

Research Article

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The multinomial convolution sum of a generalized divisor function

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Abstract: The main theorem of this article is to evaluate and express the multinomial convolution sum of the divisor function $\sigma_r^k(n; N/4, N)$ in as a simple form as possible, where $N/4$ is an arbitrary odd positive integer. This generalizes previous result in combination with Cho and Kim, which is about the case $N = 4$. While obtaining our main theorem, we derive some generalizations of other identities to the case that we are dealing with.

Keywords: divisor functions, convolution sums

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1 Introduction

For a nonnegative integer r and a positive integer n , let $\sigma_r(n)$ be the usual divisor function defined by the equation $\sigma_r(n) = \sum_{d|n} d^r$. The divisor function is an important arithmetic function playing a fundamental role in number theory. This appears naturally as the coefficients of (quasi) modular forms, a number of integer solutions as quadratic forms and in relation to geometry, etc. Let $r_k(n)$ be the number of representations of a nonnegative number n as the sum of k squares. Jacobi showed in 1834 that $r_4(n) = 8\sigma_1(n) - 32\sigma_1(n/4)$. Using the result of Jacobi, to compute $r_8(n)$ it is necessary to evaluate the following convolution sum of the divisor function:

$$\sum_{m=1}^{n-1} \sigma_1(m) \sigma_1(n-m),$$

which was presented by Besge [1]. After that, convolution sums for such divisor functions have become the subject of interest to many mathematicians. Liouville first evaluated the binomial convolution sum of $\sigma_r(n)$ as follows.

Theorem 1.1. [2] For every $k, n \in \mathbb{N}$ we have

$$\begin{aligned} \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1}^{n-1} \sigma_{2k-2s-1}(m) \sigma_{2s+1}(n-m) &= \frac{2k+3}{4k+2} \sigma_{2k+1}(n) + \left(\frac{k}{6} - n\right) \sigma_{2k-1}(n) \\ &+ \frac{1}{2k+1} \sum_{j=2}^k \binom{2k+1}{2j} B_{2j} \sigma_{2k+1-2j}(n), \end{aligned}$$

where B_j is the j th Bernoulli number.

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A slightly different version of the binomial convolution sum of $\sigma_r(n)$ was evaluated by Kim and Bayad. Note that this slight change gives rise to a simpler expression on the right hand side.

Theorem 1.2. [3] For every $k, n \in \mathbb{N}$ we have

$$\sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1}^n \sigma_{2k-2s-1}(2m-1) \sigma_{2s+1}(2n-2m+1) = \frac{1}{4} \sigma_{2k+1}^*(2n),$$

where $\sigma_r^*(n) := \sum_{2 \nmid n/d} d^r = \sigma_r(n) - \sigma_r(n/2)$.

In the same article, they also provided several results about the binomial convolution sums of $\sigma_r^*(n)$. Here, we introduce just two of them for the sake of simplicity.

Theorem 1.3. [3] For each $k, n \in \mathbb{N}$ we obtain

$$\begin{aligned} (i) \quad & \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1}^{n-1} \sigma_{2k-2s-1}^*(m) \sigma_{2s+1}^*(n-m) = \frac{1}{2} \sigma_{2k+1}^*(n) - \frac{n}{2} \sigma_{2k-1}^*(n), \\ (ii) \quad & \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1}^{2n-1} (-1)^{m+1} \sigma_{2k-2s-1}^*(m) \sigma_{2s+1}^*(2n-m) = n \sigma_{2k-1}^*(2n). \end{aligned}$$

After that, Kim and Park succeeded in evaluating the trinomial and quadrinomial convolution sums of the same divisor function as follows.

Theorem 1.4. [4] Let $n \geq 4$ be an even integer and let $k \in \mathbb{N}$. Then we have

$$\begin{aligned} (i) \quad & \sum_{\substack{a+b+c=2k+1 \\ a,b,c \text{ odd}}} a \binom{2k+1}{a, b, c} \sum_{\substack{m_1+m_2+m_3=n \\ m_3 \text{ even}}} (-1)^{m_1+1} \sigma_a^*(m_1) \sigma_b^*(m_2) \sigma_c^*(m_3) = \frac{(2k-1)n}{32} (\sigma_{2k+1}^*(n) - 2n \sigma_{2k-1}^*(n)), \\ (ii) \quad & \sum_{\substack{a+b+c+d=2k \\ a,b,c,d \text{ odd}}} \frac{a+b}{c+d+1} \binom{2k}{a, b, c, d} \sum_{\substack{m_1+m_2+m_3+m_4=n \\ m_3, m_4 \text{ odd}}} (-1)^{m_1+1} \sigma_a^*(m_1) \sigma_b^*(m_2) \sigma_c^*(m_3) \sigma_d^*(m_4) \\ & = \frac{1}{64} \left(n \sigma_{2k+1}^*(n) - 2n^2 \sigma_{2k-1}^*(n) - 64 \sum_{m < n/2} (n/2 - m) \sigma_1^*(m) \sigma_{2k-1}^*(n-2m) \right). \end{aligned}$$

Kim and Bayad [5] dealt with a similar divisor function

$$S(n) := \sum_{\substack{d|n \\ 2 \nmid d}} d$$

and introduced the orders, m -gonal shape number, type, and convexity derived from $S(n)$.

Let us consider another divisor function as follows [6]:

$$\begin{aligned} \sigma_r^\sharp(n) &= \sum_{\substack{d|n \\ \frac{n}{d} \equiv 1 \pmod{4}}} d^r - (-1)^r \sum_{\substack{d|n \\ \frac{n}{d} \equiv -1 \pmod{4}}} d^r, \\ \sigma_r^\flat(n) &= \sum_{\substack{d|n \\ d \equiv 1 \pmod{4}}} d^r - (-1)^r \sum_{\substack{d|n \\ d \equiv -1 \pmod{4}}} d^r. \end{aligned}$$

Note that if r is even, then

$$\sigma_r^\sharp(n) = \sum_{d|n} \left(\frac{-4}{n/d} \right) d^r \quad \text{and} \quad \sigma_r^\flat(n) = \sum_{d|n} \left(\frac{-4}{d} \right) d^r,$$

where

$$\left(\frac{-4}{d}\right) = \begin{cases} 1, & \text{if } d \equiv 1 \pmod{4} \\ -1, & \text{if } d \equiv -1 \pmod{4} \\ 0, & \text{otherwise,} \end{cases}$$

is a Kronecker symbol.

These divisor functions are related to the coefficients of (quasi) modular forms. In fact, it is well known that they appear as coefficients of Fourier series of modular forms for $\Gamma_0(16)$ (see Section 4.5 in [7]). Recently, Aygin and Hong obtained the results for convolution sums of $\sigma_r^\sharp(n)$ and $\sigma_r^\flat(n)$ twisted by Dirichlet characters [8]. Also, $\sigma_r^\sharp(n) = \sigma_r^*(n)$ if r is odd, so all the above results about $\sigma_r^*(n)$ can be interpreted as the results about $\sigma_r^\sharp(n)$ with “odd” indices. Kim, Bayad, and Park also gave the following result about the binomial convolution sum of divisor function $\sigma_r^\sharp(n)$ as follows.

Theorem 1.5. [6] *For each $k, n \in \mathbb{N}$ we obtain*

$$\sum_{s=0}^k \binom{2k}{2s} \sum_{m=1}^{n-1} \sigma_{2k-2s}^\sharp(m) \sigma_{2s}^\sharp(n-m) = \frac{1}{2} \sigma_{2k+1}^\sharp(n) - \frac{1}{2} \sigma_{2k}^\sharp(n).$$

Inspired by this work, the author together with Cho and Kim could evaluate the following “multinomial” convolution sum of $\sigma_r^\sharp(n)$ with “even” indices.

Theorem 1.6. [9] *For every $t, k, n \in \mathbb{N}$ we have*

$$\sum_{\substack{(s_1, s_2, \dots, s_t) \in \mathbb{N}_0^t \\ s_1 + s_2 + \dots + s_t = k}} \binom{2k}{2s_1, 2s_2, \dots, 2s_t} \sum_M (-1)^\alpha \sigma_{2s_1}^\sharp(m_1) \sigma_{2s_2}^\sharp(m_2) \cdots \sigma_{2s_t}^\sharp(m_t) = (-1)^{t-1} 2^{(2k-1)(t-1)} \sigma_{2k}^\sharp(n),$$

where

$$M = \{(m_1, \dots, m_t) \in \mathbb{N}^t \mid m_1 + \dots + m_t = 2^{t-1}n, m_1 + \dots + m_i \equiv 0 \pmod{2^{i-1}} \text{ for all } i\}$$

and

$$\alpha = m_2 + \frac{m_3}{2} + \dots + \frac{m_t}{2^{t-2}}.$$

Ramanujan [10] computed the following convolution sum:

$$\sum_{m=0}^n \sigma_r(m) \sigma_s(n-m),$$

where r and s are odd and $r + s = 2, 4, 6, 8, 12$. Note that the summation begins at $m = 0$ and ends at $m = n$. For $m = 0$ or n , $\sigma_r(0)$ is defined to be $\frac{1}{2}\zeta(-r)$, where $\zeta(s)$ is the Riemann zeta function. Recently, Bayad and Hajli [11] studied about the multidimensional zeta function and proved that the multidimensional Appell polynomials are special values at the nonpositive integers of these zeta functions.

In the present article, we generalize some of the above results including Theorem 1.6. For this purpose, we define the generalized divisor function $\sigma_r^\sharp(n)$ as for positive integers $N \geq 3$, a nonnegative integer r , and $1 \leq i \leq N-1$

$$\sigma_r^\sharp(n; i, N) = \sum_{\substack{d|n \\ \frac{n}{d} \equiv i \pmod{N}}} d^r - (-1)^r \sum_{\substack{d|n \\ \frac{n}{d} \equiv -i \pmod{N}}} d^r.$$

If there does not exist a positive divisor d of n such that $\frac{n}{d} \equiv i \pmod{N}$, then $\sum_{\substack{d|n \\ \frac{n}{d} \equiv i \pmod{N}}} d^r$ is assumed to be 0. It is obvious that $\sigma_r^\sharp(n; 1, 4) = \sigma_r^\sharp(n)$ and $\sigma_r^\sharp(n; -i, N) = (-1)^{r+1} \sigma_r^\sharp(n; i, N)$. Now we state our main theorem.

Theorem 1.7. For every $t, k, N \in \mathbb{N}$ such that $t \geq 2$, $4|N$, and $N/4$ is odd, we have

$$\sum_{\substack{(s_1, \dots, s_t) \in \mathbb{N}_0^t \\ s_1 + \dots + s_t = k}} \binom{2k}{2s_1, \dots, 2s_t} \sum_M (-1)^\alpha \sigma_{2s_1}^\#(m_1; N/4, N) \cdots \sigma_{2s_t}^\#(m_t; N/4, N) = (-1)^{t-1} 2^{(2k-1)(t-1)} \sigma_{2k}^\#(n; N/4, N),$$

where

$$M = \{(m_1, \dots, m_t) \in \mathbb{N}^t \mid m_1 + \dots + m_t = 2^{t-1}n, m_1 + \dots + m_i \equiv 0 \pmod{2^{i-1}} \text{ for all } i \leq t\}$$

and

$$\alpha = m_2 + \frac{m_3}{2} + \dots + \frac{m_t}{2^{t-2}}.$$

Theorem 1.6 was proved by the use of Theorem 1.5. So, in order to prove Theorem 1.7, we need to generalize Theorem 1.5 to the case of $\sigma_r^\#(n; N/4, N)$ and it will be done in Corollary 2.4(i). To obtain this corollary, we derive a new result (Theorem 2.2) about the binomial convolution sum of $\sigma_r^\#(n; i, N)$ and then generalize Theorems 1.3(i) and 1.5 as its corollary (see Corollary 2.4). We comment that the technique used in the proof of Corollary 2.4(i) is different from the one used in the proof of Theorem 1.5 given in [6]. Using Corollary 2.4(i) we can obtain Proposition 2.6, which can be thought of as an even-indexed version of Theorem 1.3(ii). It can also be viewed as a special case of our main theorem for $t = 2$. In Section 3, we evaluate three trinomial convolution sums of $\sigma_r^\#(n; N/4, N)$ and finally present the proof of Theorem 1.7.

2 Binomial convolution sums of $\sigma_r^\#$

In combination with Cho and Kim, the author obtained a result about the binomial convolution sum of $\sigma_r^\#(n; i, N)$.

Proposition 2.1. [9] Suppose $N \geq 3$ and $1 \leq i \leq N-1$. For all $k, n \in \mathbb{N}$ we have

$$\sum_{s=0}^{2k} \binom{2k}{s} \sum_{m=1}^{n-1} \sigma_{2k-s}^\#(m; i, N) \sigma_s^\#(n-m; i, N) = \sigma_{2k+1}^\#(n; i, N) - \left(1 - \frac{2i}{N}\right) \sigma_{2k}^\#(n; i, N) - \frac{2n}{N} \sigma_{2k-1}^\#(n; i, N).$$

To prove our main theorem we need the following result.

Theorem 2.2. Let $N \geq 4$ be an even integer. For any $k, n, i \in \mathbb{N}$ with $0 < i < N/2$, we have

$$\begin{aligned} & \sum_{s=0}^{2k} \binom{2k}{s} \sum_{m=1}^{n-1} \sigma_{2k-s}^\#(m; i, N) \sigma_s^\#(n-m; N/2 + i, N) \\ &= -\frac{n}{N} \sigma_{2k-1}^\#(n; i, N) - \frac{n}{N} \sigma_{2k-1}^\#(n; N/2 + i, N) \\ &+ \frac{i}{N} \sigma_{2k}^\#(n; i, N) + \left(\frac{i}{N} - \frac{1}{2}\right) \sigma_{2k}^\#(n; N/2 + i, N). \end{aligned}$$

We recall the identity of Huard, Ou, Spearman, and Williams to prove Theorem 2.2.

Theorem 2.3. ([12], Theorem 1) Let $f: \mathbb{Z}^4 \rightarrow \mathbb{C}$ be a function such that

$$f(a, b, x, y) - f(x, y, a, b) = f(-a, -b, x, y) - f(x, y, -a, -b)$$

for all $a, b, x, y \in \mathbb{Z}$. Let $n \in \mathbb{N}$. Then

$$\begin{aligned} & \sum_{\substack{(a,b,x,y) \in \mathbb{N}^4 \\ ax+by=n}} (f(a, b, x, -y) - f(a, -b, x, y) + f(a, a-b, x+y, y) - f(a, a+b, y-x, y) \\ & \quad + f(b-a, b, x, x+y) - f(a+b, b, x, x-y)) \\ &= \sum_{d|n} \sum_{\substack{x \in \mathbb{N} \\ x < d}} (f(0, n/d, x, d) + f(n/d, 0, d, x) + f(n/d, n/d, d-x, -x) - f(x, x-d, n/d, n/d) \\ & \quad - f(x, d, 0, n/d) - f(d, x, n/d, 0)). \end{aligned}$$

Proof of Theorem 2.2. Let

$$F_{i,N}(a) = \begin{cases} 1 & \text{if } a \equiv i \pmod{N} \\ 0 & \text{otherwise,} \end{cases}$$

$$F_N(n) = F_{0,N}(n).$$

We take $f(a, b, x, y) = (F_{i,N}(a) + F_{-i,N}(a))F_{\frac{N}{2},N}(a-b)(x-y)^{2k}$ ($k \in \mathbb{N}$) in Theorem 2.3. Since $F_{i,N}(a) = F_{-i,N}(-a)$ and $F_{\frac{N}{2},N}(a) = F_{\frac{N}{2},N}(-a)$ for $a \in \mathbb{Z}$ and even $N \geq 2$, we obtain

$$\begin{aligned} f(a, b, x, y) - f(x, y, a, b) &= (F_{i,N}(a) + F_{-i,N}(a))F_{\frac{N}{2},N}(a-b)(x-y)^{2k} \\ & \quad - (F_{i,N}(x) + F_{-i,N}(x))F_{\frac{N}{2},N}(x-y)(a-b)^{2k} \\ &= (F_{i,N}(-a) + F_{-i,N}(-a))F_{\frac{N}{2},N}(-a+b)(x-y)^{2k} \\ & \quad - (F_{i,N}(x) + F_{-i,N}(x))F_{\frac{N}{2},N}(x-y)(-a+b)^{2k} \\ &= f(-a, -b, x, y) - f(x, y, -a, -b). \end{aligned}$$

Then the left hand side is equal to

$$\begin{aligned} & \sum_{\substack{(a,b,x,y) \in \mathbb{N}^4 \\ ax+by=n}} ((F_{i,N}(a) + F_{-i,N}(a))F_{\frac{N}{2},N}(a-b)(x+y)^{2k} - (F_{i,N}(a) + F_{-i,N}(a))F_{\frac{N}{2},N}(a+b)(x-y)^{2k}) \\ &= \sum_{\substack{(a,b,x,y) \in \mathbb{N}^4 \\ ax+by=n}} \left((F_{i,N}(a) + F_{-i,N}(a))F_{\frac{N}{2},N}(a-b) \sum_{s=0}^{2k} \binom{2k}{s} x^{2k-s} y^s \right. \\ & \quad \left. - (F_{i,N}(a) + F_{-i,N}(a))F_{\frac{N}{2},N}(a+b) \sum_{s=0}^{2k} \binom{2k}{s} (-1)^s x^{2k-s} y^s \right) \\ &= \sum_{s=0}^{2k} \binom{2k}{s} \sum_{m=1}^{n-1} \left(\sum_{\substack{x|m \\ \frac{m}{x} \equiv i(N)}} x^{2k-s} \sum_{\substack{y|n-m \\ \frac{n-m}{y} \equiv \frac{N}{2}+i(N)}} y^s + \sum_{\substack{x|m \\ \frac{m}{x} \equiv -i(N)}} x^{2k-s} \sum_{\substack{y|n-m \\ \frac{n-m}{y} \equiv \frac{N}{2}-i(N)}} y^s \right. \\ & \quad \left. - (-1)^s \left(\sum_{\substack{x|m \\ \frac{m}{x} \equiv i(N)}} x^{2k-s} \sum_{\substack{y|n-m \\ \frac{n-m}{y} \equiv \frac{N}{2}-i(N)}} y^s + \sum_{\substack{x|m \\ \frac{m}{x} \equiv -i(N)}} x^{2k-s} \sum_{\substack{y|n-m \\ \frac{n-m}{y} \equiv \frac{N}{2}+i(N)}} y^s \right) \right) \\ &= \sum_{s=0}^{2k} \binom{2k}{s} \sum_{m=1}^{n-1} \left(\sum_{\substack{x|m \\ \frac{m}{x} \equiv i(N)}} x^{2k-s} - (-1)^s \sum_{\substack{x|m \\ \frac{m}{x} \equiv -i(N)}} x^{2k-s} \right) \left(\sum_{\substack{y|n-m \\ \frac{n-m}{y} \equiv \frac{N}{2}+i(N)}} y^s - (-1)^s \sum_{\substack{y|n-m \\ \frac{n-m}{y} \equiv \frac{N}{2}-i(N)}} y^s \right) \\ &= \sum_{s=0}^{2k} \binom{2k}{s} \sum_{m=1}^{n-1} \sigma_{2k-s}^{\#}(m; i, N) \sigma_s^{\#}(n-m; N/2+i, N). \end{aligned}$$

The right hand side is $-(U_1 + U_2)$, where

$$U_1 = \sum_{d|n} \sum_{x=1}^{d-1} (F_{i,N}(x) + F_{-i,N}(x)) F_{\frac{N}{2},N}(x-d)(n/d)^{2k},$$

$$U_2 = \sum_{d|n} \sum_{x=1}^{d-1} (F_{i,N}(d) + F_{-i,N}(d)) F_{\frac{N}{2},N}(d-x)(n/d)^{2k}.$$

Replacing x by $d-x$, U_1 is equal to

$$\begin{aligned} & \sum_{d|n} \sum_{x=1}^{d-1} (F_{i,N}(d-x) + F_{-i,N}(d-x)) F_{\frac{N}{2},N}(x)(n/d)^{2k} \\ &= \sum_{d|n} (n/d)^{2k} (F_{\frac{N}{2}+i,N}(d) + F_{\frac{N}{2}-i,N}(d)) \sum_{x=1}^{d-1} F_{\frac{N}{2},N}(x) \\ &= \sum_{d|n} (n/d)^{2k} (F_{\frac{N}{2}+i,N}(d) + F_{\frac{N}{2}-i,N}(d)) \sum_{x=1}^{d-1} (F_{\frac{N}{2}}(x) - F_N(x)) \\ &= \sum_{d|n} (n/d)^{2k} (F_{\frac{N}{2}+i,N}(d) + F_{\frac{N}{2}-i,N}(d)) \left(\left[\frac{2(d-1)}{N} \right] - \left[\frac{d-1}{N} \right] \right) = U_{1,1} - U_{1,2}, \end{aligned}$$

where

$$U_{1,1} = \sum_{\substack{d|n \\ d \equiv \frac{N}{2} + i(N)}} (n/d)^{2k} \left[\frac{2(d-1)}{N} \right] + \sum_{\substack{d|n \\ d \equiv \frac{N}{2} - i(N)}} (n/d)^{2k} \left[\frac{2(d-1)}{N} \right],$$

$$U_{1,2} = \sum_{\substack{d|n \\ d \equiv \frac{N}{2} + i(N)}} (n/d)^{2k} \left[\frac{d-1}{N} \right] + \sum_{\substack{d|n \\ d \equiv \frac{N}{2} - i(N)}} (n/d)^{2k} \left[\frac{d-1}{N} \right].$$

Then $U_{1,1}$ is

$$\begin{aligned} & \sum_{\substack{d|n \\ d \equiv \frac{N}{2} + i(N)}} d^{2k} \left[\frac{2(n/d-1)}{N} \right] + \sum_{\substack{d|n \\ d \equiv \frac{N}{2} - i(N)}} d^{2k} \left[\frac{2(n/d-1)}{N} \right] \\ &= \sum_{\substack{d|n \\ d \equiv \frac{N}{2} + i(N)}} d^{2k} \frac{2n/d-2i}{N} + \sum_{\substack{d|n \\ d \equiv \frac{N}{2} - i(N)}} d^{2k} \frac{2n/d-N+2i}{N} \\ &= \frac{2n}{N} \left(\sum_{\substack{d|n \\ d \equiv \frac{N}{2} + i(N)}} d^{2k-1} + \sum_{\substack{d|n \\ d \equiv \frac{N}{2} - i(N)}} d^{2k-1} \right) - \frac{2i}{N} \left(\sum_{\substack{d|n \\ d \equiv \frac{N}{2} + i(N)}} d^{2k} - \sum_{\substack{d|n \\ d \equiv \frac{N}{2} - i(N)}} d^{2k} \right) - \sum_{\substack{d|n \\ d \equiv \frac{N}{2} - i(N)}} d^{2k} \\ &= \frac{2n}{N} \sigma_{2k-1}^{\#}(n; N/2 + i, N) - \frac{2i}{N} \sigma_{2k}^{\#}(n; N/2 + i, N) - \sum_{\substack{d|n \\ d \equiv \frac{N}{2} - i(N)}} d^{2k}. \end{aligned}$$

Also, $U_{1,2}$ equals

$$\begin{aligned} & \sum_{\substack{d|n \\ d \equiv \frac{N}{2} + i(N)}} d^{2k} \left[\frac{n/d-1}{N} \right] + \sum_{\substack{d|n \\ d \equiv \frac{N}{2} - i(N)}} d^{2k} \left[\frac{n/d-1}{N} \right] \\ &= \sum_{\substack{d|n \\ d \equiv \frac{N}{2} + i(N)}} d^{2k} \frac{n/d-N/2-i}{N} + \sum_{\substack{d|n \\ d \equiv \frac{N}{2} - i(N)}} d^{2k} \frac{n/d-N/2+i}{N} \\ &= \sum_{\substack{d|n \\ d \equiv \frac{N}{2} + i(N)}} d^{2k} \frac{n/d-N/2-i}{N} + \sum_{\substack{d|n \\ d \equiv \frac{N}{2} - i(N)}} d^{2k} \left(\frac{n/d+N/2+i}{N} - 1 \right) \end{aligned}$$

$$\begin{aligned}
&= \frac{n}{N} \left(\sum_{\substack{d|n \\ \frac{n}{d} \equiv \frac{N}{2} + i(N)}} d^{2k-1} + \sum_{\substack{d|n \\ \frac{n}{d} \equiv \frac{N}{2} - i(N)}} d^{2k-1} \right) - \left(\frac{1}{2} + \frac{i}{N} \right) \left(\sum_{\substack{d|n \\ \frac{n}{d} \equiv \frac{N}{2} + i(N)}} d^{2k} - \sum_{\substack{d|n \\ \frac{n}{d} \equiv \frac{N}{2} - i(N)}} d^{2k} \right) - \sum_{\substack{d|n \\ \frac{n}{d} \equiv \frac{N}{2} - i(N)}} d^{2k} \\
&= \frac{n}{N} \sigma_{2k-1}^{\#}(n; N/2 + i, N) - \left(\frac{1}{2} + \frac{i}{N} \right) \sigma_{2k}^{\#}(n; N/2 + i, N) - \sum_{\substack{d|n \\ \frac{n}{d} \equiv \frac{N}{2} - i(N)}} d^{2k}.
\end{aligned}$$

Hence, we obtain

$$U_1 = U_{1,1} - U_{1,2} = \frac{n}{N} \sigma_{2k-1}^{\#}(n; N/2 + i, N) - \left(\frac{i}{N} - \frac{1}{2} \right) \sigma_{2k}^{\#}(n; N/2 + i, N).$$

Next, replacing x by $d - x$, we compute U_2 as follows:

$$\begin{aligned}
&\sum_{d|n} \sum_{x=1}^{d-1} (F_{i,N}(d) + F_{-i,N}(d)) F_{\frac{N}{2},N}(x) (n/d)^{2k} \\
&= \sum_{d|n} \sum_{x=1}^{d-1} (F_{i,N}(d) + F_{-i,N}(d)) (n/d)^{2k} \left(\left\lfloor \frac{2(d-1)}{N} \right\rfloor - \left\lfloor \frac{d-1}{N} \right\rfloor \right) \\
&= \sum_{\substack{d|n \\ d \equiv i(N)}} (n/d)^{2k} \left(\left\lfloor \frac{2(d-1)}{N} \right\rfloor - \left\lfloor \frac{d-1}{N} \right\rfloor \right) + \sum_{\substack{d|n \\ d \equiv -i(N)}} (n/d)^{2k} \left(\left\lfloor \frac{2(d-1)}{N} \right\rfloor - \left\lfloor \frac{d-1}{N} \right\rfloor \right).
\end{aligned}$$

We define $U_{2,1}$ and $U_{2,2}$ as

$$\begin{aligned}
U_{2,1} &= \sum_{\substack{d|n \\ d \equiv i(N)}} (n/d)^{2k} \left\lfloor \frac{2(d-1)}{N} \right\rfloor + \sum_{\substack{d|n \\ d \equiv -i(N)}} (n/d)^{2k} \left\lfloor \frac{2(d-1)}{N} \right\rfloor, \\
U_{2,2} &= \sum_{\substack{d|n \\ d \equiv i(N)}} (n/d)^{2k} \left\lfloor \frac{d-1}{N} \right\rfloor + \sum_{\substack{d|n \\ d \equiv -i(N)}} (n/d)^{2k} \left\lfloor \frac{d-1}{N} \right\rfloor
\end{aligned}$$

and we evaluate $U_{2,1}$ as follows:

$$\begin{aligned}
&\sum_{\substack{d|n \\ \frac{n}{d} \equiv i(N)}} d^{2k} \left\lfloor \frac{2(n/d-1)}{N} \right\rfloor + \sum_{\substack{d|n \\ \frac{n}{d} \equiv -i(N)}} d^{2k} \left\lfloor \frac{2(n/d-1)}{N} \right\rfloor \\
&= \sum_{\substack{d|n \\ \frac{n}{d} \equiv i(N)}} d^{2k} \frac{2(n/d-i)}{N} + \sum_{\substack{d|n \\ \frac{n}{d} \equiv -i(N)}} d^{2k} \left(\frac{2(n/d+i)}{N} - 1 \right) \\
&= \frac{2n}{N} \left(\sum_{\substack{d|n \\ \frac{n}{d} \equiv i(N)}} d^{2k-1} + \sum_{\substack{d|n \\ \frac{n}{d} \equiv -i(N)}} d^{2k-1} \right) - \frac{2i}{N} \left(\sum_{\substack{d|n \\ \frac{n}{d} \equiv i(N)}} d^{2k} - \sum_{\substack{d|n \\ \frac{n}{d} \equiv -i(N)}} d^{2k} \right) - \sum_{\substack{d|n \\ \frac{n}{d} \equiv -i(N)}} d^{2k} \\
&= \frac{2n}{N} \sigma_{2k-1}^{\#}(n; i, N) - \frac{2i}{N} \sigma_{2k}^{\#}(n; i, N) - \sum_{\substack{d|n \\ \frac{n}{d} \equiv -i(N)}} d^{2k}.
\end{aligned}$$

Finally, in the same way we can obtain that

$$U_{2,2} = \sum_{\substack{d|n \\ \frac{n}{d} \equiv i(N)}} d^{2k} \left\lfloor \frac{n/d-1}{N} \right\rfloor + \sum_{\substack{d|n \\ \frac{n}{d} \equiv -i(N)}} d^{2k} \left\lfloor \frac{n/d-1}{N} \right\rfloor = \frac{n}{N} \sigma_{2k-1}^{\#}(n; i, N) - \frac{i}{N} \sigma_{2k}^{\#}(n; i, N) - \sum_{\substack{d|n \\ \frac{n}{d} \equiv -i(N)}} d^{2k}.$$

Hence, we see that

$$U_2 = U_{2,1} - U_{2,2} = \frac{n}{N} \sigma_{2k-1}^{\sharp}(n; i, N) - \frac{i}{N} \sigma_{2k}^{\sharp}(n; i, N).$$

Finally, we conclude that $-(U_1 + U_2)$ is equal to

$$-\frac{n}{N} \sigma_{2k-1}^{\sharp}(n; i, N) - \frac{n}{N} \sigma_{2k-1}^{\sharp}(n, N/2 + i, N) + \frac{i}{N} \sigma_{2k}^{\sharp}(n; i, N) + \left(\frac{i}{N} - \frac{1}{2}\right) \sigma_{2k}^{\sharp}(n; N/2 + i, N). \quad \square$$

Corollary 2.4. For any $k, n, N \in \mathbb{N}$ with $4|N$, we have

$$\begin{aligned} (i) \quad & \sum_{s=0}^k \binom{2k}{2s} \sum_{m=1}^{n-1} \sigma_{2k-2s}^{\sharp}(m; N/4, N) \sigma_{2s}^{\sharp}(n-m; N/4, N) = \frac{1}{2} \sigma_{2k+1}^{\sharp}(n; N/4, N) - \frac{1}{2} \sigma_{2k}^{\sharp}(n; N/4, N), \\ (ii) \quad & \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1}^{n-1} \sigma_{2k-2s-1}^{\sharp}(m; N/4, N) \sigma_{2s+1}^{\sharp}(n-m; N/4, N) = \frac{1}{2} \sigma_{2k+1}^{\sharp}(n; N/4, N) - \frac{2n}{N} \sigma_{2k-1}^{\sharp}(n; N/4, N). \end{aligned}$$

Proof. By Proposition 2.1 and Theorem 2.2, we obtain that

$$\sum_{s=0}^{2k} \binom{2k}{s} \sum_{m=1}^{n-1} \sigma_{2k-s}^{\sharp}(m; N/4, N) \sigma_s^{\sharp}(n-m; N/4, N) = \sigma_{2k+1}^{\sharp}(n; N/4, N) - \frac{1}{2} \sigma_{2k}^{\sharp}(n; N/4, N) - \frac{2n}{N} \sigma_{2k-1}^{\sharp}(n; N/4, N),$$

and

$$\sum_{s=0}^{2k} \binom{2k}{s} \sum_{m=1}^{n-1} (-1)^s \sigma_{2k-s}^{\sharp}(m; N/4, N) \sigma_s^{\sharp}(n-m; N/4, N) = \frac{2n}{N} \sigma_{2k-1}^{\sharp}(n; N/4, N) - \frac{1}{2} \sigma_{2k}^{\sharp}(n; N/4, N).$$

Adding these identities we obtain the identity (i) because $\sigma_r^{\sharp}(n; i, N) = -(-1)^r \sigma_r^{\sharp}(n; -i, N)$. Similarly, we easily deduce (ii). $\square$

Lemma 2.5. Let $k, n, N \in \mathbb{N}$ and assume that $N/4$ is an odd integer. Then

$$\sigma_k^{\sharp}(2n; N/4, N) = 2^k \sigma_k^{\sharp}(n; N/4, N).$$

Proof. If $d|2n$ and $\frac{2n}{d} \equiv \frac{N}{4} \pmod{N}$ (resp., $\frac{2n}{d} \equiv -\frac{N}{4} \pmod{N}$), then d must be even. Put $d' = \frac{d}{2}$. Then $d'|n$ and $\frac{n}{d'} \equiv \frac{N}{4} \pmod{N}$ (resp., $\frac{n}{d'} \equiv -\frac{N}{4} \pmod{N}$). Hence,

$$\sigma_k^{\sharp}(2n; N/4, N) = \sum_{\substack{d|2n \\ \frac{2n}{d} \equiv \frac{N}{4} \pmod{N}}} d^k - (-1)^k \sum_{\substack{d|2n \\ \frac{2n}{d} \equiv -\frac{N}{4} \pmod{N}}} d^k = \sum_{\substack{d|n \\ \frac{n}{d} \equiv \frac{N}{4} \pmod{N}}} (2d)^k - (-1)^k \sum_{\substack{d|n \\ \frac{n}{d} \equiv -\frac{N}{4} \pmod{N}}} (2d)^k = 2^k \sigma_k^{\sharp}(n; N/4, N). \quad \square$$

Moreover, we can obtain another result about the binomial convolution sum, which turns out to be a special case of our main theorem.

Proposition 2.6. Let k, n, N be positive integers with $N/4$ odd. Then

$$\begin{aligned} (i) \quad & \sum_{s=0}^k \binom{2k}{2s} \sum_{m=1}^{2n-1} (-1)^m \sigma_{2k-2s}^{\sharp}(m; N/4, N) \sigma_{2s}^{\sharp}(2n-m; N/4, N) = -2^{2k-1} \sigma_{2k}^{\sharp}(n; N/4, N), \\ (ii) \quad & \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1}^{2n-1} (-1)^m \sigma_{2k-2s-1}^{\sharp}(m; N/4, N) \sigma_{2s+1}^{\sharp}(2n-m; N/4, N) = 2^{2k+2} \frac{n}{N} \sigma_{2k-1}^{\sharp}(n; N/4, N). \end{aligned}$$

Proof. (i) We have

$$\begin{aligned}
 & \sum_{s=0}^k \binom{2k}{2s} \sum_{m=1}^{2n-1} (-1)^m \sigma_{2k-2s}^{\#}(m; N/4, N) \sigma_{2s}^{\#}(2n-m; N/4, N) \\
 &= 2 \sum_{s=0}^k \binom{2k}{2s} \sum_{\substack{m=1 \\ m \text{ even}}}^{2n-1} \sigma_{2k-2s}^{\#}(m; N/4, N) \sigma_{2s}^{\#}(2n-m; N/4, N) \\
 &\quad - \sum_{s=0}^k \binom{2k}{2s} \sum_{m=1}^{2n-1} \sigma_{2k-2s}^{\#}(m; N/4, N) \sigma_{2s}^{\#}(2n-m; N/4, N) \\
 &= 2 \sum_{s=0}^k \binom{2k}{2s} \sum_{m=1}^{n-1} \sigma_{2k-2s}^{\#}(2m; N/4, N) \sigma_{2s}^{\#}(2n-2m; N/4, N) \\
 &\quad - \sum_{s=0}^k \binom{2k}{2s} \sum_{m=1}^{2n-1} \sigma_{2k-2s}^{\#}(m; N/4, N) \sigma_{2s}^{\#}(2n-m; N/4, N).
 \end{aligned}$$

From (i) of Corollary 2.4 and Lemma 2.5, we easily deduce that this is equal to

$$\begin{aligned}
 & 2^{2k+1} \sum_{s=0}^k \binom{2k}{2s} \sum_{m=1}^{n-1} \sigma_{2k-2s}^{\#}(m; N/4, N) \sigma_{2s}^{\#}(n-m; N/4, N) - \sum_{s=0}^k \binom{2k}{2s} \sum_{m=1}^{2n-1} \sigma_{2k-2s}^{\#}(m; N/4, N) \sigma_{2s}^{\#}(2n-m; N/4, N) \\
 &= 2^{2k} (\sigma_{2k+1}^{\#}(n; N/4, N) - \sigma_{2k}^{\#}(n; N/4, N)) - \frac{1}{2} (\sigma_{2k+1}^{\#}(2n; N/4, N) - \sigma_{2k}^{\#}(2n; N/4, N)) \\
 &= -2^{2k-1} \sigma_{2k}^{\#}(n; N/4, N).
 \end{aligned}$$

(ii) From (ii) of Corollary 2.4 and Lemma 2.5, we obtain

$$\begin{aligned}
 & \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1}^{2n-1} (-1)^m \sigma_{2k-2s-1}^{\#}(m; N/4, N) \sigma_{2s+1}^{\#}(2n-m; N/4, N) \\
 &= 2 \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{\substack{m=1 \\ m \text{ even}}}^{2n-1} \sigma_{2k-2s-1}^{\#}(m; N/4, N) \sigma_{2s+1}^{\#}(2n-m; N/4, N) \\
 &\quad - \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1}^{2n-1} \sigma_{2k-2s-1}^{\#}(m; N/4, N) \sigma_{2s+1}^{\#}(2n-m; N/4, N) \\
 &= 2 \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1}^{n-1} \sigma_{2k-2s-1}^{\#}(2m; N/4, N) \sigma_{2s+1}^{\#}(2n-2m; N/4, N) \\
 &\quad - \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1}^{2n-1} \sigma_{2k-2s-1}^{\#}(m; N/4, N) \sigma_{2s+1}^{\#}(2n-m; N/4, N) \\
 &= 2^{2k+1} \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1}^{n-1} \sigma_{2k-2s-1}^{\#}(m; N/4, N) \sigma_{2s+1}^{\#}(n-m; N/4, N) \\
 &\quad - \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1}^{2n-1} \sigma_{2k-2s-1}^{\#}(m; N/4, N) \sigma_{2s+1}^{\#}(2n-m; N/4, N) \\
 &= 2^{2k+1} \left(\frac{1}{2} \sigma_{2k+1}^{\#}(n; N/4, N) - \frac{2n}{N} \sigma_{2k-1}^{\#}(n; N/4, N) \right) - \left(\frac{1}{2} \sigma_{2k+1}^{\#}(2n; N/4, N) - \frac{4n}{N} \sigma_{2k-1}^{\#}(2n; N/4, N) \right) \\
 &= 2^{2k+2} \frac{n}{N} \sigma_{2k-1}^{\#}(n; N/4, N). \quad \square
 \end{aligned}$$

3 Applications of Theorem 2.2 to trinomial and multinomial convolution sums of $\sigma_r^{\#}$

We present three identities Proposition 3.1(i)–(iii) about trinomial convolution sums of $\sigma_r^{\#}(n; N/4, N)$ and Part (ii) will be generalized to the case of multinomial convolution sum.

Proposition 3.1. Let k, n, N be positive integers with $N/4$ odd. Then we have

$$\begin{aligned}
 (i) \quad & \sum_{\substack{(a,b,c) \in \mathbb{N}_0^3 \\ a+b+c=k}} \binom{2k}{2a, 2b, 2c} \sum_{\substack{(m_1, m_2, m_3) \in \mathbb{N}^3 \\ m_1+m_2+m_3=2n \\ m_3 \text{ even}}} (-1)^{m_1} \sigma_{2a}^\#(m_1; N/4, N) \sigma_{2b}^\#(m_2; N/4, N) \sigma_{2c}^\#(m_3; N/4, N) \\
 &= -2^{2k-2} (\sigma_{2k+1}^\#(n; N/4, N) - \sigma_{2k}^\#(n; N/4, N)), \\
 (ii) \quad & \sum_{\substack{(a,b,c) \in \mathbb{N}_0^3 \\ a+b+c=k}} \binom{2k}{2a, 2b, 2c} \sum_{\substack{(m_1, m_2, m_3) \in \mathbb{N}^3 \\ m_1+m_2+m_3=4n \\ m_3 \text{ even}}} (-1)^{m_1+\frac{m_3}{2}} \sigma_{2a}^\#(m_1; N/4, N) \sigma_{2b}^\#(m_2; N/4, N) \sigma_{2c}^\#(m_3; N/4, N) \\
 &= 2^{4k-2} \sigma_{2k}^\#(n; N/4, N), \\
 (iii) \quad & \sum_{\substack{(a,b,c) \in \mathbb{N}_0^3 \\ a+b+c=k}} \binom{2k}{2a, 2b, 2c} \sum_{\substack{(m_1, m_2, m_3) \in \mathbb{N}^3 \\ m_1+m_2+m_3=4n \\ m_3=2(4)}} (-1)^{m_1} \sigma_{2a}^\#(m_1; N/4, N) \sigma_{2b}^\#(m_2; N/4, N) \sigma_{2c}^\#(m_3; N/4, N) \\
 &= -2^{4k-2} \sigma_{2k+1}^\#(n; N/4, N).
 \end{aligned}$$

Proof of (i). Note that

$$\begin{aligned}
 & \sum_{a+b+c=k} \binom{2k}{2a, 2b, 2c} \sum_{\substack{(m_1, m_2, m_3) \in \mathbb{N}^3 \\ m_1+m_2+m_3=2n \\ m_3 \text{ even}}} (-1)^{m_1} \sigma_{2a}^\#(m_1; N/4, N) \sigma_{2b}^\#(m_2; N/4, N) \sigma_{2c}^\#(m_3; N/4, N) \\
 &= \sum_{a+b+c=k} \frac{(2k)!}{(2a)!(2b)!(2c)!} \frac{(2a+2b)!}{(2a)!(2b)!} \sum_{\substack{(m_1, m_2, m_3) \in \mathbb{N}^3 \\ m_1+m_2+m_3=2n \\ m_3 \text{ even}}} (-1)^{m_1} \sigma_{2a}^\#(m_1; N/4, N) \sigma_{2b}^\#(m_2; N/4, N) \sigma_{2c}^\#(m_3; N/4, N).
 \end{aligned}$$

We set $l = a + b$ and $2t = m_1 + m_2$. Then we have

$$\sum_{l=0}^k \binom{2k}{2l} \sum_{t=1}^{n-1} \left(\sum_{a=0}^l \binom{2l}{2a} \sum_{m_1=1}^{2t-1} (-1)^{m_1} \sigma_{2a}^\#(m_1; N/4, N) \sigma_{2l-2a}^\#(2t - m_1; N/4, N) \right) \sigma_{2k-2l}^\#(2n - 2t; N/4, N).$$

By (i) of Corollary 2.4 and Lemma 2.5, we obtain that

$$\begin{aligned}
 -\frac{1}{2} \sum_{l=0}^k \binom{2k}{2l} \sum_{t=1}^{n-1} \sigma_{2l}^\#(2t; N/4, N) \sigma_{2k-2l}^\#(2n - 2t; N/4, N) &= -2^{2k-1} \sum_{l=0}^k \binom{2k}{2l} \sum_{t=1}^{n-1} \sigma_{2l}^\#(t; N/4, N) \sigma_{2k-2l}^\#(n - t; N/4, N) \\
 &= -2^{2k-2} (\sigma_{2k+1}^\#(n; N/4, N) - \sigma_{2k}^\#(n; N/4, N)). \quad \square
 \end{aligned}$$

Proof of (ii). Putting $l = a + b$ and $2t = m_1 + m_2$, we have

$$\begin{aligned}
 & \sum_{a+b+c=k} \binom{2k}{2a, 2b, 2c} \sum_{\substack{(m_1, m_2, m_3) \in \mathbb{N}^3 \\ m_1+m_2+m_3=4n \\ m_3 \text{ even}}} (-1)^{m_1+\frac{m_3}{2}} \sigma_{2a}^\#(m_1; N/4, N) \sigma_{2b}^\#(m_2; N/4, N) \sigma_{2c}^\#(m_3; N/4, N) \\
 &= \sum_{l=0}^k \binom{2k}{2l} \sum_{t=1}^{2n-1} (-1)^t \left(\sum_{a=0}^l \binom{2l}{2a} \sum_{m_1=1}^{2t-1} (-1)^{m_1} \sigma_{2a}^\#(m_1; N/4, N) \sigma_{2l-2a}^\#(2t - m_1; N/4, N) \right) \sigma_{2k-2l}^\#(4n - 2t; N/4, N).
 \end{aligned}$$

From Lemma 2.5 and Proposition 2.6, we immediately infer

$$\begin{aligned}
 & -\frac{1}{2} \sum_{l=0}^k \binom{2k}{2l} \sum_{t=1}^{2n-1} (-1)^t \sigma_{2l}^\#(2t; N/4, N) \sigma_{2k-2l}^\#(4n - 2t; N/4, N) \\
 &= -2^{2k-1} \sum_{l=0}^k \binom{2k}{2l} \sum_{t=1}^{2n-1} (-1)^t \sigma_{2l}^\#(t; N/4, N) \sigma_{2k-2l}^\#(2n - t; N/4, N) \\
 &= 2^{4k-2} \sigma_{2k}^\#(n; N/4, N). \quad \square
 \end{aligned}$$

Proof of (iii). Replace $2n$ by $4n$ in (i) of Proposition 3.1. Then we have

$$\begin{aligned} & \sum_{\substack{(a,b,c) \in \mathbb{N}_0^3 \\ a+b+c=k}} \binom{2k}{2a, 2b, 2c} \sum_{\substack{(m_1, m_2, m_3) \in \mathbb{N}^3 \\ m_1+m_2+m_3=4n \\ m_3 \text{ even}}} (-1)^{m_1} \sigma_{2a}^\#(m_1; N/4, N) \sigma_{2b}^\#(m_2; N/4, N) \sigma_{2c}^\#(m_3; N/4, N) \\ &= -2^{2k-2} (\sigma_{2k+1}^\#(2n; N/4, N) - \sigma_{2k}^\#(2n; N/4, N)). \end{aligned}$$

The (iii) in Proposition 3.1 is correct if subtract (ii) in Proposition 3.1 from the above term. $\square$

We now generalize Proposition 3.1(ii) to the case of multinomial convolution sum as follows.

Proof of Theorem 1.7. We proceed by induction on t . The assertion for the case $t = 2$ is immediate due to Proposition 2.6. Suppose that the assertion is true for some $t - 1 \geq 2$. Namely, we assume that for all $l, m, N \in \mathbb{N}$ with $N/4$ odd, we have

$$\sum_{s_1+\dots+s_{t-1}=l} \binom{2l}{2s_1, \dots, 2s_{t-1}} \times \sum_{M'} (-1)^{\alpha'} \sigma_{2s_1}^\#(m_1; N/4, N) \cdots \sigma_{2s_{t-1}}^\#(m_{t-1}; N/4, N) = (-1)^t 2^{(2l-1)(t-2)} \sigma_{2l}^\#(m),$$

where $\alpha' = m_2 + \frac{m_3}{2} + \dots + \frac{m_{t-1}}{2^{t-3}}$ and

$$M' = \{(m_1, \dots, m_{t-1}) \in \mathbb{N}^t \mid m_1 + \dots + m_{t-1} = 2^{t-1}m, m_1 + \dots + m_i \equiv 0 \pmod{2^{i-1}} \text{ for all } 2 \leq i \leq t-1\}.$$

Fix $l = s_1 + \dots + s_{t-1}$ and $m = \frac{m_1 + \dots + m_{t-1}}{2^{t-2}}$. Applying Lemma 2.5 and Proposition 2.6, we obtain

$$\begin{aligned} & \sum_{s_1+\dots+s_t=k} \binom{2k}{2s_1, \dots, 2s_t} \sum_M (-1)^\alpha \sigma_{2s_1}^\#(m_1; N/4, N) \cdots \sigma_{2s_t}^\#(m_t; N/4, N) \\ &= \sum_{s_1+\dots+s_t=k} \frac{(2k)!}{(2s_1 + \dots + 2s_{t-1})! (2s_t)!} \frac{(2s_1 + \dots + 2s_{t-1})!}{(2s_1)! \cdots (2s_{t-1})!} \sum_M (-1)^\alpha \sigma_{2s_1}^\#(m_1; N/4, N) \cdots \sigma_{2s_t}^\#(m_t; N/4, N) \\ &= \sum_{l=0}^k \binom{2k}{2l} \sum_{m=1}^{2n-1} (-1)^m \sigma_{2k-2l}^\#(2^{t-2}(2n-m); N/4, N) \sum_{s_1+\dots+s_{t-1}=l} \binom{2l}{2s_1, \dots, 2s_{t-1}} \sum_{M'} (-1)^{\alpha'} \sigma_{2s_1}^\#(m_1; N/4, N) \cdots \\ &\quad \times \sigma_{2s_{t-1}}^\#(m_{t-1}; N/4, N) \\ &= \sum_{l=0}^k \binom{2k}{2l} \sum_{m=1}^{2n-1} (-1)^m (-2^{2l-1})^{t-2} \sigma_{2l}^\#(m; N/4, N) \sigma_{2k-2l}^\#(2^{t-1}n - 2^{t-2}m; N/4, N) \\ &= (-2^{2k-1})^{t-2} \sum_{l=0}^k \binom{2k}{2l} \sum_{m=1}^{2n-1} (-1)^m \sigma_{2l}^\#(m; N/4, N) \sigma_{2k-2l}^\#(2n-m; N/4, N) \\ &= (-2^{2k-1})^{t-1} \sigma_{2k}^\#(n; N/4, N). \end{aligned} \quad \square$$

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