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Research Article

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Remarks on the generalized interpolative contractions and some fixed-point theorems with application

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Abstract: In this manuscript, some remarks on the papers [H. A. Hammad, P. Agarwal, S. Momani, and F. Alsharari, *Solving a fractional-order differential equation using rational symmetric contraction mappings*, Fractal Fract. **5** (2021), 159] and [A. Hussain, F. Jarad, and E. Karapinar, *A study of symmetric contractions with an application to generalized fractional differential equations*, Adv. Differ. Equ. **2021** (2021), 300] are given. In the light of remarks, we introduce a new property that makes it convenient to investigate the existence of fixed points of the interpolative contractions in the orthogonal metric spaces. We derive several new results based on known contractions from the main theorems. As an application, we resolve a Urysohn integral equation.

Keywords: fixed point, generalized interpolative fractional contractions, complete O-metric space, Urysohn integral equation

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1 Introduction

The investigation of generalized contraction principles to present new and useful fixed-point theorems is the key element in the metric fixed-point theory. The founder of this field was Banach (1922) who introduced the contraction principle. Boyd and Wong [1] generalized the well-known Banach contraction principle (BCP) [2] by introducing the control function $\Psi : [0, \infty) \to [0, \infty)$ verifying the following conditions for each $y > 0$:
(1) $\Psi(y) < y$;
(2) $\lim_{x \to y+} \Psi(x) < y$.

Boyd and Wong [1] contraction principle generalizes the contraction principle introduced by Rakotch [3]. The Boyd-Wong idea has been generalized by Matkowski [4], Samet et al. [5], Karapinar and Samet [6], and Pasicki [7].

Karapinar [8] introduced interpolative contractions and presented a method to obtain fixed points of such contractions. Agarwal and Karapinar also introduced the interpolative Rus-Reich-Ćirić type contractions [9], w-interpolative Ćirić-Reich-Rus-type contractions [10], interpolative Hardy-Rogers type contractions [11], and interpolative Boyd-Wong and Matkowski type contractions [12] to ensure the existence of fixed points in

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(generalized) metric spaces. Gautam et al. [13] presented some fixed-point results for Chatterjea and cyclic Chatterjea interpolative contractions in complete quasi-partial b-metric spaces. Debnath et al. [14] proved some fixed-point theorems for Rus-Reich-Ćirić and Hardy-Rogers type interpolative contractions in b-metric spaces. Recently, Hussain et al. [15], introduced the following generalized interpolative contractions:

$$d_F(Sx, Sy) \le \lambda(\check{F}_i(x, y)), \quad i = 1, 2, 3, 4, \tag{1.1}$$

where d_F is an F metric, $S : \mathcal{A} \to \mathcal{A}$, and for positive real numbers a, b, c, the mappings $\check{F}_1, \check{F}_2, \check{F}_3, \check{F}_4$: $\mathcal{A} \times \mathcal{A} \to [0, \infty)$ are defined by

$$\begin{aligned}
\check{F}_1(x, y) &= d_F(x, y)[d_F(x, Sx)]^{\frac{1}{(a-b)(a-c)}}[d_F(y, Sy)]^{\frac{1}{(a-b)(a-c)}}\\
&\quad \times [d_F(x, Sx) + d_F(y, Sy)]^{\frac{1}{(b-a)(b-c)}}[d_F(x, Sy) + d_F(y, Sx)]^{\frac{1}{(c-a)(c-b)}},\\
\check{F}_2(x, y) &= d_F(x, y)[d_F(x, Sx)]^{\frac{a}{(a-b)(a-c)}}[d_F(y, Sy)]^{\frac{a}{(a-b)(a-c)}}\\
&\quad \times [d_F(x, Sx) + d_F(y, Sy)]^{\frac{b}{(b-a)(b-c)}}[d_F(x, Sy) + d_F(y, Sx)]^{\frac{c}{(c-a)(c-b)}},\\
\check{F}_3(x, y) &= \max\left\{\begin{array}{l} d_F(x, y), [d_F(x, Sx)]^{\frac{a^2}{(a-b)(a-c)}}[d_F(y, Sy)]^{\frac{a^2}{(a-b)(a-c)}}\\ [d_F(x, Sx) + d(y, Sy)]^{\frac{b^2}{(b-a)(b-c)}}\\ [d_F(x, Sy) + d_F(y, Sx)]^{\frac{c^2}{(c-a)(c-b)}} \end{array}\right\},\\
\check{F}_4(x, y) &= d_F(x, y)^{\frac{a^3}{(a-b)(a-c)}}d_F(y, Sy)^{\frac{a^3}{(a-b)(a-c)}}\\
&\quad \times [d_F(x, Sx) + d_F(y, Sy)]^{\frac{b^3}{(b-a)(b-c)}}[d_F(x, Sy) + d_F(y, Sx)]^{\frac{c^3}{(c-a)(c-b)}}.
\end{aligned}$$

It is important to note that despite $a, b, c > 0$, some exponents are negative, for example, if $a > b$, $a > c$, and $b > c$, then $\frac{1}{(b-a)(b-c)} < 0$. If any one of a, b, and c goes to ∞, then $\check{F}_1(x, y) = d_F(x, y)$. Moreover, we have the following interesting facts about the exponents, which can be proved by using basic algebraic tools:

$$\begin{aligned}
\frac{1}{(a-b)(a-c)} + \frac{1}{(b-a)(b-c)} + \frac{1}{(c-a)(c-b)} &= 0,\\
\frac{a}{(a-b)(a-c)} + \frac{b}{(b-a)(b-c)} + \frac{c}{(c-a)(c-b)} &= 0,\\
\frac{a^2}{(a-b)(a-c)} + \frac{b^2}{(b-a)(b-c)} + \frac{c^2}{(c-a)(c-b)} &= 1,\\
\frac{a^3}{(a-b)(a-c)} + \frac{b^3}{(b-a)(b-c)} + \frac{c^3}{(c-a)(c-b)} &= a + b + c.
\end{aligned}$$

These contractions are named as "symmetric fractional contractions of type I, II, III, and IV" for $i = 1, 2, 3, 4$, respectively. The existence of fixed points of these contractions has been shown in [15]. In the proof of [15, Theorem 2.3], the following simplifications was involved.

$$\begin{aligned}
\check{F}_1(\hbar_{n-1}, \hbar_n) &= d_F(\hbar_{n-1}, \hbar_n)d_F(\hbar_{n-1}, S\hbar_{n-1})^{\frac{1}{(a-b)(a-c)}}d_F(\hbar_n, S\hbar_n)^{\frac{1}{(a-b)(a-c)}}\\
&\quad \times [d_F(\hbar_{n-1}, S\hbar_{n-1}) + d_F(\hbar_n, S\hbar_n)]^{\frac{1}{(b-a)(b-c)}}[d_F(\hbar_{n-1}, S\hbar_n) + d_F(\hbar_n, S\hbar_{n-1})]^{\frac{1}{(c-a)(c-b)}}\\
&\le d_F(\hbar_{n-1}, \hbar_n)d_F(\hbar_{n-1}, \hbar_n)^{\frac{1}{(a-b)(a-c)}}d_F(\hbar_n, \hbar_{n+1})^{\frac{1}{(a-b)(a-c)}}\\
&\quad \times [d_F(\hbar_{n-1}, \hbar_n) + d_F(\hbar_n, \hbar_{n+1})]^{\frac{1}{(b-a)(b-c)}}[d_F(\hbar_{n-1}, \hbar_{n+1}) + d_F(\hbar_n, \hbar_n)]^{\frac{1}{(c-a)(c-b)}}\\
&\le d_F(\hbar_{n-1}, \hbar_n)d_F(\hbar_{n-1}, \hbar_n)^{\frac{1}{(a-b)(a-c)}}d_F(\hbar_n, \hbar_{n+1})^{\frac{1}{(a-b)(a-c)}}\\
&\quad \times [d_F(\hbar_{n-1}, \hbar_n) + d_F(\hbar_n, \hbar_{n+1})]^{\frac{1}{(b-a)(b-c)}}[d_F(\hbar_{n-1}, \hbar_n) + d_F(\hbar_n, \hbar_{n+1})]^{\frac{1}{(c-a)(c-b)}}\\
&= d_F(\hbar_{n-1}, \hbar_n)d_F(\hbar_{n-1}, \hbar_n)^{\frac{1}{(a-b)(a-c)}}d_F(\hbar_n, \hbar_{n+1})^{\frac{1}{(a-b)(a-c)}}\\
&\quad \times [d_F(\hbar_{n-1}, \hbar_n) + d_F(\hbar_n, \hbar_{n+1})]^{\frac{1}{(b-a)(b-c)} + \frac{1}{(c-a)(c-b)}}\\
&\le d_F(\hbar_{n-1}, \hbar_n)d_F(\hbar_{n-1}, \hbar_n)^{\frac{1}{(a-b)(a-c)}}d_F(\hbar_n, \hbar_{n+1})^{\frac{1}{(a-b)(a-c)}}\\
&\quad \times [d_F(\hbar_{n-1}, \hbar_n)d_F(\hbar_n, \hbar_{n+1})]^{\frac{1}{(b-a)(b-c)} + \frac{1}{(c-a)(c-b)}}\\
&= d_F(\hbar_{n-1}, \hbar_n)^{1+\frac{1}{(a-b)(a-c)} + \frac{1}{(b-a)(b-c)} + \frac{1}{(c-a)(c-b)}}d_F(\hbar_n, \hbar_{n+1})^{\frac{1}{(a-b)(a-c)} + \frac{1}{(b-a)(b-c)} + \frac{1}{(c-a)(c-b)}}\\
&= d_F(\hbar_{n-1}, \hbar_n).
\end{aligned}$$

We observe that the inequality in the red color, that is,

$$[d_F(\hbar_{n-1},\hbar_n)+d_F(\hbar_n,\hbar_{n+1})]^{\frac{1}{(b-a)(b-c)}+\frac{1}{(c-a)(c-b)}}\leq[d_F(\hbar_{n-1},\hbar_n)d_F(\hbar_n,\hbar_{n+1})]^{\frac{1}{(b-a)(b-c)}+\frac{1}{(c-a)(c-b)}},$$

is not true in general due to the following fact:

$$(a+b)^r\geq(ab)^r\quad\text{for all } a,b\leq 2\quad\text{and } r\leq 1.$$

Also, in the proof of [15, Theorems 3.2, 4.2, and 5.2], the simplifications of $\check{F}_2(\hbar_{n-1},\hbar_n)$, $\check{F}_3(\hbar_{n-1},\hbar_n)$, and $\check{F}_4(\hbar_{n-1},\hbar_n)$s have some gaps.

Similarly, Hammad et al. [16] defined the so-called rational symmetric contractions of type I, II, III and IV as follows:

$$d_\theta(Sx,Sy)^q\leq\lambda(\check{A}_i(x,y)),\quad i=1,2,3,4,\tag{1.2}$$

where $q\geq 1$, $\lambda\in[0,1)$, d_θ is a θ-metric, $S:\mathcal{A}\to\mathcal{A}$ a self-mapping, and

$$\check{A}_1(x,y)=d_\theta(x,y)^q\frac{[d_\theta(x,Sx)]^{\frac{q}{(a-b)(a-c)}}[d_\theta(y,Sy)]^{\frac{q}{(a-b)(a-c)}}}{1+d_\theta(x,y)^q}\times\frac{[d_\theta(x,Sx)+d_\theta(y,Sy)]^{\frac{q}{(b-a)(b-c)}}}{1+d_\theta(x,y)^q}\frac{[d_\theta(x,Sy)+d_\theta(y,Sx)]^{\frac{q}{(c-a)(c-b)}}}{1+d_\theta(x,y)^q}.$$

$$\check{A}_2(x,y)=d_\theta(x,y)^q\frac{[d_\theta(x,Sx)]^{\frac{qa}{(a-b)(a-c)}}[d_\theta(y,Sy)]^{\frac{qa}{(a-b)(a-c)}}}{1+d_\theta(x,y)^q}\times\frac{[d_\theta(x,Sx)+d_\theta(y,Sy)]^{\frac{qb}{(b-a)(b-c)}}}{1+d_\theta(x,y)^q}\frac{[d_\theta(x,Sy)+d_\theta(y,Sx)]^{\frac{qc}{(c-a)(c-b)}}}{1+d_\theta(x,y)^q}.$$

$$\check{A}_3(x,y)=\max\left\{\begin{array}{l}d_\theta(x,y)^q,\ \dfrac{[d_\theta(x,Sx)]^{\frac{qa^2}{(a-b)(a-c)}}[d_\theta(y,Sy)]^{\frac{qa^2}{(a-b)(a-c)}}}{1+d_\theta(x,y)^q}\\ \dfrac{[d_\theta(x,Sx)+d_\theta(y,Sy)]^{\frac{qb^2}{(b-a)(b-c)}}}{1+d_\theta(x,y)^q}\dfrac{[d_\theta(x,Sy)+d_\theta(y,Sx)]^{\frac{qc^2}{(c-a)(c-b)}}}{1+d_\theta(x,y)^q}\end{array}\right\}.$$

$$\check{A}_4(x,y)=d_\theta(x,y)^{\frac{qa^3}{(a-b)(a-c)}}\frac{[d_\theta(y,Sy)]^{\frac{qa^3}{(a-b)(a-c)}}}{1+d_\theta(x,y)^q}\times\frac{[d_\theta(x,Sx)+d_\theta(y,Sy)]^{\frac{qb^3}{(b-a)(b-c)}}}{1+d_\theta(x,y)^q}\frac{[d_\theta(x,Sy)+d_\theta(y,Sx)]^{\frac{qc^3}{(c-a)(c-b)}}}{1+d_\theta(x,y)^q}.$$

We note that in the proof of [16, Theorem 2], the following inequality:

$$[d_\theta(\hbar_{n-1},\hbar_n)+d_\theta(\hbar_n,\hbar_{n+1})]^{\frac{q}{(b-a)(b-c)}+\frac{q}{(c-a)(c-b)}}\leq[d_\theta(\hbar_{n-1},\hbar_n)d_\theta(\hbar_n,\hbar_{n+1})]^{\frac{q}{(b-a)(b-c)}+\frac{q}{(c-a)(c-b)}},\tag{1.3}$$

considered true to obtain the desired result; however, it is not true in general due to the following fact:

$$(a+b)^r\geq(ab)^r\quad\text{for all } a,b\leq 2\quad\text{and for any positive real number } r.$$

We deduce that the inequality (1.3) is true for $d_\theta(\hbar_i,\hbar_{i+1})\geq 2$ for all $i\in\{0,1,2,3,\ldots,n\}$ and not true if $d_\theta(\hbar_i,\hbar_{i+1})<2$.

Also, the proof of [16, Theorems 4, 6, and 8] contains some gaps.

To obtain the refinements of the proofs given in [15,16], we begin with the following observation.

Observation 1.1. The following inequality holds for all $a,b\geq 2$, and $r\geq 1$,

$$(a+b)^r\leq(ab)^r.$$

Proof. We note that the equality holds for $a=b=2$. We can assume that $a\geq b$, then $a=\eta b$; $\eta\geq 1$. Let $b=t$, so that $a=\eta t$, $t\geq 2$. Define the function $f:[2,\infty)\to(-\infty,\infty)$ by

$$f(t) = (\eta t^2)^r - (\eta t + t)^r, \quad \forall t \in [2, \infty).$$

This implies that

$$f'(t) = \frac{\mathrm{d}}{\mathrm{d}t}(f(t)) = \frac{rt^{r-1}}{(\eta + 1)^r}\left[2t^r\left(\frac{\eta}{\eta + 1}\right)^r - 1\right].$$

Since $2t^r\left(\frac{\eta}{\eta+1}\right)^r > 1$ (otherwise $t < 1$), we have $f'(t) > 0$. This implies that $f(t) \geq 0$, and hence, $(\eta t^2)^r - (\eta t + t)^r \geq 0$, that is, $(a + b)^r \leq (ab)^r$. □

Observation 1.2. Let $K \geq 2$. For any nonempty set $\mathcal{A}$, define the mapping $d : \mathcal{A} \times \mathcal{A} \to [0, \infty)$ by

$$d(u, v) = \begin{cases} K & \text{if } u \neq v, \\ 0 & \text{if } u = v. \end{cases}$$

Then the pair $(\mathcal{A}, d)$ is a metric space.

Now, we introduce a new property (P), which plays a key role in the refinement of the results given in [15,16].

Definition 1.3. Let $(\mathcal{A}, d)$ be a metric space. A mapping $T : \mathcal{A} \to \mathcal{A}$ is said to have property (P), if for any real number r, it satisfies the following inequality:

$$(d(x, T(x)) + d(T(x), T^2(x)))^r \leq (d(x, T(x))d(T(x), T^2(x)))^r, \quad \forall x \in \mathcal{A}.$$

Example 1.4. Let $\mathcal{A} = [1, \infty)$ and consider the metric d defined by $d(u, v) = |u - v|$ for all $u, v \in \mathcal{A}$. The mapping $T : \mathcal{A} \to \mathcal{A}$ defined by $T(x) = Kx$ for all $x \in \mathcal{A}$ and $K \geq \frac{5}{2}$ satisfies the property (P). Indeed,

$$\begin{aligned}(d(x, T(x)) + d(T(x), T^2(x)))^r &= [(K - 1)|x| + (K - 1)|Kx|]^r \\ &\leq [(K - 1)(K + 1)|x|]^r \leq [(K - 1)^2K|x|^2]^r \\ &= (d(x, T(x))d(T(x), T^2(x)))^r.\end{aligned}$$

Example 1.5. Every identity mapping satisfies the property (P). The constant mapping does not satisfy the property (P). The mapping $T : \mathcal{A} \to \mathcal{A}$ defined by $T(x) = 0$ for all $x \in \mathcal{A}$ satisfies the property P only for $x = 0$.

Example 1.6. Let $\mathcal{A} = (-\infty, \infty)$. The mapping $T : \mathcal{A} \to \mathcal{A}$ defined by $T(x) = 2 - 3x$ for all $x \in \mathcal{A}$ satisfies the property (P). In fact, the mapping $T : \mathcal{A} \to \mathcal{A}$ defined by $T(x) = a - bx$ for all $x \in \mathcal{A}$ for $b > a$ satisfies the property (P).

Example 1.7. Let $\mathcal{A} = [2.5, \infty)$. The mapping $T : \mathcal{A} \to \mathcal{A}$ defined by $T(x) = 2x - 1$ for all $x \in \mathcal{A}$ satisfies the property (P).

Example 1.8. Let $\mathcal{A} = [1, \infty)$. The mapping $T : \mathcal{A} \to \mathcal{A}$ defined by $T(x) = \frac{1}{x^2}$ for all $x \in \mathcal{A}$ satisfies the property (P).

Remark 1.9. The corrected proof of [15, Theorems 2.3, 3.2, 4.2, and 5.2] and [16, Theorems 2, 4, 6, and 8] can be obtained by assuming either "Observation 1.1 and Observation 1.2" or "Property (P)."

The main contribution of this paper is the introduction of property (P) that proved to be a good tool to address the gaps found in [15,16]. Second, we show that interpolative contraction implies orthogonal interpolative contraction, and orthogonal interpolative contraction implies (Ψ, Φ)-orthogonal interpolative contraction but converse is not true. Thus, presented theorems generalize the results in [9–12,17].

2 Orthogonal relations

In this section, we define orthogonal set (a non-empty set whose elements obey the orthogonal relation), $\perp$-regular space, and O-sequence (a sequence whose terms are pairwise orthogonal). The binary relation $\perp$ (orthogonal relation) is a generalization of a partial order, α-admissible function, and directed graph. It also contains the notion of orthogonality in the inner product spaces. The following definition is one of the key notions of this paper.

Definition 2.1. [18] Let $\perp$ be a binary relation defined on a nonempty set $\mathcal{A}$ (i.e., $\perp \subset \mathcal{A} \times \mathcal{A}$). If $\perp$ satisfies the property (O), then we call it orthogonal relation, and the pair $(\mathcal{A}, \perp)$ is called orthogonal set (in short, O-set).

$$(\mathrm{O}): \exists\, x_0 \in \mathcal{A}: \quad \text{either } (\forall y, x_0 \perp y) \quad \text{or} \quad (\forall y, y \perp x_0).$$

To illustrate the orthogonal set, we have the following examples.

Example 2.2. Let $\mathcal{A}$ be the set of integers and define the relation $a \perp \theta$ if and only if $a \equiv 1(\bmod \theta)$. Then $(\mathcal{A}, \perp)$ is an O-set. Indeed, $1 \perp \theta$ for each θ.

Example 2.3. Let $\mathcal{A}$ be the set of all persons in the word. Define $x \perp y$ if x can give blood to y. According to the blood transfusion protocol, if x_0 is a person such that his (her) blood type is $O-$, then we have $x_0 \perp y$ for all $y \in \mathcal{A}$. This means that $(\mathcal{A}, \perp)$ is an O-set. In this O-set, x_0 is not unique. Note that x_0 may be a person with blood type AB+. In this case, we have $y \perp x_0$ for all $y \in \mathcal{A}$.

Example 2.4. In the graph theory, a wheel graph W_n is a graph with n vertices for each $n \geq 4$, formed by connecting a single vertex to all vertices of an $(n-1)$-cycle. Let $\mathcal{A}$ be the set of all vertices of W_n for each $n \geq 4$. Define $x \perp y$ if there is a connection from x to y. Then, $(\mathcal{A}, \perp)$ is an O-set.

Example 2.5. Let $\mathcal{A}$ be a inner product space with the inner product $\langle\cdot,\cdot\rangle$. Define $x \perp y$ if $\langle x, y\rangle = 0$. It is easy to see that $0 \perp y$ for all $y \in \mathcal{A}$. Hence, $(\mathcal{A}, \perp)$ is an O-set.

Definition 2.6. [18] A sequence $\{\hbar_n : n \in \mathbb{N}\}$ is said to be an O-sequence if either $x_n \perp x_{n+1}$ or $x_{n+1} \perp x_n$ for all n.

Definition 2.7. [18] The O-set $(\mathcal{A}, \perp)$ endowed with a metric d is called an O-metric space (in short, OMS) denoted by $(\mathcal{A}, \perp, d)$.

Definition 2.8. [18] The O-sequence $\{x_n\} \subset \mathcal{A}$ is said to be O-Cauchy if $\lim_{n,m\to\infty} d(x_n, x_m) = 0$. If each O-Cauchy sequence converges in $\mathcal{A}$, then $(\mathcal{A}, \perp, d)$ is called O-complete.

Definition 2.9. [18] Let $(\mathcal{A}, \perp, d)$ be an orthogonal metric space. A mapping $f: \mathcal{A} \to \mathcal{A}$ is said to be an orthogonal contraction if there exists $k \in [0, 1)$ such that

$$d(fx, fy) \leq k d(x, y) \quad \forall x, y \in \mathcal{A} \quad \text{with } x \perp y.$$

In the following, we give some comparisons between fundamental notions.

(1) The continuity implies orthogonal continuity, but converse is not true. If $f: \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = [x], \quad \forall x \in \mathbb{R}$ and the relation $\perp \subseteq \mathbb{R} \times \mathbb{R}$ is defined by

$$x \perp y \quad \text{if } x, y \in \left(i + \frac{1}{3}, i + \frac{2}{3}\right), \quad i \in \mathbb{Z} \quad \text{or} \quad x = 0.$$

Then f is $\perp$-continuous, while f is discontinuous on $\mathbb{R}$.

(2) The completeness of the metric space implies O-completeness, but the converse is not true. We know that $\mathcal{A} = [0, 1)$ with Euclidean metric d is not complete metric space. If we define the relation $\perp \subseteq \mathcal{A} \times \mathcal{A}$ by

$$x \perp y \Leftrightarrow x \leq y \leq \frac{1}{2} \quad \text{or} \quad x = 0.$$

Then, $(\mathcal{A}, \perp, d)$ is an O-complete.

(3) The Banach contraction implies orthogonal contraction, but converse is not true. Let $\mathcal{A} = [0, 10)$ with Euclidean metric d, so that $(\mathcal{A}, d)$ is a metric space. If we define the relation $\perp \subseteq \mathcal{A} \times \mathcal{A}$ by

$$x \perp y \quad \text{if } xy \leq x \vee y.$$

Then $(\mathcal{A}, \perp, d)$ is an O-metric space. Define $f : \mathcal{A} \to \mathcal{A}$ by $f(x) = \frac{x}{2}$ (if $x \leq 2$) and $f(x) = 0$ (if $x > 2$). Since $d(f(3), f(2)) > kd(3, 2)$, so, f is not a contraction while it is an orthogonal contraction.

We will use the following lemma to support the proofs.

Lemma 2.10. [17] *Let (X, d) be a metric space and $\{x_n\} \subset X$ be a sequence verifying* $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$. *If the sequence $\{x_n\}$ is not Cauchy, then there are $\{x_{n_k}\}$, $\{x_{m_k}\}$ and $\xi > 0$ such that*

$$\lim_{k\to\infty} d\left(x_{n_k+1}, x_{m_k+1}\right) = \xi + . \tag{2.1}$$

$$\lim_{k\to\infty} d\left(x_{n_k}, x_{m_k}\right) = d\left(x_{n_k+1}, x_{m_k}\right) = d\left(x_{n_k}, x_{m_k+1}\right) = \xi. \tag{2.2}$$

Let $(\mathcal{A}, \perp, d)$ be an OMS and $P(\mathcal{A})$ form the set of nonempty subsets of $\mathcal{A}$, $CB(\mathcal{A})$ form the set of all nonempty bounded and closed subsets of $\mathcal{A}$. The set of nonempty compact subsets of $\mathcal{A}$ is denoted by $K(\mathcal{A})$. For $\Lambda \in CB(\mathcal{A})$ and $\chi \in \mathcal{A}$, consider $d(\chi, \Lambda) = \inf_{\varsigma\in\Lambda} d(\chi, \varsigma)$. Let $H : CB(\mathcal{A}) \times CB(\mathcal{A}) \to [0, \infty)$ be given as follows:

$$H(\Lambda_1, \Lambda_2) = \max\left\{\sup q \in \Lambda_1 \; d(q, \Lambda_2), \sup_{b\in\Lambda_2} d(b, \Lambda_1)\right\} \quad \text{for all } \Lambda_1, \Lambda_2 \in CB(\mathcal{A}). \tag{2.3}$$

Such a function H verifies all the axioms of O-metric. It is known as a Pompieu-Hausdorff O-metric induced by the O-metric d.

Definition 2.11. Let $T : \mathcal{A} \to P(\mathcal{A})$ be a set-valued mapping. An element $\upsilon \in \mathcal{A}$ is said to be a fixed point of T if $\upsilon \in T\upsilon$.

Definition 2.12. Let $(\mathcal{A}, \perp, d)$ be an OMS and $\perp \subset \mathcal{A} \times \mathcal{A}$ be a binary relation. The space $(\mathcal{A}, \perp, d)$ is called $\perp$-regular if for each sequence $\{x_n\} \subset \mathcal{A}$ so that $x_n \perp x_{n+1}$ for each $n \geq 0$ and $x_n \to x$ as $n \to \infty$, we have either $x_n \perp x$, or $x \perp x_n$ for all $n \geq 0$.

Definition 2.13. A mapping $T : \mathcal{A} \to \mathcal{A}$ is said to be asymptotically regular at a point υ of $\mathcal{A}$ if

$$\lim_{n\to\infty} d(T^n\upsilon, T^{n+1}\upsilon) = 0.$$

If T is asymptotically regular at each point in $\mathcal{A}$, then it is named as an asymptotically regular mapping.

The following, lemma is useful in the sequel.

Lemma 2.14. [19] *Let Λ_1 and Λ_2 be nonempty bounded and closed subsets of a metric space $(\mathcal{A}, d)$ and $\nu > 1$. Then, for all $x \in \Lambda_1$, there is $y \in \Lambda_2$ so that $d(x, y) \leq \nu H(\Lambda_1, \Lambda_2)$.*

In the following, we define $\perp$-admissible mapping, $\perp$-preserving mapping and illustrate them with examples. Let $\Lambda = \{(x, y) \in \mathcal{A} \times \mathcal{A} : x \perp y\}$.

Definition 2.15. A mapping $f : \mathcal{A} \times \mathcal{A} \to [1, \infty)$ is said to be strictly $\perp$-admissible if $f(a, \theta) > 1$ for all $a, \theta \in \mathcal{A}$ with $a \perp \theta$ and $f(a, \theta) = 1$ otherwise.

Example 2.16. Let $\mathcal{A} = [0, 1)$ and define the relation $\perp \subset \mathcal{A} \times \mathcal{A}$ by

$$a \perp \theta \quad \text{if } a\theta \in \{a, \theta\} \subset \mathcal{A}.$$

Then, $\mathcal{A}$ is an O-set. Define $f : \mathcal{A} \times \mathcal{A} \to [1, \infty)$ by

$$f(a, \theta) = \begin{cases} a + \dfrac{2}{1 + \theta} & \text{if } a \perp \theta, \\ 1 & \text{otherwise}. \end{cases}$$

Then, f is $\perp$-admissible.

Definition 2.17. Let $T : \mathcal{A} \to P(\mathcal{A})$ and $\perp \subset \mathcal{A} \times \mathcal{A}$ be an orthogonal relation. The mapping T is called $\perp$-preserving if for each $q \in \mathcal{A}$ and $p \in T(q)$ such that $q \perp p$ or $p \perp q$, there is $\omega \in T(p)$ satisfying $p \perp \omega$ or $\omega \perp p$.

Example 2.18. Let $\mathcal{A} = [0, 1)$ and define the relation $\perp \subset \mathcal{A} \times \mathcal{A}$ by

$$a \perp \theta \quad \text{if } a\theta \in \{a, \theta\} \subset \mathcal{A}.$$

Then, $\mathcal{A}$ is an O-set. Define $S : \mathcal{A} \to P(\mathcal{A})$ by

$$S(a) = \begin{cases} \left[\dfrac{a}{15}, \dfrac{a+1}{7}\right] & \text{if } a \in \mathbb{Q} \cap \mathcal{A}, \\ \{0\} & \text{if } a \in \mathbb{Q}^c \cap \mathcal{A}. \end{cases}$$

Then, S is a $\perp$-preserving mapping. Indeed, for $a = 0$, there is $\theta \in S(0) = \left[0, \frac{1}{7}\right]$ such that either $a \perp \theta$ or $\theta \perp a$, and then there is $x \in S(\theta)$ such that either $x \perp \theta$ or $\theta \perp x$.

3 GIFC of type I, II, III, and IV

In this section, we generalize contractions (1.1) and (1.2) by employing two functions $\Psi, \Phi : (0, \infty) \to (-\infty, \infty)$ and the property (P) with some necessary conditions and hence obtaining some new fixed-point theorems in the orthogonal metric spaces. These fixed-point theorems generalize some results presented by Proinov [17], Nazam et al. [20], and Karapinar et al. [9–12].

Let d be the metric on $\mathcal{A}$ and $S : \mathcal{A} \to CB(\mathcal{A})$ be a set-valued mapping. Let the mappings $\check{I}_1, \check{I}_2, \check{I}_3, \check{I}_4$: $\mathcal{A} \times \mathcal{A} \to [0, \infty)$ be defined by

$$\begin{aligned}\check{I}_1(x, y) = d(x, y)[d(x, Sx)]^{\frac{1}{(a-b)(a-c)}}[d(y, Sy)]^{\frac{1}{(a-b)(a-c)}} \\ \times [d(x, Sx) + d(y, Sy)]^{\frac{1}{(b-a)(b-c)}}[d(x, Sy) + d(y, Sx)]^{\frac{1}{(c-a)(c-b)}}\Big] \quad (\text{for some } a, b, c > 0).\end{aligned}$$

$$\begin{aligned}\check{I}_2(x, y) = d(x, y)[d(x, Sx)]^{\frac{a}{(a-b)(a-c)}}[d(y, Sy)]^{\frac{a}{(a-b)(a-c)}} \\ \times [d(x, Sx) + d(y, Sy)]^{\frac{b}{(b-a)(b-c)}}[d(x, Sy) + d(y, Sx)]^{\frac{c}{(c-a)(c-b)}}\Big] \quad (\text{for some } a, b, c > 0).\end{aligned}$$

$$\check{I}_3(x, y) = \max\left\{\begin{matrix} d(x, y), [d(x, Sx)]^{\frac{a^2}{(a-b)(a-c)}}[d(y, Sy)]^{\frac{a^2}{(a-b)(a-c)}} \\ [d(x, Sx) + d(y, Sy)]^{\frac{b^2}{(b-a)(b-c)}} \\ [d(x, Sy) + d(y, Sx)]^{\frac{c^2}{(c-a)(c-b)}} \end{matrix}\right\} \quad (\text{for some } a, b, c > 0).$$

$$\begin{aligned}\check{I}_4(x, y) = d(x, y)^{\frac{a^3}{(a-b)(a-c)}}d(y, Sy)^{\frac{a^3}{(a-b)(a-c)}} \\ \times [d(x, Sx) + d(y, Sy)]^{\frac{b^3}{(b-a)(b-c)}}[d(x, Sy) + d(y, Sx)]^{\frac{c^3}{(c-a)(c-b)}} \quad (\text{for some } a, b, c > 0).\end{aligned}$$

We begin with the following definition. The novelty of generalized interpolative fractional contraction (GIFC) is that it allows the exponents of the distances to be fractions. The classical contractions contain sum of distances, while GIFC contains the product of distances with fractional exponents. The studies on GIFC are independents of the results appeared in [21–23].

Definition 3.1. Let $(\mathcal{A}, \perp, d)$ be an OMS. A mapping $S : \mathcal{A} \to CB(\mathcal{A})$ is said to be a GIFC of type I, II, III, and IV for $i = 1, 2, 3, 4$ respectively, if there exist a strictly $\perp$-admissible mapping f and $a, b, c \in (0, \infty]$ for $i = 1$ and $a, b, c \in (0, \infty)$ for $i = 2, 3, 4$ such that

$$\Psi(f(x, y)H(Sx, Sy)) \leq \Phi(\check{I}_i(x, y)), \tag{3.1}$$

for all $(x, y) \in \Lambda$, $H(Sx, Sy) > 0$, $x \notin Sx$ and $y \notin Sy$.

If either $a = \infty$ or $b = \infty$ or $c = \infty$ in GIFC of type I, we receive the recently announced (ψ, ϕ)-contraction by Proinov [17] provided $(x, y) \notin \Lambda$.

We also note that for $\Phi(x) = \Psi(x) - \tau$ and $\Psi(x) = \ln(x)$ for all $x \in (0, \infty)$, $\tau > 0$, the contraction (3.1 $i = 1$) can be written as follows:

$$\begin{aligned}\tau + \ln(f(x, y)H(Sx, Sy)) \leq{} & \ln(d(x, y)) + \frac{1}{(a-b)(a-c)}\ln(d(x, Sx)) + \frac{1}{(a-b)(a-c)}\ln(d(y, Sy)) \\ & + \frac{1}{(a-b)(a-c)}\ln[d(x, Sx) + d(y, Sy)] + \frac{1}{(a-b)(a-c)}\ln[d(x, Sy) + d(y, Sx)],\end{aligned}$$

and then, we have

$$\begin{aligned}\tau + \Psi(f(x, y)H(Sx, Sy)) \leq{} & \Psi(d(x, y)) + \frac{1}{(a-b)(a-c)}\Psi(d(x, Sx)) + \frac{1}{(a-b)(a-c)}\Psi(d(y, Sy)) \\ & + \frac{1}{(a-b)(a-c)}\Psi[d(x, Sx) + d(y, Sy)] + \frac{1}{(a-b)(a-c)}\Psi[d(x, Sy) + d(y, Sx)].\end{aligned}$$

This represents a general version of the contraction introduced in [24], and if either $a = \infty$ or $b = \infty$ or $c = \infty$ and $(x, y) \notin \Lambda$, we have exactly same contraction that was introduced in [24].

Remark 3.2. It is very important to note that the set of self-mappings satisfying property P and contraction (3.1) is not empty. For example, the mappings $S(x) = 2 - 3x$ for all $x \in (-\infty, \infty)$ and $S(x) = 2x - 1$ for all $x \in [2.5, \infty)$ satisfy both the property P and contraction (3.1) with $\Phi(x) = \Psi(x) - \tau$ and $\Psi(x) = \ln(x)$ for all $x \in (0, \infty)$, where $\tau > 0$.

The following example explains (3.1 $i = 1$).

Example 3.3. Let $\mathcal{A} = [1, 7)$ and define the relation $\perp$ on $\mathcal{A}$ by

$$x \perp y \quad \text{if } xy \in \{x, y\}.$$

Then $\perp$ is an orthogonal relation, and so $(\mathcal{A}, \perp)$ is an O-set. Let d be the Euclidean metric on $\mathcal{A}$, and then, $(\mathcal{A}, d)$ is an incomplete metric space. Define $\Psi, \Phi : \mathbb{R}^+ \to \mathbb{R}$ by

$$\Psi(x) = \begin{cases} x + 1 & \text{if } x \in \left\{\frac{10}{3}, 6, \frac{16}{3}, \frac{28}{5}\right\} \\ x + 10 & \text{if } x \in \mathbb{R}^+ - \left\{\frac{10}{3}, 6, \frac{16}{3}, \frac{28}{5}\right\} \end{cases} \qquad \Phi(x) = \begin{cases} \frac{x}{2} & \text{if } x \in \left\{\frac{10}{3}, 6, \frac{16}{3}, \frac{28}{5}\right\} \\ x + 5 & \text{if } x \in \mathbb{R}^+ - \left\{\frac{10}{3}, 6, \frac{16}{3}, \frac{28}{5}\right\}. \end{cases}$$

Let $S : \mathcal{A} \to CB(\mathcal{A})$ be defined by

$$S(x) = \begin{cases} \{5, 6\} & \text{if } 1 \leq x < 2 \\ \{3, 4\} & \text{if } 2 \leq x < 3 \\ \{1, 6\} & \text{if } 3 \leq x < 7. \end{cases}$$

Define $f : \mathcal{A} \times \mathcal{A} \to [1, \infty)$ by

$$f(x,y) = \begin{cases} x + \dfrac{2}{1+y} & \text{if } x \perp y, \\ 1 & \text{otherwise}. \end{cases}$$

Then f is $\perp$-admissible. Our calculations show that $H(Sx, Sy) = 2$, $f(x, y) = \frac{5}{3}$, $d(x, Sx) = 4$, $d(y, Sy) = 1$, $d(x, Sy) = 2$, and $d(y, Sx) = 3$if $x = 1$, and $y = 2(1 \perp 2)$. Thus, there exist $a, b, c \in (0, \infty]$ and $\lambda \in (0, 1)$ such that

$$\frac{10}{3} = f(x,y)H(Sx,Sy) > \lambda \check{I}_1(x,y) = \lambda 4^{\frac{1}{(a-b)(a-c)}} \cdot 5^{\frac{1}{(b-a)(b-c)} + \frac{1}{(c-a)(c-b)}}.$$

Thus, S is not a $\perp$-admissible orthogonal interpolative set-valued contraction. However, S is a GIFC. Indeed,

$$\frac{13}{3} = \Psi(f(x,y)H(Sx,Sy)) \le \Phi(\check{I}_1(x,y)) = 4^{\frac{1}{(a-b)(a-c)}} \cdot 5^{\frac{1}{(b-a)(b-c)} + \frac{1}{(c-a)(c-b)}} + 5.$$

We obtain the same conclusions for $x = 1$, $y = 3(1 \perp 3)$; $x = 1$, $y = 4(1 \perp 4)$, and $x = 1$, $y = 5$.

The following main theorem states some conditions that guarantee the existence of a fixed point of a mapping S satisfying (3.1).

Theorem 3.4. *Let* $(\mathcal{A}, \perp, d)$ *be an* $\perp$*-regular O-complete metric space (in short, OCMS). Let* $S : \mathcal{A} \to CB(\mathcal{A})$ *be an* $\perp$*-preserving mapping verifying* (3.1) *for* $i = 1$ *and property (P). Suppose the relation* $\perp$ *is transitive and the functions* $\Psi, \Phi : (0, \infty) \to (-\infty, \infty)$ *are so that:*

(i) *for each* $\hbar_0 \in \mathcal{A}$*, there is* $\hbar_1 \in S(\hbar_0)$ *such that* $\hbar_1 \perp \hbar_0$ *or* $\hbar_0 \perp \hbar_1$;
(ii) Ψ, Φ *are nondecreasing and* $\Phi(y) < \Psi(y)$ *for all* $y > 0$;
(iii) $\limsup_{y\to\delta+}\Phi(y) < \Psi(\delta+)$ *for all* $\delta > 0$;
(iv) $\limsup_{a\to 0}\Phi(a) \le \liminf_{a\to\xi+}\Psi(a)$.

Then, S has a fixed point in $\mathcal{A}$.

Proof. By (i), for an arbitrary $\hbar_0 \in \mathcal{A}$, there is $\hbar_1 \in S(\hbar_0)$ so that $\hbar_0 \perp \hbar_1$ or $\hbar_1 \perp \hbar_0$. Since S is $\perp$-preserving, there is $\hbar_2 \in S(\hbar_1)$ so that $\hbar_1 \perp \hbar_2$ or $\hbar_2 \perp \hbar_1$ and then $\hbar_3 \in S(\hbar_2)$ so that $\hbar_2 \perp \hbar_3$ or $\hbar_3 \perp \hbar_2$. In general, there is $\hbar_{n+1} \in S(\hbar_n)$ in order that $\hbar_n \perp \hbar_{n+1}$ or $\hbar_{n+1} \perp \hbar_n$ for all $n \ge 0$. Hence, $f(\hbar_n, \hbar_{n+1}) > 1$. Note that if $\hbar_n \in S(\hbar_n)$, then $\hbar_n$ is a fixed point of S. Suppose that $\hbar_n \notin S(\hbar_n)$ for all $n \ge 0$. Thus, $H(S\hbar_{n-1}, S\hbar_n) > 0$ (otherwise $\hbar_n \in S\hbar_n$). Since $f(\hbar_n, \hbar_{n+1}) > 1$ and $S(\hbar_n), S(\hbar_{n+1})$ are bounded and closed sets, by Lemma 2.14, there is $\hbar_{n+1} \in S(\hbar_n)(\hbar_n \ne \hbar_{n+1})$ so that $d(\hbar_n, \hbar_{n+1}) \le f(\hbar_{n-1}, \hbar_n)H(S(\hbar_{n-1}), S(\hbar_n))$ for all $n \ge 1$. By (ii) and (3.1), one writes

$$\Psi(h_n) < \Psi(f(\hbar_{n-1}, \hbar_n)H(S(\hbar_{n-1}), S(\hbar_n))) \le \Phi(\check{I}_1(\hbar_{n-1}, \hbar_n)) \le \Phi(h_{n-1}).$$

In view of second part of (ii), we write

$$\Psi(h_n) \le \Phi(h_{n-1}) < \Psi(h_{n-1}). \tag{3.2}$$

Since Ψ is non decreasing, one obtains $h_n < h_{n-1}$ for each $n \ge 1$. This shows that the sequence $\{h_n\}$ is decreasing, so that there is $L \ge 0$ such that $\lim_{n\to\infty} h_n = L+$. If $L > 0$, by (3.2), one obtains

$$\Psi(L+) = \lim_{n\to\infty}\Psi(h_n) \le \lim_{n\to\infty}\sup\Phi(h_{n-1}) \le \lim_{a\to L+}\sup\Phi(a).$$

This contradicts (iii), so $L = 0$, i.e., S is asymptotically regular.

We claim that $\{\hbar_n\}$ is a Cauchy sequence. If not, suppose that $\{\hbar_n\}$ is not a Cauchy sequence. Here, by Lemma 2.10, there are $\{\hbar_{n_k}\}$, $\{\hbar_{m_k}\}$ of $\{\hbar_n\}$, and $\xi > 0$, such that (2.1) and (2.2) hold. By (2.1), we infer that $d(\hbar_{n_k+1}, \hbar_{m_k+1}) > \xi$. Since $\hbar_n \perp \hbar_{n+1}$ for all $n \ge 0$, by transitivity of $\perp$, we have $\hbar_{n_k} \perp \hbar_{m_k}$, and hence, $f(\hbar_{n_k}, \hbar_{m_k}) > 1$ for all $k \ge 1$. Letting $x = \hbar_{n_k}$ and $y = \hbar_{m_k}$ in (3.1 $i = 1$), we have for each $k \ge 1$,

$$\Psi\Big(d\Big(\hbar_{n_k+1},\hbar_{m_k+1}\Big)\Big)\le\Psi(f\Big(\hbar_{n_k},\hbar_{m_k}\Big)H\Big(S\hbar_{n_k},S\hbar_{m_k}\Big)\Big)\le\Phi(\check{I}_1\Big(\hbar_{n_k},\hbar_{m_k}\Big)).$$

We note that

$$\begin{aligned}\check{I}_1\Big(\hbar_{n_k},\hbar_{m_k}\Big)&=d\Big(\hbar_{n_k},\hbar_{m_k}\Big)d\Big(\hbar_{n_k},S\hbar_{n_k}\Big)^{\frac{1}{(a-b)(a-c)}}d\Big(\hbar_{m_k},S\hbar_{m_k}\Big)^{\frac{1}{(a-b)(a-c)}}\\&\quad\times\Big[d\Big(\hbar_{n_k},S\hbar_{n_k}\Big)+d\Big(\hbar_{m_k},S\hbar_{m_k}\Big)\Big]^{\frac{1}{(b-a)(b-c)}}\Big[d\Big(\hbar_{n_k},S\hbar_{m_k}\Big)+d\Big(\hbar_{m_k},S\hbar_{n_k}\Big)\Big]^{\frac{1}{(c-a)(c-b)}}\Big]\\&\le d\Big(\hbar_{n_k},\hbar_{m_k}\Big)d\Big(\hbar_{n_k},\hbar_{n_k+1}\Big)^{\frac{1}{(a-b)(a-c)}}d\Big(\hbar_{m_k},\hbar_{m_k+1}\Big)^{\frac{1}{(a-b)(a-c)}}\Big[d\Big(\hbar_{n_k},\hbar_{n_k+1}\Big)+d\Big(\hbar_{m_k},\hbar_{m_k+1}\Big)\Big]^{\frac{1}{(b-a)(b-c)}}\\&\quad\times\Big[d\Big(\hbar_{n_k},\hbar_{m_k+1}\Big)+d\Big(\hbar_{m_k},\hbar_{n_k+1}\Big)\Big]^{\frac{1}{(c-a)(c-b)}}=B_k.\end{aligned}$$

If $\hbar_k=d(\hbar_{n_k+1},\hbar_{m_k+1})$, we have

$$\Psi(\hbar_k)\le\Phi(B_k),\quad\text{for all }k\ge1.\tag{3.3}$$

By (2.1), we have $\lim_{k\to\infty}\hbar_k=\xi+$ and (3.3) implies

$$\liminf_{a\to\xi+}\Psi(a)\le\liminf_{k\to\infty}\Psi(\hbar_k)\le\limsup_{k\to\infty}\Phi(B_k)\le\limsup_{a\to0}\Phi(a).$$

It is a contradiction to (iv), so $\{\hbar_n\}$ is a Cauchy sequence in the OCMS $(\mathcal{A},\perp,d)$, and hence, there is $a^*\in\mathcal{A}$ so that $\hbar_n\to a^*$ as $n\to\infty$, and the $\perp$-regularity of $(\mathcal{A},\perp,d)$ yields that $\hbar_n\perp a^*$ or $a^*\perp\hbar_n$. Thus, $f(\hbar_n,a^*)>1$. We claim that $d(a^*,S(a^*))=0$. Assume that $d(\hbar_{n+1},S(a^*))>0$ for infinitely many values of n. By (3.1 $i=1$),

$$\Psi(d(\hbar_{n+1},S(a^*)))\le\Psi(f(\hbar_n,a^*)H(S(\hbar_n),S(a^*)))\le\Phi(\check{I}_1(\hbar_n,a^*)).$$

By the first part of (ii), we get $d(\hbar_{n+1},S(a^*))<\check{I}_1(\hbar_n,a^*)$. Applying limit $n\to\infty$, we obtain $d(a^*,S(a^*))\le0$. This implies that $d(a^*,S(a^*))=0$, and hence, $a^*\in S(a^*)$. □

Next result states new conditions ensuring the existence of the fixed points of mapping S verifying (3.1 $i=1$).

Theorem 3.5. *Let $(\mathcal{A},\perp,d)$ be an $\perp$-regular OCMS. Let $S:\mathcal{A}\to CB(\mathcal{A})$ be an $\perp$-preserving mapping verifying* (3.1 $i=1$) *and property* (P)*. Assume the relation $\perp$ is transitive and the functions $\Psi,\Phi:(0,\infty)\to(-\infty,\infty)$ are so that*

(*i*) *for each $\hbar_0\in\mathcal{A}$, there is $\hbar_1\in S(\hbar_0)$ such that $\hbar_0\perp\hbar_1$ or $\hbar_1\perp\hbar_0$;*
(*ii*) $\Phi(y)<\Psi(y)$ *for all* $y>0$;
(*iii*) $\inf_{a>\xi>0}\Psi(a)>-\infty$;
(*iv*) *if $\{\Psi(\hbar_n)\}$ and $\{\Phi(\hbar_n)\}$ are converging to same limit and $\{\Psi(\hbar_n)\}$ is strictly decreasing, then* $\lim_{n\to\infty}\hbar_n=0$;
(*v*) $\limsup_{a\to0}\Phi(a)<\liminf_{a\to\xi+}\Psi(a)$ *for all* $\xi>0$;
(*vi*) $\limsup_{a\to0}\Phi(a)<\liminf_{a\to\xi}\Psi(a)$ *for all* $\xi>0$.

Then, S possesses a fixed point in $\mathcal{A}$.

Proof. By (i), for an arbitrary $\hbar_0\in\mathcal{A}$, there is $\hbar_1\in S(\hbar_0)$ so that $\hbar_0\perp\hbar_1$ or $\hbar_1\perp\hbar_0$. Since S is $\perp$-preserving, there is $\hbar_2\in S(\hbar_1)$ so that $\hbar_1\perp\hbar_2$ or $\hbar_2\perp\hbar_1$ and then, $\hbar_3\in S(\hbar_2)$ so that $\hbar_2\perp\hbar_3$ or $\hbar_3\perp\hbar_2$. In general, there is $\hbar_{n+1}\in S(\hbar_n)$ in order that $\hbar_n\perp\hbar_{n+1}$ or $\hbar_{n+1}\perp\hbar_n$ for all $n\ge0$. Hence, $f(\hbar_n,\hbar_{n+1})>1$. Note that if $\hbar_n\in S(\hbar_n)$, then $\hbar_n$ is a fixed point of S. Suppose that $\hbar_n\notin S(\hbar_n)$ for all $n\ge0$. Thus, $H(S\hbar_{n-1},S\hbar_n)>0$ (otherwise $\hbar_n\in S\hbar_n$). Since $f(\hbar_n,\hbar_{n+1})>1$ and $S(\hbar_n),S(\hbar_{n+1})$ are bounded and closed sets, by Lemma 2.14, there is $\hbar_{n+1}\in S(\hbar_n)(\hbar_n\ne\hbar_{n+1})$ so that $d(\hbar_n,\hbar_{n+1})\le f(\hbar_{n-1},\hbar_n)H(S(\hbar_{n-1}),S(\hbar_n))$ for all $n\ge1$. By (ii) and (3.1), one writes

$$\Psi(d(\hbar_n,\hbar_{n+1}))\le\Psi(f(\hbar_{n-1},\hbar_n)H(S(\hbar_{n-1}),S(\hbar_n)))\le\Phi(\check{I}_1(\hbar_{n-1},\hbar_n))\le\Psi(d(\hbar_{n-1},\hbar_n)).\tag{3.4}$$

The inequality (3.4) shows that $\{\Psi(d(\hbar_{n-1},\hbar_n))\}$ is strictly decreasing. If it is not bounded below, in view of (iii), we obtain $\inf_{d(\hbar_{n-1},\hbar_n)>\xi}\Psi(d(\hbar_{n-1},\hbar_n))>-\infty$. This implies that

$$\lim \inf_{d(\hbar_{n-1},\hbar_n)\to\xi+} \Psi(d(\hbar_{n-1},\hbar_n)) > -\infty.$$

Thus, $\lim_{n\to\infty} d(\hbar_{n-1},\hbar_n) = 0$, otherwise we have

$$\lim \inf_{d(\hbar_{n-1},\hbar_n)\to\xi+} \Psi(d(\hbar_{n-1},\hbar_n)) = -\infty$$

(i.e., a contradiction to (iii)). If it is bounded below, then $\{\Psi(d(\hbar_{n-1},\hbar_n))\}$ is a convergent sequence and by (3.4), $\{\Phi(d(\hbar_{n-1},\hbar_n))\}$ also converges, and both have same limit. Thus, by (iv), one obtains $\lim_{n\to\infty} d(\hbar_{n-1},\hbar_n) = 0$. Hence, S is asymptotically regular.

Now, we claim that $\{\hbar_n\}$ is a Cauchy sequence. If $\{\hbar_n\}$ is not a Cauchy sequence, so by Lemma 2.10, there exist $\{\hbar_{n_k}\}$, $\{\hbar_{m_k}\}$, and $\xi > 0$ such that (2.1) and (2.2) hold. By (2.1), we infer that $d(\hbar_{n_k+1},\hbar_{m_k+1}) > \xi$. Since $\hbar_n \perp \hbar_{n+1}$ for all $n \geq 0$ so by transitivity of $\perp$, we have $\hbar_{n_k} \perp \hbar_{m_k}$, and hence, $f(\hbar_{n_k},\hbar_{m_k}) > 1$ for all $k \geq 1$. Letting $x = \hbar_{n_k}$ and $y = \hbar_{m_k}$ in (3.1), one writes for all $k \geq 1$,

$$\Psi\left(d\left(\hbar_{n_k+1},\hbar_{m_k+1}\right)\right) \leq \Psi(f\left(\hbar_{n_k},\hbar_{m_k}\right)H\left(S\hbar_{n_k},S\hbar_{m_k}\right)) \leq \Phi(\left(\check{I}_1\left(\hbar_{n_k},\hbar_{m_k}\right)\right)).$$

We note that

$$\begin{aligned}\check{I}_1\left(\hbar_{n_k},\hbar_{m_k}\right) &= d\left(\hbar_{n_k},\hbar_{m_k}\right) d\left(\hbar_{n_k},S\hbar_{n_k}\right)^{\frac{1}{(a-b)(a-c)}} d\left(\hbar_{m_k},S\hbar_{m_k}\right)^{\frac{1}{(a-b)(a-c)}} \\ &\quad \times \left[d\left(\hbar_{n_k},S\hbar_{n_k}\right) + d\left(\hbar_{m_k},S\hbar_{m_k}\right)\right]^{\frac{1}{(b-a)(b-c)}} \left[d\left(\hbar_{n_k},S\hbar_{m_k}\right) + d\left(\hbar_{m_k},S\hbar_{n_k}\right)\right]^{\frac{1}{(c-a)(c-b)}} \\ &\leq d\left(\hbar_{n_k},\hbar_{m_k}\right) d\left(\hbar_{n_k},\hbar_{n_k+1}\right)^{\frac{1}{(a-b)(a-c)}} d\left(\hbar_{m_k},\hbar_{m_k+1}\right)^{\frac{1}{(a-b)(a-c)}} \\ &\quad \times \left[d\left(\hbar_{n_k},\hbar_{n_k+1}\right) + d\left(\hbar_{m_k},\hbar_{m_k+1}\right)\right]^{\frac{1}{(b-a)(b-c)}} \left[d\left(\hbar_{n_k},\hbar_{m_k+1}\right) + d\left(\hbar_{m_k},\hbar_{n_k+1}\right)\right]^{\frac{1}{(c-a)(c-b)}} = B_k.\end{aligned}$$

If $\hbar_k = d(\hbar_{n_k+1},\hbar_{m_k+1})$, we have

$$\Psi(\hbar_k) \leq \Phi(B_k), \quad \text{for all } k \geq 1. \tag{3.5}$$

By (2.1), we have $\lim_{k\to\infty} \hbar_k = \xi+$ and (3.5) implies

$$\liminf_{a\to\xi+} \Psi(a) \leq \liminf_{k\to\infty} \Psi(\hbar_k) \leq \limsup_{k\to\infty} \Phi(B_k) \leq \limsup_{a\to 0} \Phi(a).$$

It contradicts (v), so $\{\hbar_n\}$ is a Cauchy sequence in the OCMS $\mathcal{A}$. Hence, there is $a^* \in \mathcal{A}$ in order that $\hbar_n \to a^*$ as $n \to \infty$.

To show that $Sa^* = a^*$, we have two cases:

Case 1. If $d(\hbar_{n+1}, Sa^*) = 0$ for some $n \geq 0$, then since

$$d(a^*, Sa^*) \leq d(a^*, \hbar_{n+1}) + d(\hbar_{n+1}, Sa^*) = d(a^*, \hbar_{n+1})$$

taking limit $n \to \infty$ on both sides, we have $d(a^*, Sa^*) \leq 0$. This implies $d(a^*, S(a^*)) = 0$; thus, $a^* = S(a^*)$.

Case 2. If for all $n \geq 0$, $d(\hbar_{n+1}, Sa^*) > 0$, then by $\perp$-regularity of $\mathcal{A}$, we find $\hbar_n \perp a^*$ or $a^* \perp \hbar_n$, so $f(\hbar_n, a^*) > 1$. By (3.1 $i = 1$), one writes

$$\Psi(d(\hbar_{n+1}, Sa^*)) \leq \Psi(f(\hbar_n, a^*)H(S\hbar_n, Sa^*)) \leq \Phi(\check{I}_1(\hbar_n, a^*)) \quad \text{for all } n \geq 0.$$

By taking $H_n = d(\hbar_{n+1}, Sa^*)$ and $b_n = \check{I}_1(\hbar_n, a^*)$, one writes

$$\Psi(H_n) \leq \Phi(b_n) \quad \text{for all } n \geq 0. \tag{3.6}$$

Take $\xi = d(a^*, Sa^*)$. Note that $H_n \to \xi$ and $b_n \to 0$ as $n \to \infty$. Applying limits on (3.6), we have

$$\lim \inf_{a\to\xi} \Psi(a) \leq \lim \inf_{n\to\infty} \Psi(H_n) \leq \lim \sup_{n\to\infty} \Phi(b_n) \leq \lim \inf_{a\to 0} \Phi(a).$$

This contradicts (vi) if $\xi > 0$. Thus, we have $d(a^*, Sa^*) = 0$, i.e., $a^* \in Sa^*$, that is, a^* is a fixed point of S. □

Now using the property (P) and doing some simplifications, we have

$$\check{I}_2(\hbar_{n-1},\hbar_n) \leq d(\hbar_{n-1},\hbar_n). \tag{3.7}$$

$$\check{I}_3(\hbar_{n-1},\hbar_n) \leq \max\{d(\hbar_{n-1},\hbar_n), d(\hbar_n,\hbar_{n+1})\}. \tag{3.8}$$

$$\check{I}_4(\hbar_{n-1}, \hbar_n) \leq \max\{d(\hbar_{n-1}, \hbar_n), d(\hbar_n, \hbar_{n+1})\}. \tag{3.9}$$

The next two results address the GIFC of type II and III.

Theorem 3.6. *Let $(\mathcal{A}, \perp, d)$ be an $\perp$-regular OCMS. Let $S : \mathcal{A} \to CB(\mathcal{A})$ be an $\perp$-preserving mapping verifying (3.1) for $i = 2, 3$ and property (P). Suppose the relation $\perp$ is transitive and the functions $\Psi, \Phi : (0, \infty) \to (-\infty, \infty)$ are so that*

(*i*) *for each $\hbar_0 \in \mathcal{A}$, there is $\hbar_1 \in S(\hbar_0)$ such that $\hbar_1 \perp \hbar_0$ or $\hbar_0 \perp \hbar_1$;*
(*ii*) *Ψ, Φ are non decreasing and $\Phi(y) < \Psi(y)$ for all $y > 0$;*
(*iii*) $\limsup_{y\to\delta+}\Phi(y) < \Psi(\delta+)$ *for all $\delta > 0$;*
(*iv*) $\limsup_{a\to 0}\Phi(a) \leq \liminf_{a\to\xi+}\Psi(a)$.

Then S has a fixed point in $\mathcal{A}$.

Proof. Keeping in view inequalities (3.7) and (3.8) with the fact that $d(\hbar_{n-1}, \hbar_n) > d(\hbar_n, \hbar_{n+1})$ and following the proof of Theorem 3.4, we assert that S admits a fixed point in $\mathcal{A}$. If $d(\hbar_{n-1}, \hbar_n) < d(\hbar_n, \hbar_{n+1})$, then we have a contradiction to the definition of function Ψ. □

Theorem 3.7. *Let $(\mathcal{A}, \perp, d)$ be an $\perp$-regular OCMS. Let $S : \mathcal{A} \to CB(\mathcal{A})$ be an $\perp$-preserving mapping verifying (3.1 $i = 2, 3$) and property (P). Assume the relation $\perp$ is transitive and the functions $\Psi, \Phi : (0, \infty) \to (-\infty, \infty)$ are so that*

(*i*) *for each $\hbar_0 \in \mathcal{A}$, there is $\hbar_1 \in S(\hbar_0)$ such that $\hbar_0 \perp \hbar_1$ or $\hbar_1 \perp \hbar_0$;*
(*ii*) *$\Phi(y) < \Psi(y)$ for all $y > 0$;*
(*iii*) $\inf_{a>\xi>0}\Psi(a) > -\infty$;
(*iv*) *if $\{\Psi(\hbar_n)\}$ and $\{\Phi(\hbar_n)\}$ are converging to same limit and $\{\Psi(\hbar_n)\}$ is strictly decreasing, then $\lim_{n\to\infty}\hbar_n = 0$;*
(*v*) $\limsup_{a\to 0}\Phi(a) < \liminf_{a\to\xi+}\Psi(a)$ *for all $\xi > 0$;*
(*vi*) $\limsup_{a\to 0}\Phi(a) < \liminf_{a\to\xi}\Psi(a)$ *for all $\xi > 0$.*

Then, S possesses a fixed point in $\mathcal{A}$.

Proof. Keeping in view inequalities (3.7) and (3.8) with the fact that $d(\hbar_{n-1}, \hbar_n) > d(\hbar_n, \hbar_{n+1})$ and following the proof of Theorem 3.4, we assert that S admits a fixed point in $\mathcal{A}$. If $d(\hbar_{n-1}, \hbar_n) < d(\hbar_n, \hbar_{n+1})$, then we have a contradiction to the definition of function Ψ. □

The next two results address GIFC of type IV.

Theorem 3.8. *Let $(\mathcal{A}, \perp, d)$ be an $\perp$-regular OCMS. Let $S : \mathcal{A} \to CB(\mathcal{A})$ be an $\perp$-preserving mapping verifying (3.1) for $i = 4$ with $a + b + c < 0.5$ and property (P). Suppose the relation $\perp$ is transitive and the functions $\Psi, \Phi : (0, \infty) \to (-\infty, \infty)$ are so that*

(*i*) *for each $\hbar_0 \in \mathcal{A}$, there is $\hbar_1 \in S(\hbar_0)$ such that $\hbar_1 \perp \hbar_0$ or $\hbar_0 \perp \hbar_1$;*
(*ii*) *Ψ, Φ are non decreasing and $\Phi(y) < \Psi(y)$ for all $y > 0$;*
(*iii*) $\limsup_{y\to\delta+}\Phi(y) < \Psi(\delta+)$ *for all $\delta > 0$;*
(*iv*) $\limsup_{a\to 0}\Phi(a) \leq \liminf_{a\to\xi+}\Psi(a)$.

Then S has a fixed point in $\mathcal{A}$.

Proof. Keeping in view inequality (3.9) and following the proof of Theorem 3.6, we assert that S admits a fixed point in $\mathcal{A}$. □

Theorem 3.9. *Let $(\mathcal{A}, \perp, d)$ be an $\perp$-regular OCMS. Let $S : \mathcal{A} \to CB(\mathcal{A})$ be an $\perp$-preserving mapping verifying (3.1 $i = 4$) with $a + b + c < 0.5$ and property (P). Assume the relation $\perp$ is transitive and the functions $\Psi, \Phi : (0, \infty) \to (-\infty, \infty)$ are so that*

(*i*) *for each $\hbar_0 \in \mathcal{A}$, there is $\hbar_1 \in S(\hbar_0)$ such that $\hbar_0 \perp \hbar_1$ or $\hbar_1 \perp \hbar_0$;*

(*ii*) $\Phi(y) < \Psi(y)$ *for all* $y > 0$;
(*iii*) $\inf_{a>\xi>0}\Psi(a) > -\infty$;
(*iv*) *if* $\{\Psi(\hbar_n)\}$ *and* $\{\Phi(\hbar_n)\}$ *are converging to same limit and* $\{\Psi(\hbar_n)\}$ *is strictly decreasing, then* $\lim_{n\to\infty}\hbar_n = 0$;
(*v*) $\limsup_{a\to 0}\Phi(a) < \liminf_{a\to\xi^+}\Psi(a)$ *for all* $\xi > 0$;
(*vi*) $\limsup_{a\to 0}\Phi(a) < \liminf_{a\to\xi}\Psi(a)$ *for all* $\xi > 0$.

Then S *possesses a fixed point in* $\mathcal{A}$.

Proof. Keeping in view the inequality (3.9) and following the proof of Theorem 3.7, we assert that S admits a fixed-point in $\mathcal{A}$. □

4 Corollaries

Let us define $\Psi(y) = y$ for all $y > 0$, in any one of Theorems 3.4 and 3.5, we receive a general version of the interpolative Boyd-Wong fixed-point theorem proved in [12], and defining $\Phi(y) = \Psi(y) - \tau$ in Theorem 3.4, we receive the general versions of the results appeared in [25,26].

Corollary 4.1. *Let* $(\mathcal{A}, d)$ *be a complete metric space. Let* $S : \mathcal{A} \to CB(\mathcal{A})$ *be a mapping so that*

$$\Psi(H(Sx, Sy)) \leq \Psi(\check{I}_i(x, y)) - \tau \quad \forall x, y \in \mathcal{A},\ \ i = 1, 2, 3, 4 \text{ provided } H(Sx, Sy) > 0,$$

where $\Psi : (0, \infty) \to \mathbb{R}$ *is non decreasing and* $\tau > 0$. *Then there is a fixed point of* S *in* $\mathcal{A}$.

If we define $\Phi(y) = \Psi(y) - \tau(y)$ in Theorem 3.4, we obtain an interpolative fractional version of the fixed-point theorem presented in [27].

Corollary 4.2. *Let* $(\mathcal{A}, d)$ *be a complete metric space. Let* $S : \mathcal{A} \to CB(\mathcal{A})$ *be a mapping so that*

$$\tau(d(x, y)) + \Psi(H(Sx, Sy)) \leq \Psi(\check{I}_i(x, y)) \quad \forall x, y \in \mathcal{A},\ \ i = 1, 2, 3, 4 \text{ provided } H(Sx, Sy) > 0,$$

where $\Psi : (0, \infty) \to \mathbb{R}$ *is non decreasing and* $\liminf_{a\to t}\tau(a) > 0,\ \forall t \geq 0$. *Then* S *has a fixed point in* $\mathcal{A}$.

We receive the following interpolative fractional version of Moradi theorem [28], if we take $\Phi(y) = h(\Psi(y))$ in Theorem 3.4.

Corollary 4.3. *Let* $(\mathcal{A}, \perp, d)$ *be an* $\perp$*-regular OCMS. Assume that* $S : \mathcal{A} \to CB(\mathcal{A})$ *is* $\perp$*-preserving mapping satisfying* (P) *so that*

$$\Psi(f(x, y)H(Sx, Sy)) \leq h(\Psi(\check{I}_i(x, y))) \quad \forall (x, y) \in \Lambda,\ \ i = 1, 2, 3, 4 \text{ provided } H(Sx, Sy) > 0,$$

where
(*i*) $h : I \to [0, \infty)$ *is an upper semi-continuous function with* $h(y) < y$ *for all* $y \in I \subset \mathbb{R}$;
(*ii*) $\Psi : (0, \infty) \to I$ *is non decreasing.*

Assume that for each $\hbar_0 \in \mathcal{A}$ *there is* $\hbar_1 \in S(\hbar_0)$ *such that* $\hbar_0 \perp \hbar_1$ *or* $\hbar_1 \perp \hbar_0$. *Then,* S *has a unique fixed point in* $\mathcal{A}$.

Defining $h(y) = y^{\delta}$; $\delta \in (0, 1)$ in Corollary 4.3, we have the next result.

Corollary 4.4. *Let* $(\mathcal{A}, \perp, d)$ *be an* $\perp$*-regular and OCMS. Assume that* $S : \mathcal{A} \to CB(\mathcal{A})$ *is* $\perp$*-preserving mapping satisfying* (P) *so that*

$$\Psi(f(x, y)H(Sx, Sy)) \leq (\Psi(\check{I}_i(x, y)))^{r} \quad \forall (x, y) \in \Lambda,\ \ i = 1, 2, 3, 4 \text{ provided } H(Sx, Sy) > 0,$$

where $\Psi : (0, \infty) \to (0, 1)$ *is non decreasing. Assume that for each* $\hbar_0 \in \mathcal{A}$ *there is* $\hbar_1 \in S(\hbar_0)$ *such that* $\hbar_0 \perp \hbar_1$ or $\hbar_1 \perp \hbar_0$. *Then* S *has a fixed point in* $\mathcal{A}$.

Observe that Corollary 4.4 is an improvement of Jleli-Samet fixed-point theorem [29], and the results of Li and Jiang [30] and Ahmad et al. [31].

An improvement of Skof fixed-point theorem [32] may be stated by putting $\Phi(y) = \lambda\Psi(y)$ in Theorem 3.4 for $i = 1$ with $a = \infty$, or $b = \infty$, or $c = \infty$.

Corollary 4.5. *Let* $(\mathcal{A}, \perp, d)$ *be an* $\perp$*-regular OCMS and* $S : \mathcal{A} \to CB(\mathcal{A})$ *be an* $\perp$*-preserving mapping satisfying* (P) *so that*

$$\Psi(f(x, y)H(Sx, Sy)) \le \lambda\Psi(\check{I}_1(x, y)) \quad \forall (x, y) \in \Lambda, \text{ provided } H(Sx, Sy) > 0,$$

where $\Psi : (0, \infty) \to (0, \infty)$ *is non-decreasing and* $\lambda \in (0, 1)$. *Assume that for each* $\hbar_0 \in \mathcal{A}$ *there is* $\hbar_1 \in S(\hbar_0)$ *so that* $\hbar_0 \perp \hbar_1$ *or* $\hbar_1 \perp \hbar_0$. *Then* S *has a unique fixed point in* $\mathcal{A}$.

5 GRIFC of type I, II, III, and IV

Definition 5.1. Let $(\mathcal{A}, \perp, d)$ be an OMS. A mapping $S : \mathcal{A} \to CB(\mathcal{A})$ is said to be a generalized rational interpolative fractional contraction (GRIFC) of type I, II, III, and IV for $i = 1, 2, 3$, and 4, respectively, if there exist a strictly $\perp$-admissible mapping f, $q \ge 1$ and $a, b, c \in (0, \infty]$ for $i = 1$ and $a, b, c \in (0, \infty)$ for $i = 2, 3, 4$ such that

$$\Psi(f(x, y)H(Sx, Sy)^q) \le \Phi(\check{I}_i(x, y)), \tag{5.1}$$

for all $(x, y) \in \Lambda$, $H(Sx, Sy) > 0$, $x \notin Sx$ and $y \notin Sy$.

The contraction (5.1) represents a general version of the contraction introduced in [24]. If $q = 1$ and either $a = \infty$ or $b = \infty$ or $c = \infty$ and $(x, y) \notin \Lambda$, then (5.1) is identical to the contraction introduced in [24].

By using the property (P), we have the following information:

$$\check{I}_1(\hbar_{n-1}, \hbar_n) \le d(\hbar_{n-1}, \hbar_n)^q. \tag{5.2}$$

$$\check{I}_2(\hbar_{n-1}, \hbar_n) \le d(\hbar_{n-1}, \hbar_n)^q. \tag{5.3}$$

$$\check{I}_3(\hbar_{n-1}, \hbar_n) \le \max\{d(\hbar_{n-1}, \hbar_n)^q, d(\hbar_n, \hbar_{n+1})^q\}. \tag{5.4}$$

$$\check{I}_4(\hbar_{n-1}, \hbar_n) \le \max\{d(\hbar_{n-1}, \hbar_n)^q, d(\hbar_n, \hbar_{n+1})^q\}. \tag{5.5}$$

Keeping in view inequalities (5.2), (5.3), (5.4), and (5.5), we note that Theorems 3.4–3.9 hold true for GRIFC of type I, II, III, and IV.

6 The existence of the solution to Urysohn integral equation

In this section, we will apply Theorem 3.4 subject to the self-mapping S defined on the non empty set $\mathcal{A}$ for the existence of the unique solution to Urysohn integral equation (UIE):

$$x(\hbar) = f(\hbar) + \int_{\mathrm{IR}} K_1(\hbar, s, x(s))\mathrm{d}s. \tag{6.1}$$

This integral equation encapsulates both Volterra integral equation (VIE) and Fredholm integral equation (FIE), depending on the region of integration (IR). If IR = (a, x), where a is fixed, then UIE is VIE, and for

IR = (a, b), where a, and b are fixed, UIE is FIE. In the literature, one can find many approaches to find a unique solution to UIE (see [33–35] and the references therein). We are interested to use a fixed-point technique for this purpose. The fixed-point technique is simple and elegant to show the existence of a unique solution to further mathematical models.

Let IR be a set of finite measure and $\mathcal{L}^2_{\mathrm{IR}} = \left\{x \mid \int_{\mathrm{IR}} |x(s)|^2 \mathrm{d}s < \infty\right\}$. Define the norm $\|.\| : \mathcal{L}^2_{\mathrm{IR}} \to [0, \infty)$ by

$$\|x\|_2 = \sqrt{\int_{\mathrm{IR}} |x(s)|^2 \mathrm{d}s}, \quad \text{for all } x, y \in \mathcal{L}^2_{\mathrm{IR}}.$$

An equivalent norm can be defined as follows:

$$\|x\|_{2,\nu} = \sqrt{\sup\left\{e^{-\nu \int_{\mathrm{IR}} \alpha(s)\mathrm{d}s} \int_{\mathrm{IR}} |x(s)|^2 \mathrm{d}s\right\}}, \quad \text{for all } x \in \mathcal{L}^2_{\mathrm{IR}}; \nu > 1.$$

Then $(\mathcal{L}^2_{\mathrm{IR}}, \|.\|_{2,\nu})$ is a Banach space. Let $\mathcal{A} = \{x \in \mathcal{L}^2_{\mathrm{IR}} : x(s) > 0 \quad \text{for almost every s}\}$. The metric d_ν associated with norm $\|.\|_{2,\nu}$ is given by $d_\nu(x, y) = \|x - y\|_{2,\nu}$ for all $x, \ y \in \mathcal{A}$. Define an orthogonal relation $\perp$ on $\mathcal{A}$ by

$$a \perp \upsilon \quad \text{if and only if } a(s)\upsilon(s) \geq \upsilon(s), \quad \text{for all } a, \upsilon \in \mathcal{A}.$$

Then, $(\mathcal{A}, \perp, d)$ is an OCMS (see [18, Theorem 4.1]). Let $L : \mathcal{A} \times \mathcal{A} \to (1, \infty)$ be defined by

$$L(\delta, t) = e^{\|\delta + t\|_{\mathcal{L}^2}} \quad \text{for all } \delta, t \in \mathcal{A} \quad \text{with } \delta \perp t.$$

Then, L is a strictly $\perp$-admissible mapping. Put $M = \inf\{L(\delta, t), \quad \forall \delta, t \in \mathcal{A} \text{ with } \delta \perp t\}$.

(A1) The kernel $K_1 : \mathrm{IR} \times \mathrm{IR} \times \mathbb{R} \to \mathbb{R}$ satisfies Carathéodory conditions and

$$|K_1(\hbar, s, x(s))| \leq w(\hbar, s) + e(\hbar, s)|x(s)|; \ w, e \in \mathcal{L}^2(\mathrm{IR} \times \mathrm{IR}), e(\hbar, s) > 0.$$

(A2) The function $f : \mathrm{IR} \to [1, \infty)$ is continuous and bounded on IR.

(A3) There exists a positive constant C such that

$$\sup_{\hbar \in \mathrm{IR}} \int_{\mathrm{IR}} |K_1(\hbar, s)| \mathrm{d}s \leq C.$$

(A4) For any $x_0 \in \mathcal{L}^2_{\mathrm{IR}}$, there is $x_1 = R(x_0)$ such that $x_1 \perp x_0$ or $x_0 \perp x_1$.

(A5) There exists a non negative and measurable function $q : \mathrm{IR} \times \mathrm{IR} \to \mathbb{R}$ such that

$$\alpha(\hbar) := \int_{\mathrm{IR}} q^2(\hbar, s) \mathrm{d}s \leq \frac{1}{\nu M^2}$$

and integrable over IR with

$$|K_1(\hbar, s, x(s)) - K_1(\hbar, s, y(s))| \leq q(\hbar, s)|x(s) - y(s)|$$

for all $\hbar, s \in \mathrm{IR}$ and $x, y \in \mathcal{A}$ with $x \perp y$.

Theorem 6.1. *Suppose that the mappings f and K_1 mentioned earlier satisfy conditions* (A1)–(A5), *then the UIE* (6.1) *has a unique solution.*

Proof. Let $\mathcal{A} = \{x \in \mathcal{L}^2_{\mathrm{IR}} : x(s) > 0 \text{ for almost every s}\}$ and define the mapping $R : \mathcal{A} \to \mathcal{A}$, in accordance with the afore-mentioned notations, by

$$(Rx)(\hbar) = f(\hbar) + \int_{\mathrm{IR}} K_1(\hbar, s, x(s)) \mathrm{d}s.$$

The operator R is $\perp$-preserving:

Let $x, y \in \mathcal{A}$ with $x \perp y$, then $x(s)y(s) \geq y(s)$. Since, for almost every $\hbar \in$ IR,

$$(Rx)(\hbar) = f(\hbar) + \int_{\mathrm{IR}} K_1(\hbar, s, x(s))\mathrm{d}s \geq 1,$$

and this implies that $(Rx)(\hbar)(Ry)(\hbar) \geq (Ry)(\hbar)$. Thus, $(Rx) \perp (Ry)$.

Self-operator:

The conditions (A1) and (A3) imply that R is continuous and compact mapping from $\mathcal{A}$ to $\mathcal{A}$ (see [33, Lemma 3]).

By (A4), for any $x_0 \in \mathcal{A}$, there is $x_1 = R(x_0)$ such that $x_1 \perp x_0$ or $x_0 \perp x_1$ and using the fact that R is $\perp$-preserving, we have $x_n = R^n(x_0)$ with $x_n \perp x_{n+1}$ or $x_{n+1} \perp x_n$ for all $n \geq 0$. We will check the contractive condition (3.1) of Theorem 3.4 in the following lines. By (A5) and Holder inequality, we have

$$\begin{aligned}
|(Rx)(\hbar) - (Ry)(\hbar)|^2 &= \left| \int_{\mathrm{IR}} K_1(\hbar, s, x(s))\mathrm{d}s - \int_{\mathrm{IR}} K_1(\hbar, s, y(s))\mathrm{d}s \right|^2 \\
&\leq \left(\int_{\mathrm{IR}} |K_1(\hbar, s, x(s)) - K_1(\hbar, s, y(s))|\mathrm{d}s \right)^2 \\
&\leq \left(\int_{\mathrm{IR}} q(\hbar, s)|x(s) - y(s)|\mathrm{d}s \right)^2 \\
&\leq \int_{\mathrm{IR}} q^2(\hbar, s)\mathrm{d}s \cdot \int_{\mathrm{IR}} |x(s) - y(s)|^2\mathrm{d}s \\
&= \alpha(\hbar) \int_{\mathrm{IR}} |x(s) - y(s)|^2\mathrm{d}s.
\end{aligned}$$

This implies, by integrating with respect to $\hbar$,

$$\begin{aligned}
\int_{\mathrm{IR}} |(Rx)(\hbar) - (Ry)(\hbar)|^2 \, d\hbar &\leq \int_{\mathrm{IR}} \left(\alpha(\hbar) \int_{\mathrm{IR}} |x(s) - y(s)|^2\mathrm{d}s \right) \mathrm{d}\hbar \\
&= \int_{\mathrm{IR}} \left(\alpha(\hbar) e^{\nu \int_{\mathrm{IR}} \alpha(s)\mathrm{d}s} \cdot e^{-\nu \int_{\mathrm{IR}} \alpha(s)\mathrm{d}s} \int_{\mathrm{IR}} |x(s) - y(s)|^2\mathrm{d}s \right) \mathrm{d}\hbar \\
&\leq \|x - y\|_{2,\nu}^2 \int_{\mathrm{IR}} \alpha(\hbar) e^{\nu \int_{\mathrm{IR}} \alpha(s)\mathrm{d}s} \mathrm{d}\hbar \\
&\leq \frac{1}{\nu M^2} \|x - y\|_{2,\nu}^2 e^{\nu \int_{\mathrm{IR}} \alpha(s)\mathrm{d}s}.
\end{aligned}$$

Thus, we have

$$M^2 e^{-\nu \int_{\mathrm{IR}} \alpha(s)\mathrm{d}s} \int_{\mathrm{IR}} |(Rx)(\hbar) - (Ry)(\hbar)|^2 \mathrm{d}\hbar \leq \frac{1}{\nu} \|x - y\|_{2,\nu}^2.$$

This implies that

$$M^2 \|(Rx) - (Ry)\|_{2,\nu}^2 \leq \frac{1}{\nu} \|x - y\|_{2,\nu}^2.$$

That is,

$$L(x, y) d_\nu((Rx), (Ry)) \leq \sqrt{\frac{1}{\nu}} \, d_\nu(x, y).$$

Taking ln on both sides and defining $\Psi(t) = \ln(t)$ with $\Phi(t) = \Psi(t) - \tau$, $\tau > 0$, we have

$$\Psi(L(x,y)d_\nu((Rx),(Ry))) \le \Phi(\check{I}_1(x,y)), \quad \tau = -\ln\left(\sqrt{\frac{1}{\nu}}\right), \; a = \infty.$$

The defined Ψ and Φ satisfy the remaining conditions of Theorem 3.4. Hence, by Theorem 3.4, the operator R has a unique fixed point. This means that the UIE (6.1) has a unique solution. □

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