Research Article

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Initial-boundary value problem of fifth-order Korteweg-de Vries equation posed on half line with nonlinear boundary values

https://doi.org/10.1515/math-2022-0040
received January 23, 2022; accepted April 7, 2022

Abstract: In this paper, we study the initial boundary problem of fifth-order Korteweg-de Vries equation with nonlinear boundary values. First, we establish a so-called sharp boundary trace regularity associated with the linearized fifth-order Korteweg-de Vries equation. Then, aided by the sharp boundary trace regularity, we verify that initial-boundary value problem of fifth-order Korteweg-de Vries equation with nonlinear boundary conditions is locally well-posed when initial and boundary values are properly chosen.

Keywords: fifth-order Korteweg-de Vries equation, initial-boundary value problem, nonlinear boundary condition, sharp boundary trace regularity, local well-posedness

MSC 2020: 35G16, 35Q53

1 Introduction

Korteweg-de Vries equation

$$\partial_t u \pm \partial_x^3 u + u\partial_x u = 0$$

origins from shallow water waves that are weakly and nonlinearly interacting, and nowadays is extensively applied in ion acoustic waves in plasma, long internal waves in a density-stratified ocean, and acoustic waves on a crystal lattice, etc. However, under certain conditions, the third-order dispersive term is too weak to describe the physical facts, then fifth-order dispersive term is introduced to strength it (if the angle between the propagation direction and the magnetic-acoustic wave in a cold collision-free plasma and the external magnetic field become critical value, then the third-order dispersive term vanishes and is replaced by the fifth-order dispersive term [1]; a fifth-order term was necessary to model capillary-gravity waves for Bond number near $\frac{1}{3}$ [2])

$$\partial_t u \pm \partial_x^5 u + u\partial_x u = 0.$$

The Cauchy problem for the fifth-order KdV equation has been extensively studied after Kato smoothing effect discovered in the early 1980s, see for example [3–8]. Compared with pure initial value problems, initial boundary value problems posed on part of entire line with boundaries are more applicable to the reality and can provide more accurate data to physical experiments or practical problems. Although there is

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less research on initial boundary value problems than that on pure initial value problems, nowadays, more and more attention has been paid for initial boundary value problems [9–13].

Different from the former works (say, for example, [9–13]) which deal with the linear boundary values, we investigate the well-posedness in the sense of Hadamard of the initial-boundary value problem

$$\begin{cases} \partial_t u + \partial_x^5 u = u\partial_x u, \quad x > 0, \ t > 0, \\ u(x,0) = \phi(x), \quad x > 0, \\ \partial_x u(0,t) = h_1(t), \ \partial_x^2 u(0,t) = u(0,t)^2 + h_2(t), \ \partial_x^4 u(0,t) = u(0,t)^2 + h_3(t), \ t > 0, \end{cases} \tag{1.1}$$

with nonlinear boundary feedback (This often happens in application [14,15], or theory, i.e., feedback stabilization [16]). Due to the presence of the nonlinear boundary condition, the Kato smoothing effect is not strong enough to deal with (1.1). Instead, a so-called sharp boundary trace regularity is needed. More precisely, for any $h_1 \in H_0^{\frac{s+1}{5}}$, $h_2 \in H_0^{\frac{s}{5}}$, $h_3 \in H_0^{\frac{s-2}{5}}$ the corresponding solution u of

$$\begin{cases} \partial_t u + \partial_x^5 u = 0, \quad x > 0, \ t > 0, \\ u(x,0) = 0, \quad x > 0, \\ \partial_x u(0,t) = h_1(t), \ \partial_x^2 u(0,t) = h_2(t), \ \partial_x^4 u(0,t) = h_3(t), \ t > 0, \end{cases} \tag{1.2}$$

satisfies:

$$\partial_x^k u \in L^\infty(\mathbb{R}^+; H^{\frac{2-k}{5}}(0,T)), \quad k = 0, 1, 2, 3, 4.$$

Taking account of sharp boundary trace regularity, we define the solution space as

$$Y_T^s \equiv \left\{ u \in C(0,T; H^s(\mathbb{R}^+)) \,\middle|\, \sum_{k=0}^{4} \|u\|_{L^\infty\left(\mathbb{R}^+; H^{\frac{s+2-k}{5}}(0,T)\right)} < \infty \right\}.$$

For the vector consists of initial and boundary values $(\varphi, \overrightarrow{h}) \equiv (\varphi, (h_1, h_2, h_3))$, we defined the corresponding function space as

$$X_T^s \equiv H^s(\mathbb{R}^+) \times \mathcal{H}^s(0,T),$$

with

$$\mathcal{H}^s(0,T) \equiv H_0^{\frac{s+1}{5}}(0,T) \times H_0^{\frac{s}{5}}(0,T) \times H_0^{\frac{s-2}{5}}(0,T).$$

The main results can be stated as:

Theorem 1.1. *Let $s \in [0,5]$, $T > 0$ be given. For any s-compatible $(\varphi, \overrightarrow{h}) \in X_T^s$, $f \in L^1(0,T; H^s(R^+))$, there exists $T^* \in (0,T]$ such that the initial-boundary value problem*

$$\begin{cases} \partial_t u + \partial_x^5 u = f, \quad x > 0, \ t > 0, \\ u(x,0) = \phi(x), \quad x > 0, \\ \partial_x u(0,t) = h_1(t), \ \partial_x^2 u(0,t) = u(0,t)^2 + h_2(t), \ \partial_x^4 u(0,t) = u(0,t)^2 + h_3(t), \ t > 0, \end{cases} \tag{1.3}$$

admits a unique solution $u \in Y_{T^*}^s$. Moreover, the solution depends Lipschitz continuously on $(\varphi, \overrightarrow{h})$ and f in the corresponding space.

Remark 1.2. However, initial boundary value problem (1.1) includes both nonlinear damping and nonlinear feedback, Kato smoothing effect, and sharp boundary trace regularity are not enough to deal with both nonlinearities, and Bourgain's regularity is needed. Thus, the solution space should be taken as:

$$Y_T^s \equiv \left\{ u \in C(0,T; H^s(\mathbb{R}^+)) \cap X^{s,b} \,\middle|\, \sum_{k=0}^{4} \|u\|_{L^\infty\left(\mathbb{R}^+; H^{\frac{s+2-k}{5}}(0,T)\right)} < \infty \right\},$$

where $X^{s,b}$ is the Bourgain space.

The paper is organized as follows. In Section 2, we will first derive an explicit integral representation of (1.2), then established sharp boundary trace regularity. The proof of our main result in this paper (Theorem 1.1) will be presented in Section 3. Section 4 includes remarks which show the readers all the possible cases can be proved similarly.

2 Linear estimates

In this part, we will establish linear estimate as follows:

Proposition 2.1. *Let $T > 0$ and $0 \le s \le 5$ be given. For any $\phi \in H^s(\mathbb{R}^+)$, $\vec{h} \in \mathcal{H}^s(\mathbb{R}^+)$, $f \in L^1(0, T; H^s(\mathbb{R}^+))$, the initial boundary value problem*

$$\begin{cases} \partial_t u + \partial_x^5 u = f(x,t), \quad x > 0, \ t > 0, \\ u(x,0) = \phi(x), \quad x > 0, \\ \partial_x u(0,t) = h_1(t), \ \partial_x^2 u(0,t) = h_2(t), \ \partial_x^4 u(0,t) = h_3(t), \ t > 0 \end{cases} \tag{2.1}$$

admits a solution $u \in C(0, T; H^s(\mathbb{R}^+)) \cap L^2(0, T; H^{2+s}(\mathbb{R}^+))$ satisfying

$$\|u\|_{Y_T^s} \le C(\|(\phi, \vec{h})\|_{X_T^s} + \|f\|_{L^1(0,T;\, H^s(\mathbb{R}^+))}).$$

2.1 Representation of the solution

Applying Laplace transformation to (2.1) we obtain

$$\begin{cases} s\hat{u}(x,s) + \dfrac{d^5\hat{u}}{dx^5}(x,s) = 0, \\ \dfrac{d}{dx}\hat{u}(0,s) = \hat{h}_1(s), \ \dfrac{d^2}{dx^2}\hat{u}(0,s) = \hat{h}_2(s), \ \dfrac{d^4}{dx^4}\hat{u}(0,s) = \hat{h}_3(s). \end{cases} \tag{2.2}$$

Then, the solution $\hat{u}(x,s)$ of (2.2) can be written in the form

$$\hat{u}(x,s) = \sum_{j=1}^{3} c_j(s) e^{\lambda_j(s)x},$$

where $\lambda_j(s), \ j = 1, 2, 3$, are the solutions of the characteristic equation:

$$s = \lambda^5, \ \text{ with } \operatorname{Re} s > 0 \tag{2.3}$$

and $c_j(s), \ j = 1, 2, 3$, solve the linear system

$$\begin{cases} \lambda_1 c_1 + \lambda_2 c_2 + \lambda_3 c_3 = \hat{h}_1, \\ \lambda_1^2 c_1 + \lambda_2^2 c_2 + \lambda_3^2 c_3 = \hat{h}_2, \\ \lambda_1^4 c_1 + \lambda_2^4 c_2 + \lambda_3^4 c_3 = \hat{h}_3. \end{cases}$$

Then, Cramer's rule implies that

$$c_j(s) = \frac{\Delta_j(s)}{\Delta(s)}, \ j = 1, 2, 3,$$

where $\Delta(s)$ is the determinant of the coefficient matrix

$$\Delta = \begin{vmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \\ \lambda_1^4 & \lambda_2^4 & \lambda_3^4 \end{vmatrix}$$

and $\Delta_j(s)$ $(j = 1, 2, 3)$ are determinants of the matrices that are obtained by replacing the ith-column $(j = 1, 2, 3)$ of $\Delta(s)$ by the column vector $(\hat{h}_1(s), \hat{h}_2(s), \hat{h}_3(s))$.

Inverse Laplace transform of $\hat{u}(x, s)$ yields

$$u(x, t) = \frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty} e^{st}\hat{u}(x, s)\mathrm{d}s = \sum_{j=1}^{3}\frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty} e^{st}\frac{\Delta_j(s)}{\Delta(s)}e^{\lambda_j(s)x}\mathrm{d}s.$$

In order to find the smoothing effect associated with each boundary value function, we divide the solution u of (2.1) into

$$u(x, t) = u_1(x, t) + u_2(x, t) + u_3(x, t) \equiv \sum_{m=1}^{3} u_m(x, t),$$

where $u_m(x, t)$ solves (2.1) with $h_j = 0$ when $j \neq m$ $(m, j = 1, 2, 3)$. Thus, each $u_m(x, t)$ has the representation:

$$u_m(x, t) = \sum_{j=1}^{3}\frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty} e^{st}\hat{u}(x, s)\mathrm{d}s = \sum_{j=1}^{3}\frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty} e^{st}\frac{\Delta_{jm}(s)}{\Delta(s)}e^{\lambda_j(s)x}\hat{h}_m(s),\, \mathrm{d}s,$$

where $\Delta_{jm}(s)$ is obtained from $\Delta_j(s)$ by letting $\hat{h}_m(s) = 1$ and $\hat{h}_k(s) = 0$ for $k \neq m$ $(k, m = 1, 2, 3)$.

In the last formulas, the right-hand sides are continuous with respect to r for $r \geq 0$. As the left-hand sides do not depend on r, we may take $r = 0$ in these formulas. Hence,

$$\begin{aligned} u_m(x, t) &= \sum_{j=1}^{3}\frac{1}{2\pi i}\int_{0}^{i\infty} e^{st}\frac{\Delta_{jm}(s)}{\Delta(s)}e^{\lambda_j(s)x}\hat{h}_m(s)\mathrm{d}s + \sum_{j=1}^{3}\frac{1}{2\pi i}\int_{-i\infty}^{0} e^{st}\frac{\Delta_{jm}(s)}{\Delta(s)}e^{\lambda_j(s)x}\hat{h}_m(s)\mathrm{d}s \\ &= \sum_{j=1}^{3}\frac{1}{2\pi}\int_{0}^{\infty} e^{i\rho^5 t}\frac{\Delta_{jm}^{+}(\rho)}{\Delta^{+}(\rho)}e^{\lambda_j^{+}(\rho)x}\hat{h}_m^{+}(\rho)5\rho^4\mathrm{d}\rho + \sum_{j=1}^{3}\frac{1}{2\pi}\int_{0}^{\infty} e^{-i\rho^5 t}\frac{\Delta_{jm}(\rho)}{\Delta^{-}(\rho)}e^{\lambda_j^{-}(\rho)x}\hat{h}_m^{-}(\rho)5\rho^4\mathrm{d}\rho, \end{aligned}$$

where $\hat{h}^{\pm}(\rho) = \hat{h}(\pm i\rho^5)$, and $\Delta_{jm}^{\pm}(\rho)$, $\Delta^{\pm}(\rho)$, $\lambda_j^{\pm}(\rho)$ are defined the same way. Thus, $\overline{\hat{h}^{+}(\rho)} = \hat{h}^{-}(\rho)$, $\overline{\Delta^{+}(\rho)} = \Delta^{-}(\rho)$, $\overline{\lambda_j^{+}(\rho)} = \lambda_j^{-}(\rho)$.

When taking $s = i\rho^5$, the roots (characteristic roots) of (2.3) are as follows:

$$\lambda_1(\rho) = i\rho, \quad \lambda_2(\rho) = \rho\left(\cos\frac{9\pi}{10} + i\sin\frac{9\pi}{10}\right), \quad \lambda_3(\rho) = \rho\left(\cos\frac{13\pi}{10} + i\sin\frac{13\pi}{10}\right).$$

2.2 Boundary smoothing effect

We introduce the following technical lemma

Lemma 2.2. *For any $f \in L^2(a, \infty)$, let Kf be the function defined by*

$$Kf(x) = \int_{a}^{\infty} e^{\gamma(\mu)x}f(\mu)\mathrm{d}\mu,$$

where $a \in \mathbb{R}$, $\gamma(\mu)$ is a continuous complex-valued function defined on (a, ∞) satisfying the following two conditions

(*i*) $\mathrm{Re}\gamma(\mu) < 0$, *for* $\mu > a$;
(*ii*) *There exist* $\delta > 0$ *and* $b > 0$ *such that*

$$\sup_{a<\mu<a+\delta} \frac{|\mathrm{Re}\gamma(\mu)|}{\mu - a} \geq b;$$

(*iii*) *There exists a complex number* $\alpha + i\beta$ *such that*

$$\lim_{\mu\to\infty} \frac{\gamma(\mu)}{\mu} = \alpha + i\beta.$$

Then there exists a constant C *such that for all* $f \in L^2(0, \infty)$,

$$\|Kf\|_{L^2(\mathbb{R}^+)} \leq C\|f\|_{L^2(a,\infty)}.$$

Proof. See Lemma 2.5 in [17]. □

Proposition 2.3. *Let* $s > 0$. *There exists a constant* C *such that* $\forall \overrightarrow{h} \in \mathcal{H}^s(\mathbb{R}^+)$, *the solution*

$$u(x, t) \equiv W_{bdr}(\overrightarrow{h})$$

of initial boundary value problem

$$\begin{cases} \partial_t u + \partial_x^5 u = 0, \quad x > 0, \ t > 0, \\ u(x, 0) = 0, \quad x > 0, \\ \partial_x u(0, t) = h_1(t), \ \partial_x^2 u(0, t) = h_2(t), \ \partial_x^4 u(0, t) = h_3(t), \ t > 0, \end{cases} \tag{2.4}$$

satisfying

$$\sup_{0<t<T} \|u(\cdot,t)\|_{H^s(\mathbb{R}^+)} + \sum_{k=0}^{4} \|\partial_x^k u\|_{L^\infty\left(\mathbb{R}^+;\, H^{\frac{s+2-k}{5}}(0,T)\right)} \leq C\|\overrightarrow{h}\|_{\mathcal{H}^s(\mathbb{R}^+)}. \tag{2.5}$$

Proof. We have for $\rho \to \infty$,

$$\frac{\Delta_{11}^+(\rho)}{\Delta^+(\rho)} \sim \rho^{-1}, \ \frac{\Delta_{21}^+(\rho)}{\Delta^+(\rho)} \sim \rho^{-1}, \ \frac{\Delta_{31}^+(\rho)}{\Delta^+(\rho)} \sim \rho^{-1}, \tag{2.6}$$

$$\frac{\Delta_{12}^+(\rho)}{\Delta^+(\rho)} \sim \rho^{-2}, \ \frac{\Delta_{22}^+(\rho)}{\Delta^+(\rho)} \sim \rho^{-2}, \ \frac{\Delta_{32}^+(\rho)}{\Delta^+(\rho)} \sim \rho^{-2}, \tag{2.7}$$

and

$$\frac{\Delta_{13}^+(\rho)}{\Delta^+(\rho)} \sim \rho^{-4}, \ \frac{\Delta_{23}^+(\rho)}{\Delta^+(\rho)} \sim \rho^{-4}, \ \frac{\Delta_{33}^+(\rho)}{\Delta^+(\rho)} \sim \rho^{-4} \tag{2.8}$$

(the tedious proof of the aforementioned estimates (2.6)–(2.8) will be postponed to the appendix).

The solution of (2.4) can be divided into

$$u = u_1 + u_2 + u_3,$$

where u_i is the solution with respect to h_i ($i = 1, 2, 3$). For example, u_1 is the solution of

$$\begin{cases} \partial_t u - \partial_x^5 u = 0, \quad x > 0, \ t > 0, \\ u(x, 0) = 0, \quad x > 0, \\ \partial_x u(0, t) = h_1(t), \ \partial_x^2 u(0, t) = \partial_x^4 u(0, t) = 0, \ t > 0. \end{cases} \tag{2.9}$$

It is enough to verify the estimate (2.5) for u_1

$$\sup_{0<t<T} \|u_1(\cdot,t)\|_{H^s(\mathbb{R}^+)} + \sum_{k=0}^{4} \|\partial_x^k u_1\|_{L^\infty([0,\infty);\, H^{\frac{s+2-k}{5}}(0,T))} \le C\|h_1\|_{H^{\frac{s+1}{5}}(\mathbb{R}^+)}. \tag{2.10}$$

The other cases are similar, so we omit the details here.

When $s = 0$. Observing that

$$u_1(x,t) = \sum_{j=1}^{3} \frac{1}{2\pi} \int_0^\infty e^{i\rho^5 t} \frac{\Delta_{j1}^+(\rho)}{\Delta^+(\rho)} e^{\lambda_j^+(\rho)x} \hat{h}_1^+(\rho) 5\rho^4 \mathrm{d}\rho + \sum_{j=1}^{3} \frac{1}{2\pi} \int_0^\infty e^{-i\rho^5 t} \frac{\Delta_{jm}^-(\rho)}{\Delta^-(\rho)} e^{\lambda_j^-(\rho)x} \hat{h}_1^-(\rho) 5\rho^4 \mathrm{d}\rho \equiv u_{11}(x,t) + u_{12}(x,t).$$

By (2.6), it follows from Lemma 2.2 that for any $t \ge 0$,

$$\begin{aligned} \|u_{11}(\cdot,t)\|^2_{L^2(\mathbb{R}^+)} &\le C \sum_{j=1}^{3} \int_0^\infty \left| \frac{\Delta_{j1}^+(\rho)}{\Delta^+(\rho)} \right|^2 (e^{\mathrm{Re}\lambda_j(\rho)} + 1)^2 |\hat{h}_1(\rho)(5\rho^4)|^2 \mathrm{d}\rho \\ &\le C \int_0^\infty |\hat{h}_1(\rho)|^2 \rho^6 \mathrm{d}\rho = C \int_0^\infty |\hat{h}_1(i\rho^5)|^2 \rho^2 \mathrm{d}\rho^5 \\ &\le C\|h_1\|^2_{H^{\frac{1}{5}}(\mathbb{R}^+)}. \end{aligned}$$

The same argument applied to u_{12} to obtain the same estimate. Thus,

$$\|u_1\|_{C(0,T;\, L^2(\mathbb{R}^+))} \le C\|h_1\|_{H^{\frac{1}{5}}(\mathbb{R}^+)}. \tag{2.11}$$

Noting that for $0 \le k \le 4$, we have

$$\begin{aligned} \partial_x^k u_{11}(x,t) &= \sum_{j=1}^{3} \frac{1}{2\pi i} \int_0^{i\infty} e^{i\rho^3 t} \frac{\Delta_{j1}^+(\rho)}{\Delta^+(\rho)} [\lambda_j(\rho)]^k e^{\lambda_j(\rho)x} \hat{h}_1(\rho) 5\rho^4 \mathrm{d}\rho \\ &= \sum_{j=1}^{3} \frac{1}{2\pi i} \int_0^{i\infty} e^{i\mu t} \frac{\Delta_{j1}^+(\theta(\mu))}{\Delta^+(\theta(\mu))} [\lambda_j(\theta(\mu))]^k e^{\lambda_j(\theta(\mu))x} \hat{h}_1(\theta(\mu)) \mathrm{d}\mu, \end{aligned}$$

where $\theta(\mu)$ is the real solution of $\mu = \rho^5$ for $\rho \ge 0$. By using the Plancherel theorem (with respect to t), it yields that for any $x > 0$,

$$\|\partial_x^k u_{11}\|^2_{H^{\frac{2-k}{5}}(0,T)} \le C \sum_{j=1}^{3} \int_0^\infty (1 + |\mu|^2)^{\frac{2-k}{5}} \left| [\lambda_j(\theta(\mu))]^k e^{\lambda_j(\theta(\mu))x} \frac{\Delta_{j1}^+(\rho)}{\Delta^+(\rho)} \right|^2 |\hat{h}_1(i\mu)|^2 \mathrm{d}\mu.$$

Thus, one finds there is a constant C such that

$$\begin{aligned} &\sup_{x\in(0,\infty)} \|\partial_x^k u_{11}\|^2_{H^{\frac{2-k}{5}}(\mathbb{R}^+)} \\ &\le C \sum_{j=1}^{3} \int_0^\infty (1 + |\mu|^2)^{\frac{2-k}{5}} |[\lambda_j(\theta(\mu))]^k|^2 \sup_{x\in(0,\infty)} |e^{\lambda_j(\theta(\mu))x}|^2 \left| \frac{\Delta_{j1}^+(\rho)}{\Delta^+(\rho)} \right|^2 |\hat{h}_1(i\mu)|^2 \mathrm{d}\mu \\ &\le C\|h_1\|_{H^{\frac{1}{5}}(\mathbb{R}^+)}. \end{aligned} \tag{2.12}$$

The following estimates were used in obtaining the last inequality:

$$\sup_{x\in(0,\infty)} |e^{\lambda_1(\rho)x}|^2 \left| \frac{\Delta_{11}^+(\rho)}{\Delta^+(\rho)} \right|^2 \le \rho^{-1}, \quad \sup_{x\in(0,\infty)} |e^{\lambda_2(\rho)x}|^2 \left| \frac{\Delta_{21}^+(\rho)}{\Delta^+(\rho)} \right|^2 \le \rho^{-1}, \quad \sup_{x\in(0,\infty)} |e^{\lambda_3(\rho)x}|^2 \left| \frac{\Delta_{31}^+(\rho)}{\Delta^+(\rho)} \right|^2 \le \rho^{-1}.$$

(2.12) leads to

$$\sum_{k=0}^{4} \|\partial_x^k u_1\|_{L^\infty\left(\mathbb{R}^+;\, H^{\frac{2-k}{5}}(0,T)\right)} \le C\|h_1\|_{H^{\frac{1}{5}}(\mathbb{R}^+)}. \tag{2.13}$$

Collecting (2.11) and (2.13) to obtain (2.10) for u_1.

When $s = 5$. For the solution u of (2.9), let $v = \partial_t u$. Then v is a solution of

$$\begin{cases} \partial_t v - \partial_x^5 v = 0, \quad x > 0,\ t > 0, \\ v(x,0) = 0, \quad x > 0, \\ \partial_x v(0,t) = h_1'(t),\ \partial_x^2 v(0,t) = \partial_x^4 v(0,t) = 0,\ t > 0. \end{cases} \tag{2.14}$$

Applying (2.13) to (2.14) yields that

$$\|v\|_{C(0,T;\, L^2(\mathbb{R}^+))} + \sum_{k=0}^{4} \|v\|_{L^\infty(\mathbb{R}^+;\, H^{\frac{2-k}{5}}(\mathbb{R}^+))} \le C\|h_1'\|_{H^{\frac{1}{5}}(\mathbb{R}^+)}. \tag{2.15}$$

Define

$$u_1(x,t) = \int_0^t v(x,\tau)\mathrm{d}\tau,$$

then we have

$$\partial_x u_1(0,t) = \int_0^t \partial_x v(0,\tau)\mathrm{d}\tau = \int_0^t h_1'(\tau)\mathrm{d}\tau = h_1(t).$$

Furthermore, it is easy to verify that

$$\partial_t u_1 + \partial_x^5 u_1 = \int_0^t \partial_t v(x,\tau)\mathrm{d}\tau - \int_0^t \partial_x^5 v(x,\tau)\mathrm{d}\tau = \int_0^t (\partial_t v - \partial_x^5 v)\mathrm{d}\tau = 0.$$

Thus, u_1 solves the initial-boundary value problem (2.9). Since

$$\partial_t u_1 = \partial_x^5 u_1,$$

it follows that

$$\|u_1\|_{C(0,T;\, H^5(\mathbb{R}^+))} + \sum_{k=0}^{4} \|u_1\|_{L^\infty((\mathbb{R}^+);\, H^{\frac{7-k}{5}}(\mathbb{R}^+))} \le C\|h_1\|_{H^{\frac{1+5}{5}}(\mathbb{R}^+)},$$

which is the result of Proposition 2.5 for $s = 5$.

When $0 < s < 5$. Interpolation leads to the desired result.

For $s \in [5n, 5(n+1)]$ $(n = 1, 2, \dots)$, it is enough to repeat the procedure of the case $s \in [0,5]$. □

2.3 Extension strategy

For the solution

$$u(x,t) = W_R(t)\phi(x) + \int_0^t W_R(t-s)f(s)\mathrm{d}s$$

of the initial value problem

$$\begin{cases} \partial_t u + \partial_x^5 u = f(x,t), & x \in \mathbb{R}, \ t > 0, \\ u(x,0) = \phi(x), & x \in \mathbb{R}, \end{cases} \tag{2.16}$$

we have the Kato smoothing effects and hidden regularity as follows:

Lemma 2.4. *Let* $T > 0$ *be given*, $\forall s \in [0,5]$. *Then* $\forall \phi \in H^s(\mathbb{R})$, $\forall f \in L^1(0,T;\, H^s(\mathbb{R}))$, *the solution u of Cauchy problem* (2.16) *satisfying*

$$\|u\|_{L^2(0,T;\, H^{2+s}_{\mathrm{loc}}(\mathbb{R}))} + \sup_{0<t<\infty} \|u(\cdot,t)\|_{H^s(\mathbb{R})} \le C(\|\phi\|_{H^s(\mathbb{R})} + \|f\|_{L^1(\mathbb{R}^+;\, H^s(\mathbb{R}))}).$$

Proof. See [11]. □

Lemma 2.5. *Let* $T > 0$ *be given*, $\forall s \in [0,5]$. *Then* $\forall \phi \in H^s(\mathbb{R})$, $\forall f \in L^1(0,T;\, H^s(\mathbb{R}))$, *the solution u of Cauchy problem* (2.16) *satisfying*

$$\sum_{k=0}^{4} \sup_{x\in\mathbb{R}} \|\partial_x^k u\|_{H^{\frac{s+2-k}{5}}(0,T)} \le C(\|\phi\|_{H^s(\mathbb{R})} + \|f\|_{L^1((0,T),H^s(\mathbb{R}))}).$$

Proof. See [11]. □

The solution of the initial-boundary value problem

$$\begin{cases} \partial_t u + \partial_x^5 u = f(x,t), & x > 0, \ t > 0, \\ u(x,0) = \phi(x), & x > 0, \\ \partial_x u(0,t) = \partial_x^2 u(0,t) = \partial_x^4 u(0,t) = 0, & t > 0 \end{cases} \tag{2.17}$$

can be represented by

$$u(x,t) = W_{\mathbb{R}}(t)\phi^* + \int_0^t W_{\mathbb{R}}(t-\tau) f^*(\cdot,\tau)\mathrm{d}\tau - W_{bdr}\vec{p} - W_{bdr}\vec{q},$$

where $\phi^* = E\phi$ and $f^* = Ef$ are bounded extension of ϕ and f from $\mathbb{R}^+$ to $\mathbb{R}$ respectively, and $\vec{q} = (q_1, q_2, q_3)$, $\vec{p} = (p_1, p_2, p_3)$ with

$$q_1(t) = W_{\mathbb{R}}(t)\phi^*\_x = 0, \quad q_2(t) = \partial_x W_{\mathbb{R}}(t)\phi^*\|_{x=0}, \quad q_3(t) = \partial_x^2 W_{\mathbb{R}}(t)\phi^*\|_{x=0}$$

and

$$p_1(t) = \left.\int_0^t W_{\mathbb{R}}(t-\tau) f^*(\cdot,\tau)\mathrm{d}\tau\right|_{x=0},$$

$$p_2(t) = \partial_x \left.\int_0^t W_{\mathbb{R}}(t-\tau) f^*(\cdot,\tau)\mathrm{d}\tau\right|_{x=0},$$

$$p_3(t) = \partial_x^2 \left.\int_0^t W_{\mathbb{R}}(t-\tau) f^*(\cdot,\tau)\mathrm{d}\tau\right|_{x=0}.$$

Then, establish the estimates on $W_{bdr}\overrightarrow{q}$, $W_{bdr}\overrightarrow{p}$:

Lemma 2.6. *Let $T > 0$ be given, $\forall s \in [0, 5]$. Then $\forall \phi \in H^s(\mathbb{R}^+)$, we have*

$$\sup_{0<t<T} \|W_{bdr}\overrightarrow{q}(\cdot,t)\|_{H^s(\mathbb{R}^+)} + \sum_{k=0}^{4} \sup_{x\in\mathbb{R}} \|W_{bdr}\overrightarrow{q}\|_{H^{\frac{s+2-k}{5}}(0,T)} \le C\|\phi\|_{H^s(\mathbb{R}^+)}.$$

Proof. By Proposition 2.3 and Lemmas 2.4–2.5, we have

$$\begin{aligned}
&\sup_{0<t<T} \|W_{bdr}\overrightarrow{q}(\cdot,t)\|_{H^s(\mathbb{R})} + \sum_{k=0}^{4} \sup_{x\in\mathbb{R}^+} \|W_{bdr}\overrightarrow{q}\|_{H^{\frac{s+2-k}{5}}(0,T)}\\
&\le \|W_{\mathbb{R}}\phi^*\|_{H^{\frac{s+2}{5}}(0,T)} + \|\partial_x W_{\mathbb{R}}\phi^*\|_{H^{\frac{s+1}{5}}(0,T)} + \|\partial_x^2 W_{\mathbb{R}}\phi^*\|_{H^{\frac{s}{5}}(0,T)}\\
&\le \|\phi^*\|_{H^s(\mathbb{R})}\\
&\le C\|\phi\|_{H^s(\mathbb{R}^+)}.
\end{aligned}$$

□

Lemma 2.7. *Let $T > 0$ be given, $\forall s \in [0, 5]$. Then $\forall f \in L^1((0, T), H^s(\mathbb{R}^+))$, we have*

$$\sup_{0<t<T} \|W_{bdr}\overrightarrow{p}(\cdot,t)\|_{H^s(\mathbb{R}^+)} + \sum_{k=0}^{4} \sup_{x\in\mathbb{R}} \|W_{bdr}\overrightarrow{p}\|_{H^{\frac{s+2-k}{5}}(0,T)} \le \|f\|_{L^1(0,T;\, H^s(\mathbb{R}^+))}.$$

Proof. By Proposition 2.3 and Lemmas 2.4–2.5, we have

$$\begin{aligned}
&\sup_{0<t<\infty} \|W_{bdr}\overrightarrow{p}(\cdot,t)\|_{H^s(\mathbb{R})} + \sum_{k=0}^{4} \sup_{x\in\mathbb{R}} \|W_{bdr}\overrightarrow{p}\|_{H^{\frac{s+2-k}{5}}(0,T)}\\
&\le \left\| \int_0^t W_{\mathbb{R}}(t-\tau)f^*(\cdot,\tau)\mathrm{d}\tau \right\|_{H^{\frac{s+2}{5}}(0,T)} + \left\| \partial_x \int_0^t W_{\mathbb{R}}(t-\tau)f^*(\cdot,\tau)\mathrm{d}\tau \right\|_{H^{\frac{s+1}{5}}(0,T)} + \left\| \partial_x^2 \int_0^t W_{\mathbb{R}}(t-\tau)f^*(\cdot,\tau)\mathrm{d}\tau \right\|_{H^{\frac{s}{5}}(0,T)}\\
&\le \|f^*\|_{L^1(0,T;\, H^s(\mathbb{R}))}\\
&\le C\|f\|_{L^1(0,T;\, H^s(\mathbb{R}^+))}.
\end{aligned}$$

□

2.4 Proof of Proposition 2.1.

Proof. Collecting all of the results of Proposition 2.3, Lemma 2.6, Lemma 2.7 together, we obtain

$$\|u\|_{C(0,T;\, H^s(\mathbb{R}^+))} + \sum_{k=0}^{4} \sup_{x\in\mathbb{R}^+} \|u\|_{H^{\frac{s+2-k}{5}}(0,T)} \le C(\|\phi\|_{H^s(\mathbb{R}^+)} + \|\overrightarrow{h}\|_{\mathcal{H}^s(\mathbb{R}^+)} + \|f\|_{L^1(0,T;\, H^s(\mathbb{R}^+))}),$$

which is

$$\|u\|_{Y_T^s} \le C(\|(\phi, \overrightarrow{h})\|_{X_T^s} + \|f\|_{L^1(0,T;\, H^s(\mathbb{R}^+))}).$$

□

3 Well-posedness

For the nonlinear boundary feedback $u(0, t)^2$, we need the following nonlinear estimates:

Lemma 3.1. *Let $T > 0$ be given. Then $\forall s \in [0, 5]$, we have*

$$\|uv\|_{H^{\frac{s}{5}}(0,T)} \le CT^{\frac{3}{10}\left(1-\frac{s}{5}\right)}\|u\|_{H^{\frac{s+2}{5}}(0,T)}\|v\|_{H^{\frac{s+2}{5}}(0,T)}, \tag{3.1}$$

$$\|uv\|_{H^{\frac{s-1}{5}}(0,T)} \le CT^{\frac{5}{10}\left(1-\frac{s}{5}\right)}\|u\|_{H^{\frac{s+2}{5}}(0,T)}\|v\|_{H^{\frac{s+2}{5}}(0,T)}, \tag{3.2}$$

and

$$\|uv\|_{H^{\frac{s-2}{5}}(0,T)} \le CT^{\frac{7}{10}\left(1-\frac{s}{5}\right)}\|u\|_{H^{\frac{s+2}{5}}(0,T)}\|v\|_{H^{\frac{s+2}{5}}(0,T)}. \tag{3.3}$$

Proof. We apply the interpolation skill to prove the inequalities (3.1)–(3.3).

Proof of (3.1). When $s = 0$. Since $H^{\frac{2}{5}} \hookrightarrow L^{10}$, then Hölder's inequality leads to

$$\|uv\|_{L^2(0,T)} \le T^{\frac{3}{10}}\|u\|_{L^{10}(0,T)}\|v\|_{L^{10}(0,T)} \le T^{\frac{7}{10}}\|u\|_{H^{\frac{2}{5}}(0,T)}\|v\|_{H^{\frac{2}{5}}(0,T)}.$$

When $s = 5$. Since $H^1(0,T)$ is Banach algebra, we have

$$\|uv\|_{H^1(0,T)} \le \|u\|_{H^1(0,T)}\|v\|_{H^1(0,T)} \le \|u\|_{H^{\frac{7}{5}}(0,T)}\|v\|_{H^{\frac{7}{5}}(0,T)}.$$

Taking $\theta = 1 - \frac{s}{5}$, interpolation leads to

$$[L^2(0,T), H^1(0,T)]_\theta = H^{\frac{s}{5}}, \quad \left[H^{\frac{2}{5}}(0,T), H^{\frac{7}{5}}(0,T)\right]_\theta = H^{\frac{s+2}{5}}$$

and

$$\|uv\|_{H^{\frac{s}{5}}(0,T)} \le CT^{\frac{3}{10}\left(1-\frac{s}{5}\right)}\|u\|_{H^{\frac{s+2}{5}}(0,T)}\|v\|_{H^{\frac{s+2}{5}}(0,T)}.$$

Proof of (3.2), (3.3) is similar, we omit the details. □

Proof of Theorem 1.1.

Proof. Let $r > 0$ and $0 < \tau \le \max\{1, T\}$ be constant to be determined. Take a ball from Y_τ^s:

$$\mathcal{B}_{\tau,r} = \{v \in Y_\tau^s \le r\},$$

which is bounded closed convex subset of Y_τ^s. Define a map Γ on $\mathcal{B}_{\tau,r}$ by

$$u = \Gamma(v)$$

being the unique solution of

$$\begin{cases} \partial_t u + \partial_x^5 u = f, \quad x > 0,\ t > 0, \\ u(x,0) = \phi, \quad x > 0, \\ \partial_x u(0,t) = h_1(t),\ \partial_x^2 u(0,t) = v(0,t)^2 + h_2(t),\ \partial_x^4 u(0,t) = v(0,t)^2 + h_3(t) \quad t > 0, \end{cases}$$

for $v \in \mathcal{B}_{\tau,r}$.

Applying Proposition 2.1, Lemma 3.1, we have

$$\begin{aligned} \|\Gamma(v)\|_{Y_\tau^s} &\le C\left(\|\phi\|_{H^s(R^+)} + \|\vec{h}\|_{\mathcal{H}^s} + \|v(0,t)^2\|_{H^{\frac{s-2}{5}}}(0,\tau) + \|f\|_{L^1([0,T];\ H^s(R^+))}\right) \\ &\le C\left(\|\phi\|_{H^s(R^+)} + \|\vec{h}\|_{\mathcal{H}^s} + \tau^{\frac{7}{10}\left(1-\frac{s}{5}\right)}\|v(0,t)\|^2_{H^{\frac{s+2}{5}}}(0,\tau) + \|f\|_{L^1([0,T];\ H^s(R^+))}\right) \\ &< C(\|\phi\|_{H^s(R^+)} + \|\vec{h}\|_{\mathcal{H}^s} + \|f\|_{L^1([0,T];\ H^s(R^+))}) + C\tau^{\frac{7}{10}\left(1-\frac{s}{5}\right)}\|v\|^2_{L^\infty(\mathbb{R}^+;\ H^{\frac{s+2}{5}}(0,\tau))} \\ &< C(\|\phi\|_{H^s(R^+)} + \|\vec{h}\|_{\mathcal{H}^s} + \|f\|_{L^1([0,\tau];\ H^s(R^+))}) + C\tau^{\frac{7}{10}\left(1-\frac{s}{5}\right)}\|v\|^2_{Y_\tau^s}. \end{aligned}$$

Take $r = 2C(\|\phi\|_{H^s(R^+)} + \|\vec{h}\|_{\mathcal{H}^s} + \|f\|_{L^1([0,\tau];\ H^s(R^+))})$, choose $0 < \tau \le 1$ so small that $C\tau^{\frac{7}{10}\left(1-\frac{s}{5}\right)}r \le \frac{1}{3}$. Then for any $v \in \mathcal{B}_{\tau,r}$, we have

$$\|\Gamma(v)\|_{Y_\tau^s} \le \frac{r}{2} + C_2\tau^{\frac{1}{4}}r^2 < \frac{r}{2} + \frac{r}{3} = \frac{5}{6}r,$$

which implies that $\Gamma : \mathcal{B}_{\tau,r} \to \mathcal{B}_{\tau,r}$.

For any $u, v \in \mathcal{B}_{\tau,r}$, $\Gamma(u) - \Gamma(v)$ satisfies

$$\begin{cases} \partial_t U - \partial_x^5 U = 0, & x > 0,\ t > 0, \\ U(x,0) = 0, & x > 0, \\ \partial_x U(0,t) = 0,\ \partial_x^2 U(0,t) = u(0,t)^2 - v(0,t)^2,\ \partial_x^4 U(0,t) = u(0,t)^2 - v(0,t)^2, & t > 0. \end{cases}$$

Thus, by Lemma 3.2, then Lemma 3.1, we obtain

$$\begin{aligned} \|\Gamma(u) - \Gamma(v)\|_{Y_\tau^s} &\le C\|u(0,t)^2 - v(0,t)^2\|_{H^{\frac{s-2}{5}}}(0,\tau) \\ &\le C\tau^{\frac{7}{10}\left(1-\frac{s}{5}\right)}\|u(0,t) + v(0,t)\|_{H^{\frac{s+2}{5}}}(0,\tau)\|u(0,t) - v(0,t)\|_{H^{\frac{s+2}{5}}}(0,\tau) \\ &\le C\tau^{\frac{7}{10}\left(1-\frac{s}{5}\right)}\left(\|u\|_{L^\infty(\mathbb{R}^+;\ H^{\frac{s+2}{5}}(0,\tau))} + \|v\|_{L^\infty(\mathbb{R}^+;\ H^{\frac{s+2}{5}}(0,\tau))}\right)\|u - v\|_{L^\infty(\mathbb{R}^+;\ H^{\frac{s+2}{5}}(0,\tau))} \\ &\le C\tau^{\frac{7}{10}\left(1-\frac{s}{5}\right)}(\|u\|_{Y_\tau^s} + \|v\|_{Y_\tau^s})\|u - v\|_{Y_\tau^s} \\ &< \frac{2}{3}\|u - v\|_{Y_\tau^s}, \end{aligned}$$

which implies that $\Gamma : \mathcal{B}_{\tau,r} \to \mathcal{B}_{\tau,r}$ is the contraction mapping.

By Banach's contraction mapping principle, there exists a unique fixed point of Γ in $\mathcal{B}_{\tau,r}$ which is the desired solution to be found. □

4 Concluding remarks

For differential operator $Au = -\partial_x^5$ associated with $\partial_t u + \partial_x^5 u = 0$, we have

$$\langle Au, u\rangle = \langle A^* u, u\rangle = -\int_0^\infty u\partial_x^5 u = -u\partial_x^4\Big|_0^\infty + \partial_x u\partial_x^3 u|_0^\infty - \frac{1}{2}(\partial_x^2 u)^2\Big|_0^\infty = 0$$

provided that

a. $\partial_x u(0,t) = \partial_x^2 u(0,t) = \partial_x^4 u(0,t) = 0$;
b. $u(0,t) = \partial_x^2 u(0,t) = \partial_x^3 u(0,t) = 0$;
c. $\partial_x^2 u(0,t) = \partial_x^3 u(0,t) = \partial_x^4 u(0,t) = 0$;
d. $u(0,t) = \partial_x u(0,t) = \partial_x^2 u(0,t) = 0$.

Phillips-Lumper theorem implies that the operator A generates an operator group under any one of the boundary values: a, b, c, d. Based on this observation, we point out that

Remark 4.1. The analysis in this paper is applicable with the initial boundary value problem

$$\begin{cases} \partial_t u + \partial_x^5 u = u\partial_x u, & x > 0,\ t > 0, \\ u(x,0) = \phi(x), & x > 0, \\ \bar{a} \text{ or } \bar{b} \text{ or } \bar{c} \text{ or } \bar{d}, & t > 0, \end{cases} \tag{4.4}$$

where

ā. $\partial_x u(0,t) = h_1(t)$, $\partial_x^2 u(0,t) = u(0,t)^2 + h_2(t)$, $\partial_x^4 u(0,t) = u(0,t)^2 + h_3(t)$;
b̄. $u(0,t) = h_1(t)$, $\partial_x^2 u(0,t) = u(0,t)^2 + h_2(t)$, $\partial_x^3 u(0,t) = u(0,t)^2 + h_3(t)$;
c̄. $\partial_x^2 u(0,t) = u(0,t)^2 + h_1(t)$, $\partial_x^3 u(0,t) = u(0,t)^2 + h_2(t)$, $\partial_x^4 u(0,t) = u(0,t)^2 + h_3(t)$;
d̄. $u(0,t) = h_1(t)$, $\partial_x u(0,t) = h_2(t)$, $\partial_x^2 u(0,t) = u(0,t)^2 + h_3(t)$.

For differential operator $Bu = \partial_x^5$ associated with $\partial_t u - \partial_x^5 u = 0$, we have

$$\langle Bu, u\rangle = \langle B^* u, u\rangle = -\int_0^\infty u\partial_x^5 u = u\partial_x^4\Big|_0^\infty - \partial_x u\partial_x^3 u|_0^\infty + \frac{1}{2}(\partial_x^2 u)^2\Big|_0^\infty = -\frac{1}{2}(\partial_x^2 u(0,t))^2 \le 0,$$

provided that

e. $\partial_x u(0,t) = \partial_x^4 u(0,t) = 0$;
f. $u(0,t) = \partial_x^3 u(0,t) = 0$;
g. $\partial_x^3 u(0,t) = \partial_x^4 u(0,t) = 0$;
h. $u(0,t) = \partial_x u(0,t) = 0$.

Phillips-Lumper theorem implies that the operator B generates an operator group under any one of the boundary values: e, f, g, h. Based on this observation, we point out that

Remark 4.2. The analysis in this paper is applicable to initial boundary value problem

$$\begin{cases} \partial_t u - \partial_x^5 u = u\partial_x u, & x > 0,\ t > 0, \\ u(x,0) = \phi(x), & x > 0, \\ \bar{e} \text{ or } \bar{f} \text{ or } \bar{g}, & t > 0, \end{cases} \tag{4.5}$$

where

$\bar{\mathbf{e}}$. $\partial_x u(0,t) = h_1(t)$, $\partial_x^4 u(0,t) = u(0,t)^2 + h_3(t)$;
$\bar{\mathbf{f}}$. $u(0,t) = h_1(t)$, $\partial_x^3 u(0,t) = u(0,t)^2 + h_3(t)$;
$\bar{\mathbf{g}}$. $\partial_x^3 u(0,t) = u(0,t)^2 + h_2(t)$, $\partial_x^4 u(0,t) = u(0,t)^2 + h_3(t)$.

Acknowledgements: The authors would like to thank the anonymous referees for their careful reading and useful comments.

Funding information: This work was financially supported by the Natural Science Foundation of Zhejiang Province (No. LY18A010024) and National Natural Science Foundation of China (No. 12075208).

Author contributions: Xaingqing Zhao conceived and designed the study. Xiangqing Zhao and Chengqiang Wang wrote the manuscript. Jifeng Bao helped to prepare the revision. All authors have accepted responsibility for the entire content of this manuscript and approved its submission.

Conflict of interest: The authors state no conflict of interest.

Data availability statement: Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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Appendix

Proof. In this appendix, we give the proof of (2.8)–(2.10).

$$\lambda_j \sim \rho, \quad \lambda_j^2 \sim \rho^2, \quad \lambda_j^4 \sim \rho^4,$$

where $j = 1, 2, 3$. We have

$$\Delta^+ = \begin{vmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \\ \lambda_1^4 & \lambda_2^4 & \lambda_3^4 \end{vmatrix}.$$

Firstly, we consider h_1, we have

$$\Delta_{11}^+ = \begin{vmatrix} 1 & \lambda_2 & \lambda_3 \\ 0 & \lambda_2^2 & \lambda_3^2 \\ 0 & \lambda_2^4 & \lambda_3^4 \end{vmatrix} = \lambda_2^2\lambda_3^4 - \lambda_3^2\lambda_2^4 \sim \rho^6, \quad (\rho \to \infty).$$

Similarly, we have

$$\Delta_{21}^+ = \begin{vmatrix} \lambda_1 & 1 & \lambda_3 \\ \lambda_1^2 & 0 & \lambda_3^2 \\ \lambda_1^4 & 0 & \lambda_3^4 \end{vmatrix} \sim \rho^6, \quad (\rho \to \infty)$$

and

$$\Delta_{31}^+ = \begin{vmatrix} \lambda_1 & \lambda_2 & 1 \\ \lambda_1^2 & \lambda_2^2 & 0 \\ \lambda_1^4 & \lambda_2^4 & 0 \end{vmatrix} \sim \rho^6, \quad (\rho \to \infty).$$

We deduce that when $\rho \to \infty$,

$$\Delta^+(\rho) \sim \lambda_{11}\Delta_{11}^+ + \lambda_{12}\Delta_{21}^+ + \lambda_{13}\Delta_{31}^+ \sim \rho^7.$$

Thus, we have

$$\lim_{\rho\to\infty} \left| \frac{\rho\Delta_{11}^+(\rho)}{\Delta^+(\rho)} \right| < \infty.$$

We obtain that when $\rho \to \infty$,

$$\frac{\Delta_{11}^+(\rho)}{\Delta^+(\rho)} \sim \rho^{-1}, \quad \frac{\Delta_{21}^+(\rho)}{\Delta^+(\rho)} \sim \rho^{-1}, \quad \frac{\Delta_{31}^+(\rho)}{\Delta^+(\rho)} \sim \rho^{-1}.$$

Secondly, we consider h_2, we have

$$\Delta_{12}^+ = \begin{vmatrix} 0 & \lambda_2 & \lambda_3 \\ 1 & \lambda_2^2 & \lambda_3^2 \\ 0 & \lambda_2^4 & \lambda_3^4 \end{vmatrix} = \lambda_3\lambda_2^4 - \lambda_2\lambda_3^4 \sim \rho^5, \quad (\rho \to \infty).$$

Similarly, we obtain that for $\rho \to \infty$,

$$\Delta_{22}^+(\rho) \sim \rho^5, \quad \Delta_{32}^+(\rho) \sim \rho^5.$$

We obtain for $\rho \to \infty$,

$$\frac{\Delta_{12}^+(\rho)}{\Delta^+(\rho)} \sim \rho^{-2}, \quad \frac{\Delta_{22}^+(\rho)}{\Delta^+(\rho)} \sim \rho^{-2}, \quad \frac{\Delta_{32}^+(\rho)}{\Delta^+(\rho)} \sim \rho^{-2}.$$

Thirdly, we consider h_3.

$$\Delta_{13}^{+} = \begin{vmatrix} 0 & \lambda_2 & \lambda_3 \\ 0 & \lambda_2^2 & \lambda_3^2 \\ 1 & \lambda_2^4 & \lambda_3^4 \end{vmatrix} = \lambda_2\lambda_3^2 - \lambda_3\lambda_2^2 \sim \rho^3, \quad (\rho \to \infty).$$

Similarly, we obtain that for $\rho \to \infty$,

$$\Delta_{23}^{+}(\rho) \sim \rho^3, \; \Delta_{33}^{+}(\rho) \sim \rho^3.$$

We obtain for $\rho \to \infty$,

$$\frac{\Delta_{13}^{+}(\rho)}{\Delta^{+}(\rho)} \sim \rho^{-4}, \quad \frac{\Delta_{23}^{+}(\rho)}{\Delta^{+}(\rho)} \sim \rho^{-4}, \quad \frac{\Delta_{33}^{+}(\rho)}{\Delta^{+}(\rho)} \sim \rho^{-4}.$$

□