Research Article

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Unicity of meromorphic functions concerning differences and small functions

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Abstract: In this paper, we study the unicity of meromorphic functions concerning differences and small functions and mainly prove two results: 1. Let f be a transcendental entire function of finite order with a Borel exceptional entire small function $a(z)$, and let η be a constant such that $\Delta_\eta^2 f \not\equiv 0$. If $\Delta_\eta^2 f$ and $\Delta_\eta f$ share $\Delta_\eta a$ CM, then $a(z)$ is a constant a and $f(z) = a + Be^{Az}$, where A, B are two nonzero constants; 2. Let f be a transcendental meromorphic function with $\rho_2(f) < 1$, let a_1, a_2 be two distinct small functions of f, let $L(z, f)$ be a linear difference polynomial, and let $a_1 \not\equiv L(z, a_2)$. If $\delta(a_2, f) > 0$, and f and $L(z, f)$ share a_1 and ∞ CM, then $\frac{L(z,f)-a_1}{f-a_1} = c$, for some constant $c \neq 0$. The results improve some results following C. X. Chen and R. R. Zhang [*Uniqueness theorems related difference operators of entire functions*, Chinese Ann. Math. Ser. A **42** (2021), no. 1, 11–22] and R. R. Zhang, C. X. Chen, and Z. B. Huang [*Uniqueness on linear difference polynomials of meromorphic functions*, AIMS Math. **6** (2021), no. 4, 3874–3888].

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1 Introduction and main results

In this paper, we assume that the reader is familiar with the basic notions of Nevanlinna's value distribution theory, see [1–4]. In the following, a meromorphic function always means meromorphic in the whole complex plane.

By $S(r, f)$, we denote any quantity satisfying $S(r, f) = o(T(r, f))$ as $r \to \infty$ possible outside of an exceptional set E with finite logarithmic measure $\int_E \mathrm{d}r/r < \infty$. A meromorphic function a is said to be a small function of f if it satisfies $T(r, a) = S(r, f)$.

Let f be a nonconstant meromorphic function. The order and the hyper-order of f are defined by

$$\rho(f) = \varlimsup_{r\to\infty} \frac{\log^+ T(r, f)}{\log r}$$

and

$$\rho_2(f) = \varlimsup_{r\to\infty} \frac{\log^+ \log^+ T(r, f)}{\log r}.$$

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Let f be a transcendental meromorphic function, and let a be a small function of f. We define

$$\lambda(f-a)=\overline{\lim_{r\to\infty}}\frac{\log^{+}N\left(r,\frac{1}{f-a}\right)}{\log r},$$

$$\delta(a,f)=\underline{\lim_{r\to\infty}}\frac{m\left(r,\frac{1}{f-a}\right)}{T(r,f)}=1-\overline{\lim_{r\to\infty}}\frac{N\left(r,\frac{1}{f-a}\right)}{T(r,f)}.$$

It is clear that $0\le\delta(a,f)\le 1$. If $\delta(a,f)>0$, then a is called a deficient function of f and $\delta(a,f)$ is its deficiency. If a is a constant, then a is called a deficient value of f. In this paper, deficiency possible outside of an exceptional set E with finite logarithmic measure.

If

$$\overline{\lim_{r\to\infty}}\frac{\log^{+}N\left(r,\frac{1}{f-a}\right)}{\log r}<\rho(f),$$

for $\rho(f)>0$; and $N\left(r,\frac{1}{f-a}\right)=O(\log r)$ for $\rho(f)=0$, then a is called a Borel exceptional function of f. If a is a constant, then a is called a Borel exceptional value of f.

Let f and g be two meromorphic functions, and let a be a small function of both f and g. We say that f and g share a small function a CM(IM) if $f-a$ and $g-a$ have the same zeros counting multiplicities (ignoring multiplicities).

Let η be a nonzero finite complex number, and let n be a positive integer. We define the difference operators of f as $\Delta_\eta f(z)=f(z+\eta)-f(z)$ and $\Delta_\eta^n f(z)=\Delta_\eta(\Delta_\eta^{n-1}f(z))$, $n\ge 2$.

Let $\eta_1,\eta_2,\ldots,\eta_n$ be distinct complex numbers, and let $b_i(\not\equiv 0)$ $(i=1,2,\ldots,n)$ be small functions of f. We define the linear difference polynomial of f as follows:

$$L(z,f)=b_1(z)f(z+\eta_1)+b_2(z)f(z+\eta_2)+\cdots+b_n(z)f(z+\eta_n). \tag{1.1}$$

Nevanlinna [4] proved the following famous five-value theorem.

Theorem A. *Let f and g be two nonconstant meromorphic functions, and let a_j $(j=1,2,3,4,5)$ be five distinct values in the extended complex plane. If f and g share a_j $(j=1,2,3,4,5)$ IM, then $f\equiv g$.*

Li and Qiao [5] improved Theorem A as follows:

Theorem B. *Let f and g be two nonconstant meromorphic functions, and let a_j $(j=1,2,3,4,5)$ (one of them can be identically infinite) be five distinct small functions of both f and g. If f and g share a_j $(j=1,2,3,4,5)$ IM, then $f\equiv g$.*

In 1986, Jank et al. [6] proved.

Theorem C. *Let f be a nonconstant entire function, and let a be a nonzero finite complex number. If f, f' and f'' share a CM, then $f\equiv f'$.*

Recently, the uniqueness in difference analogs of meromorphic functions has become a subject of some interests, see [7–18].

Chen et al. [10] and Farissi et al. [11] obtained the difference analog to Theorem C and proved

Theorem D. [11] *Let f be a nonconstant entire function of finite order, let η be a nonzero constant, and let $a(\not\equiv 0)$ be an entire small function of f satisfying $a(z+\eta)=a(z)$, If f, $\Delta_\eta f$ and $\Delta_\eta^2 f$ share a CM, then $f\equiv\Delta_\eta f$.*

In 2021, Chen and Zhang [8] proved.

Theorem E. *Let f be a transcendental entire function of finite order with $\lambda(f-a)<\rho(f)$, where $a(z)$ is an entire small function of $f(z)$ satisfying $\rho(a)<1$, and let η be a nonzero constant such that $\Delta_\eta^2 f \not\equiv 0$. If $\Delta_\eta^2 f$ and $\Delta_\eta f$ share $\Delta_\eta a$ CM, where $\Delta_\eta a$ is a small function of $\Delta_\eta^2 f$, then $f(z)=a(z)+Be^{Az}$, where A, B are two nonzero constants and $a(z)$ is reduced to a constant.*

In [8], the authors pointed out that $\rho(a)<1$ is reasonable. According to the aforementioned theorems, we naturally pose the following problem.

Problem 1. Whether $\rho(a)<1$ can be deleted in Theorem E?

In this paper, we give a positive answer to Problem 1 and prove the following result.

Theorem 1. *Let f be a transcendental entire function of finite order with a Borel exceptional entire small function $a(z)$, and let η be a constant such that $\Delta_\eta^2 f \not\equiv 0$. If $\Delta_\eta^2 f$ and $\Delta_\eta f$ share $\Delta_\eta a$ CM, then $a(z)$ is a constant a and $f(z)=a+Be^{Az}$, where A, B are two nonzero constants.*

Remark. If $\lambda(f-a)<\rho(f)$, then $a(z)$ is a Borel exceptional function of $f(z)$. Hence, Theorem 1 improves and extends Theorem E.

The following example shows that there exists a transcendental entire function f satisfying Theorem 1.

Example 1. [8] Suppose $f=e^{z\ln 2}+1$, then it is easy to obtain 1 is a Borel exceptional value of f. Let $\eta=1$, we obtain $\Delta_\eta^2 f \equiv \Delta_\eta f$. Thus, we see $\Delta_\eta^2 f$ and $\Delta_\eta f$ share 0 CM.

In 1996, Brück [19] posed the following conjecture.

Conjecture. *Let f be a nonconstant entire function such that $\rho_2(f)<\infty$, which is not a positive integer. If f and f' share one finite value a* CM, *then*

$$\frac{f'-a}{f-a}=c,$$

for some constant $c\neq 0$.

In 2009, Heittokangas et al. [20] proved the following result.

Theorem F. *Let f be a meromorphic function with $\rho(f)<2$, let η be a nonzero complex number, and let a be a finite complex number. If f and $f(z+\eta)$ share a and ∞ CM, then*

$$\frac{f(z+\eta)-a}{f(z)-a}=c,$$

for some constant $c\neq 0$.

In 2021, Zhang et al. [18] proved

Theorem G. *Let f be a transcendental meromorphic function with $\rho_2(f)<1$, let a_1, a_2 be two distinct small functions of f satisfying $\rho(a_j)<1$ $(j=1,2)$, and let $L(z,f)$ be a linear difference polynomial of the form* (1.1) *with $\rho(b_i)<1$ $(i=1,2,\ldots,n)$ and $a_1\not\equiv L(z,a_2)$. If $\delta(a_2,f)>0$, and f and $L(z,f)$ share a_1 and ∞ CM, then*

$$\frac{L(z,f)-a_1}{f-a_1}=c,$$

for some constant c. In particular, if the deficient function $a_2\equiv 0$, then $L(z,f)\equiv f$.

Naturally, we pose the following problem.

Problem 2. Whether $\rho(a_j) < 1$ $(j = 1, 2)$, $\rho(b_i) < 1$ $(i = 1, 2, \ldots, n)$ can be deleted or not in Theorem G?

In this paper, we give a positive answer to Problem 2 and prove the following result.

Theorem 2. *Let f be a transcendental meromorphic function with $\rho_2(f) < 1$, let a_1, a_2 be two distinct small functions of f, let $L(z,f)$ be a linear difference polynomial of the form* (1.1), *and let $a_1 \not\equiv L(z, a_2)$. If $\delta(a_2, f) > 0$, and f and $L(z,f)$ share a_1 and ∞ CM, then*

$$\frac{L(z,f) - a_1}{f - a_1} = c,$$

for some constant $c \neq 0$. In particular, if the deficient function $a_2 \equiv 0$, then $L(z,f) \equiv f$.

The following example shows that there exists a transcendental meromorphic function f with $\rho_2(f) < 1$ satisfying Theorem 2.

Example 2. [18] Let $f = e^{\pi iz} + 6$, and let $L(z,f) = \Delta_1 f = -2e^{\pi iz}$. Then, we have $L(z,f)$ and f share $4, \infty$ CM and $\delta(6,f) = 1 > 0$. Thus,

$$\frac{L(z,f) - 4}{f - 4} = -2.$$

2 Lemmas

In order to prove our results, we need the following lemmas.

Lemma 1. [13] *Let f be a nonconstant entire function of finite order. If a is a Borel exceptional entire small function of f, then $\delta(a,f) = 1$.*

Lemma 2. [21–23] *Let f be a nonconstant mermorphic function with $\rho_2(f) < 1$, and let η be a nonzero finite complex number. Then*

$$m\left(r, \frac{f(z+\eta)}{f(z)}\right) = S(r,f).$$

If f is of finite order, then for any $\varepsilon > 0$, we have

$$m\left(r, \frac{f(z+\eta)}{f(z)}\right) = O(r^{\rho(f)-1+\varepsilon}).$$

Lemma 3. [7] *Let a be a finite complex number, let f be a transcendental meromorphic function of finite order with two Borel exceptional values a and ∞, and let η be a nonzero constant such that $\Delta_\eta f \not\equiv 0$. If f and $\Delta_\eta f$ share a, ∞ CM, then $a = 0$, $f(z) = e^{Az+B}$, where $A(\neq 0)$* and *B are two constants.*

Lemma 4. [21] *Let f be a nonconstant meromorphic function of finite order, and let η be a nonzero finite complex number. Then*

$$N(r, f(z+\eta)) = N(r, f(z)) + S(r,f).$$

Lemma 5. *Let η be a nonzero finite complex number, let n be a positive integer, and let f be a transcendental meromorphic function of finite order satisfying $\delta(a,f)=1$, $\delta(\infty,f)=1$, where a is a small function of f. If $\Delta_\eta^n f \not\equiv 0$, then*

(1) $T(r,\Delta_\eta^n f) = T(r,f) + S(r,f)$;

(2) $\delta(\Delta_\eta^n a, \Delta_\eta^n f) = \delta(\infty, \Delta_\eta^n f) = 1$.

Proof. By Lemma 2 and Nevanlinna's first fundamental theorem, we have

$$\begin{aligned} m\left(r,\frac{1}{f-a}\right) &= m\left(r,\frac{\Delta_\eta^n(f-a)}{f-a}\right) + m\left(r,\frac{1}{\Delta_\eta^n(f-a)}\right) \\ &\le m\left(r,\frac{1}{\Delta_\eta^n(f-a)}\right) + S(r,f) \le T\left(r,\frac{1}{\Delta_\eta^n(f-a)}\right) + S(r,f) \\ &\le T(r,\Delta_\eta^n(f-a)) + S(r,f) \le T(r,\Delta_\eta^n f) + S(r,f). \end{aligned} \tag{2.1}$$

It follows from (2.1) and Lemmas 2 and 4 that

$$\begin{aligned} &\frac{m\left(r,\frac{1}{f-a}\right)}{T(r,f)} \le \frac{T(r,\Delta_\eta^n f)}{T(r,f)} + \frac{S(r,f)}{T(r,f)}, \\ 1 = \delta(a,f) &\le \varliminf_{r\to\infty} \frac{T(r,\Delta_\eta^n f)}{T(r,f)} + \varlimsup_{r\to\infty} \frac{S(r,f)}{T(r,f)} \\ &\le \varliminf_{r\to\infty} \frac{T(r,\Delta_\eta^n f)}{T(r,f)} \le \varlimsup_{r\to\infty} \frac{T(r,\Delta_\eta^n f)}{T(r,f)} \\ &\le \varlimsup_{r\to\infty} \frac{m(r,\Delta_\eta^n f) + N(r,\Delta_\eta^n f)}{T(r,f)} \\ &\le \varlimsup_{r\to\infty} \frac{m\left(r,\frac{\Delta_\eta^n f}{f}\right) + m(r,f) + (n+1)N(r,f)}{T(r,f)} \\ &\le \varlimsup_{r\to\infty} \frac{T(r,f) + nN(r,f) + S(r,f)}{T(r,f)} \\ &\le 1 + \varlimsup_{r\to\infty} n\frac{N(r,f)}{T(r,f)} + \varlimsup_{r\to\infty} \frac{S(r,f)}{T(r,f)} = 1. \end{aligned}$$

Then we have $T(r,\Delta_\eta^n f) = T(r,f) + S(r,f)$.

By (2.1) and $\lim_{r\to\infty}\frac{T(r,\Delta_\eta^n f)}{T(r,f)} = 1$, we obtain

$$\begin{aligned} 1 = \delta(a,f) &\le \varliminf_{r\to\infty} \frac{m\left(r,\frac{1}{\Delta_\eta^n(f-a)}\right)}{T(r,\Delta_\eta^n f)} \cdot \varlimsup_{r\to\infty} \frac{T(r,\Delta_\eta^n f)}{T(r,f)} + \varlimsup_{r\to\infty} \frac{S(r,f)}{T(r,f)} \\ &\le \varliminf_{r\to\infty} \frac{m\left(r,\frac{1}{\Delta_\eta^n(f-a)}\right)}{T(r,\Delta_\eta^n f)} = \delta(\Delta_\eta^n a, \Delta_\eta^n f) \le 1. \end{aligned}$$

It follows that $\delta(\Delta_\eta^n a, \Delta_\eta^n f) = 1$.

Combining $\delta(\infty,f)=1$, $N(r,\Delta_\eta^n f) \le (n+1)N(r,f)$ with $\lim_{r\to\infty}\frac{T(r,\Delta_\eta^n f)}{T(r,f)} = 1$, we obtain $\delta(\infty,\Delta_\eta^n f) = 1$. □

Lemma 6. *Let f be a meromorphic function of finite order, and let η, c, d be three nonzero finite complex numbers. If $f(z+\eta) = cf(z)$, then either $T(r,f) \ge \mathrm{d}r$ for sufficiently large r or f is a constant.*

Proof. In the following, we consider three cases.

Case 1. There exists z_0 such that $f(z_0) = \infty$. Without loss of generality, we assume that $z_0 = 0$, and then we deduce that for all positive integers j, $f(j\eta) = \infty$. Thus, for sufficiently large r and $2n|\eta| \le r < (2n+1)|\eta|$, we have

$$\begin{aligned} T(r,f) \ge N(r,f) &= \int_0^r \frac{n(t,f) - n(0,f)}{t} \mathrm{d}t + n(0,f)\log r \\ &\ge \sum_{j=1}^{2n-1} j \int_{j|\eta|}^{(j+1)|\eta|} \frac{\mathrm{d}t}{t} = \sum_{j=1}^{2n-1} j \log\left(1 + \frac{1}{j}\right) \\ &\ge \sum_{j=1}^{2n-1} j \log\left(1 + \frac{1}{2n-1}\right) = n \log\left(1 + \frac{1}{2n-1}\right)^{2n-1} \\ &\ge n\log 2 > \frac{\log 2}{4|\eta|} r. \end{aligned}$$

It follows that $T(r,f) \ge \mathrm{d}r$, where $d = \frac{\log 2}{4\,|\,\eta\,|}$.

Case 2. There exists z_0 such that $f(z_0) = 0$ and $f \not\equiv 0$. Set $g = \frac{1}{f}$. Then by $f(z+\eta) = cf(z)$, we obtain that $g(z+\eta) = \frac{1}{c}g(z)$ and $g(z_0) = \infty$. Thus, by the proof of Case 1, we deduce that $T(r,f) \ge \mathrm{d}r$.

Case 3. $f \ne 0, \infty$. Since f is of finite order, then $f = e^p$, where p is a polynomial. If $\deg p \ge 1$, then $T(r,f) \ge \mathrm{d}r$; if $\deg p = 0$, then f is a nonzero constant. □

3 Proof of Theorem 1

Firstly, we prove $\rho(f) > 0$. Suppose on the contrary that $\rho(f) = 0$.

Set $F(z) = f(z) - a(z)$. Since $a(z)$ is a Borel exceptional entire small function of $f(z)$, we obtain

$$N\left(r, \frac{1}{F}\right) = N\left(r, \frac{1}{f-a}\right) = O(\log r).$$

Hence, F has finitely many zeros. Thus, we assume that $z_1, z_2, \ldots, z_n$ are zeros of F, where n is a positive integer.

Hence, by $\rho(f) = 0$, we deduce that $\frac{F}{(z-z_1)(z-z_2)\cdots(z-z_n)} = e^p$, where p is a constant.

It follows that $F(z) = c(z-z_1)(z-z_2)\cdots(z-z_n)$, where c is a nonzero constant. Thus, we have

$$T(r,F) = n\log r + O(1).$$

Since $T(r,a) = S(r,F)$, we obtain that $a(z)$ is a constant and $f(z)$ is a nonconstant polynomial, which contradicts with $\Delta_\eta^2 f$ and $\Delta_\eta f$ share $\Delta_\eta a$ CM. It follows $\rho(f) > 0$.

Obviously, $\delta(\infty, f) = 1$. Since $a(z)$ is a Borel exceptional entire small function of $f(z)$, then by Lemma 1, we obtain $\delta(a,f) = 1$.

By Lemma 5, we obtain

$$\delta(\Delta_\eta a, \Delta_\eta f) = 1, \quad \delta(\Delta_\eta^2 a, \Delta_\eta^2 f) = 1, \tag{3.1}$$

$$\delta(\infty, \Delta_\eta f) = 1, \quad \delta(\infty, \Delta_\eta^2 f) = 1. \tag{3.2}$$

We claim that $\Delta_\eta a \equiv \Delta_\eta^2 a$. Otherwise, since $\Delta_\eta^2 f$ and $\Delta_\eta f$ share $\Delta_\eta a$ CM, then by Nevanlinna's second fundamental theorem and Lemma 5 and (3.1), we have

$$\begin{aligned}T(r,f)&=T(r,\Delta_\eta^2 f)+S(r,f)\\&\le \overline{N}(r,\Delta_\eta^2 f)+\overline{N}\left(r,\frac{1}{\Delta_\eta^2 f-\Delta_\eta a}\right)+\overline{N}\left(r,\frac{1}{\Delta_\eta^2 f-\Delta_\eta^2 a}\right)+S(r,f)\\&\le \overline{N}\left(r,\frac{1}{\Delta_\eta f-\Delta_\eta a}\right)+S(r,f)\le S(r,f),\end{aligned}$$

a contradiction.

Obviously, $\delta(0,F)=\delta(a,f)=1$, $\delta(\infty,F)=1$. Since f is a transcendental entire function and $\Delta_\eta^2 f$ and $\Delta_\eta f$ share $\Delta_\eta a$ CM , we have $\Delta_\eta^2 F$ and $\Delta_\eta F$ share $0,\infty$ CM.

It follows from (3.1) and (3.2) that

$$\delta(0,\Delta_\eta F)=1,\quad \delta(0,\Delta_\eta^2 F)=1, \tag{3.3}$$

$$\delta(\infty,\Delta_\eta F)=1,\quad \delta(\infty,\Delta_\eta^2 F)=1. \tag{3.4}$$

Set

$$G=\Delta_\eta F.$$

Since $\Delta_\eta^2 F$ and $\Delta_\eta F$ share $0,\infty$ CM, we obtain $\Delta_\eta G$ and G share $0,\infty$ CM. By (3.3), (3.4), we obtain

$$\delta(0,G)=1,\quad \delta(0,\Delta_\eta G)=1, \tag{3.5}$$

$$\delta(\infty,G)=1,\quad \delta(\infty,\Delta_\eta G)=1. \tag{3.6}$$

By Lemma 5, we have

$$T(r,G)=T(r,f)+S(r,f). \tag{3.7}$$

Since $a(z)$ is a Borel exceptional entire small function of $f(z)$, we obtain $\lambda(f-a)<\rho(f)$. It follows that

$$\overline{\lim_{r\to\infty}}\frac{\log^+ N\left(r,\frac{1}{F}\right)}{\log r}=\overline{\lim_{r\to\infty}}\frac{\log^+ N\left(r,\frac{1}{f-a}\right)}{\log r}<\rho(f). \tag{3.8}$$

By Nevanlinna's first fundamental theorem, we have

$$\begin{aligned}m\left(r,\frac{1}{F}\right)&\le m\left(r,\frac{1}{\Delta_\eta F}\right)+m\left(r,\frac{\Delta_\eta F}{F}\right),\\T(r,F)-N\left(r,\frac{1}{F}\right)&\le T(r,\Delta_\eta F)-N\left(r,\frac{1}{\Delta_\eta F}\right)+S(r,F),\\N\left(r,\frac{1}{\Delta_\eta F}\right)&\le N\left(r,\frac{1}{F}\right)+T(r,\Delta_\eta F)-T(r,F)+S(r,F)\\&\le N\left(r,\frac{1}{F}\right)+m\left(r,\frac{\Delta_\eta F}{F}\right)+m(r,F)-m(r,F)+S(r,F)\\&\le N\left(r,\frac{1}{F}\right)+S(r,F).\end{aligned} \tag{3.9}$$

By Lemma 2, set $\varepsilon=\frac{1}{2}$, we obtain

$$S(r,F)\le Mr^{\rho(f)-\frac{1}{2}}, \tag{3.10}$$

where M is a positive constant.

It follows from (3.8) that

$$N\left(r,\frac{1}{F}\right)<r^{\frac{\rho(f)+\lambda(F)}{2}}. \tag{3.11}$$

By (3.10) and (3.11), we obtain

$$N\left(r,\frac{1}{F}\right)+S(r,F)<(1+M)r^{M_1}, \tag{3.12}$$

where $M_1=\max\left\{\rho(f)-\frac{1}{2},\frac{\rho(f)+\lambda(F)}{2}\right\}$.

By (3.9) and (3.12), we obtain

$$\frac{\log^+ N\left(r,\frac{1}{\Delta_\eta F}\right)}{\log r}\le\frac{\log(1+M)r^{M_1}}{\log r}\le M_1+\frac{\log(1+M)}{\log r}.$$

Thus, we have

$$\overline{\lim_{r\to\infty}}\frac{\log^+ N\left(r,\frac{1}{G}\right)}{\log r}=\overline{\lim_{r\to\infty}}\frac{\log^+ N\left(r,\frac{1}{\Delta_\eta F}\right)}{\log r}\le M_1<\rho(f). \tag{3.13}$$

It follows from (3.7) and (3.13), we deduce that 0 is a Borel exceptional value of G.

By Lemma 3, we obtain $G(z)=e^{A_1z+B_1}$, where $A_1(\neq 0)$, B_1 are two constants. That is,

$$F(z+\eta)-F(z)=e^{A_1z+B_1}. \tag{3.14}$$

By Hadamard's factorization theorem, we have

$$F(z)=\alpha(z)e^{p_1(z)}, \tag{3.15}$$

where α is an entire function such that $\rho(\alpha)=\lambda(\alpha)<\rho(F)$, and p_1 is a nonconstant polynomial with $\deg p_1=\rho(F)$.

Hence, we obtain

$$T(r,\alpha)=S\left(r,e^{p_1}\right). \tag{3.16}$$

It follows from (3.14) and (3.15) that

$$\alpha(z+\eta)e^{p_1(z+\eta)}-\alpha(z)e^{p_1(z)}=e^{A_1z+B_1}. \tag{3.17}$$

Next, we consider two cases.

Case 1. $\deg p_1\ge 2$. By (3.17), we have

$$\frac{\alpha(z+\eta)}{e^{A_1z+B_1}}e^{p_1(z+\eta)}-\frac{\alpha(z)}{e^{A_1z+B_1}}e^{p_1(z)}\equiv 1. \tag{3.18}$$

Obviously, $T(r,e^{A_1z+B_1})=S(r,e^{p_1})$. It follows from (3.16), (3.18), and Nevanlinna's second fundamental theorem that

$$\begin{aligned}T\left(r,e^{p_1}\right)&\le T\left(r,\frac{\alpha}{e^{A_1z+B_1}}e^{p_1}\right)+S\left(r,e^{p_1}\right)\\&\le\overline{N}\left(r,\frac{\alpha}{e^{A_1z+B_1}}e^{p_1}\right)+\overline{N}\left(r,\frac{1}{\frac{\alpha}{e^{A_1z+B_1}}e^{p_1}}\right)+\overline{N}\left(r,\frac{1}{\frac{\alpha}{e^{A_1z+B_1}}e^{p_1}+1}\right)+S\left(r,\frac{\alpha}{e^{A_1z+B_1}}e^{p_1}\right)\\&\le S\left(r,e^{p_1}\right),\end{aligned}$$

a contradiction.

Case 2. $\deg p_1=1$. Let $p_1(z)=mz+n$, where $m(\neq 0)$ and n are two complex numbers.

Now, we consider two subcases.

Case 2.1. $A_1\neq m$. Thus, by (3.17), we obtain

$$c_1\alpha(z+\eta)e^{(m-A_1)z}+c_2\alpha(z)e^{(m-A_1)z}\equiv 1, \tag{3.19}$$

where $c_1=e^{m\eta+n-B_1}$ and $c_2=-e^{n-B_1}$.

Obviously, $T(r,\alpha) = S(r, e^{(m-A_1)z})$. It follows from (3.19) and Nevanlinna's second fundamental theorem that

$$\begin{aligned} T\left(r, e^{(m-A_1)z}\right) &\le T(r, c_2\alpha e^{(m-A_1)z}) + S(r, e^{(m-A_1)z}) \\ &\le \overline{N}(r, c_2\alpha e^{(m-A_1)z}) + \overline{N}\left(r, \frac{1}{c_2\alpha e^{(m-A_1)z}}\right) + \overline{N}\left(r, \frac{1}{c_2\alpha e^{(m-A_1)z} - 1}\right) + S(r, c_2\alpha e^{(m-A_1)z}) \\ &\le S(r, e^{(m-A_1)z}), \end{aligned}$$

a contradiction.

Case 2.2. $A_1 = m$. Thus, by (3.17), we obtain

$$c_1\alpha(z+\eta) + c_2\alpha(z) \equiv 1, \tag{3.20}$$

where $c_1 = e^{m\eta+n-B_1}$, $c_2 = -e^{n-B_1}$.

Next, we consider two subcases.

Case 2.2.1. $c_1 + c_2 = 0$. Hence,

$$e^{m\eta+n-B_1} - e^{n-B_1} = e^{n-B_1}(e^{m\eta} - 1) = 0.$$

It follows $e^{m\eta} = 1$.

By Lemma 6 and $\rho(\alpha) < \rho(F) = 1$, we deduce

$$\alpha = \frac{z}{\eta c_1} + c_3,$$

where c_3 is a constant.

Hence, $f(z) = a(z) + \left(\frac{z}{\eta c_1} + c_3\right)e^{mz+n}$. It follows

$$T(r,f) = T\left(r, a + \left(\frac{z}{\eta c_1} + c_3\right)e^{mz+n}\right) \le T(r, e^{mz+n}) + S(r,f) \le \frac{m}{\pi}r + S(r,f). \tag{3.21}$$

Since $\Delta_\eta a \equiv \Delta_\eta^2 a$, then $b \equiv \Delta_\eta b$, where $b = \Delta_\eta a$. It follows $b(z+\eta) \equiv 2b(z)$. By Lemma 6, we deduce that either $T(r,b) \ge \mathrm{d}r$ or $b(z)$ is a constant. If $T(r,b) \ge \mathrm{d}r$, by (3.21), we know that $b(z)$ is not a small function of $f(z)$, a contradiction. Then $b(z)$ is a constant, obviously $b(z) \equiv 0$. It follows $a(z+\eta) \equiv a(z)$. By Lemma 6 and (3.21), we deduce that $a(z)$ is a constant.

Thus, we have

$$\begin{aligned} \Delta_\eta f &= \left(\frac{z+\eta}{\eta c_1} + c_3\right)e^{m(z+\eta)+n} - \left(\frac{z}{\eta c_1} + c_3\right)e^{mz+n} \\ &= \left(\frac{z}{\eta c_1} + c_3 + \frac{1}{c_1}\right)e^{mz+n}e^{m\eta} - \left(\frac{z}{\eta c_1} + c_3\right)e^{mz+n} \\ &= \frac{1}{c_1}e^{mz+n}, \end{aligned}$$

and

$$\Delta_\eta^2 f = \Delta_\eta(\Delta_\eta f) = \frac{1}{c_1}e^{m(z+\eta)+n} - \frac{1}{c_1}e^{mz+n} = \frac{1}{c_1}e^{mz+n}e^{m\eta} - \frac{1}{c_1}e^{mz+n} = 0.$$

This contradicts with $\Delta_\eta^2 f \not\equiv 0$. Hence, this case cannot occur.

Case 2.2.2. $c_1 + c_2 \ne 0$.

By Lemma 6 and $\rho(\alpha) < \rho(F) = 1$, we deduce

$$\alpha = c,$$

where c is a constant.

It follows that $f(z) = a(z) + ce^{mz+n}$. Obviously, $c \neq 0$, we obtain

$$T(r,f) = T(r, a + ce^{mz+n}) \leq T(r, e^{mz+n}) + S(r,f) \leq \frac{m}{\pi}r + S(r,f). \tag{3.22}$$

Since $\Delta_\eta a \equiv \Delta_\eta^2 a$, by (3.22) and using the same argument as used in case 2.2.1, we can prove that $a(z)$ is a constant a. Therefore, we have $f(z) = a + Be^{Az}$, where A, B are nonzero constants.

This completes the proof of Theorem 1.

4 Proof of Theorem 2

Since f and $L(z,f)$ share a_1 and ∞ CM, we obtain

$$\frac{L(z,f) - a_1(z)}{f(z) - a_1(z)} = h(z), \tag{4.1}$$

where h is a meromorphic function satisfying $N(r,h) = S(r,f)$, $N\left(r, \frac{1}{h}\right) = S(r,f)$.

It follows from (4.1) that

$$\frac{1}{a_1 - L(z,a_2) - (a_1 - a_2)h}\left(\frac{L(z, f - a_2)}{f - a_2} - h\right) = \frac{1}{f - a_2}. \tag{4.2}$$

By Lemma 2 and Nevanlinna's first fundamental theorem, we have

$$\begin{aligned} T(r,h) &= m(r,h) + S(r,f) \\ &= m\left(r, \frac{L(z,f) - a_1}{f - a_1}\right) + S(r,f) \\ &\leq m\left(r, \frac{L(z, f - a_1)}{f - a_1}\right) + m\left(r, \frac{L(z,a_1) - a_1}{f - a_1}\right) + S(r,f) \\ &\leq m\left(r, \frac{1}{f - a_1}\right) + S(r,f) \leq T(r,f) + S(r,f). \end{aligned}$$

It follows

$$S(r,h) = S(r,f). \tag{4.3}$$

Since $\delta(a_2, f) > 0$, we deduce that $m\left(r, \frac{1}{f - a_2}\right) \geq c_1 T(r,f)$ for sufficiently large r, where c_1 is some positive constant. Then, by (4.2), we have

$$\begin{aligned} T(r,f) &\leq \frac{1}{c_1} m\left(r, \frac{1}{f - a_2}\right) \\ &= \frac{1}{c_1} m\left(r, \frac{1}{a_1 - L(z,a_2) - (a_1 - a_2)h}\left(\frac{L(z, f - a_2)}{f - a_2} - h\right)\right) \\ &\leq \frac{1}{c_1} m\left(r, \frac{1}{a_1 - L(z,a_2) - (a_1 - a_2)h}\right) + \frac{1}{c_1} m\left(r, \frac{L(z, f - a_2)}{f - a_2}\right) + \frac{1}{c_1} m(r,h) + S(r,f) \\ &\leq \frac{2}{c_1} T(r,h) + S(r,f). \end{aligned}$$

It follows

$$S(r,f) = S(r,h). \tag{4.4}$$

Since $a_1(z) \not\equiv a_2(z)$, we have

$$\begin{aligned} N\left(r, \frac{1}{\frac{a_1 - L(z,a_2)}{a_1 - a_2} - h}\right) &= N\left(r, \frac{a_1 - a_2}{a_1 - L(z, a_2) - (a_1 - a_2)h}\right) \\ &\le N\left(r, \frac{1}{a_1 - L(z, a_2) - (a_1 - a_2)h}\right) + S(r, f) \\ &= N\left(r, \frac{1}{a_1 - a_2}\frac{1}{\frac{a_1 - L(z,a_2)}{a_1 - a_2} - h}\right) + S(r, f) \\ &\le N\left(r, \frac{1}{\frac{a_1 - L(z,a_2)}{a_1 - a_2} - h}\right) + S(r, f). \end{aligned}$$

Thus, we have

$$N\left(r, \frac{1}{a_1 - L(z, a_2) - (a_1 - a_2)h}\right) = N\left(r, \frac{1}{\frac{a_1 - L(z,a_2)}{a_1 - a_2} - h}\right) + S(r, f). \tag{4.5}$$

It follows from (4.3), (4.4), $a_1 \not\equiv L(z, a_2)$, and Nevanlinna's second fundamental theorem that

$$\begin{aligned} T(r, h) &\le \overline{N}(r, h) + \overline{N}\left(r, \frac{1}{h}\right) + \overline{N}\left(r, \frac{1}{h - \frac{a_1 - L(z,a_2)}{a_1 - a_2}}\right) + S(r, f) \\ &\le \overline{N}\left(r, \frac{1}{h - \frac{a_1 - L(z,a_2)}{a_1 - a_2}}\right) + S(r, f) \le T(r, h) + S(r, f). \end{aligned} \tag{4.6}$$

By (4.5) and (4.6), we obtain

$$N\left(r, \frac{1}{a_1 - L(z, a_2) - (a_1 - a_2)h}\right) = T(r, h) + S(r, f).$$

It follows

$$m\left(r, \frac{1}{a_1 - L(z, a_2) - (a_1 - a_2)h}\right) = S(r, f). \tag{4.7}$$

By (4.7), we have

$$\begin{aligned} & m\left(r, \frac{h}{a_1 - L(z, a_2) - (a_1 - a_2)h}\right) \\ &= m\left(r, \frac{1}{a_2 - a_1} + \frac{L(z, a_2) - a_1}{a_2 - a_1}\frac{1}{a_1 - L(z, a_2) - (a_1 - a_2)h}\right) \\ &\le m\left(r, \frac{1}{a_2 - a_1}\right) + m\left(r, \frac{L(z, a_2) - a_1}{a_2 - a_1}\right) + m\left(r, \frac{1}{a_1 - L(z, a_2) - (a_1 - a_2)h}\right) \\ &\le S(r, f). \end{aligned} \tag{4.8}$$

It follows from (4.2), (4.7), (4.8), and Lemma 2 that

$$\begin{aligned} m\left(r, \frac{1}{f - a_2}\right) &= m\left(r, \frac{1}{a_1 - L(z, a_2) - (a_1 - a_2)h}\left(\frac{L(z, f - a_2)}{f - a_2} - h\right)\right) \\ &\le m\left(r, \frac{1}{a_1 - L(z, a_2) - (a_1 - a_2)h}\frac{L(z, f - a_2)}{f - a_2}\right) + m\left(r, \frac{h}{a_1 - L(z, a_2) - (a_1 - a_2)h}\right) \end{aligned}$$

$$\leq m\left(r, \frac{1}{a_1 - L(z, a_2) - (a_1 - a_2)h}\right) + m\left(r, \frac{L(z, f - a_2)}{f - a_2}\right) + m\left(r, \frac{h}{a_1 - L(z, a_2) - (a_1 - a_2)h}\right)$$
$$\leq S(r, f),$$

which contradicts with $\delta(a_2, f) > 0$. Hence, h is a constant c. That is,

$$\frac{L(z, f) - a_1}{f - a_1} = c,$$

obviously $c \neq 0$.

Next, we consider the case: $a_2 \equiv 0$. Then, by (4.2) and $h = c$, we have

$$\frac{1}{a_1(1 - c)}\left(\frac{L(z, f)}{f} - c\right) = \frac{1}{f}.$$

We claim that $c = 1$. Suppose on the contrary that $c \neq 1$, then we obtain

$$m\left(r, \frac{1}{f}\right) = m\left(r, \frac{1}{a_1(1 - c)}\left(\frac{L(z, f)}{f} - c\right)\right) \leq S(r, f),$$

which contradicts with $\delta(0, f) > 0$. Hence, $c = 1$. That is, $L(z, f) \equiv f$.

Thus, Theorem 2 is proved.

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