P_{1,r}(\mathcal{A}) = \frac{102}{11},\\
&P_{2,r}(\mathcal{A}) = \frac{183}{11}, \quad P_{3,r}(\mathcal{A}) = \frac{150}{11}, \quad r_1 = \frac{132}{257}, \quad \mu = \frac{11}{21},\\
&P_{1,r_1}(\mathcal{A}) = \frac{1974}{257}, \quad P_{2,r_1}(\mathcal{A}) = \frac{4071}{257}, \quad P_{3,r_1}(\mathcal{A}) = \frac{3300}{257}, \quad h = \frac{2827}{4137}.
\end{aligned}$$

When $i = 4$, we obtain

$$\begin{aligned}
&\mu \sum_{i_2i_3i_4 \in N_0^3} |a_{4i_2i_3i_4}| + \sum_{\substack{i_2i_3i_4 \in N_2^3\\ \delta_{4i_2i_3i_4}=0}} |a_{4i_2i_3i_4}| w + h \sum_{i_2i_3i_4 \in N_3^3} \max_{j \in \{i_2,i_3,i_4\}} \frac{P_{j,r_1}(\mathcal{A})}{|a_{jjjj}|} |a_{4i_2i_3i_4}|\\
&= \frac{11}{21} \times 0 + \frac{11}{21} \times 0 + \frac{2827}{4137} \times \frac{132}{257} \times 11 = \frac{5324}{1379}\\
&< \frac{121}{21} = 11 \times \frac{11}{21} = |a_{4444}| \times w_4.
\end{aligned}$$

So, $\mathcal{A}$ is positive definite by Theorem 5, that is, $f(x)$ is positive definite.

4 Conclusions

In this paper, we introduced some new criterions for identifying $\mathcal{H}$-tensors, which only depended on the elements of tensor. They were well used to check the positive definiteness of an even-order homogeneous polynomial form $f(x) \equiv \mathcal{A}x^m$. We also verified the effectiveness of new criteria by several numerical examples.

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