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Research Article

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On a more accurate half-discrete Hilbert-type inequality involving hyperbolic functions

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Abstract: In this work, by the introduction of a new kernel function composed of exponent functions with several parameters, and using the method of weight coefficient, Hermite-Hadamard's inequality, and some other techniques of real analysis, a more accurate half-discrete Hilbert-type inequality including both the homogeneous and non-homogeneous cases is established. Furthermore, by introducing the Bernoulli number and the rational fraction expansion of tangent function, some special and interesting Hilbert-type inequalities and their equivalent hardy-type inequalities are presented at the end of the paper.

Keywords: Hilbert-type inequality, hyperbolic function, Hermite-Hadamard's inequality, rational fraction expansion, Bernoulli number

MSC 2020: 26D15, 41A17

1 Introduction

Suppose $p > 1$, $f(x), v(x)$ are measurable functions defined on Ω, and $f(x), v(x) > 0$. Define a function space as follows:

$$L_{p,v}(\Omega) := \left\{ f : \|f\|_{p,v} := \left(\int_{\Omega} f^p(x)v(x)\mathrm{d}x \right)^{\frac{1}{p}} < \infty \right\}.$$

Particularly, if $v(x) = 1$, then we have the following abbreviated forms: $\|f\|_p := \|f\|_{p,v}$ and $L_p(\Omega) := L_{p,v}(\Omega)$.

Suppose $p > 1$, $a_n, v_n > 0$, and $\boldsymbol{a} = \{a_n\}_{n=0}^{\infty}$. Define a sequence space as follows:

$$l_{p,v} := \left\{ \boldsymbol{a} : \|\boldsymbol{a}\|_{p,v} := \left(\sum_{n=0}^{\infty} a_n^p v_n \right)^{\frac{1}{p}} < \infty \right\}.$$

Particularly, we abbreviate $\|a\|_{p,v}$ to $\|a\|_p$ and $l_{p,v}$ to l_p for $v_n = 1$.

Assuming that $f(x)$, $g(x)$ are two nonnegative real-valued functions, and $f, g \in L_2(\mathbb{R}^+)$, we have [1]

$$\int_{\mathbb{R}^+} \int_{\mathbb{R}^+} \frac{f(x)g(y)}{x + y} \mathrm{d}x\mathrm{d}y < \pi \|f\|_2 \|g\|_2. \tag{1.1}$$

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Similarly, let $\boldsymbol{a} = \{a_m\}_{m=1}^{\infty} \in l_2$ and $\boldsymbol{b} = \{b_n\}_{n=1}^{\infty} \in l_2$. Then

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n} < \pi\|\boldsymbol{a}\|_2\|\boldsymbol{b}\|_2. \tag{1.2}$$

The constant factor π in (1.1) and (1.2) is the best possible, and inequalities (1.1) and (1.2) are usually named as Hilbert inequality. By introducing a pair of conjugate parameters (p, q), $p > 1$, and $\frac{1}{p} + \frac{1}{q} = 1$, (1.1) and (1.2) can be extended to more general forms:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n} < \frac{\pi}{\sin\frac{\pi}{p}}\|\boldsymbol{a}\|_p\|\boldsymbol{b}\|_q, \tag{1.3}$$

$$\int_{\mathbb{R}^+}\int_{\mathbb{R}^+}\frac{f(x)g(y)}{x+y}\mathrm{d}x\mathrm{d}y < \frac{\pi}{\sin\frac{\pi}{p}}\|f\|_p\|g\|_q, \tag{1.4}$$

where the constant factor $\frac{\pi}{\sin\frac{\pi}{p}}$ is the best possible. In addition, we also have some classical inequalities similar to inequalities (1.3) and (1.4), such as [1]

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{\max\{m, n\}} < pq\|\boldsymbol{a}\|_p\|\boldsymbol{b}\|_q, \tag{1.5}$$

$$\int_{\mathbb{R}^+}\int_{\mathbb{R}^+}\frac{f(x)g(y)}{\max\{x, y\}}\mathrm{d}x\mathrm{d}y < pq\|f\|_p\|g\|_q. \tag{1.6}$$

In general, such inequalities as (1.3), (1.4), (1.5), and (1.6) are known as Hilbert-type inequalities. Although these classical inequalities were proposed for more than 100 years, considerable attention has been paid to their parameter extensions, strengthened forms, and higher dimensional generalizations by researchers all over the world in recent years, and some new valuable Hilbert-type inequalities were established. The following inequality is a classical extension of (1.3) established by Krnić and Pečarić [2]: If $p > 1$, and $\frac{1}{p} + \frac{1}{q} = 1$, $0 < \beta_1, \beta_2 \le 2$, $\beta_1 + \beta_2 = \beta$, $\mu_m = m^{p(1-\beta_1)-1}$, and $\nu_n = n^{q(1-\beta_2)-1}$, then

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{(m+n)^{\beta}} < B(\beta_1, \beta_2)\|\boldsymbol{a}\|_{p,\mu}\|\boldsymbol{b}\|_{q,\nu}, \tag{1.7}$$

where $B(u, v)$ is the beta function. In addition, Yang [3] gave an extension of (1.4) as follows:

$$\int_{\mathbb{R}^+}\int_{\mathbb{R}^+}\frac{f(x)g(y)}{x^{\beta}+y^{\beta}}\mathrm{d}x\mathrm{d}y < \frac{\pi}{\beta\sin\frac{\pi}{r}}\|f\|_{p,\mu}\|g\|_{q,\nu}, \tag{1.8}$$

where $\beta > 0$, $\mu(x) = x^{p(1-\frac{\beta}{r})-1}$, $\nu(y) = y^{q(1-\frac{\beta}{s})-1}$, and $\frac{1}{r} + \frac{1}{s} = 1$.

For other extensions of classical discrete Hilbert-type inequalities, we can refer to [4–12], and some extended results of integral version can be found in [10,11,13–17]. Furthermore, by introducing various new kernel functions, special constants, and special functions, and considering discrete and integral forms, many Hilbert-type inequalities with new kernel functions were established in the past 20 years (see [18–26]). In addition to such types of Hilbert-type inequalities mentioned above, some new results on time scales were also established in recent years (see [27,28]). In what follows, we present the following two integral Hilbert-type inequalities, which involve the kernels related to hyperbolic functions, and are closely related to our research in the present paper, that is [25,26],

$$\int_{\mathbb{R}^+}\int_{\mathbb{R}^+}\operatorname{csch}(x^{\delta}y)f(x)g(y)\mathrm{d}x\mathrm{d}y < \frac{\pi^2}{4}\|f\|_{2,\mu}\|g\|_{2,\nu}, \tag{1.9}$$

$$\int_{\mathbb{R}^+}\int_{\mathbb{R}^+}(\coth(xy)-1)f(x)g(y)\mathrm{d}x\mathrm{d}y < \frac{\pi^2}{12}\|f\|_{2,\nu}\|g\|_{2,\nu}, \tag{1.10}$$

where $\delta \in \{1, -1\}$, $\mu(x) = x^{1-4\delta}$, $\nu(y) = y^{-3}$.

Besides the integral and discrete Hilbert-type inequalities, it should be pointed out that Hilbert-type inequalities sometimes appear in half-discrete form, such as the following two [29,30]:

$$\int_{\mathbb{R}^+} f(x)\sum_{n=1}^{\infty}\frac{a_n}{x+n}\mathrm{d}x < \frac{\pi}{\sin\frac{\pi}{p}}\|f\|_p\|\boldsymbol{a}\|_q, \tag{1.11}$$

$$\int_{\mathbb{R}^+} f(x)\sum_{n=1}^{\infty}\frac{\log\frac{x}{n}}{x-n}a_n\mathrm{d}x < \left(\frac{\pi}{\sin\frac{\pi}{p}}\right)^2\|f\|_p\|\boldsymbol{a}\|_q. \tag{1.12}$$

With regard to some other half-discrete inequalities, we refer to [31–34].

In this paper, by using techniques of real analysis, particularly Hermite-Hadamard's inequality, we consider the half-discrete forms of (1.9) and (1.10), then the following Hilbert-type inequalities involving some hyperbolic functions will be established:

$$\int_0^a f(x)\sum_{n=0}^{\infty}\operatorname{csch}(\sqrt[2m]{x(2n+1)})a_n\mathrm{d}x < 2^{-\frac{1}{p}}(2^{2m}-1)B_m\pi^{2m}\|f\|_{p,\mu_1}\|\boldsymbol{a}\|_{q,\nu}\ (a\ge 1), \tag{1.13}$$

$$\int_a^{\infty} f(x)\sum_{n=0}^{\infty}\left(\coth\sqrt[2m]{\frac{2n+1}{x}}-1\right)a_n\mathrm{d}x < 2^{-\frac{1}{p}}B_m\pi^{2m}\|f\|_{p,\mu_2}\|\boldsymbol{a}\|_{q,\nu}\ (0\le a\le 1), \tag{1.14}$$

$$\int_0^a f(x)\sum_{n=s}^{\infty}\cosh(\sqrt[2m]{xn})\operatorname{csch}(3\sqrt[2m]{xn})a_n\mathrm{d}x < \frac{2m\pi^{2m}}{6^{2m}}\psi^{(2m-1)}\left(\frac{\pi}{6}\right)\|f\|_{p,\mu_1}\|\boldsymbol{a}\|_{q,\bar{\nu}}\ (a\ge 1), \tag{1.15}$$

where $m, s \in \mathbb{N}^+$, B_m is the Bernoulli number, $\mu_1(x) = \frac{1}{x}$, $\mu_2(x) = x^{2p-1}$, $\nu_n = \frac{1}{2n+1}$, and $\bar{\nu}_n = \frac{1}{n}$.

More generally, a new kernel function in more general form is constructed in Section 2, and then a Hilbert-type inequality including both the homogeneous and non-homogeneous cases is established. It will be shown that the newly obtained inequality is a unified extension of inequalities (1.13), (1.14), and (1.15), and some other special cases of the newly obtained inequality are presented in Section 4.

2 Definitions and lemmas

Definition 2.1. For $t > 0$, define Γ-function as follows:

$$\Gamma(t) := \int_{\mathbb{R}^+} z^{t-1}e^{-z}\mathrm{d}z.$$

In particular, we have $\Gamma(t) = (t-1)!$ for $t \in \mathbb{N}^+$.

Lemma 2.2. *Let* $-\lambda_3 \le \lambda_2 \le \lambda_1 < \lambda_3$, *and* $\lambda > 1$. *Define*

$$k(z) := \frac{e^{\lambda_1 z}+e^{\lambda_2 z}}{e^{\lambda_3 z}-e^{-\lambda_3 z}},\quad z>0, \tag{2.1}$$

and

$$C(\lambda_1,\lambda_2,\lambda_3,\lambda) := \sum_{j=0}^{\infty}\left[\frac{1}{(2\lambda_3 j-\lambda_1+\lambda_3)^{\lambda}}+\frac{1}{(2\lambda_3 j-\lambda_2+\lambda_3)^{\lambda}}\right]. \tag{2.2}$$

Then

$$\int_{\mathbb{R}^+} k(z)z^{\lambda-1}\mathrm{d}z = \Gamma(\lambda)C(\lambda_1,\lambda_2,\lambda_3,\lambda). \tag{2.3}$$

Proof. It can be easy to show that $\lambda_3 > 0$ and

$$k(z) = \sum_{j=0}^{\infty} \left(e^{(-2\lambda_3 j + \lambda_1 - \lambda_3)z} + e^{(-2\lambda_3 j + \lambda_2 - \lambda_3)z} \right). \tag{2.4}$$

By Lebesgue term-by-term integration theorem, we obtain

$$\int_{\mathbb{R}^+} k(z) z^{\lambda-1} \mathrm{d}z = \sum_{j=0}^{\infty} \left(\int_{\mathbb{R}^+} e^{(-2\lambda_3 j + \lambda_1 - \lambda_3)z} z^{\lambda-1} \mathrm{d}z + \int_{\mathbb{R}^+} e^{(-2\lambda_3 j + \lambda_2 - \lambda_3)z} z^{\lambda-1} \mathrm{d}z \right). \tag{2.5}$$

Setting $(2\lambda_3 j - \lambda_1 + \lambda_3)z = u$, we obtain

$$\int_{\mathbb{R}^+} e^{(-2\lambda_3 j + \lambda_1 - \lambda_3)z} z^{\lambda-1} \mathrm{d}z = \frac{\int_{\mathbb{R}^+} e^{-u} u^{\lambda-1} \mathrm{d}u}{(2\lambda_3 j - \lambda_1 + \lambda_3)^{\lambda}} = \frac{\Gamma(\lambda)}{(2\lambda_3 j - \lambda_1 + \lambda_3)^{\lambda}}. \tag{2.6}$$

Similarly, setting $(2\lambda_3 j - \lambda_2 + \lambda_3)z = u$, we obtain

$$\int_{\mathbb{R}^+} e^{(-2\lambda_3 j + \lambda_2 - \lambda_3)z} z^{\lambda-1} \mathrm{d}z = \frac{\int_{\mathbb{R}^+} e^{-u} u^{\lambda-1} \mathrm{d}u}{(2\lambda_3 j - \lambda_2 + \lambda_3)^{\lambda}} = \frac{\Gamma(\lambda)}{(2\lambda_3 j - \lambda_2 + \lambda_3)^{\lambda}}. \tag{2.7}$$

Plugging (2.6) and (2.7) back into (2.5), and using (2.2), we arrive at (2.3). Lemma 2.2 is proved. ☐

Lemma 2.3. *Suppose that $\beta \neq 0$, $b \geq \frac{1}{2}$, and $-\lambda_3 \leq \lambda_2 \leq \lambda_1 < \lambda_3$. Let $0 \leq a \leq 1$, and $\Omega = (a, \infty)$ when $\beta < 0$. Let $a \geq 1$, and $\Omega = (0, a)$ when $\beta > 0$. Assume that $\lambda > 1$, $0 < \gamma \leq 1$, and $0 < \lambda\gamma \leq 1$. Define*

$$K(x, y) := \frac{e^{\lambda_1 x^{\beta}(y+b)^{\gamma}} + e^{\lambda_2 x^{\beta}(y+b)^{\gamma}}}{e^{\lambda_3 x^{\beta}(y+b)^{\gamma}} - e^{-\lambda_3 x^{\beta}(y+b)^{\gamma}}}, \quad x > 0, y \geq 0. \tag{2.8}$$

Then

$$\omega(n) := \int_{\Omega} K(x, n) x^{\beta\lambda - 1} \mathrm{d}x \leq \frac{\Gamma(\lambda)}{|\beta|(n + b)^{\gamma\lambda}} C(\lambda_1, \lambda_2, \lambda_3, \lambda), \quad n \in \mathbb{N}^+, \tag{2.9}$$

$$\varpi(x) := \sum_{n=0}^{\infty} K(x, n)(n + b)^{\gamma\lambda - 1} < \frac{\Gamma(\lambda)}{\gamma x^{\beta\lambda}} C(\lambda_1, \lambda_2, \lambda_3, \lambda), \quad x > 0. \tag{2.10}$$

Proof. Setting $x^{\beta}(n + b)^{\gamma} = z$, and using (2.3), we obtain

$$\int_{\Omega} K(x, n) x^{\beta\lambda - 1} \mathrm{d}x = \frac{1}{|\beta|(n + b)^{\gamma\lambda}} \int_{\mathbb{R}^+} k(z) z^{\lambda-1} \mathrm{d}z = \frac{\Gamma(\lambda)}{|\beta|(n + b)^{\gamma\lambda}} C(\lambda_1, \lambda_2, \lambda_3, \lambda). \tag{2.11}$$

It follows therefore that (2.9) holds true.

Since $-\lambda_3 \leq \lambda_2 \leq \lambda_1 < \lambda_3$, it follows from (2.1) that

$$\frac{\mathrm{d}k}{\mathrm{d}z} = \left[(\lambda_1 - \lambda_3) e^{(\lambda_1 + \lambda_3)z} - (\lambda_1 + \lambda_3) e^{(\lambda_1 - \lambda_3)z} + (\lambda_2 - \lambda_3) e^{(\lambda_2 + \lambda_3)z} - (\lambda_2 + \lambda_3) e^{(\lambda_2 - \lambda_3)z} \right] (e^{\lambda_3 z} - e^{-\lambda_3 z})^{-2} < 0. \tag{2.12}$$

In view of that

$$6\lambda_3^2 - 2\lambda_2^2 > 2(\lambda_3 - \lambda_2)(\lambda_3 + \lambda_2) \geq 0$$

and

$$6\lambda_3^2 - 2\lambda_1^2 > 2(\lambda_3 - \lambda_1)(\lambda_3 + \lambda_1) \geq 0,$$

we obtain

$$\frac{d^2k}{dz^2} = \Big[(\lambda_1-\lambda_3)^2 e^{(\lambda_1+2\lambda_3)z} + (\lambda_1+\lambda_3)^2 e^{(\lambda_1-2\lambda_3)z} + (\lambda_2-\lambda_3)^2 e^{(\lambda_2+2\lambda_3)z} + (\lambda_2+\lambda_3)^2 e^{(\lambda_2-2\lambda_3)z} + (6\lambda_3^2-2\lambda_1^2)e^{\lambda_1 z} + (6\lambda_3^2-2\lambda_2^2)e^{\lambda_2 z}\Big]\left(e^{\lambda_3 z}-e^{-\lambda_3 z}\right)^{-3} > 0. \tag{2.13}$$

For arbitrary $x > 0$, let $z := x^\beta(y+b)^\gamma$. In view of $0 < \gamma \le 1$, we obtain $\frac{dz}{dy} > 0$ and $\frac{d^2z}{dy^2} \le 0$. It follows therefore that

$$\frac{dK}{dy} = \frac{dk}{dz}\frac{dz}{dy} < 0 \tag{2.14}$$

and

$$\frac{d^2K}{dy^2} = \frac{d^2k}{dz^2}\left(\frac{dz}{dy}\right)^2 + \frac{dk}{dz}\frac{d^2z}{dy^2} > 0. \tag{2.15}$$

Let

$$H(y) := K(x,y)(y+b)^{\gamma\lambda-1} := K(x,y)h(y).$$

Observing that $0 < \gamma\lambda \le 1$, and using (2.14) and (2.15), we have

$$\frac{dH}{dy} = \frac{dK}{dy}h(y) + \frac{dh}{dy}K(x,y) < 0$$

and

$$\frac{d^2H}{dy^2} = \frac{d^2K}{dy^2}h(y) + 2\frac{dK}{dy}\frac{dh}{dy} + \frac{d^2h}{dy^2}K(x,y) > 0.$$

Hence, by Hermite-Hadamard's inequality (see [35,36]), we obtain

$$\varpi(x) = \sum_{n=0}^{\infty} H(n) < \sum_{n=0}^{\infty} \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} H(y)dy = \int_{-\frac{1}{2}}^{\infty} H(y)dy \le \int_{-b}^{\infty} H(y)dy. \tag{2.16}$$

Setting $x^\beta(y+b)^\gamma = z$, and using (2.3), we obtain

$$\int_{-b}^{\infty} H(y)dy = \frac{1}{\gamma x^{\beta\lambda}} \int_{\mathbb{R}^+} k(z)z^{\lambda-1}dz = \frac{\Gamma(\lambda)}{\gamma x^{\beta\lambda}} C(\lambda_1,\lambda_2,\lambda_3,\lambda). \tag{2.17}$$

Applying (2.17) to (2.16), we obtain (2.10), and the proof of Lemma 2.3 is completed. □

Lemma 2.4. *Suppose that $\beta \ne 0$, $b \ge \frac{1}{2}$, and $-\lambda_3 \le \lambda_2 \le \lambda_1 < \lambda_3$. Let $0 \le a \le 1$, and $\Omega = (a,\infty)$ when $\beta < 0$. Let $a \ge 1$, and $\Omega = (0,a)$ when $\beta > 0$. Assume that $\lambda > 1$, $0 < \gamma \le 1$, and $0 < \lambda\gamma \le 1$. Let $p > 1$, $\frac{1}{p}+\frac{1}{q} = 1$, $\mu(x) = x^{p(1-\beta\lambda)-1}$, and $\nu_n = (n+b)^{q(1-\gamma\lambda)-1}$. $K(x,y)$ is defined via* Lemma 2.3. *For an arbitrary positive integer l, which is sufficiently large, set*

$$\tilde{\boldsymbol{a}} := \{\tilde{a}_n\}_{n=0}^{\infty} := \left\{(n+b)^{\gamma\lambda-1-\frac{\gamma}{ql}}\right\}_{n=0}^{\infty}, \tag{2.18}$$

$$\tilde{f}(x) := \begin{cases} x^{\beta\lambda-1+\frac{\beta}{pl}}, & x \in E, \\ 0, & x \in \Omega\backslash E, \end{cases} \tag{2.19}$$

where $E = \{x : x > 0, \quad x^{\operatorname{sgn}\beta} < 1\}$. Then

$$\tilde{J} := \sum_{n=0}^{\infty} \tilde{a}_n \int_E K(x,n)\tilde{f}(x)dx = \int_E \tilde{f}(x)\sum_{n=0}^{\infty} K(x,n)\tilde{a}_n dx > \frac{l}{|\beta\gamma|}\left(\int_1^{\infty} k(z)z^{\lambda-\frac{1}{ql}-1}dz + b^{\frac{-\gamma}{l}}\int_0^1 k(z)z^{\lambda+\frac{1}{pl}-1}dz\right). \tag{2.20}$$

Proof. Observing that $0 < \lambda\gamma \le 1$, and using (2.14), we obtain

$$\tilde{J} > \int_E x^{\beta\lambda-1+\frac{\beta}{pl}} \int_0^\infty K(x,y)(y+b)^{\gamma\lambda-1-\frac{\gamma}{ql}}\mathrm{d}y\mathrm{d}x. \tag{2.21}$$

Setting $x^\beta(y+b)^\gamma = z$, we obtain

$$\begin{aligned}\tilde{J} &> \frac{1}{\gamma}\int_E x^{\frac{\beta}{l}-1}\left[\int_{x^\beta b^\gamma}^\infty k(z)z^{\lambda-\frac{1}{ql}-1}\mathrm{d}z\right]\mathrm{d}x\\ &= \frac{1}{\gamma}\int_E x^{\frac{\beta}{l}-1}\left[\int_1^\infty k(z)z^{\lambda-\frac{1}{ql}-1}\mathrm{d}z\right]\mathrm{d}x + \frac{1}{\gamma}\int_E x^{\frac{\beta}{l}-1}\left[\int_{x^\beta b^\gamma}^1 k(z)z^{\lambda-\frac{1}{ql}-1}\mathrm{d}z\right]\mathrm{d}x\\ &= \frac{l}{|\beta\gamma|}\int_1^\infty k(z)z^{\lambda-\frac{1}{ql}-1}\mathrm{d}z + \frac{1}{\gamma}\int_E x^{\frac{\beta}{l}-1}\left[\int_{x^\beta b^\gamma}^1 k(z)z^{\lambda-\frac{1}{ql}-1}\mathrm{d}z\right]\mathrm{d}x.\end{aligned} \tag{2.22}$$

For $\beta > 0$, by the use of Fubini's theorem, we obtain

$$\int_E x^{\frac{\beta}{l}-1}\left[\int_{x^\beta b^\gamma}^1 k(z)z^{\lambda-\frac{1}{ql}-1}\mathrm{d}z\right]\mathrm{d}x = \int_0^1 k(z)z^{\lambda-\frac{1}{ql}-1}\left(\int_0^{z^{\frac{1}{\beta}}b^{\frac{-\gamma}{\beta}}} x^{\frac{\beta}{l}-1}\mathrm{d}x\right)\mathrm{d}z = \frac{lb^{\frac{-\gamma}{l}}}{\beta}\int_0^1 k(z)z^{\lambda+\frac{1}{pl}-1}\mathrm{d}z. \tag{2.23}$$

Applying (2.23) to (2.22), it follows (2.20). Similarly, for $\beta < 0$, by the use of Fubini's theorem again, we have

$$\int_E x^{\frac{\beta}{l}-1}\left[\int_{x^\beta b^\gamma}^1 k(z)z^{\lambda-\frac{1}{ql}-1}\mathrm{d}z\right]\mathrm{d}x = \int_0^1 k(z)z^{\lambda-\frac{1}{ql}-1}\left(\int_{z^{\frac{1}{\beta}}b^{\frac{-\gamma}{\beta}}}^\infty x^{\frac{\beta}{l}-1}\mathrm{d}x\right)\mathrm{d}z = \frac{lb^{\frac{-\gamma}{l}}}{|\beta|}\int_0^1 k(z)z^{\lambda+\frac{1}{pl}-1}\mathrm{d}z. \tag{2.24}$$

Plug (2.24) back into (2.22), then (2.20) follows obviously. Lemma 2.4 is proved. □

Lemma 2.5. *Let* $-1 < u < 1$, $\psi(x) = \tan x$, *and* $m \in \mathbb{N}^+$. *Then*

$$\psi^{(2m-1)}\left(\frac{u\pi}{2}\right) = \frac{2^{2m}(2m-1)!}{\pi^{2m}}\sum_{j=0}^\infty\left[\frac{1}{(2j+1-u)^{2m}} + \frac{1}{(2j+1+u)^{2m}}\right]. \tag{2.25}$$

Proof. Observing that $\psi(x) = \tan x$ can be expressed in the form of rational fraction expansion as follows [37]:

$$\psi(x) = \tan x = 2\sum_{j=0}^\infty\left[\frac{1}{(2j+1)\pi - 2x} - \frac{1}{(2j+1)\pi + 2x}\right]. \tag{2.26}$$

Finding the $(2m-1)$th derivative of $\psi(x)$, we have

$$\psi^{(2m-1)}(x) = 2^{2m}(2m-1)!\sum_{j=0}^\infty\left\{\frac{1}{[(2j+1)\pi - 2x]^{2m}} + \frac{1}{[(2j+1)\pi + 2x]^{2m}}\right\}. \tag{2.27}$$

Setting $x = \frac{u\pi}{2}$ in (2.27), we arrive at (2.25). Lemma 2.5 is proved. □

3 Main results

Theorem 3.1. *Suppose that* $\beta \ne 0$, $b \ge \frac{1}{2}$, *and* $-\lambda_3 \le \lambda_2 \le \lambda_1 < \lambda_3$. *Let* $0 \le a \le 1$, *and* $\Omega = (a, \infty)$ *when* $\beta < 0$. *Let* $a \ge 1$, *and* $\Omega = (0, a)$ *when* $\beta > 0$. *Assume that* $\lambda > 1$, $0 < \gamma \le 1$, *and* $0 < \lambda\gamma \le 1$. *Let* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$,

$\mu(x) = x^{p(1-\beta\lambda)-1}$, *and* $\nu_n = (n+b)^{q(1-\gamma\lambda)-1}$. *Let* $f(x)$, $a_n \geq 0$ *with* $f(x) \in L_{p,\mu}(\Omega)$, *and* $\boldsymbol{a} = \{a_n\}_{n=0}^{\infty} \in l_{q,\nu}$. $C(\lambda_1, \lambda_2, \lambda_3, \lambda)$ *and* $K(x, y)$ *are defined via* Lemmas 2.2 *and* 2.3, *respectively. Then the following three inequalities are equivalent*:

$$J_1 := \sum_{n=0}^{\infty}(n+b)^{p\gamma\lambda-1}\left(\int_{\Omega}K(x,n)f(x)\mathrm{d}x\right)^p < \left[|\beta|^{-\frac{1}{q}}|\gamma|^{-\frac{1}{p}}\Gamma(\lambda)C(\lambda_1,\lambda_2,\lambda_3,\lambda)\right]^p\|f\|_{p,\mu}^p, \tag{3.1}$$

$$J_2 := \int_{\Omega}x^{q\beta\lambda-1}\left(\sum_{n=0}^{\infty}K(x,n)a_n\right)^q\mathrm{d}x < \left[|\beta|^{-\frac{1}{q}}\gamma^{-\frac{1}{p}}\Gamma(\lambda)C(\lambda_1,\lambda_2,\lambda_3,\lambda)\right]^q\|\boldsymbol{a}\|_{q,\nu}^q, \tag{3.2}$$

$$J := \sum_{n=0}^{\infty}a_n\int_{\Omega}K(x,n)f(x)\mathrm{d}x = \int_{\Omega}f(x)\sum_{n=0}^{\infty}K(x,n)a_n\mathrm{d}x < |\beta|^{-\frac{1}{q}}\gamma^{-\frac{1}{p}}\Gamma(\lambda)C(\lambda_1,\lambda_2,\lambda_3,\lambda)\|f\|_{p,\mu}\|\boldsymbol{a}\|_{q,\nu}, \tag{3.3}$$

where the constant $|\beta|^{-\frac{1}{p}}\gamma^{-\frac{1}{q}}\Gamma(\lambda)C(\lambda_1,\lambda_2,\lambda_3,\lambda)$ *in* (3.1), (3.2), *and* (3.3) *is the best possible.*

Proof. By Hölder's inequality, and using (2.9), we obtain

$$\begin{aligned}\left(\int_{\Omega}K(x,n)f(x)\mathrm{d}x\right)^p &= \left[\int_{\Omega}K(x,n)\left(x^{\frac{1-\beta\lambda}{q}}f(x)\right)x^{\frac{\beta\lambda-1}{q}}\mathrm{d}x\right]^p\\ &\leq \int_{\Omega}K(x,n)x^{\frac{p(1-\beta\lambda)}{q}}f^p(x)\mathrm{d}x\left(\int_{\Omega}K(x,n)x^{\beta\lambda-1}\mathrm{d}x\right)^{p-1}\\ &= [\omega(n)]^{p-1}\int_{\Omega}K(x,n)x^{\frac{p(1-\beta\lambda)}{q}}f^p(x)\mathrm{d}x\\ &\leq \left[\frac{\Gamma(\lambda)}{|\beta|(n+b)^{\gamma\lambda}}C(\lambda_1,\lambda_2,\lambda_3,\lambda)\right]^{p-1}\int_{\Omega}K(x,n)x^{\frac{p(1-\beta\lambda)}{q}}f^p(x)\mathrm{d}x.\end{aligned} \tag{3.4}$$

Plugging (3.4) back into the left hand side of (3.1), and using Lebesgue term-by-term integration theorem as well as inequality (2.10), we have

$$J_1 \leq \left[\frac{\Gamma(\lambda)}{|\beta|}C(\lambda_1,\lambda_2,\lambda_3,\lambda)\right]^{p-1}\int_{\Omega}f^p(x)x^{\frac{p(1-\beta\lambda)}{q}}\sum_{n=0}^{\infty}K(x,n)(n+b)^{\gamma\lambda-1}\mathrm{d}x < \left[|\beta|^{-\frac{1}{q}}|\gamma|^{-\frac{1}{p}}\Gamma(\lambda)C(\lambda_1,\lambda_2,\lambda_3,\lambda)\right]^p\|f\|_{p,\mu}^p.$$

The proof of inequality (3.1) is completed. Similarly, by the use of Hölder's inequality again and (2.10), we have

$$\begin{aligned}\left(\sum_{n=0}^{\infty}K(x,n)a_n\right)^q &= \left[\sum_{n=0}^{\infty}K(x,n)(n+b)^{\frac{\gamma\lambda-1}{p}}\left((n+b)^{\frac{1-\gamma\lambda}{p}}a_n\right)\right]^q\\ &\leq [\varpi(x)]^{q-1}\sum_{n=0}^{\infty}K(x,n)(n+b)^{\frac{q(1-\gamma\lambda)}{p}}a_n^q\\ &< \left[\frac{\Gamma(\lambda)}{\gamma x^{\beta\lambda}}C(\lambda_1,\lambda_2,\lambda_3,\lambda)\right]^{q-1}\sum_{n=0}^{\infty}K(x,n)(n+b)^{\frac{q(1-\gamma\lambda)}{p}}a_n^q.\end{aligned} \tag{3.5}$$

It follows from Lebesgue term-by-term integration theorem and (2.9) that

$$J_2 < \left[\frac{\Gamma(\lambda)}{|\beta|}C(\lambda_1,\lambda_2,\lambda_3,\lambda)\right]^{q-1}\sum_{n=0}^{\infty}(n+b)^{\frac{q(1-\gamma\lambda)}{p}}a_n^q\int_{\Omega}K(x,n)x^{\beta\lambda-1}\mathrm{d}x < \left[|\beta|^{-\frac{1}{q}}\gamma^{-\frac{1}{p}}\Gamma(\lambda)C(\lambda_1,\lambda_2,\lambda_3,\lambda)\right]^q\|\boldsymbol{a}\|_{q,\nu}^q.$$

Inequality (3.2) is proved. Additionally, we will prove (3.3) by (3.1). In fact, we can first obtain two representations of J by Lebesgue term-by-term integration theorem, and then by the use of Hölder's inequality, we obtain

$$J = \sum_{n=0}^{\infty}\left[(n+b)^{\gamma\lambda-\frac{1}{p}}\int_{\Omega}K(x,n)f(x)\mathrm{d}x\right]\left[a_n(n+b)^{-\gamma\lambda+\frac{1}{p}}\right] \le J_1^{\frac{1}{p}}\left[\sum_{n=0}^{\infty}a_n^q(n+b)^{q(1-\gamma\lambda)-1}\right]^{\frac{1}{q}} = J_1^{\frac{1}{p}}\|\boldsymbol{a}\|_{q,\nu}. \quad (3.6)$$

Applying (3.1) to (3.6), we have (3.3). On the contrary, we assume that (3.3) holds true, and let $\boldsymbol{b} = \{b_n\}_{n=0}^{\infty}$, where

$$b_n := (n+b)^{p\gamma\lambda-1}\left(\int_{\Omega}K(x,n)f(x)\mathrm{d}x\right)^{p-1}.$$

By the use of (3.3), it follows that

$$\begin{aligned}J_1 &= \sum_{n=0}^{\infty}b_n\int_{\Omega}K(x,n)f(x)\mathrm{d}x < |\beta|^{-\frac{1}{q}}\gamma^{-\frac{1}{p}}\Gamma(\lambda)C(\lambda_1,\lambda_2,\lambda_3,\lambda)\|f\|_{p,\mu}\|\boldsymbol{b}\|_{q,\nu}\\ &= |\beta|^{-\frac{1}{q}}\gamma^{-\frac{1}{p}}\Gamma(\lambda)C(\lambda_1,\lambda_2,\lambda_3,\lambda)\|f\|_{p,\mu}J_1^{\frac{1}{q}}.\end{aligned} \quad (3.7)$$

By (3.7), inequality (3.1) follows naturally. Hence, inequalities (3.1) and (3.3) are equivalent. In order to prove the equivalence of inequalities (3.1), (3.2), and (3.3), it suffices to prove that (3.2) and (3.3) are equivalent. In fact, if (3.2) is assumed to be true, then

$$J = \int_{\Omega}\left[x^{-\beta\lambda+\frac{1}{q}}f(x)\right]\left[x^{\beta\gamma-\frac{1}{q}}\sum_{n=0}^{\infty}K(x,n)a_n\right]\mathrm{d}x \le \|f\|_{p,\mu}J_2^{\frac{1}{q}}. \quad (3.8)$$

Applying (3.2) to (3.8), we have (3.3). Conversely, assume (3.3) holds true, and let

$$g(x) := x^{q\beta\lambda-1}\left(\sum_{n=0}^{\infty}K(x,n)a_n\right)^{q-1}.$$

Then

$$\begin{aligned}J_2 &= \int_{\Omega}g(x)\sum_{n=0}^{\infty}K(x,n)a_n\mathrm{d}x < |\beta|^{-\frac{1}{q}}\gamma^{-\frac{1}{p}}\Gamma(\lambda)C(\lambda_1,\lambda_2,\lambda_3,\lambda)\|g\|_{p,\mu}\|\boldsymbol{a}\|_{q,\nu}\\ &= |\beta|^{-\frac{1}{q}}\gamma^{-\frac{1}{p}}\Gamma(\lambda)C(\lambda_1,\lambda_2,\lambda_3,\lambda)\|\boldsymbol{a}\|_{q,\nu}J_2^{\frac{1}{p}}.\end{aligned} \quad (3.9)$$

Therefore, inequality (3.2) holds true, and the equivalence of inequalities (3.1), (3.2), and (3.3) is proved. Finally, it will be proved that the constant $|\beta|^{-\frac{1}{q}}\gamma^{-\frac{1}{p}}\Gamma(\lambda)C(\lambda_1,\lambda_2,\lambda_3,\lambda)$ in (3.1), (3.2), and (3.3) is optimal. Assuming that the constant factor $|\beta|^{-\frac{1}{q}}\gamma^{-\frac{1}{p}}\Gamma(\lambda)C(\lambda_1,\lambda_2,\lambda_3,\lambda)$ in (3.3) is not optimal, there must be a real number c satisfying

$$0 < c < |\beta|^{-\frac{1}{q}}\gamma^{-\frac{1}{p}}\Gamma(\lambda)C(\lambda_1,\lambda_2,\lambda_3,\lambda), \quad (3.10)$$

so that (3.3) still holds if $|\beta|^{-\frac{1}{q}}\gamma^{-\frac{1}{p}}\Gamma(\lambda)C(\lambda_1,\lambda_2,\lambda_3,\lambda)$ is replaced by c, that is,

$$\sum_{n=0}^{\infty}a_n\int_{\Omega}K(x,n)f(x)\mathrm{d}x = \int_{\Omega}f(x)\sum_{n=0}^{\infty}K(x,n)a_n\mathrm{d}x < c\|f\|_{p,\mu}\|\boldsymbol{a}\|_{q,\nu}. \quad (3.11)$$

Replace f and a_n in (3.11) with $\tilde{f}$ and $\tilde{a}_n$ defined in Lemma 2.4, respectively. It implies that

$$\begin{aligned}\int_E \tilde{f}(x)\sum_{n=0}^{\infty}K(x,n)\tilde{a}_n\mathrm{d}x < c\|\tilde{f}\|_{p,\mu}\|\tilde{\boldsymbol{a}}\|_{q,\nu} &= c\left(\int_E x^{\frac{\beta}{l}-1}\mathrm{d}x\right)^{\frac{1}{p}}\left(\sum_{n=0}^{\infty}(n+b)^{-\frac{\gamma}{l}-1}\right)^{\frac{1}{q}}\\ &= c\left(\frac{l}{|\beta|}\right)^{\frac{1}{p}}\left[b^{-\frac{\gamma}{l}-1}+\sum_{n=1}^{\infty}(n+b)^{-\frac{\gamma}{l}-1}\right]^{\frac{1}{q}}\\ &< c\left(\frac{l}{|\beta|}\right)^{\frac{1}{p}}\left[b^{-\frac{\gamma}{l}-1}+\int_0^{\infty}(y+b)^{-\frac{\gamma}{l}-1}\mathrm{d}y\right]^{\frac{1}{q}}\\ &= c\left(\frac{l}{|\beta|}\right)^{\frac{1}{p}}\left(b^{-\frac{\gamma}{l}-1}+\frac{lb^{-\frac{\gamma}{l}}}{\gamma}\right)^{\frac{1}{q}}.\end{aligned} \tag{3.12}$$

Combining (2.20) and (3.12), we have

$$\int_1^{\infty}k(z)z^{\lambda-\frac{1}{ql}-1}\mathrm{d}z + b^{\frac{-\gamma}{l}}\int_0^1 k(z)z^{\lambda+\frac{1}{pl}-1}\mathrm{d}z < c\gamma|\beta|^{\frac{1}{q}}\left(\frac{b^{-\frac{\gamma}{l}-1}}{l}+\frac{b^{-\frac{\gamma}{l}}}{\gamma}\right)^{\frac{1}{q}}. \tag{3.13}$$

Letting $l \to +\infty$ in (3.13), and using (2.3), we have

$$c \geq |\beta|^{-\frac{1}{q}}\gamma^{-\frac{1}{p}}\Gamma(\lambda)C(\lambda_1,\lambda_2,\lambda_3,\lambda). \tag{3.14}$$

Inequalities (3.10) and (3.14) are apparently contradictory, and therefore $|\beta|^{-\frac{1}{q}}\gamma^{-\frac{1}{p}}\Gamma(\lambda)C(\lambda_1,\lambda_2,\lambda_3,\lambda)$ in (3.3) is the best possible. It can also be proved $|\beta|^{-\frac{1}{q}}\gamma^{-\frac{1}{p}}\Gamma(\lambda)C(\lambda_1,\lambda_2,\lambda_3,\lambda)$ in (3.1) and (3.2) is the best possible from the equivalence of (3.1), (3.2), and (3.3). Theorem 3.1 is proved. □

4 Some corollaries

Let $\lambda_1 = \lambda_2 = 0$, $\lambda_3 = \rho$ ($\rho > 0$), and $\lambda = 2m$ ($m \in \mathrm{N}^+$) in Theorem 3.1. In view of the equation [37]:

$$\sum_{j=0}^{\infty}\frac{2}{(2j+1)^{2m}} = \frac{B_m}{(2m)!}(2^{2m}-1)\pi^{2m}, \quad m \in \mathrm{N}^+, \tag{4.1}$$

where B_m is the Bernoulli number, $B_1 = \frac{1}{6}$, $B_2 = \frac{1}{30}$, $B_3 = \frac{1}{42}, \ldots$, we obtain the following corollary.

Corollary 4.1. *Suppose that* $\rho > 0$, $\beta \neq 0$, $b \geq \frac{1}{2}$, *and* $0 < 2m\gamma \leq 1$ ($m \in \mathrm{N}^+$). *Let* $0 \leq a \leq 1$, *and* $\Omega = (a,\infty)$ *when* $\beta < 0$. *Let* $a \geq 1$, *and* $\Omega = (0,a)$ *when* $\beta > 0$. *Assume that* $p > 1$, $\frac{1}{p}+\frac{1}{q} = 1$, $\mu(x) = x^{p(1-2m\beta)-1}$, *and* $\nu_n = (n+b)^{q(1-2m\gamma)-1}$. *Let* $f(x)$, $a_n \geq 0$ *with* $f(x) \in L_{p,\mu}(\Omega)$, *and* $\boldsymbol{a} = \{a_n\}_{n=0}^{\infty} \in l_{q,\nu}$. *Then the following three inequalities are equivalent*:

$$\sum_{n=0}^{\infty}(n+b)^{2mp\gamma-1}\left[\int_{\Omega}\operatorname{csch}(\rho x^{\beta}(n+b)^{\gamma})f(x)\mathrm{d}x\right]^p < \left[|\beta|^{-\frac{1}{q}}\gamma^{-\frac{1}{p}}(2^{2m}-1)\frac{B_m}{2m}\left(\frac{\pi}{\rho}\right)^{2m}\right]^p\|f\|_{p,\mu}^p, \tag{4.2}$$

$$\int_{\Omega}x^{2mq\beta-1}\left[\sum_{n=0}^{\infty}\operatorname{csch}(\rho x^{\beta}(n+b)^{\gamma})a_n\right]^q\mathrm{d}x < \left[|\beta|^{-\frac{1}{q}}\gamma^{-\frac{1}{p}}(2^{2m}-1)\frac{B_m}{2m}\left(\frac{\pi}{\rho}\right)^{2m}\right]^q\|\boldsymbol{a}\|_{q,\nu}^q, \tag{4.3}$$

$$\int_{\Omega}f(x)\sum_{n=0}^{\infty}\operatorname{csch}(\rho x^{\beta}(n+b)^{\gamma})a_n\mathrm{d}x < |\beta|^{-\frac{1}{q}}\gamma^{-\frac{1}{p}}(2^{2m}-1)\frac{B_m}{2m}\left(\frac{\pi}{\rho}\right)^{2m}\|f\|_{p,\mu}\|\boldsymbol{a}\|_{q,\nu}, \tag{4.4}$$

where the constant $|\beta|^{-\frac{1}{q}}\gamma^{-\frac{1}{p}}(2^{2m}-1)\frac{B_m}{2m}\left(\frac{\pi}{\rho}\right)^{2m}$ *in* (4.2), (4.3), *and* (4.4) *is the best possible.*

Setting $\beta = \gamma$ $(0 < 2m\gamma \le 1)$, $b = s$ $(s \in \mathrm{N}^+)$ in Corollary 4.1, we obtain the following inequalities with a non-homogeneous kernel $(a \ge 1)$:

$$\sum_{n=s}^{\infty} n^{2mp\gamma-1}\left[\int_0^a \operatorname{csch}(\rho(xn)^\gamma)f(x)\mathrm{d}x\right]^p < \left[(2^{2m}-1)\frac{B_m}{2m\gamma}\left(\frac{\pi}{\rho}\right)^{2m}\right]^p \|f\|_{p,\mu}^p, \tag{4.5}$$

$$\int_0^a x^{2mq\gamma-1}\left[\sum_{n=s}^{\infty}\operatorname{csch}(\rho(xn)^\gamma)a_n\right]^q \mathrm{d}x < \left[(2^{2m}-1)\frac{B_m}{2m\gamma}\left(\frac{\pi}{\rho}\right)^{2m}\right]^q \|\boldsymbol{a}\|_{q,\nu}^q, \tag{4.6}$$

$$\int_0^a f(x)\sum_{n=s}^{\infty}\operatorname{csch}(\rho(xn)^\gamma)a_n\mathrm{d}x < (2^{2m}-1)\frac{B_m}{2m\gamma}\left(\frac{\pi}{\rho}\right)^{2m}\|f\|_{p,\mu}\|\boldsymbol{a}\|_{q,\nu}, \tag{4.7}$$

where $\mu(x) = x^{p(1-2m\gamma)-1}$ and $\nu_n = n^{q(1-2m\gamma)-1}$.

Setting $\beta = -\gamma$ $(0 < 2m\gamma \le 1)$, $b = s$ $(s \in \mathrm{N}^+)$ in Corollary 4.1, we obtain the following inequalities with a homogeneous kernel $(0 \le a \le 1)$:

$$\sum_{n=s}^{\infty} n^{2mp\gamma-1}\left[\int_a^\infty \operatorname{csch}\left(\rho\left(\frac{n}{x}\right)^\gamma\right)f(x)\mathrm{d}x\right]^p < \left[(2^{2m}-1)\frac{B_m}{2m\gamma}\left(\frac{\pi}{\rho}\right)^{2m}\right]^p \|f\|_{p,\mu}^p, \tag{4.8}$$

$$\int_a^\infty x^{-2mq\gamma-1}\left[\sum_{n=s}^{\infty}\operatorname{csch}\left(\rho\left(\frac{n}{x}\right)^\gamma\right)a_n\right]^q \mathrm{d}x < \left[(2^{2m}-1)\frac{B_m}{2m\gamma}\left(\frac{\pi}{\rho}\right)^{2m}\right]^q \|\boldsymbol{a}\|_{q,\nu}^q, \tag{4.9}$$

$$\int_a^\infty f(x)\sum_{n=s}^{\infty}\operatorname{csch}\left(\rho\left(\frac{n}{x}\right)^\gamma\right)a_n\mathrm{d}x < (2^{2m}-1)\frac{B_m}{2m\gamma}\left(\frac{\pi}{\rho}\right)^{2m}\|f\|_{p,\mu}\|\boldsymbol{a}\|_{q,\nu}, \tag{4.10}$$

where $\mu(x) = x^{p(1+2m\gamma)-1}$ and $\nu_n = n^{q(1-2m\gamma)-1}$.

Setting $\beta = \gamma$ $(0 < 2m\gamma \le 1)$, $b = \frac{1}{2}$, and replacing ρ with $\rho 2^\gamma$ in Corollary 4.1, it follows that

$$\sum_{n=0}^{\infty}(2n+1)^{2mp\gamma-1}\left[\int_0^a \operatorname{csch}(\rho(x(2n+1))^\gamma)f(x)\mathrm{d}x\right]^p < \left[2^{-1/p}(2^{2m}-1)\frac{B_m}{2m\gamma}\left(\frac{\pi}{\rho}\right)^{2m}\right]^p \|f\|_{p,\mu}^p, \tag{4.11}$$

$$\int_0^a x^{2mq\gamma-1}\left[\sum_{n=0}^{\infty}\operatorname{csch}(\rho(x(2n+1))^\gamma)a_n\right]^q \mathrm{d}x < \left[2^{-1/p}(2^{2m}-1)\frac{B_m}{2m\gamma}\left(\frac{\pi}{\rho}\right)^{2m}\right]^q \|\boldsymbol{a}\|_{q,\nu}^q, \tag{4.12}$$

$$\int_0^a f(x)\sum_{n=0}^{\infty}\operatorname{csch}(\rho(x(2n+1))^\gamma)a_n\mathrm{d}x < 2^{-1/p}(2^{2m}-1)\frac{B_m}{2m\gamma}\left(\frac{\pi}{\rho}\right)^{2m}\|f\|_{p,\mu}\|\boldsymbol{a}\|_{q,\nu}, \tag{4.13}$$

where $a \ge 1$, $\mu(x) = x^{p(1-2m\gamma)-1}$, and $\nu_n = (2n+1)^{q(1-2m\gamma)-1}$. Letting $\gamma = \frac{1}{2m}$, $\rho = 1$ in (4.13), we obtain inequality (1.13).

Setting $\beta = -\gamma$ $(0 < 2m\gamma \le 1)$, $b = \frac{1}{2}$, and replacing ρ with $\rho 2^\gamma$ in Corollary 4.1, we can obtain the homogeneous forms corresponding to (4.11), (4.12), and (4.13) with the same constant factors, such as

$$\int_a^\infty f(x)\sum_{n=0}^{\infty}\operatorname{csch}\left(\rho\left(\frac{2n+1}{x}\right)^\gamma\right)a_n\mathrm{d}x < 2^{-1/p}(2^{2m}-1)\frac{B_m}{2m\gamma}\left(\frac{\pi}{\rho}\right)^{2m}\|f\|_{p,\mu}\|\boldsymbol{a}\|_{q,\nu}, \tag{4.14}$$

where $0 < a \le 1$, $\mu(x) = x^{p(1+2m\gamma)-1}$, and $\nu_n = (2n+1)^{q(1-2m\gamma)-1}$.

Let $\lambda_1 = 0$, $\lambda_2 = -2\rho$, $\lambda_3 = 2\rho$ $(\rho > 0)$, and $\lambda = 2m$ $(m \in \mathrm{N}^+)$ in Theorem 3.1. It follows from (2.1) that

$$k(t) = \frac{1+e^{-2\rho t}}{e^{2\rho t}-e^{-2\rho t}} = \frac{1}{2}(\coth(\rho t)-1).$$

Additionally, by the equation [37]:

$$\sum_{j=0}^{\infty}\frac{2}{(j+1)^{2m}}=\frac{B_m}{(2m)!}(2\pi)^{2m},\quad m\in\mathrm{N}^+,\tag{4.15}$$

and (4.1), we have

$$C(\lambda_1,\lambda_2,\lambda_3,\lambda)=\frac{1}{\rho^{2m}}\sum_{j=0}^{\infty}\left[\frac{1}{(4j+4)^{2m}}+\frac{1}{(4j+2)^{2m}}\right]=\frac{B_m}{2(2m)!}\left(\frac{\pi}{\rho}\right)^{2m}.$$

Therefore, we obtain Corollary 4.2.

Corollary 4.2. *Suppose that* $\rho>0$, $\beta\neq 0$, $b\geq\frac{1}{2}$, *and* $0<2m\gamma\leq 1$ $(m\in\mathrm{N}^+)$. *Let* $0\leq a\leq 1$, *and* $\Omega=(a,\infty)$ *when* $\beta<0$. *Let* $a\geq 1$, *and* $\Omega=(0,a)$ *when* $\beta>0$. *Let* $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, $\mu(x)=x^{p(1-2m\beta)-1}$, *and* $\nu_n=(n+b)^{q(1-2m\gamma)-1}$. *Suppose that* $f(x)$, $a_n\geq 0$ *with* $f(x)\in L_{p,\mu}(\Omega)$, *and* $\boldsymbol{a}=\{a_n\}_{n=0}^{\infty}\in l_{q,\nu}$. *Then the following three inequalities are equivalent*:

$$\sum_{n=0}^{\infty}(n+b)^{2mp\gamma-1}\left[\int_{\Omega}(\coth(\rho x^{\beta}(n+b)^{\gamma})-1)f(x)\mathrm{d}x\right]^p<\left[|\beta|^{-\frac{1}{q}}\gamma^{-\frac{1}{p}}\frac{B_m}{2m}\left(\frac{\pi}{\rho}\right)^{2m}\right]^p\|f\|_{p,\mu}^p,\tag{4.16}$$

$$\int_{\Omega}x^{2mq\beta-1}\left[\sum_{n=0}^{\infty}(\coth(\rho x^{\beta}(n+b)^{\gamma})-1)a_n\right]^q\mathrm{d}x<\left[|\beta|^{-\frac{1}{q}}\gamma^{-\frac{1}{p}}\frac{B_m}{2m}\left(\frac{\pi}{\rho}\right)^{2m}\right]^q\|\boldsymbol{a}\|_{q,\nu}^q,\tag{4.17}$$

$$\int_{\Omega}f(x)\sum_{n=0}^{\infty}(\coth(\rho x^{\beta}(n+b)^{\gamma})-1)a_n\mathrm{d}x<|\beta|^{-\frac{1}{q}}\gamma^{-\frac{1}{p}}\frac{B_m}{2m}\left(\frac{\pi}{\rho}\right)^{2m}\|f\|_{p,\mu}\|\boldsymbol{a}\|_{q,\nu},\tag{4.18}$$

where the constant $|\beta|^{-\frac{1}{q}}\gamma^{-\frac{1}{p}}\frac{B_m}{2m}\left(\frac{\pi}{\rho}\right)^{2m}$ *in* (4.16), (4.17), *and* (4.18) *is the best possible*.

Remark 4.3. Letting $\lambda_1=\lambda_2=-\lambda_3=-\rho$ $(\rho>0)$, and $\lambda=2m$ $(m\in\mathrm{N}^+)$ in Theorem 3.1, we can also obtain Corollary 4.2. In fact, by (2.2) and (4.15), we obtain

$$C(\lambda_1,\lambda_2,\lambda_3,\lambda)=\left(\frac{1}{2\rho}\right)^{2m}\sum_{j=0}^{\infty}\frac{2}{(j+1)^{2m}}=\frac{B_m}{(2m)!}\left(\frac{\pi}{\rho}\right)^{2m}.$$

Therefore, Theorem 3.1 reduces to Corollary 4.2 obviously.

Setting $\beta=\gamma$, $b=\frac{1}{2}$, and replacing ρ with $\rho 2^{\gamma}$ in Corollary 4.2, it follows that

$$\sum_{n=0}^{\infty}(2n+1)^{2mp\gamma-1}\left[\int_0^a(\coth(\rho(x(2n+1))^{\gamma})-1)f(x)\mathrm{d}x\right]^p<\left[2^{-1/p-1}\frac{B_m}{m\gamma}\left(\frac{\pi}{\rho}\right)^{2m}\right]^p\|f\|_{p,\mu}^p,\tag{4.19}$$

$$\int_0^a x^{2mq\gamma-1}\left[\sum_{n=0}^{\infty}(\coth(\rho(x(2n+1))^{\gamma})-1)a_n\right]^q\mathrm{d}x<\left[2^{-1/p-1}\frac{B_m}{m\gamma}\left(\frac{\pi}{\rho}\right)^{2m}\right]^q\|\boldsymbol{a}\|_{q,\nu}^q,\tag{4.20}$$

$$\int_0^a f(x)\sum_{n=0}^{\infty}[\coth(\rho(x(2n+1))^{\gamma})-1]a_n\mathrm{d}x<2^{-1/p-1}\frac{B_m}{m\gamma}\left(\frac{\pi}{\rho}\right)^{2m\gamma}\|f\|_{p,\mu}\|\boldsymbol{a}\|_{q,\nu},\tag{4.21}$$

where $a\geq 1$, $\mu(x)=x^{p(1-2m\gamma)-1}$, and $\nu_n=(2n+1)^{q(1-2m\gamma)-1}$.

Setting $\beta=-\gamma$, $b=\frac{1}{2}$, and replacing ρ with $\rho 2^{\gamma}$ in Corollary 4.2, we obtain the homogeneous forms corresponding to (4.19), (4.20), and (4.21) with the same constant factors, such as

$$\int_a^\infty f(x)\sum_{n=0}^\infty\left(\coth\left(\rho\left(\frac{2n+1}{x}\right)^\gamma\right)-1\right)a_n\mathrm{d}x < 2^{-1/p-1}\frac{B_m}{m\gamma}\left(\frac{\pi}{\rho}\right)^{2m\gamma}\|f\|_{p,\mu}\|\boldsymbol{a}\|_{q,\nu}, \tag{4.22}$$

where $0 \le a \le 1$, $\mu(x) = x^{p(1+2m\gamma)-1}$, and $\nu_n = (2n+1)^{q(1-2m\gamma)-1}$. If we set $\gamma = \frac{1}{2m}\rho = 1$ in (4.22), then (4.22) reduces to (1.14).

Setting $\beta = \gamma = \frac{1}{2m}$, $b = s$ $(m, s \in \mathrm{N}^+)$, and $\beta = -\frac{1}{2m}$, $\gamma = \frac{1}{2m}$, $b = s$ $(m, s \in \mathrm{N}^+)$ in Corollary 4.2, respectively, we obtain

$$\int_0^a f(x)\sum_{n=s}^\infty[\coth(\rho\sqrt[2m]{xn})-1]a_n\mathrm{d}x < B_m\left(\frac{\pi}{\rho}\right)^{2m}\|f\|_{p,\mu_1}\|\boldsymbol{a}\|_{q,\nu}\ (a \ge 1), \tag{4.23}$$

$$\int_a^\infty f(x)\sum_{n=s}^\infty\left[\coth\left(\rho\sqrt[2m]{\frac{n}{x}}\right)-1\right]a_n\mathrm{d}x < B_m\left(\frac{\pi}{\rho}\right)^{2m}\|f\|_{p,\mu_2}\|\boldsymbol{a}\|_{q,\nu}\ (0 \le a \le 1), \tag{4.24}$$

where $\mu_1(x) = \frac{1}{x}$, $\mu_2(x) = x^{2p-1}$, $\nu_n = \frac{1}{n}$.

Let $\lambda_2 = -\lambda_1$, and $\lambda = 2m$ $(m \in \mathrm{N}^+)$ in Theorem 3.1. By Lemma 2.5, we obtain the following corollary.

Corollary 4.4. *Suppose that* $\beta \ne 0$, $b \ge \frac{1}{2}$, $-\lambda_3 < 0 \le \lambda_1 < \lambda_3$, *and* $0 < 2m\gamma \le 1$ $(m \in \mathrm{N}^+)$. *Let* $0 \le a \le 1$, *and* $\Omega = (a, \infty)$ *when* $\beta < 0$. *Let* $a \ge 1$, *and* $\Omega = (0, a)$ *when* $\beta > 0$. *Let* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\psi(x) = \tan x$, $\mu(x) = x^{p(1-2m\beta)-1}$, *and* $\nu_n = (n+b)^{q(1-2m\gamma)-1}$. *Suppose that* $f(x)$, $a_n \ge 0$ *with* $f(x) \in L_{p,\mu}(\Omega)$, *and* $\boldsymbol{a} = \{a_n\}_{n=0}^\infty \in l_{q,\nu}$. *Then the following three inequalities are equivalent*:

$$\begin{aligned}&\sum_{n=0}^\infty (n+b)^{2mp\gamma-1}\left[\int_\Omega \cosh(\lambda_1 x^\beta(n+b)^\gamma)\mathrm{csch}(\lambda_3 x^\beta(n+b)^\gamma)f(x)\mathrm{d}x\right]^p\\ &\quad< \left[|\beta|^{-\frac{1}{q}}\gamma^{-\frac{1}{p}}\left(\frac{\pi}{2\lambda_3}\right)^{2m}\psi^{(2m-1)}\left(\frac{\lambda_1\pi}{2\lambda_3}\right)\right]^p\|f\|_{p,\mu}^p,\end{aligned} \tag{4.25}$$

$$\int_\Omega x^{2mq\beta-1}\left[\sum_{n=0}^\infty \cosh(\lambda_1 x^\beta(n+b)^\gamma)\mathrm{csch}(\lambda_3 x^\beta(n+b)^\gamma)a_n\right]^q\mathrm{d}x < \left[|\beta|^{-\frac{1}{q}}\gamma^{-\frac{1}{p}}\left(\frac{\pi}{2\lambda_3}\right)^{2m}\psi^{(2m-1)}\left(\frac{\lambda_1\pi}{2\lambda_3}\right)\right]^q\|\boldsymbol{a}\|_{q,\nu}^q, \tag{4.26}$$

$$\int_\Omega f(x)\sum_{n=0}^\infty \cosh(\lambda_1 x^\beta(n+b)^\gamma)\mathrm{csch}(\lambda_3 x^\beta(n+b)^\gamma)a_n\mathrm{d}x < |\beta|^{-\frac{1}{q}}\gamma^{-\frac{1}{p}}\left(\frac{\pi}{2\lambda_3}\right)^{2m}\psi^{(2m-1)}\left(\frac{\lambda_1\pi}{2\lambda_3}\right)\|f\|_{p,\mu}\|\boldsymbol{a}\|_{q,\nu}, \tag{4.27}$$

where the constant $|\beta|^{-\frac{1}{q}}\gamma^{-\frac{1}{p}}\left(\frac{\pi}{2\lambda_3}\right)^{2m}\psi^{(2m-1)}\left(\frac{\lambda_1\pi}{2\lambda_3}\right)$ *in* (4.25), (4.26), *and* (4.27) *is the best possible.*

In Corollary 4.4, let $\beta = \gamma$, $b = \frac{1}{2}$, and replace λ_1 and λ_3 with $\lambda_1 2^\gamma$ and $\lambda_3 2^\gamma$, respectively, then we obtain

$$\begin{aligned}&\sum_{n=0}^\infty (2n+1)^{2mp\gamma-1}\left[\int_0^a \cosh(\lambda_1(x(2n+1))^\gamma)\mathrm{csch}(\lambda_3(x(2n+1))^\gamma)f(x)\mathrm{d}x\right]^p\\ &\quad< \left[\frac{1}{\gamma 2^{1/p}}\left(\frac{\pi}{2\lambda_3}\right)^{2m}\psi^{(2m-1)}\left(\frac{\lambda_1\pi}{2\lambda_3}\right)\right]^p\|f\|_{p,\mu}^p,\end{aligned} \tag{4.28}$$

$$\int_0^a x^{2mq\gamma-1}\left[\sum_{n=0}^\infty \cosh(\lambda_1(x(2n+1))^\gamma)\mathrm{csch}(\lambda_3(x(2n+1))^\gamma)a_n\right]^q\mathrm{d}x < \left[\frac{1}{\gamma 2^{1/p}}\left(\frac{\pi}{2\lambda_3}\right)^{2m}\psi^{(2m-1)}\left(\frac{\lambda_1\pi}{2\lambda_3}\right)\right]^q\|\boldsymbol{a}\|_{q,\nu}^q, \tag{4.29}$$

$$\int_0^a f(x)\sum_{n=0}^\infty \cosh(\lambda_1(x(2n+1))^\gamma)\mathrm{csch}(\lambda_3(x(2n+1))^\gamma)a_n\mathrm{d}x < \frac{1}{\gamma 2^{1/p}}\left(\frac{\pi}{2\lambda_3}\right)^{2m}\psi^{(2m-1)}\left(\frac{\lambda_1\pi}{2\lambda_3}\right)\|f\|_{p,\mu}\|\boldsymbol{a}\|_{q,\nu}, \tag{4.30}$$

where $a \ge 1$, $\mu(x) = x^{p(1-2m\gamma)-1}$, and $\nu_n = (2n+1)^{q(1-2m\gamma)-1}$.

In Corollary 4.4, let $\beta = -\gamma$, $b = \frac{1}{2}$, and replace λ_1 and λ_3 with $\lambda_1 2^\gamma$ and $\lambda_3 2^\gamma$, respectively, then we can obtain the homogeneous forms corresponding to (4.28), (4.29), and (4.30).

In Corollary 4.4, let $\beta = \gamma$, $b = s$ ($s \in \mathrm{N}^+$), then we obtain

$$\sum_{n=s}^{\infty} n^{2mp\gamma-1}\left[\int_0^a \cosh(\lambda_1(xn)^\gamma)\operatorname{csch}(\lambda_3(xn)^\gamma)f(x)\mathrm{d}x\right]^p < \left[\frac{1}{\gamma}\left(\frac{\pi}{2\lambda_3}\right)^{2m}\psi^{(2m-1)}\left(\frac{\lambda_1\pi}{2\lambda_3}\right)\right]^p \|f\|_{p,\mu}^p, \tag{4.31}$$

$$\int_0^a x^{2mq\beta-1}\left[\sum_{n=s}^{\infty}\cosh(\lambda_1(xn)^\gamma)\operatorname{csch}(\lambda_3(xn)^\gamma)a_n\right]^q \mathrm{d}x < \left[\frac{1}{\gamma}\left(\frac{\pi}{2\lambda_3}\right)^{2m}\psi^{(2m-1)}\left(\frac{\lambda_1\pi}{2\lambda_3}\right)\right]^q \|\boldsymbol{a}\|_{q,\nu}^q, \tag{4.32}$$

$$\int_0^a f(x)\sum_{n=s}^{\infty}\cosh(\lambda_1(xn)^\gamma)\operatorname{csch}(\lambda_3(xn)^\gamma)a_n\mathrm{d}x < \frac{1}{\gamma}\left(\frac{\pi}{2\lambda_3}\right)^{2m}\psi^{(2m-1)}\left(\frac{\lambda_1\pi}{2\lambda_3}\right)\|f\|_{p,\mu}\|\boldsymbol{a}\|_{q,\nu}, \tag{4.33}$$

where $a \geq 1$, $\mu(x) = x^{p(1-2m\gamma)-1}$, and $\nu_n = n^{q(1-2m\gamma)-1}$. Letting $\lambda_1 = 1$, $\lambda_3 = 3$, and $\gamma = \frac{1}{2m}$ in (4.33), we obtain (1.15).

Remark 4.5. Setting $\lambda_1 = \rho$, $\lambda_3 = 2\rho$ ($\rho > 0$) in Corollary 4.4, and observing that

$$\cosh(u)\operatorname{csch}(2u) = \frac{1}{2}\operatorname{csch}(u),$$

it implies that

$$\sum_{n=0}^{\infty}(n+b)^{2mp\gamma-1}\left[\int_\Omega \operatorname{csch}(\rho x^\beta(n+b)^\gamma)f(x)\mathrm{d}x\right]^p < \left[|\beta|^{-\frac{1}{q}}\gamma^{-\frac{1}{p}}2^{1-4m}\left(\frac{\pi}{\rho}\right)^{2m}\psi^{(2m-1)}\left(\frac{\pi}{4}\right)\right]^p \|f\|_{p,\mu}^p, \tag{4.34}$$

$$\int_\Omega x^{2mq\beta-1}\left[\sum_{n=0}^{\infty}\operatorname{csch}(\rho x^\beta(n+b)^\gamma)a_n\right]^q \mathrm{d}x < \left[|\beta|^{-\frac{1}{q}}\gamma^{-\frac{1}{p}}2^{1-4m}\left(\frac{\pi}{\rho}\right)^{2m}\psi^{(2m-1)}\left(\frac{\pi}{4}\right)\right]^q \|\boldsymbol{a}\|_{q,\nu}^q, \tag{4.35}$$

$$\int_\Omega f(x)\sum_{n=0}^{\infty}\operatorname{csch}(\rho x^\beta(n+b)^\gamma)a_n\mathrm{d}x < |\beta|^{-\frac{1}{q}}\gamma^{-\frac{1}{p}}2^{1-4m}\left(\frac{\pi}{\rho}\right)^{2m}\psi^{(2m-1)}\left(\frac{\pi}{4}\right)\|f\|_{p,\mu}\|\boldsymbol{a}\|_{q,\nu}. \tag{4.36}$$

Inequalities (4.2), (4.3), (4.4), (4.34), (4.35), and (4.36) are equivalent. In fact, by Lemma 2.5 and (4.1), we obtain

$$\psi^{(2m-1)}\left(\frac{\pi}{4}\right) = (2m-1)!\left(\frac{4}{\pi}\right)^{2m}\sum_{j=0}^{\infty}\left[\frac{1}{(4j+1)^{2m}} + \frac{1}{(4j+3)^{2m}}\right] = \left(\frac{4}{\pi}\right)^{2m}\sum_{j=0}^{\infty}\frac{(2m-1)!}{(2j+1)^{2m}} = \frac{B_m}{m}(2^{2m}-1)2^{4m-2},$$

and it follows therefore that (4.2), (4.3), (4.3), (4.34), (4.35), and (4.36) are equivalent.

Remark 4.6. Setting $\lambda_1 = \rho$, $\lambda_3 = 4\rho$ ($\rho > 0$) in Corollary 4.4, and observing that

$$\cosh(u)\operatorname{csch}(4u) = \frac{1}{4}\operatorname{csch}(u)\operatorname{sech}(2u),$$

we obtain the following Hilbert-type inequalities involving hyperbolic secant and hyperbolic cosecant functions:

$$\begin{aligned}&\sum_{n=0}^{\infty}(n+b)^{2mp\gamma-1}\left[\int_\Omega \operatorname{csch}(\rho x^\beta(n+b)^\gamma)\operatorname{sech}(2\rho x^\beta(n+b)^\gamma)f(x)\mathrm{d}x\right]^p \\ &\quad < \left[|\beta|^{-\frac{1}{q}}\gamma^{-\frac{1}{p}}4^{1-3m}\left(\frac{\pi}{\rho}\right)^{2m}\psi^{(2m-1)}\left(\frac{\pi}{8}\right)\right]^p \|f\|_{p,\mu}^p,\end{aligned} \tag{4.37}$$

$$\int_{\Omega} x^{2mq\beta-1}\left[\sum_{n=0}^{\infty}\operatorname{csch}(\rho x^{\beta}(n+b)^{\gamma})\operatorname{sech}(2\rho x^{\beta}(n+b)^{\gamma})a_n\right]^q \mathrm{d}x < \left[|\beta|^{-\frac{1}{q}}\gamma^{-\frac{1}{p}}4^{1-3m}\left(\frac{\pi}{\rho}\right)^{2m}\psi^{(2m-1)}\left(\frac{\pi}{8}\right)\right]^q \|\boldsymbol{a}\|_{q,\nu}^q, \quad (4.38)$$

$$\int_{\Omega} f(x)\sum_{n=0}^{\infty}\operatorname{csch}(\rho x^{\beta}(n+b)^{\gamma})\operatorname{sech}(2\rho x^{\beta}(n+b)^{\gamma})a_n \mathrm{d}x < |\beta|^{-\frac{1}{q}}\gamma^{-\frac{1}{p}}4^{1-3m}\left(\frac{\pi}{\rho}\right)^{2m}\psi^{(2m-1)}\left(\frac{\pi}{8}\right)\|f\|_{p,\mu}\|\boldsymbol{a}\|_{q,\nu}. \quad (4.39)$$

Letting $\beta = \gamma$, $\rho = 1$, $b = s$ ($s \in \mathbb{N}^{+}$), and $m = 1$ in (4.39), we have

$$\int_{0}^{a} f(x)\sum_{n=s}^{\infty}\operatorname{csch}((xn)^{\gamma})\operatorname{sech}(2(xn)^{\gamma})a_n \mathrm{d}x < \frac{\pi^2}{16\gamma}\psi\left(\frac{\pi}{8}\right)\|f\|_{p,\mu}\|\boldsymbol{a}\|_{q,\nu},$$

where $a \ge 1$, $0 < \gamma \le \frac{1}{2}$, $\mu(x) = x^{p(1-2\gamma)-1}$, and $\nu_n = n^{q(1-2\gamma)-1}$.

5 Conclusion

The main objective of this work is to establish some half-discrete Hilbert-type inequalities involving hyperbolic functions. This interest is mainly motivated by [22,25,26], where the authors provided some integral inequalities with hyperbolic functions. In order to do so, we first constructed a more general kernel function composed of several exponent functions with multiple parameters. By using the Hermite-Hadamard's inequality, Hölder's inequality as well as some other techniques of real analysis, we established a half-discrete Hilbert-type with the newly constructed kernel function. Second, by constructing a special sequence and a special function (Lemma 2.4), we proved that the constant factor of the established Hilbert-type inequality is the best possible. At last, by the introduction of the Bernoulli number and the rational fraction expansion of tangent function, some special examples and their equivalent forms were considered. We need to point out that some hyperbolic functions or their combinations are not included in our results, and it is worthy of further research.

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