Research Article

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Monotonicity of solutions for fractional *p*-equations with a gradient term

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Abstract: In this paper, we consider the following fractional p-equation with a gradient term:

$$(-\Delta)_p^s u(x) = f(x, u(x), \nabla u(x)).$$

We first prove the uniqueness and monotonicity of positive solutions in a bounded domain. Then by estimating the singular integrals which define the fractional p-laplacian along a sequence of approximate maximum points, we obtain monotonicity of positive solutions in the whole space via the sliding method. In order to solve the difficulties caused by the gradient term, we introduce some new techniques which may also be applied to investigate the qualitative properties of solutions for many problems with gradient terms. Our results are extensions of Berestycki and Nirenberg [*Monotonicity, symmetry and antisymmetry of solutions of semilinear elliptic equations*, J. Geom. Phys. **5** (1988), 237–275] and Wu and Chen [*The sliding methods for the fractional p-Laplacian*, Adv. Math. **361** (2020), 106933].

Keywords: The fractional p-equation with a gradient term, monotonicity, the sliding method

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1 Introduction

In studying differential equations it is often of interest to know if the solutions have symmetry, or perhaps monotonicity, in some direction. Monotonicity and symmetry for solutions to the p-Laplace equation in bounded domains were first obtained in [1,2]. In [3], Gidas et al. obtained monotonicity and symmetry for positive solutions to the Dirichlet boundary value problem by using the maximum principle and the method of moving planes. Subsequently, Berestycki and Nirenberg [4] investigated the symmetry and monotonicity to the solutions to second-order elliptic equation with gradient term

$$-\Delta u = f(x, u, \nabla u), \quad x \in \Omega, \tag{1.1}$$

by the sliding method and the method of moving planes. In 1991, Berestycki and Nirenberg [5] considered the monotonicity and uniqueness of solutions to (1.1) with $f(x, u, \nabla u) = f(u)$.

During the last few decades, equations involving the fractional Laplacian and the fractional p-Laplacian have been extensively studied. Chen et al. [6] developed a direct method of moving planes for the fractional semilinear equation and proved the symmetry and monotonicity of the solution. Dipierro et al. [7] obtained symmetry and monotonicity of bounded solutions for fractional equation in unbounded domain Ω with the epigraph property. We also refer readers to [4,8] for some typical applications of the sliding method. Very recently, Liu [9], Wu and Chen [10,11] introduced a direct sliding method for the fractional Laplacian and the fractional p-Laplacian; Wang [12] obtained uniqueness and monotonicity of solutions to the fractional

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equation with a gradient term in a bounded domain and upper half-space; Dai et al. [13] investigated nonlinear equations involving pseudo-relativistic Schrödinger operators; Wu [14] developed a sliding method for the higher-order fractional Laplacians.

Motivated by the aforementioned papers, the goal of this paper is to extend the results in [4] to the fractional p-equation. That is, we study the monotonicity and uniqueness of solutions for the following nonlinear nonlocal equation with a gradient term

$$(-\Delta)_p^s u(x) = f(x, u(x), \nabla u(x)), \tag{1.2}$$

where $0 < s < 1$, $2 \le p < \infty$ and ∇u denotes the gradient of u, fractional p-Laplacian $(-\Delta)_p^s$ is given by

$$(-\Delta)_p^s u(x) = C_{n,\mathrm{sp}} \mathrm{P.V.} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x - y|^{n+\mathrm{sp}}} \mathrm{d}y, \tag{1.3}$$

where P.V. represents the Cauchy principal value. The fractional p-Laplacian we consider in this paper is actually a special case of the following fully nonlinear nonlocal operator, introduced in [15], with $\alpha = \mathrm{sp}$, $G(t) = |t|^{p-2}t$,

$$F_\alpha u(x) = C_{n,\alpha} \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n \setminus B_\varepsilon(x)} \frac{G(u(x) - u(y))}{|x - y|^{n+\alpha}} \mathrm{d}y.$$

In order for the integral (1.3) to make sense, we require that

$$u \in C^{1,1}_{\mathrm{loc}} \cap \mathcal{L}_{\mathrm{sp}}$$

with

$$\mathcal{L}_{\mathrm{sp}} = \left\{ u \in L^{p-1}_{\mathrm{loc}}(\mathbb{R}^n) \;\middle|\; \int_{\mathbb{R}^n} \frac{|u(x)|^{p-1}}{1 + |x|^{n+\mathrm{sp}}} \mathrm{d}x < \infty \right\}.$$

On the one hand, we extend the case $s = 1$ in [4] to the fractional p-Laplacian case $0 < s < 1$, $2 \le p < \infty$, and extend the bounded domain to $\mathbb{R}^n$. On the other hand, the nonlinear term $f(x, u, \nabla u)$ we will deal with contains the nonlinear term $f(u)$ or $f(x, u)$. The first difficulty is caused by the gradient term. And the second difficulty is that compared with the fractional Laplacian; the fractional p-Laplacian shows more complexity due to its nonlinearity.

In order to apply the sliding method, we give the exterior conditions on u. Let $u(x) = \varphi(x)$, $x \in \Omega^c$, and assume that

(C) for any three points $x = (x', x_n)$, $y = (x', y_n)$ and $z = (x', z_n)$ lying on a segment parallel to the x_n axis, $y_n < x_n < z_n$, with $y, z \in \Omega^c$, we have

$$\varphi(y) < u(x) < \varphi(z), \quad \text{if } x \in \Omega \tag{1.4}$$

and

$$\varphi(y) \le \varphi(x) \le \varphi(z), \quad \text{if } x \in \Omega^c. \tag{1.5}$$

Remark 1.1. The same monotonicity conditions (1.4) and (1.5) (with Ω^c replaced by $\partial\Omega$) appeared in [5,8,11].

For bounded domain, we prove

Theorem 1.1. *Suppose that $u \in C^{1,1}_{\mathrm{loc}}(\Omega) \cap C(\bar{\Omega})$ satisfies (C) and is a solution of problem*

$$\begin{cases} (-\Delta)_p^s u(x) = f(x, u, \nabla u), & x \in \Omega, \\ u(x) = \varphi(x), & x \in \Omega^c, \end{cases} \tag{1.6}$$

where Ω is a bounded domain which is convex in x_n direction. Assume that f is continuous in all variables, Lipschitz continuous in $(u, \nabla u)$ and nondecreasing in x_n. Then u is strictly monotone increasing with respect to x_n in Ω, i.e., for any $\tau > 0$,

$$u(x', x_n + \tau) > u(x', x_n), \quad \text{for all } (x', x_n), (x', x_n + \tau) \in \Omega.$$

Furthermore, the solution of (1.6) *is unique.*

Remark 1.2. For the finite cylinder $C = \{(x', x_n) \in \mathbb{R}^n \| x_n| < a, x' \in \omega\}$, where $a > 0$ and ω is a bounded domain in $\mathbb{R}^{n-1}$ with smooth boundary, the results of Theorem 1.1 still hold.

Remark 1.3. The conditions assumed in Theorems 1.1 and 1 of [16] are different, and there is no relation between them. Dai et al. [16] studied the positive solution u and obtained that u was strictly increasing in the left half of Ω (i.e., $\{x \in \Omega | x_n \leq 0\}$) in x_n direction with $x_n < 0$ by the method of moving planes, but the solution obtained can be negative and is strictly increasing with respect to x_n in the whole domain Ω by the sliding method.

We can immediately get a new antisymmetry result for (1.6) if bounded domain Ω is symmetric about $x_n = 0$ by Theorem 1.1.

Corollary 1.1. (*Antisymmetry*) *Assume that the conditions of* Theorem 1.1 *are satisfied and in addition that φ is odd in x_n on Ω^c. If $f(x, u, \nabla u)$ is odd in $(x_n, u, \nabla_{x'} u)$, then u is odd, i.e., antisymmetric in x_n:*

$$u(x', -x_n) = -u(x', x_n), \forall x \in \Omega.$$

This follows from the fact that $\tilde{u} = -u(x', -x_n)$ is a solution satisfying the same conditions, and so equals u.

Let

$$D_M = \{x \in \mathbb{R}^n \| x_n| \leq M\}.$$

We say that u is in $C^{1,1}(D_M)$ for $M > 0$, if there exists a constant $C(M)$ such that

$$\|u\|_{C^{1,1}(D_M)} \leq C(M).$$

For the whole space, we prove

Theorem 1.2. *Suppose that $u \in C^{1,1}(D_M) \cap \mathcal{L}_{sp}$ is a solution to*

$$(-\Delta)_p^s u(x) = f(x, u, \nabla u), x \in \mathbb{R}^n, \tag{1.7}$$

such that

$$|u(x)| \leq 1, x \in \mathbb{R}^n \tag{1.8}$$

and

$$u(x', x_n) \underset{x_n \to \pm\infty}{\longrightarrow} \pm 1, \quad \text{uniformly in } x' = (x_1, \ldots, x_{n-1}). \tag{1.9}$$

Assume that f is bounded, continuous in all variables and satisfies

$$f(x', x_n, u, \nabla u) \leq f(x', \bar{x}_n, u, \nabla u), x = (x', x_n) \in \mathbb{R}^n, x_n \leq \bar{x}_n. \tag{1.10}$$

Suppose that there exists $\delta > 0$ such that

$$f \text{ is nonincreasing on } u \in [-1, -1 + \delta] \quad \text{and} \quad u \in [1 - \delta, 1]. \tag{1.11}$$

Then u is strictly monotone increasing with respect to x_n, and furthermore, it depends on x_n only.

Remark 1.4. Theorem 1.2 is closely related to the well-known De Giorgi conjecture [17]. As an example, we may think of $f(x, u, \nabla u) = u - u^3$, $p = 2$, which yields the fractional Allen-Cahn equation, that is, a widely studied model in phase transitions in media with long-range particle interactions [18]. The conclusion of Theorem 1.2 is also valid for $p = 2$, and there is no corresponding result in [12].

In order to solve the difficulty that the nonlinear terms at the right-hand side of (1.6) and (1.7) contain the gradient term, in bounded domains when deriving the contradiction for the minimum point of the function $w^\tau(x)$ (see Section 2 for definition), we use the technique of finding the minimum value of the function $w^\tau(x)$ for the variables τ and x at the same time. This is different from the previous sliding process which only finds the minimum value of the variable x for the fixed τ. In the whole space, we estimate the singular integrals defining the fractional p-Laplacian along a sequence of approximate maximum, and τ and the sequence of approximate maximum are estimated at the same time.

For more related articles on symmetry and nonexistence results of local and nonlocal equations, we also refer readers to [19] for Laplace equations with a gradient term, [20–26] for fractional equations, [27] for weighted fractional equation, [28,29] for fractional equations with a gradient term, [30–32] for fully nonlinear equations with a gradient term, [33–36] for fractional p-Laplace equation, [37] for parabolic p-Laplace equation and [38] for p-Laplace equation. For results related to the existence of solutions for fractional p-Laplacian problems, we refer to [39] and references therein.

The paper is organized as follows. In Section 2, Theorem 1.1 is proved via the sliding method. In Section 3, we derive monotonicity for the fractional p-equation with a gradient term in $\mathbb{R}^n$.

2 Monotonicity and uniqueness of solutions in bounded domains

For simplicity, we list some notations used frequently. For $\tau \in \mathbb{R}$, denote $x = (x', x_n)$, $x' = (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}$. Set

$$u^\tau(x) = u(x', x_n + \tau), \quad w^\tau(x) = u^\tau(x) - u(x).$$

Proof of Theorem 1.1. When $p = 2$, $(-\Delta)_p^s$ reduces to fractional Laplacian $(-\Delta)^s$, see [12] for the proof of Theorem 1.1. And the following proof will omit the case of $p = 2$. When $p > 2$, the main difference between the fractional p-Laplacian and the fractional Laplacian is that the former is a nonlinear operator, hence we need to use $(-\Delta)_p^s u^\tau(x) - (-\Delta)_p^s u(x)$ instead of $(-\Delta)_p^s w^\tau(x)$, and this makes the proof different. Now we will give a detailed proof of Theorem 1.1 in the case of $p > 2$.

For $\tau > 0$, it is defined on the set $\Omega^\tau = \Omega - \tau e_n$, which is obtained from Ω by sliding it downward a distance τ parallel to the x_n axis, where $e_n = (0, \ldots, 0, 1)$. Set

$$D^\tau := \Omega^\tau \cap \Omega, \tilde{\tau} = \sup\{\tau | \tau > 0, D^\tau \neq \emptyset\}$$

and

$$w^\tau(x) = u^\tau(x) - u(x), x \in D^\tau.$$

We mainly divide the following two steps to prove that u is strictly increasing in the x_n direction, i.e.,

$$w^\tau(x) > 0, x \in D^\tau, \text{ for any } 0 < \tau < \tilde{\tau}. \tag{2.12}$$

Step 1. For τ sufficiently close to $\tilde{\tau}$, i.e., D^τ is narrow, we claim that there exists $\delta > 0$ small enough such that

$$w^\tau(x) \geq 0, \quad \forall x \in D^\tau, \quad \forall \tau \in (\tilde{\tau} - \delta, \tilde{\tau}). \tag{2.13}$$

If (2.13) is false, we set

$$A_0 = \min_{\substack{x \in \bar{D}^\tau \\ \tilde{\tau} - \delta < \tau < \tilde{\tau}}} w^\tau(x) < 0.$$

From the condition (C), A_0 can be obtained for some $(\tau^0, x^0) \in \{(\tau, x)|(\tau, x) \in (\tilde{\tau} - \delta, \tilde{\tau}) \times D^\tau\}$. Noting that $w^{\tau^0}(x) \geq 0, x \in \partial D^{\tau^0}$, we arrive at $x^0 \in D^{\tau^0}$. So $w^{\tau^0}(x^0) = A_0$. Since (τ^0, x^0) is a minimizing point, we have $\nabla w^{\tau^0}(x^0) = 0$, i.e., $\nabla u^{\tau^0}(x^0) = \nabla u(x^0)$. Since u^{τ^0} satisfies the same equation (1.6) in Ω^{τ^0} as u does in Ω, and f is nondecreasing in x_n, so we have

$$\begin{aligned}(-\Delta)_p^s u^\tau(x^0) - (-\Delta)_p^s u(x^0) &= f((x^0)', x_n^0 + \tau^0, u^{\tau^0}(x^0), \nabla u^{\tau^0}(x^0)) - f(x^0, u^{\tau^0}(x^0), \nabla u^{\tau^0}(x^0)) \\ &\geq f(x^0, u^{\tau^0}(x^0), \nabla u(x^0)) - f(x^0, u(x^0), \nabla u(x^0)) \\ &= -c^{\tau^0}(x^0) w^{\tau^0}(x^0),\end{aligned} \tag{2.14}$$

where $-c^{\tau^0}(x^0) = \frac{f(x^0, u^{\tau^0}(x^0), \nabla u(x^0)) - f(x^0, u(x^0), \nabla u(x^0))}{u^{\tau^0}(x^0) - u(x^0)}$ is a L^∞ function satisfying

$$|c^{\tau^0}(x^0)| \leq C, \quad \forall x^0 \in D^{\tau^0}.$$

Hence,

$$(-\Delta)_p^s u^{\tau^0}(x^0) - (-\Delta)_p^s u(x^0) + c^{\tau^0}(x^0) w^{\tau^0}(x^0) \geq 0. \tag{2.15}$$

On the other hand, note that $G(t) = |t|^{p-2}t$ is a strictly increasing function in t, and $G'(t) = (p-1)|t|^{p-2} \geq 0$. By the definition of the fractional p-Laplacian, we have

$$\begin{aligned}(-\Delta)_p^s u^{\tau^0}(x^0) - (-\Delta)_p^s u(x^0) &= C_{n,\mathrm{sp}} \mathrm{P.V.} \int_{\mathbb{R}^n} \frac{G(u^{\tau^0}(x^0) - u^{\tau^0}(y)) - G(u(x^0) - u(y))}{|x^0 - y|^{n+\mathrm{sp}}} \mathrm{d}y \\ &= C_{n,\mathrm{sp}} \mathrm{P.V.} \int_{D^{\tau^0}} \frac{G(u^{\tau^0}(x^0) - u^{\tau^0}(y)) - G(u(x^0) - u(y))}{|x^0 - y|^{n+\mathrm{sp}}} \mathrm{d}y \\ &\quad + C_{n,\mathrm{sp}} \mathrm{P.V.} \int_{\mathbb{R}^n \setminus D^{\tau^0}} \frac{G(u^{\tau^0}(x^0) - u^{\tau^0}(y)) - G(u(x^0) - u(y))}{|x^0 - y|^{n+\mathrm{sp}}} \mathrm{d}y \\ &:= C_{n,\mathrm{sp}}(I_1 + I_2).\end{aligned} \tag{2.16}$$

To estimate I_1, noting that x^0 is the minimum point of w^{τ^0} in $\bar{D}^{\tau^0}$, it follows

$$(u^{\tau^0}(x^0) - u^{\tau^0}(y)) - (u(x^0) - u(y)) = w^{\tau^0}(x^0) - w^{\tau^0}(y) \leq 0, \quad y \in \bar{D}^{\tau^0}.$$

By the monotonicity of G, we derive

$$G(u^{\tau^0}(x^0) - u^{\tau^0}(y)) - G(u(x^0) - u(y)) \leq 0.$$

Hence,

$$I_1 \leq 0. \tag{2.17}$$

Next we prove that $I_2 < 0$. Similar to [40], we need the following lemma.

Lemma 2.1. [40] *For $G(t) = |t|^{p-2}t$, $p > 2$, there exists a constant $C > 0$ such that*

$$G(t_2) - G(t_1) \geq C(t_2 - t_1)^{p-1}$$

for arbitrary $t_2 > t_1$.

Noting that $w^{\tau^0}(y) \geq 0$ in $(D^{\tau^0})^c$ and $w^{\tau^0}(x^0) < 0$, it implies

$$(u^{\tau^0}(x^0) - u^{\tau^0}(y)) - (u(x^0) - u(y)) = w^{\tau^0}(x^0) - w^{\tau^0}(y) < 0, \quad y \in (D^{\tau^0})^c.$$

By Lemma 2.1, we have

$$I_2 = \int_{\mathbb{R}^n \setminus D^{\tau^0}} \frac{G(u^{\tau^0}(x^0) - u^{\tau^0}(y)) - G(u(x^0) - u(y))}{|x^0 - y|^{n+\mathrm{sp}}} \mathrm{d}y \leq \int_{(D^{\tau^0})^c} \frac{-C(w^{\tau^0}(y) - w^{\tau^0}(x^0))^{p-1}}{|x^0 - y|^{n+\mathrm{sp}}} \mathrm{d}y < 0. \tag{2.18}$$

From (1.4) in the condition (C), there exists a point $y^0 \in \partial D^{\tau^0}$ such that

$$w^{\tau^0}(y^0) > 0.$$

Since function w^{τ^0} is continuous in $\mathbb{R}^n$, there exists a small $\delta > 0$ and $C_0 > 0$ such that

$$w^{\tau^0}(y) \geq C_0 > 0, \quad y \in (D^{\tau^0})^c \cap B_\delta(y^0). \tag{2.19}$$

Combining (2.17), (2.18) and (2.19), it yields

$$\begin{aligned}
&(-\Delta)_p^s u^{\tau^0}(x^0) - (-\Delta)_p^s u(x^0) + c^{\tau^0}(x^0)w^{\tau^0}(x^0) \\
&\leq C_{n,\mathrm{sp}} \int\limits_{(D^{\tau^0})^c} \frac{-C(w^{\tau^0}(y) - w^{\tau^0}(x^0))^{p-1}}{|x^0 - y|^{n+\mathrm{sp}}} \mathrm{d}y + c^{\tau^0}(x^0)w^{\tau^0}(x^0) \\
&\leq -C \int\limits_{(D^{\tau^0})^c \cap B_\delta(y^0)} \frac{(C_0 + |w^{\tau^0}(x^0)|)^{p-1}}{|x^0 - y|^{n+\mathrm{sp}}} \mathrm{d}y + c^{\tau^0}(x^0)w^{\tau^0}(x^0) \\
&\leq \frac{-C}{d_n^{\mathrm{sp}}}(C_0 + |w^{\tau^0}(x^0)|)^{p-1} + \inf_{D^{\tau^0}} c^{\tau^0}(x)w^{\tau^0}(x^0) \\
&\leq \frac{-C}{d_n^{\mathrm{sp}}} C_0^{p-1} + \frac{-C}{d_n^{\mathrm{sp}}}|w^{\tau^0}(x^0)|^{p-1} + \inf_{D^{\tau^0}} c^{\tau^0}(x)w^{\tau^0}(x^0) < 0,
\end{aligned} \tag{2.20}$$

where $p > 2$, d_n denotes the width of D^{τ^0} in the x_n direction and $c^{\tau^0}(x)$ is bounded. This contradicts (2.15).

Hence, (2.13) is true for τ sufficiently close to $\tilde{\tau}$.

Step 2. The inequality (2.13) provides a starting point, from which we can carry out the sliding. Now we decrease τ as long as (2.13) holds to its limiting position. Define

$$\tau_0 = \inf\{\tau | w^\tau(x) \geq 0, x \in D^\tau, 0 < \tau < \tilde{\tau}\}.$$

We will prove

$$\tau_0 = 0.$$

Otherwise, assume $\tau_0 > 0$, we will show that the domain Ω be slided upward a little bit more and we still have

$$w^\tau(x) \geq 0, x \in D^\tau, \quad \text{for any } \tau_0 - \varepsilon < \tau \leq \tau_0, \tag{2.21}$$

which contradicts the definition of τ_0.

Since $w^{\tau_0}(x) > 0$, $x \in \Omega \cap \partial D^{\tau_0}$ by the condition (C) and $w^{\tau_0}(x) \geq 0$, $x \in D^{\tau_0}$, it follows

$$w^{\tau_0}(x) \not\equiv 0, x \in D^{\tau_0}.$$

If there exists a point $\tilde{x} \in D^{\tau_0}$ such that $w^{\tau_0}(\tilde{x}) = 0$, then $\tilde{x}$ is the minimum point, similar to (2.20), we have

$$(-\Delta)_p^s u^{\tau_0}(\tilde{x}) - (-\Delta)_p^s u(\tilde{x}) < 0, \tag{2.22}$$

which contradicts to

$$\begin{aligned}
(-\Delta)_p^s u^{\tau_0}(\tilde{x}) - (-\Delta)_p^s u(\tilde{x}) &= f(\tilde{x}', \tilde{x}_n + \tau_0, u^{\tau_0}(\tilde{x}), \nabla u^{\tau_0}(\tilde{x})) - f(\tilde{x}, u(\tilde{x}), \nabla u(\tilde{x})) \\
&\geq f(\tilde{x}, u^{\tau_0}(\tilde{x}), \nabla u^{\tau_0}(\tilde{x})) - f(\tilde{x}, u(\tilde{x}), \nabla u(\tilde{x})) = 0.
\end{aligned}$$

Hence,

$$w^{\tau_0}(x) > 0, \quad x \in D^{\tau_0}. \tag{2.23}$$

Next we will prove (2.21). Suppose (2.21) is not true, one has

$$A_1 = \min_{\substack{x \in D^\tau \\ \tau_0 - \varepsilon < \tau < \tau_0}} w^\tau(x) < 0.$$

The minimum A_1 can be obtained for some $\mu \in (\tau_0 - \varepsilon, \tau_0)$, $\bar{x} \in D^{\mu}$ where $w^{\mu}(\bar{x}) = A_1$ by condition (C). We carve out of D^{τ_0} a closed set $K \subset D^{\tau_0}$ such that $D^{\tau_0} \backslash K$ is narrow. According to (2.23),

$$w^{\tau_0}(x) \geq C_0 > 0, x \in K.$$

From the continuity of w^{τ} in τ, we have for small $\varepsilon > 0$,

$$w^{\mu}(x) \geq 0, x \in K. \tag{2.24}$$

From (C), it follows

$$w^{\mu}(x) \geq 0, x \in (D^{\mu})^c.$$

So $\bar{x} \in D^{\mu} \backslash K$ and $\nabla w^{\mu}(\bar{x}) = 0$. Since $D^{\tau_0} \subset D^{\mu}$ and small ε, we obtain that $D^{\mu}K$ is a narrow domain. Similar to (2.15), we have

$$(-\Delta)^s_p u^{\mu}(\bar{x}) - (-\Delta)^s_p u(\bar{x}) + c(\bar{x}) w^{\mu}(\bar{x}) \geq 0.$$

Similar to (2.20), by narrow domain $D^{\mu} \backslash K$, we have

$$(-\Delta)^s_p u^{\mu}(\bar{x}) - (-\Delta)^s_p u(\bar{x}) + c(\bar{x}) w^{\mu}(\bar{x}) < 0.$$

This is a contradiction. Hence we derive (2.21), which contradicts to the definition of τ_0. So $\tau_0 = 0$. Therefore, we have shown that

$$w^{\tau}(x) \geq 0, x \in D^{\tau}, \quad \text{for any } 0 < \tau < \tilde{\tau}. \tag{2.25}$$

Let us prove (2.12). Since

$$w^{\tau}(x) \not\equiv 0, x \in D^{\tau}, \quad \text{for any } 0 < \tau < \tilde{\tau},$$

if there exists a point x^0 for some $\tau_1 \in (0, \tilde{\tau})$ such that $w^{\tau_1}(x^0) = 0$, then x^0 is the minimum point and similar to (2.20), we have

$$(-\Delta)^s_p u^{\tau_1}(x^0) - (-\Delta)^s_p u(x^0) < 0,$$

which contradicts

$$f((x^0)', x^0_n + \tau_1, u^{\tau_1}(x^0), \nabla u^{\tau_1}(x^0)) - f(x^0, u(x^0), \nabla u(x^0)) \geq 0.$$

Therefore, we arrive at (2.12).

Now we prove uniqueness. If $\bar{u}$ is another solution satisfying the same conditions, the same argument as before but replace $w^{\tau} = u^{\tau} - u$ with $w^{\tau} = \bar{u}^{\tau} - u$. Similar to (2.25), we have $\bar{u}^{\tau}(x) \geq u$ in D^{τ} for any $0 < \tau < \tilde{\tau}$. Hence, $\bar{u} \geq u$. Interchanging the roles of u and $\bar{u}$, we find the opposite inequality. Therefore, $\bar{u} = u$.

This completes the proof of Theorem 1.1. □

3 Monotonicity of solutions in $\mathbb{R}^n$

In the section, we prove Theorem 1.2 by the sliding method based on the following lemma.

Lemma 3.1. [40] *Assume that $u \in C^{1,1} \cap \mathcal{L}_{sp}$ is the bounded solution of* (1.7) *and $\psi_k \in C_0^{\infty}(\mathbb{R}^n)$. Then for any $\delta > 0$, we have*

$$|(-\Delta)^s_p (u + \varepsilon_k \psi_k)(x) - (-\Delta)^s_p u(x)| = o_{\varepsilon_k}(1),$$

where $o_{\varepsilon_k}(1) \to 0$ as $k \to 0 (\varepsilon_k \to 0)$.

Proof of Theorem 1.2. Denote $x = (x', x_n)$. For any $\tau \in \mathbb{R}$, define

$$u^{\tau}(x) = u(x', x_n + \tau), U^{\tau}(x) = u(x) - u^{\tau}(x).$$

From (1.9), there exists a constant $a > 0$ such that

$$u(x', x_n) \geq 1 - \delta, \quad \text{for } x_n \geq a \quad \text{and} \quad u(x', x_n) \leq -1 + \delta, \quad \text{for } x_n \leq -a.$$

For any $\tau \geq 2a$, no matter where x is, we have either

$$u^\tau(x', x_n) \geq 1 - \delta \quad (\text{if } x_n \geq -a) \tag{3.26}$$

or

$$u(x', x_n) \leq -1 + \delta \quad (\text{if } x_n \leq -a). \tag{3.27}$$

Outline of the proof: We will use the sliding method to prove the monotonicity of u and divide the proof into three Steps.

Step 1, we will show that for τ sufficiently large, we have $U^\tau(x) \leq 0, x \in \mathbb{R}^n$. From (3.26) and (3.27), we will show that

$$U^\tau(x) \leq 0, \quad x \in \mathbb{R}^n, \quad \forall \tau \geq 2a. \tag{3.28}$$

This provides the starting point for the sliding method. Then in Step 2, we decrease τ continuously as long as (3.28) holds to its limiting position. Define

$$\tau_0 := \inf\{\tau | U^\tau(x) \leq 0, x \in \mathbb{R}^n\}. \tag{3.29}$$

We will show that $\tau_0 = 0$. In Step 3, we deduce that the solution u must be strictly monotone increasing in x_n and only depends on x_n.

Now we will show the details in the following three steps.

Step 1. We will show that

$$U^\tau(x) \leq 0, \quad x \in \mathbb{R}^n, \quad \text{for any } \tau \geq 2a. \tag{3.30}$$

If not, then

$$\sup_{\mathbb{R}^n} U^\tau(x) = A > 0,$$

so for some $\tau_1 \geq 2a$ there exists a sequence $\{x^k\} \subset \mathbb{R}^n$, such that

$$U^{\tau_1}(x^k) \to A > 0, \quad \text{as } k \to \infty. \tag{3.31}$$

Denote $x^k = (x_1^k, \ldots, x_n^k)$. Since $U^{\tau_1}(x^k) = u(x^k) - u^{\tau_1}(x^k) \to 0$ as $x_n^k \to \pm\infty$, it implies that there exists $M_0 > 0$ such that $|x_n^k| \leq M_0$.

Let

$$\eta(x) = \begin{cases} ce^{\frac{1}{|x|^2-1}}, & |x| < 1, \\ 0, & |x| \geq 1, \end{cases} \tag{3.32}$$

where $c > 0$ is a constant, taking $c = e$ such that $\eta(0) = \max_{\mathbb{R}^n} \eta(x) = 1$. Set

$$\psi_k(x) = \eta(x - x^k).$$

From (3.31), there exists a sequence $\{\varepsilon_k\}$, with $\varepsilon_k \to 0$ such that

$$U^{\tau_1}(x^k) + \varepsilon_k \psi_k(x^k) > A.$$

Since for any $x \in \mathbb{R}^n \backslash B_1(x^k)$, $U^{\tau_1}(x) \leq A$ and $\psi_k(x) = 0$, hence

$$U^{\tau_1}(x^k) + \varepsilon_k \psi_k(x^k) > U^{\tau_1}(x) + \varepsilon_k \psi_k(x), \quad \text{for any } x \in \mathbb{R}^n \backslash B_1(x^k).$$

It follows that there exists a point $\bar{x}^k \in B_1(x^k)$ such that

$$U^{\tau_1}(\bar{x}^k) + \varepsilon_k \psi_k(\bar{x}^k) = \max_{\mathbb{R}^n}(U^{\tau_1}(x) + \varepsilon_k \psi_k(x)) > A. \tag{3.33}$$

On one hand, by (1.7), (1.10) and Lemma 3.1, we have

$$\begin{aligned}
&(-\Delta)_p^s(u+\varepsilon_k\psi_k)(\bar{x}^k)-(-\Delta)_p^s u^{\tau_1}(\bar{x}^k)\\
&= (-\Delta)_p^s(u+\varepsilon_k\psi_k)(\bar{x}^k)-(-\Delta)_p^s u(\bar{x}^k)+(-\Delta)_p^s u(\bar{x}^k)-(-\Delta)_p^s u^{\tau_1}(\bar{x}^k)\\
&\le f(\bar{x}^k,u(\bar{x}^k),\nabla u(\bar{x}^k))-f(\bar{x}^k,u^{\tau_1}(\bar{x}^k),\nabla u^{\tau_1}(\bar{x}^k))+o_{\varepsilon_k}(1)\\
&\le f(\bar{x}^k,u(\bar{x}^k),\nabla u(\bar{x}^k))-f(\bar{x}^k,u^{\tau_1}(\bar{x}^k),\nabla u(\bar{x}^k))+f(\bar{x}^k,u^{\tau_1}(\bar{x}^k),\nabla u(\bar{x}^k))-f(\bar{x}^k,u^{\tau_1}(\bar{x}^k),\nabla u^{\tau_1}(\bar{x}^k))+o_{\varepsilon_k}(1).
\end{aligned}$$

Since $\tau_1 \ge 2a$, no matter where $\bar{x}^k$ is, one of the points $\bar{x}^k$ and $\bar{x}^k+(0',\tau_1)$ is in the domain $\{x||x_n|\ge a\}$ where f is nonincreasing in u by (1.11). Since $u(\bar{x}^k)>u^{\tau_1}(\bar{x}^k)$, from (3.26), both $u(\bar{x}^k)$ and $u^{\tau_1}(\bar{x}^k)$ are close to 1, while from (3.27), both $u(\bar{x}^k)$ and $u^{\tau_1}(\bar{x}^k)$ are close to -1. So we apply the monotonicity of f to derive that

$$f(\bar{x}^k,u(\bar{x}^k),\nabla u(\bar{x}^k)) \le f(\bar{x}^k,u^{\tau_1}(\bar{x}^k),\nabla u(\bar{x}^k)).$$

From (3.33), we have $\nabla(U^{\tau_1}(\bar{x}^k)+\varepsilon_k\psi(\bar{x}^k))=0$. It follows that $\nabla U^{\tau_1}(\bar{x}^k)\to 0$ as $k\to\infty$. Let $k\to\infty$, we have

$$(-\Delta)_p^s(u+\varepsilon_k\psi_k)(\bar{x}^k)-(-\Delta)_p^s u^{\tau_1}(\bar{x}^k)\le o_{\varepsilon_k}(1). \tag{3.34}$$

On the other hand, by (3.33) and Lemma 2.1,

$$\begin{aligned}
&(-\Delta)_p^s(u+\varepsilon_k\psi_k)(\bar{x}^k)-(-\Delta)_p^s u^{\tau_1}(\bar{x}^k)\\
&= C_{n,\mathrm{sp}}\mathrm{P.V.}\int_{\mathbb{R}^n}\frac{G(u(\bar{x}^k)+\varepsilon_k\psi_k(\bar{x}^k)-u(y)-\varepsilon_k\psi_k(y))-G(u^{\tau_1}(\bar{x}^k)-u^{\tau_1}(y))}{|\bar{x}^k-y|^{n+\mathrm{sp}}}\mathrm{d}y\\
&\ge C\mathrm{P.V.}\int_{\mathbb{R}^n}\frac{|U^{\tau_1}(\bar{x}^k)-U^{\tau_1}(y)+\varepsilon_k\psi_k(\bar{x}^k)-\varepsilon_k\psi_k(y)|^{p-1}}{|\bar{x}^k-y|^{n+\mathrm{sp}}}\mathrm{d}y\\
&\ge C\int_{B_2^c(x^k)}\frac{|U^{\tau_1}(\bar{x}^k)+\varepsilon_k\psi_k(\bar{x}^k)-U^{\tau_1}(y)|^{p-1}}{|\bar{x}^k-y|^{n+\mathrm{sp}}}\mathrm{d}y\\
&\ge C\int_{B_2^c(x^k)}\frac{|A-U^{\tau_1}(y)|^{p-1}}{|x^k-y|^{n+\mathrm{sp}}}\mathrm{d}y = C\int_{B_2^c(0)}\frac{|A-U^{\tau_1}(z+x^k)|^{p-1}}{|z|^{n+\mathrm{sp}}}\mathrm{d}z,
\end{aligned} \tag{3.35}$$

where C is a positive constant which can be different from line to line. Denote

$$u_k(x)=u(x+x^k).$$

Since f is bounded, one derive that u is uniformly Hölder continuous, by the Arzelà-Ascoli theorem, up to extraction of a subsequence, we obtain

$$u_k(x)\to u_\infty(x),\quad x\in\mathbb{R}^n,\quad \text{as } k\to\infty.$$

Combining (3.34) and (3.35), letting $k\to\infty$, one arrive at

$$U^{\tau_1}(x+x^k)\to A,\quad x\in B_2^c(0),\quad \text{uniformly},$$

$$U^{\tau_1}(x+x^k)\to u_\infty(x)-u^{\tau_1}(x)\equiv A,\quad x\in B_2^c(0),$$

Since $\{x_n^k\}$ is bounded, from (1.9), we have

$$u_\infty(x',x_n)\underset{x_n\to\pm\infty}{\longrightarrow}\pm 1\quad \text{uniformly in } x'=(x_1,\dots,x_{n-1}).$$

Hence,

$$u_\infty(x',x_n)=A+u_\infty(x',x_n+\tau_1)=2A+u_\infty(x',x_n+2\tau_1)=\cdots=kA+u_\infty(x',x_n+k\tau_1), \tag{3.36}$$

for any $k\in\mathbb{N}$. This is impossible because $u_\infty(x)$ is bounded. This verifies (3.30).

Step 2. Note that (3.30) provides a starting point, from which we can carry out the sliding. We decrease τ and show that for any $0<\tau<2a$, it yields

$$U^\tau(x)\le 0,\quad x\in\mathbb{R}^n. \tag{3.37}$$

Define

$$\tau_0 = \inf\{\tau | U^\tau(x) \le 0, x \in \mathbb{R}^n\}.$$

We prove that $\tau_0 = 0$. Otherwise, we have $\tau_0 > 0$. To derive a contradiction, we prove that τ_0 can be decreased a little bit while inequality (3.37) is still valid.

We first prove that

$$\sup_{\mathbb{R}^{n-1}\times[-a,a]} U^{\tau_0}(x) < 0. \tag{3.38}$$

If not, then

$$\sup_{\mathbb{R}^{n-1}\times[-a,a]} U^{\tau_0}(x) = 0,$$

and there exists a sequence

$$\{x^k\} \subset \mathbb{R}^{n-1} \times [-a, a], \quad k = 1, 2, \dots,$$

such that

$$U^{\tau_0}(x^k) \to 0, \quad \text{as } k \to \infty.$$

Let $\psi_k(x) = \eta(x - x^k)$, where η is in (3.32). Then there exists a sequence $\varepsilon_k \to 0$ such that

$$U^{\tau_0}(x^k) + \varepsilon_k \psi_k(x^k) > 0.$$

Since for $x \in \mathbb{R}^n \backslash B_1(x^k)$, we have $U^{\tau_0}(x) \le 0$ and $U^{\tau_0}(x) + \varepsilon_k \psi_k(x) > 0$. Then there exists $\bar{x}^k \in B_1(x^k)$ such that

$$U^{\tau_0}(\bar{x}^k) + \varepsilon_k \psi_k(\bar{x}^k) = \max_{\mathbb{R}^n}(U^{\tau_0}(x) + \varepsilon_k \psi_k(x)) > 0. \tag{3.39}$$

On one hand, similar to the argument in Step 1, we have

$$(-\Delta)^s_p(u + \varepsilon_k \psi_k)(\bar{x}^k) - (-\Delta)^s_p u^{\tau_0}(\bar{x}^k) \le f(\bar{x}^k, u^{\tau_0}(\bar{x}^k), \nabla u(\bar{x}^k)) - f(\bar{x}^k, u^{\tau_0}(\bar{x}^k), \nabla u^{\tau_0}(\bar{x}^k)) + o_{\varepsilon_k}(1). \tag{3.40}$$

On the other hand, by (3.39) and Lemma 2.1, similar to (3.35)

$$(-\Delta)^s_p(u + \varepsilon_k \psi)(\bar{x}^k) - (-\Delta)^s_p u^{\tau_0}(\bar{x}^k) \ge C \int_{B_2^c(x^k)} \frac{|U^{\tau_0}(y)|^{p-1}}{|x^k - y|^{n+sp}} dy = C \int_{B_2^c(0)} \frac{|U^{\tau_0}(z + x^k)^{p-1}|}{|z|^{n+sp}} dz. \tag{3.41}$$

Denote

$$u_k(x) = u(x + x^k) \quad \text{and} \quad U_k^{\tau_0}(x) = U^{\tau_0}(x + x^k).$$

One has $u_k(x) \to u_\infty(x), x \in \mathbb{R}^n,$ as $k \to \infty$. Letting $k \to \infty$, we obtain

$$\nabla u(\bar{x}^k) = \nabla u^{\tau_0}(\bar{x}^k), \quad \text{as } k \to \infty.$$

It yields that

$$(-\Delta)^s_p(u + \varepsilon_k \psi_k)(\bar{x}^k) - (-\Delta)^s_p u^{\tau_0}(\bar{x}^k) \le o_{\varepsilon_k}(1). \tag{3.42}$$

So combining (3.42) and (3.41), letting $k \to \infty$, we have

$$U_k^{\tau_0}(x) \to 0, x \in B_2^c(0), \quad \text{uniformly, as } k \to \infty.$$

Therefore,

$$u_\infty(x) - u_\infty^{\tau_0}(x) \equiv 0, \quad x \in B_2^c(0).$$

Hence, for any $k \in \mathbb{N}$

$$u_\infty(x', x_n) = u_\infty(x', x_n + \tau_0) = u_\infty(x', x_n + 2\tau_0) = \dots = u_\infty(x', x_n + k\tau_0).$$

Take x_n sufficiently negative and k sufficiently large, hence $u_\infty(x', x_n)$ is close to -1 and $u_\infty(x', x_n + \tau_0)$ is sufficiently close to 1 by (1.9), this is impossible. Therefore, (3.38) is checked.

Next we prove that, there exists an $\varepsilon > 0$, such that

$$U^\tau(x) \le 0, \quad \forall x \in \mathbb{R}^n, \quad \tau \in (\tau_0 - \varepsilon, \tau_0]. \tag{3.43}$$

First, (3.38) implies immediately that there exists an $\varepsilon > 0$ such that

$$\sup_{\mathbb{R}^{n-1}\times[-a,a]} U^\tau(x) < 0, \quad \tau \in (\tau_0 - \varepsilon, \tau_0]. \tag{3.44}$$

So we only need to prove that

$$\sup_{\mathbb{R}^n \setminus (\mathbb{R}^{n-1}\times[-a,a])} U^\tau(x) \le 0, \quad \tau \in (\tau_0 - \varepsilon, \tau_0].$$

If not, then

$$\sup_{\mathbb{R}^n \setminus (\mathbb{R}^{n-1}\times[-a,a])} U^\tau(x) := A > 0, \quad \tau \in (\tau_0 - \varepsilon, \tau_0].$$

For some $\tau_2 \in (\tau_0 - \varepsilon, \tau_0]$, there exists a subsequence of $\{x^k\}$ such that $U^{\tau_2}(x^k) \to A$. By condition (1.9), $\{x_n^k\}$ is bounded, and we assume that $|x_n^k| \le M(M > a)$. By a similar argument to the proof of (3.38), set $\varphi(x) = \eta(x - x^k)$, there exists $\varepsilon_k \to 0$, and $\bar{x}^k \in B_1(x^k)$ such that

$$U^{\tau_2}(\bar{x}^k) + \varepsilon_k\varphi(\bar{x}^k) = \max_{\mathbb{R}^n}(U^{\tau_2}(x) + \varepsilon_k\varphi(x)) > 0$$

and

$$(-\Delta)_p^s(u + \varepsilon_k\varphi)(\bar{x}^k) - (-\Delta)_p^s u^{\tau_2}(\bar{x}^k) \le f(\bar{x}^k, u^{\tau_2}(\bar{x}^k), \nabla u(\bar{x}^k)) - f(\bar{x}^k, u^{\tau_2}(\bar{x}^k), \nabla u^{\tau_2}(\bar{x}^k)) + o_{\varepsilon_k}(1). \tag{3.45}$$

In addition,

$$(-\Delta)_p^s(u + \varepsilon_k\varphi)(\bar{x}^k) - (-\Delta)_p^s u^{\tau_2}(\bar{x}^k) \ge C \int_{B_2^c(0)} \frac{|A - U^{\tau_2}(z + x^k)|^{p-1}}{|z|^{n+\mathrm{sp}}} dz. \tag{3.46}$$

Denoting $U_k^{\tau_2}(x) = U^{\tau_2}(x + x^k)$, combining (3.45) and (3.46) and letting $k \to \infty$, we derive

$$U_\infty^{\tau_2}(x) = \lim_{k\to\infty} U^{\tau_2}(x + x^k) = A > 0.$$

Since $\{x_n^k\}$ is bounded, similar to the proof of (3.30), we have (3.36) for any $k \in \mathbb{N}$. This is impossible due to the boundedness of $u_\infty(x)$. This proves (3.43) which contradicts the definition of τ_0. Hence, we obtain $\tau_0 = 0$. So (3.37) is correct.

Step 3. We will show that u is strictly increasing with respect to x_n and $u(x)$ depends on x_n only.

We already have

$$U^\tau(x) \le 0, \quad x \in \mathbb{R}^n, \quad \forall \tau > 0. \tag{3.47}$$

Now we claim that

$$U^\tau(x) < 0, \quad x \in \mathbb{R}^n, \quad \forall \tau > 0. \tag{3.48}$$

Otherwise, from (3.47) for some $\tau_3 > 0$ there exists $x^0 \in \mathbb{R}^n$ such that $U^{\tau_3}(x^0) = 0$, then x^0 is the maximum point of U^{τ_3} in $\mathbb{R}^n$. On one hand, since $\nabla U^{\tau_3}(x^0) = 0$ we have

$$\begin{aligned}
&(-\Delta)_p^s u(x^0) - (-\Delta)_p^s u^{\tau_3}(x^0) \\
&\quad = f(x^0, u(x^0), \nabla u(x^0)) - f((x^0)', x_n^0 + \tau_3, u^{\tau_3}(x^0), \nabla u^{\tau_3}(x^0)) \\
&\quad \le f((x^0)', x_n^0 + \tau_3, u(x^0), \nabla u(x^0)) - f((x^0)', x_n^0 + \tau_3, u^{\tau_3}(x^0), \nabla u^{\tau_3}(x^0)) = 0.
\end{aligned}$$

On the other hand, similar to (2.22), if $U^{\tau_3}(y) \not\equiv 0$ in $\mathbb{R}^n$, we have

$$(-\Delta)_p^s u(x^0) - (-\Delta)_p^s u^{\tau_3}(x^0) > 0.$$

This is a contradiction. Hence, (3.48) must be true.

Next we claim that $u(x)$ depends only on x_n. In fact, it can be seen from the above process that the argument still holds if we replace $u^\tau(x)$ by $u(x + \tau\nu)$, where $\nu = (\nu_1, \ldots, \nu_n)$ with $\nu_n > 0$ being an arbitrary vector pointing upward. Applying the similar arguments as in Steps 1 and 2, we can derive that, for each of such ν,

$$u(x + \tau\nu) > u(x), \quad \forall \tau > 0, \quad x \in \mathbb{R}^n.$$

Letting $\nu_n \to 0$, from the continuity of u, we deduce that for arbitrary ν with $\nu_n = 0$,

$$u(x + \tau\nu) \geq u(x).$$

By replacing ν by $-\nu$, we obtain that

$$u(x + \tau\nu) = u(x)$$

for arbitrary ν with $\nu = 0$. It implies that u is independent of x', hence $u(x) = u(x_n)$.

The proof of Theorem 1.2 is completed. □

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