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Research Article

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An efficient finite element method based on dimension reduction scheme for a fourth-order Steklov eigenvalue problem

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Abstract: In this article, an effective finite element method based on dimension reduction scheme is proposed for a fourth-order Steklov eigenvalue problem in a circular domain. By using the Fourier basis function expansion and variable separation technique, the original problem is transformed into a series of radial one-dimensional eigenvalue problems with boundary eigenvalue. Then we introduce essential polar conditions and establish the discrete variational form for each radial one-dimensional eigenvalue problem. Based on the minimax principle and the approximation property of the interpolation operator, we prove the error estimates of approximation eigenvalues. Finally, some numerical experiments are provided, and the numerical results show the efficiency of the proposed algorithm.

Keywords: fourth-order Steklov eigenvalue problem, weighted Sobolev space, dimension reduction scheme, finite element approximation, error estimation

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1 Introduction

Fourth-order Steklov eigenvalue problems with eigenvalue parameter in boundary conditions are widely used in mathematics and physics, such as the surface wave research, the stability analysis of mechanical oscillator in a viscous fluid, the study of vibration mode of structure in contact with an incompressible fluid, and so on [1–5]. The first eigenvalue λ_1 also plays an important role in the positivity-preserving properties for the biharmonic-operator Δ^2 under the boundary conditions $\phi|_{\partial\Omega} = (\Delta\phi - \lambda\phi_\nu)|_{\partial\Omega} = 0$ in [6,7].

There are many existing results about the fourth-order Steklov eigenvalue problems, but they mainly focus on the qualitative analysis. Kuttler [8] proved that the first eigenvalue is simple and the corresponding eigenfunction does not change the sign. Ferrero et al. [7] and Bucur et al. [9] studied the spectrum on a bounded domain, and the explicit representation of the spectrum is given when the domain is a ball. Recently, the existence of an optimal convex shape among domains of a given measure is proved in [10], and the Weyl-type asymptotic formula for the counting function of the biharmonic Steklov eigenvalues also is established in [11]. For the numerical methods of the fourth-order Steklov eigenvalue problems, a conforming finite element method was first proposed in [12], then some spectral methods are also developed [13].

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As we all know, if the conforming finite element method is directly used to solve a fourth-order problem, the boundary of the element requires the continuity of the first derivative, which not only brings the difficulty of constructing the basis function but also costs a lot of calculation time and memory capacity, especially for some special regions, such as circular region, spherical region, and so on. How to efficiently solve a fourth-order Steklov eigenvalue problem in a circular domain? To the best of our knowledge, there are few reports on using some efficient numerical to solve this problem. Thus, the aim of this article is to propose an effective finite element method based on a dimension reduction scheme for a fourth-order Steklov eigenvalue problem in a circular domain. By using the Fourier basis function expansion and variable separation technique, the original problem is transformed into a series of radial one-dimensional eigenvalue problems with boundary eigenvalue. Then we introduce essential polar conditions and establish the discrete variational form for each radial one-dimensional eigenvalue problem. Based on the minimax principle and the approximation property of the interpolation operator, we prove the error estimates of approximation eigenvalues. Finally, some numerical experiments are provided, and the numerical results show the efficiency of the proposed algorithm.

This article is organized as follows. In Section 2, a reduced scheme based on polar coordinate transformation is presented. In Section 3, the weighted space and discrete variational form are derived. In Section 4, the error estimation of approximation solutions is proved. In Section 5, we present the process of effective implementation of the algorithm. We present some numerical experiments in Section 6 to illustrate the accuracy and efficiency of our proposed algorithm. Finally, we give in Section 7 some concluding remarks.

2 Reduced scheme based on polar coordinate transformation

The fourth-order Steklov eigenvalue problems read:

$$\Delta^2\phi = 0, \quad \text{in } \Omega, \tag{2.1}$$

$$\phi = 0, \quad \text{on } \partial\Omega, \tag{2.2}$$

$$\Delta\phi = \lambda\frac{\partial\phi}{\partial\nu}, \quad \text{on } \partial\Omega, \tag{2.3}$$

where $\Omega = \{(x, y) \in \mathbb{R}^2 : 0 < \sqrt{x^2 + y^2} < R\}$, ν is the unit outward normal to the boundary $\partial\Omega$. Let $x = r\cos\theta$, $y = r\sin\theta$, $\psi(r, \theta) = \phi(x, y)$. Then we derive that

$$\Delta\phi(x, y) = L\psi(r, \theta) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi(r, \theta)}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\psi(r, \theta)}{\partial\theta^2}. \tag{2.4}$$

Then the equivalent form of (2.1)–(2.3) in polar coordinates is as follows:

$$L^2\psi(r, \theta) = 0, \quad \text{in } \mathcal{D} = (0, R) \times [0, 2\pi), \tag{2.5}$$

$$\psi(R, \theta) = 0, \quad \theta \in [0, 2\pi), \tag{2.6}$$

$$L\psi(R, \theta) = \lambda\frac{\partial\psi}{\partial r}(R, \theta), \quad \theta \in [0, 2\pi). \tag{2.7}$$

Since $\psi(r, \theta)$ is 2π periodic in θ, then we have

$$\psi(r, \theta) = \sum_{|m|=0}^{\infty}\psi_m(r)\mathrm{e}^{im\theta}. \tag{2.8}$$

Substituting (2.8) into (2.4), we derive that

$$L\psi(r, \theta) = \sum_{|m|=0}^{\infty}\frac{1}{r}\left\{\frac{\partial}{\partial r}\left[r\frac{\partial\psi_m(r)}{\partial r}\right] - \frac{m^2}{r}\psi_m(r)\right\}\mathrm{e}^{im\theta}. \tag{2.9}$$

Following the discussion in [14,15], to overcome the pole singularity introduced by polar coordinate transformation, we need to introduce the essential pole conditions, which make (2.9) meaningful, as follows:

$$m^2\psi_m(0) = 0, \lim_{r\to 0^+}\left\{\frac{\partial}{\partial r}\left[r\frac{\partial\psi_m(r)}{\partial r}\right] - \frac{m^2}{r}\psi_m(r)\right\} = 0. \tag{2.10}$$

Using the fact that $\frac{\partial}{\partial r}\left[r\frac{\partial\psi_m(r)}{\partial r}\right] = \frac{\partial\psi_m(r)}{\partial r} + r\frac{\partial^2\psi_m(r)}{\partial r^2}$, (2.10) can be reduced to

$$m^2\psi_m(0) = 0, \quad (1 - m^2)\frac{\partial\psi_m(0)}{\partial r} = 0. \tag{2.11}$$

From (2.11) we can further obtain that

$$(1) \quad \frac{\partial\psi_m(0)}{\partial r} = 0, \quad (m = 0); \tag{2.12}$$

$$(2) \quad \psi_m(0) = 0, \quad (|m| = 1); \tag{2.13}$$

$$(3) \quad \psi_m(0) = 0, \quad \frac{\partial\psi_m(0)}{\partial r} = 0, \quad (|m| \geq 2). \tag{2.14}$$

Let $r = \frac{t+1}{2}R$, $u_m(t) = \psi_m(r)$, $L_m u_m(t) = \frac{1}{t+1}\frac{\partial}{\partial t}\left[t\frac{\partial u_m(t)}{\partial t}\right] - \frac{m^2}{(t+1)^2}u_m(t)$. From (2.8), (2.11)–(2.14), and the orthogonal properties of Fourier basis functions, (2.5)–(2.7) are equivalent to a series of one-dimensional eigenvalue problems

$$L_m^2 u_m(t) = 0, \quad t \in (-1, 1), \tag{2.15}$$

$$(1) \quad \frac{\partial u_m(-1)}{\partial t} = 0, \quad u_m(1) = 0, \quad L_m u_m(1) = \frac{R}{2}\lambda_m\frac{\partial u_m(1)}{\partial t}, \quad (m = 0); \tag{2.16}$$

$$(2) \quad u_m(-1) = 0, \quad u_m(1) = 0, \quad L_m u_m(1) = \frac{R}{2}\lambda_m\frac{\partial u_m(1)}{\partial t}, \quad (|m| = 1); \tag{2.17}$$

$$(3) \quad u_m(\pm 1) = 0, \quad \frac{\partial u_m(-1)}{\partial t} = 0, \quad L_m u_m(1) = \frac{R}{2}\lambda_m\frac{\partial u_m(1)}{\partial t}, \quad (|m| \geq 2). \tag{2.18}$$

3 Weighted space and discrete variational form

Without losing generality, we only consider the case of $m \geq 0$. First, we divide the solution interval $I = (-1, 1)$ as follows:

$$-1 = t_0 < t_1 < \cdots < t_i < \cdots < t_n = 1.$$

Define the usual weighted Sobolev space:

$$L_\omega^2(I) := \left\{\rho : \int_I \omega\rho^2 \mathrm{d}t < \infty\right\}$$

equipped with the following inner product and norm:

$$(\rho, v)_\omega = \int_I \omega\rho v \mathrm{d}t, \quad \|\rho\|_w = \left(\int_I \omega\rho^2 \mathrm{d}t\right)^{\frac{1}{2}},$$

where $\omega = 1 + t$, $t \in (-1, 1)$. We further introduce the following weighted Sobolev space:

$$H_{0,\omega,m}^2(I) := \left\{u_m : L_m u_m \in L_\omega^2(I), m^2 u_m(-1) = (1 - m^2)\frac{\partial u_m(-1)}{\partial t} = u_m(1) = 0\right\},$$

equipped with the inner product and norm:

$$(u_m, v_m)_{2,\omega,m} = (L_m u_m, L_m v_m)_\omega, \ \|u_m\|_{2,\omega,m} = \sqrt{(u_m, u_m)_{2,\omega,m}}.$$

Then the variational form of (2.15)–(2.18) is: Find $(\lambda_m, u_m \neq 0) \in \mathbb{R} \times H^2_{0,\omega,m}(I)$, such that

$$A_m(u_m, v_m) = \lambda_m B_m(u_m, v_m), \quad \forall v_m \in H^2_{0,\omega,m}(I), \tag{3.1}$$

where

$$A_m(u_m, v_m) = \int\limits_I (t+1) L_m u_m L_m v_m \mathrm{d}t,$$

$$B_m(u_m, v_m) = R \frac{\partial u_m(1)}{\partial t} \frac{\partial v_m(1)}{\partial t}.$$

Let us denote by U_h a piecewise cubic Hermite interpolation function space. Define the approximation space $S_h(m) = U_h \cap H^2_{0,\omega,m}(I)$. Then the discrete variational form associated with (3.1) is: Find $(\lambda_{mh}, u_{mh} \neq 0) \in \mathbb{R} \times S_h(m)$, such that

$$A_m(u_{mh}, v_{mh}) = \lambda_{mh} B_m(u_{mh}, v_{mh}), \quad \forall v_{mh} \in S_h(m). \tag{3.2}$$

4 Error estimation of approximation solutions

For the sake of brevity, we shall use the expression $a \lesssim b$ which denotes $a \leq cb$, where c is a positive constant.

Lemma 1. *For any $u_m, v_m \in H^2_{0,\omega,m}(I)$, the following equalities hold:*

$$\begin{aligned}\int\limits_I (t+1) L_m u_m L_m v_m \mathrm{d}t = \int\limits_I (t+1) u_m'' v_m'' \mathrm{d}t + (2m^2+1) \int\limits_I \frac{1}{t+1} u_m' v_m' \mathrm{d}t \\ + m^2(m^2-4) \int\limits_I \frac{1}{(t+1)^3} u_m v_m \mathrm{d}t + u_m'(1) v_m'(1)\end{aligned} \tag{4.1}$$

with $m \neq 1$, and

$$\begin{aligned}\int\limits_I (t+1) L_m u_m L_m v_m \mathrm{d}t = \int\limits_I (t+1)\left(u_m' - \frac{m}{t+1} u_m\right)'\left(v_m' - \frac{m}{t+1} v_m\right)' \mathrm{d}t \\ + (1+m)^2 \int\limits_I \frac{1}{t+1}\left(u_m' - \frac{m}{t+1} u_m\right)\left(v_m' - \frac{m}{t+1} v_m\right) \mathrm{d}t + (1+m) u_m'(1) v_m'(1)\end{aligned} \tag{4.2}$$

with $m = 1$.

Proof. Using integration by parts, pole conditions, and boundary conditions, we derive that

$$\int\limits_I (u_m'' v_m' + u_m' v_m'') \mathrm{d}x = u_m'(1) v_m'(1), \tag{4.3}$$

$$\int\limits_I \frac{1}{(t+1)^2} (u_m' v_m + u_m v_m') \mathrm{d}t = 2 \int\limits_I \frac{1}{(t+1)^3} u_m v_m \mathrm{d}t, \tag{4.4}$$

$$\int\limits_I \frac{1}{t+1} (u_m'' v_m + u_m v_m'') \mathrm{d}t = 2 \int\limits_I \frac{1}{(t+1)^3} u_m v_m \mathrm{d}t - 2 \int\limits_I \frac{1}{(t+1)} u_m' v_m' \mathrm{d}t. \tag{4.5}$$

Then when $m \neq 1$, we derive from (4.3)–(4.5) that

$$\begin{aligned}\int_I (t+1)L_m u_m L_m v_m \mathrm{d}t &= \int_I \left[(t+1)u_m'' + u_m' - \frac{m^2}{t+1}u_m\right]\left[v_m'' + \frac{v_m'}{t+1} - \frac{m^2}{(t+1)^2}v_m\right]\mathrm{d}t\\
&= \int_I (t+1)u_m''v_m''\mathrm{d}t + \int_I u_m''v_m' + u_m'v_m''\mathrm{d}t - m^2\int_I \frac{1}{(t+1)^2}(u_m'v_m + u_m v_m')\mathrm{d}t\\
&\quad - m^2\int_I \frac{1}{t+1}(u_m''v_m + u_m v_m'')\mathrm{d}t + m^4\int_I \frac{1}{(t+1)^3}u_m v_m \mathrm{d}t + \int_I \frac{1}{(t+1)}u_m'v_m'\mathrm{d}t\\
&= \int_I (t+1)u_m''v_m''\mathrm{d}t + (2m^2+1)\int_I \frac{1}{t+1}u_m'v_m'\mathrm{d}t + m^2(m^2-4)\int_I \frac{1}{(t+1)^3}u_m v_m \mathrm{d}t + u_m'(1)v_m'(1).\end{aligned}$$

When $m = 1$, we have

$$\begin{aligned}\int_I (t+1)L_m u_m L_m v_m \mathrm{d}t &= \int_I (t+1)\left[u_m'' + \frac{u_m'}{t+1} - \frac{m^2}{(t+1)^2}u_m\right]\left[v_m'' + \frac{v_m'}{t+1} - \frac{m^2}{(t+1)^2}v_m\right]\mathrm{d}t\\
&= \int_I (t+1)\left(u_m' - \frac{m}{t+1}u_m\right)'\left(v_m' - \frac{m}{t+1}v_m\right)'\mathrm{d}t + (1+m)^2\int_I \frac{1}{t+1}\left(u_m' - \frac{m}{t+1}u_m\right)\\
&\quad \times \left(v_m' - \frac{m}{t+1}v_m\right)\mathrm{d}t + (1+m)\int_I \left[\left(u_m' - \frac{m}{t+1}u_m\right)\left(v_m' - \frac{m}{t+1}v_m\right)\right]'\mathrm{d}t\\
&= \int_I (t+1)\left(u_m' - \frac{m}{t+1}u_m\right)'\left(v_m' - \frac{m}{t+1}v_m\right)'\mathrm{d}t + (1+m)^2\int_I \frac{1}{t+1}\left(u_m' - \frac{m}{t+1}u_m\right)\\
&\quad \times \left(v_m' - \frac{m}{t+1}v_m\right)\mathrm{d}t + (1+m)u_m'(1)v_m'(1).\end{aligned}$$

□

Theorem 1. *$A_m(u_m, v_m)$ is a bounded and coercive bilinear functional on $H^2_{0,\omega,m}(I) \times H^2_{0,\omega,m}(I)$, i.e.,*

$$|A_m(u_m, v_m)| \lesssim \|u_m\|_{2,\omega,m}\|v_m\|_{2,\omega,m},$$

$$A_m(u_m, u_m) \gtrsim \|u_m\|^2_{2,\omega,m}.$$

Proof. From Cauchy-Schwarz inequality, we derive that

$$\begin{aligned}|A_m(u_m, v_m)| &= \left|\int_I (t+1)L_m u_m L_m v_m \mathrm{d}t\right|\\
&\leq \left[\int_I (t+1)|L_m u_m|^2\mathrm{d}t\right]^{\frac{1}{2}}\left[\int_I (t+1)|L_m v_m|^2\mathrm{d}t\right]^{\frac{1}{2}}\\
&\lesssim \|u_m\|_{2,\omega,m}\|v\|_{2,\omega,m},\\
A_m(u_m, u_m) &= \int_I (t+1)|L_m u_m|^2\mathrm{d}t \gtrsim \|u_m\|^2_{2,\omega,m}.\end{aligned}$$

□

Lemma 3.2. *Let λ_m^l be the* lth *eigenvalue of the variational form* (3.1). *Use V_l to denote any l-dimensional subspace of $H^2_{0,\omega,m}(I)$. For $\lambda_m^1 \leq \lambda_m^2 \leq \cdots \leq \lambda_m^l \leq \cdots$, it holds*

$$\lambda_m^l = \min_{V_l \subset H^2_{0,\omega,m}(I)} \max_{v_m \in V_l} \frac{A_m(v_m, v_m)}{B_m(v_m, v_m)}. \tag{4.6}$$

Proof. See Theorem 3.1 in [17]. □

Lemma 3.3. *Let λ_m^i be the eigenvalue of the variational form* (3.1) *and be arranged in the ascending order. Define*

$$W_{i,j} = \operatorname{span}\{u_m^i, \dots, u_m^j\},$$

where u_m^i is the eigenfunction associated with λ_m^i. Then there hold

$$\lambda_m^l = \max_{v_m \in W_{k,l}} \frac{A_m(v_m, v_m)}{B_m(v_m, v_m)} \quad k \le l, \tag{4.7}$$

$$\lambda_m^l = \min_{v_m \in W_{l,n}} \frac{A_m(v_m, v_m)}{B_m(v_m, v_m)} \quad l \le n. \tag{4.8}$$

Proof. See Lemma 3.2 in [17]. □

For the discrete form (3.2), the following minimax principle is also effective (see [17]).

Lemma 3.4. *Let λ_{mh}^l be the eigenvalue of the discrete variational form* (3.2). *Use V_{lh} to denote any l-dimensional subspace of $S_h(m)$. For $\lambda_{mh}^1 \le \lambda_{mh}^2 \le \dots \le \lambda_{mh}^l \le \cdots$, it holds*

$$\lambda_{mh}^l = \min_{V_{hl} \subset S_h(m)} \max_{v_m \in V_{hl}} \frac{A_m(v_m, v_m)}{B_m(v_m, v_m)}. \tag{4.9}$$

Define an orthogonal projection $Q_h^{2,m} : H_{0,\omega,m}^2(I) \to S_h(m)$ by

$$A_m(u_m - Q_h^{2,m} u_m, v_{mh}) = 0, \quad \forall v_{mh} \in S_h(m).$$

Theorem 2. *Let λ_{mh}^l be the approximation solution of λ_m^l. Then it holds*

$$0 < \lambda_m^l \le \lambda_{mh}^l \le \lambda_m^l \max_{v_m \in W_{1,l}} \frac{B_m(v_m, v_m)}{B_m(Q_h^{2,m} v_m, Q_h^{2,m} v_m)}. \tag{4.10}$$

Proof. According to the positive definite property of $A_m(u_m, v_m)$ and $B_m(u_m, v_m)$ we derive that $\lambda_m^l > 0$. Since $S_h(m) \subset H_{0,\omega,m}^2(I)$, then from (4.6) and (4.9) we obtain $\lambda_m^l \le \lambda_{mh}^l$. Let $Q_h^{2,m} W_{1,l}$ be the space spanned by $Q_h^{2,m} u_m^1, Q_h^{2,m} u_m^2, \dots, Q_h^{2,m} u_m^l$. From the statements of Lemma 4.1 in [17], we know that $Q_h^{2,m} W_{1,l}$ is an l-dimensional subspace of $S_h(m)$. We derive from the minimax principle that

$$\lambda_{mh}^l \le \max_{v_m \in Q_h^{2,m} W_{1,l}} \frac{A_m(v_m, v_m)}{B_m(v_m, v_m)} = \max_{v_m \in W_{1,l}} \frac{A_m(Q_h^{2,m} v_m, Q_h^{2,m} v_m)}{B_m(Q_h^{2,m} v_m, Q_h^{2,m} v_m)}.$$

From the bilinear property of $A_m(v_m, v_m)$, we have $A_m(v_m, v_m) = A_m(Q_h^{2,m} v_m, Q_h^{2,m} v_m) + 2A_m(v_m - Q_h^{2,m} v_m, Q_h^{2,m} v_m) + A_m(v_m - Q_h^{2,m} v_m, v_m - Q_h^{2,m} v_m)$. Further from $A_m(v_m - Q_h^{2,m} v_m, Q_h^{2,m} v_m) = 0$ and the non-negativity of $A_m(v_m - Q_h^{2,m} v_m, v_m - Q_h^{2,m} v_m)$, we have

$$A_m(Q_h^{2,m} v_m, Q_h^{2,m} v_m) \le A_m(v_m, v_m).$$

Thus, we obtain that

$$\begin{aligned}
\lambda_{mh}^l &\le \max_{v_m \in W_{1,l}} \frac{A_m(v_m, v_m)}{B_m(Q_h^{2,m} v_m, Q_h^{2,m} v_m)} \\
&= \max_{v_m \in W_{1,l}} \frac{A_m(v_m, v_m)}{B_m(v_m, v_m)} \frac{B_m(v_m, v_m)}{B_m(Q_h^{2,m} v_m, Q_h^{2,m} v_m)} \\
&\le \lambda_m^l \max_{v_m \in W_{1,l}} \frac{B_m(v_m, v_m)}{B_m(Q_h^{2,m} v_m, Q_h^{2,m} v_m)}.
\end{aligned}$$

The proof is complete. □

Define the interpolation operator $I_{mh}: H^2_{0,\omega,m}(I) \to U_h$ by

$$I_{mh}u_m(t) = H_{3,m,i}(t), \quad t \in I_i,$$

where $I_i = [t_{i-1}, t_i]$, $H_{3,mi}(t)$ is a cubic Hermite interpolation polynomial of u_m in I_i. Let

$$u_{mi}(t) = u_m(t), \quad t \in I_i.$$

From the remainder theorem of cubic Hermite interpolation, we have

$$u_{mi}(t) - H_{3,m,i}(t) = \frac{(u_{mi})^{(4)}(\xi_{mi}(t))}{4!}(t - t_{i-1})^2(t - t_i)^2,$$

where $\xi_{mi}(t) \in I_i$ is a function depending on t.

Theorem 3. *Let* $E_{mi}(t) = \frac{(u_{mi})^{(4)}(\xi_{mi}(t))}{4!}$, $u_m \in H^2_{0,\omega,m}(I)$. *Assume that* u_m *is sufficiently smooth such that* $|\partial_t^k E_{mi}(t)| \le M (k = 0, 1, 2)$, *where* M *is a constant greater than zero. Then the following inequality holds*:

$$\|\partial_t^2(I_{mh}u_m - u_m)\| \lesssim h^2, \tag{4.11}$$

where $h = \max_{1 \le i \le n}\{h_i\}$, $h_i = t_i - t_{i-1}$, $\|u_m\| = \left[\int_I u_m^2 \mathrm{d}t\right]^{\frac{1}{2}}$.

Proof. Since

$$u_{mi}(t) - H_{3,m,i}(t) = E_{mi}(t)(t - t_{i-1})^2(t - t_i)^2,$$

then we have

$$\partial_t^2(u_{mi}(t) - H_{3,m,i}(t)) = \partial_t^2 E_{mi}(t)(t - t_{i-1})^2(t - t_i)^2 + 2\partial_t E_{mi}(t)\partial_t[(t - t_{i-1})^2(t - t_i)^2] + E_{mi}(t)\partial_t^2[(t - t_{i-1})^2(t - t_i)^2].$$

Thus, we obtain

$$\begin{aligned}
|\partial_t^2(u_{mi}(t) - H_{3,m,i}(t))|^2 &\lesssim [(t - t_{i-1})^2(t - t_i)^2]^2 + [\partial_t((t - t_{i-1})^2(t - t_i)^2)]^2 + \{\partial_t^2[(t - t_{i-1})^2(t - t_i)^2]\}^2 \\
&= [(t - t_{i-1})(t - t_i)]^4 + 4[(t - t_{i-1})(t - t_i)(2t - t_{i-1} - t_i)]^2 \\
&\quad + 4[4(t - t_{i-1})(t - t_i) + (t - t_{i-1})^2 + (t - t_i)^2]^2 \\
&\le \left(\frac{h_i}{2}\right)^8 + 4\left[h_i\left(\frac{h_i}{2}\right)^2\right]^2 + \left[8\left(\frac{h_i}{2}\right)^2 + h_i^2\right]^2 \lesssim h_i^4.
\end{aligned}$$

Thus,

$$\|\partial_t^2(I_{mh}u_m - u_m)\|^2 = \sum_{i=1}^n \int_{I_i} [\partial_t^2(u_{mi}(t) - H_{3,m,i}(t))]^2 \mathrm{d}t \lesssim \sum_{i=1}^n h_i^5 \lesssim h^4.$$

Furthermore, we have

$$\|\partial_t^2(I_{mh}u_m - u_m)\| \lesssim h^2.$$

The proof is complete. □

Theorem 4. *Let* λ^l_{mh} *be the approximate eigenvalue of* λ^l_m. *Assume that* $u_m \in H^2_{0,\omega,m}(I)$ *and satisfies the condition of* Theorem 3, *then the following inequality holds*:

$$|\lambda^l_{mh} - \lambda^l_m| \lesssim h^4,$$

where $c(l)$ *is a constant independent of* h.

Proof. For brief, we only give the proof for the case of $m \ne 1$, and it can be similarly proven for the case of $m = 1$. For $\forall q \in W_{1,l}$, we have $q = \sum_{i=1}^l q_i u^i_m$. By using the orthogonality of the characteristics function u^i_m and $B_m(u^i_m, u^i_m) = 1$, we have

$$\frac{B_m(q,q)-B_m(Q_h^{2,m}q,Q_h^{2,m}q)}{B_m(q,q)} \le \frac{2|B_m(q,q-Q_h^{2,m}q)|}{B_m(q,q)}$$
$$\le \frac{2\sum_{i,j=1}^{l}|q_iq_jB_m(u_m^i-Q_h^{2,m}u_m^i,u_m^j)|}{\sum_{i=1}^{l}|q_i|^2}$$
$$\le 2l\max_{i,j=1,\dots,l}|B_m(u_m^i-Q_h^{2,m}u_m^i,u_m^j)|.$$

From Cauchy-Schwarz inequality we have

$$\begin{aligned}
|B_m(u_m^i-Q_h^{2,m}u_m^i,u_m^j)| &= \frac{1}{\lambda_m^j}|\lambda_m^jB_m(u_m^j,u_m^i-Q_h^{2,m}u_m^i)|\\
&= \frac{1}{\lambda_m^j}|A_m(u_m^j,u_m^i-Q_h^{2,m}u_m^i)|\\
&= \frac{1}{\lambda_m^j}|A_m(u_m^j-Q_h^{2,m}u_m^j,u_m^i-Q_h^{2,m}u_m^i)|\\
&\le \frac{1}{\lambda_m^j}[A_m(u_m^j-Q_h^{2,m}u_m^j,u_m^j-Q_h^{2,m}u_m^j)]^{\frac{1}{2}}[A_m(u_m^i-Q_h^{2,m}u_m^i,u_m^i-Q_h^{2,m}u_m^i)]^{\frac{1}{2}}.
\end{aligned}$$

When $m \ge 2$, from Hardy inequality (cf. B8.6 in [16]) we derive

$$\int_I \frac{1}{(t+1)^3}(u_m^i)^2\mathrm{d}t \lesssim \int_I \frac{1}{t+1}(\partial_t u_m^i)^2\mathrm{d}t,$$
$$\int_I \frac{1}{(t+1)^2}(\partial_t u_m^i)^2\mathrm{d}t \lesssim \int_I (\partial_t^2 u_m^i)^2\mathrm{d}t.$$

Since

$$\begin{aligned}
[\partial_t u_m^i(1)]^2 &= \frac{1}{4}\left[\int_I \partial_t((t+1)\partial_t u_m^i)\mathrm{d}t\right]^2\\
&= \frac{1}{4}\left[\int_I \partial_t u_m^i + (t+1)\partial_t^2 u_m^i\mathrm{d}t\right]^2\\
&\lesssim \int_I (\partial_t u_m^i)^2\mathrm{d}t + \int_I (t+1)(\partial_t^2 u_m^i)^2\mathrm{d}t\\
&\lesssim \int_I \frac{1}{t+1}(\partial_t u_m^i)^2\mathrm{d}t + \int_I (t+1)(\partial_t^2 u_m^i)^2\mathrm{d}t,
\end{aligned}$$

from Lemma 1 we have

$$\begin{aligned}
\|u_m^i\|_{2,\omega,m}^2 &\lesssim \int_I (t+1)(\partial_t^2 u_m^i)^2\mathrm{d}t + \int_I \frac{1}{t+1}(\partial_t u_m^i)^2\mathrm{d}t + \int_I \frac{1}{(t+1)^3}(u_m^i)^2\mathrm{d}t\\
&\lesssim \int_I (\partial_t^2 u_m^i)^2\mathrm{d}t + \int_I \frac{1}{t+1}(\partial_t u_m^i)^2\mathrm{d}t\\
&\lesssim \int_I (\partial_t^2 u_m^i)^2\mathrm{d}t + \int_I \frac{1}{(t+1)^2}(\partial_t u_m^i)^2\mathrm{d}t\\
&\lesssim \int_I (\partial_t^2 u_m^i)^2\mathrm{d}t.
\end{aligned}$$

Then we derive that

$$
\begin{aligned}
|B_m(u_m^i - Q_h^{2,m}u_m^i, u_m^j)| &= \frac{1}{\lambda_m^j}|\lambda_m^j B_m(u_m^j, u_m^i - Q_h^{2,m}u_m^i)| \\
&\leq \frac{1}{\lambda_m^j}[A_m(u_m^j - Q_h^{2,m}u_m^j, u_m^j - Q_h^{2,m}u_m^j)]^{\frac{1}{2}}[A_m(u_m^i - Q_h^{2,m}u_m^i, u_m^i - Q_h^{2,m}u_m^i)]^{\frac{1}{2}} \\
&\leq \frac{1}{\lambda_m^j}[A_m(u_m^j - I_{mh}u_m^j, u_m^j - I_{mh}u_m^j)]^{\frac{1}{2}} \cdot [A_m(u_m^i - I_{mh}u_m^i, u_m^i - I_{mh}u_m^i)]^{\frac{1}{2}} \\
&\leq \frac{M}{\lambda_m^j}\|u_m^j - I_h u_m^j\|_{2,\omega,m}\|u_m^i - I_h u_m^i\|_{2,\omega,m} \\
&\lesssim \bigg(\int_I [\partial_t^2(u_m^j - I_{mh}u_m^j)]^2 \mathrm{d}t\bigg)^{\frac{1}{2}}\bigg(\int_I [\partial_t^2(u_m^i - I_{mh}u_m^i)]^2 \mathrm{d}t\bigg)^{\frac{1}{2}} \\
&\lesssim \|\partial_t^2(u_m^j - I_{mh}u_m^j)\| \cdot \|\partial_t^2(u_m^i - I_{mh}u_m^i)\|.
\end{aligned}
$$

Similarly, when $m = 0$, we derive that

$$|B_m(u_m^i - Q_h^{2,m}u_m^i, u_m^j)| \lesssim \|\partial_t^2(u_m^j - I_{mh}u_m^j)\| \cdot \|\partial_t^2(u_m^i - I_{mh}u_m^i)\|.$$

Since

$$\frac{B_m(q,q)}{B_m(Q_h^{2,m}q, Q_h^{2,m}q)} \leq \frac{1}{1 - 2l\max_{i,j=1,\dots,l}|B_m(u_m^i - Q_h^{2,m}u_m^i, u_m^j)|},$$

we obtain from Theorems 2 and 3 the desired results. □

5 Efficient implementation of the algorithm

In order to efficiently solve the problems (3.2), we start by constructing a set of basis functions which satisfy boundary conditions. Let

$$
\varphi_0^0(t) = \begin{cases} \Big(2\dfrac{t-t_0}{h_1}+1\Big)\Big(\dfrac{t-t_0}{h_1}-1\Big)^2, & t_i \leq t \leq t_{i+1}, \\ 0, & \text{others}, \end{cases}
$$

$$
\varphi_0^1(t) = \begin{cases} h_1^{-2}(t-t_0)(t-t_1)^2, & t_0 \leq t \leq t_1 \\ 0, & \text{others}, \end{cases}
$$

$$
\varphi_i^0(t) = \begin{cases} \Big(1+\dfrac{t-t_i}{h_i}\Big)^2\Big(1-2\dfrac{t-t_i}{h_i}\Big), & t_{i-1} \leq t \leq t_i, \\ \Big(2\dfrac{t-t_i}{h_{i+1}}+1\Big)\Big(\dfrac{t-t_i}{h_{i+1}}-1\Big)^2, & t_i \leq t \leq t_{i+1}, \\ 0, & \text{others}, \end{cases}
$$

$$
\varphi_i^1(t) = \begin{cases} h_i^{-2}(t-t_i)(t-t_{i-1})^2, & t_{i-1} \leq t \leq t_i, \\ h_{i+1}^{-2}(t-t_i)(t-t_{i+1})^2, & t_i \leq t \leq t_{i+1}, \\ 0, & \text{others}, \end{cases}
$$

$$
\varphi_n^1(t) = \begin{cases} h_n^{-2}(t-t_n)(t-t_{n-1})^2, & t_{n-1} \leq t \leq t_n, \\ 0, & \text{others}, \end{cases}
$$

where $i = 1, \dots, n-1$. It is clear that

$$S_h(0) = \operatorname{span}\{\varphi_0^0(t), \dots, \varphi_{n-1}^0(t), \varphi_1^1(t), \dots, \varphi_n^1(t)\};$$
$$S_h(1) = \operatorname{span}\{\varphi_1^0(t), \dots, \varphi_{n-1}^0(t), \varphi_0^1(x), \dots, \varphi_n^1(t)\};$$
$$S_h(m) = \operatorname{span}\{\varphi_1^0(t), \dots, \varphi_{n-1}^0(t), \varphi_1^1(x), \dots, \varphi_n^1(t)\}, (m \geq 2).$$

Denote

$$a_{ij}^{pq} = \int_I (t+1)(\varphi_j^p)''(\varphi_i^q)''\mathrm{d}t, \quad b_{ij}^{pq} = \int_I \frac{1}{t+1}(\varphi_j^p)'(\varphi_i^q)'\mathrm{d}t,$$
$$c_{ij}^{pq} = \int_I \frac{1}{(t+1)^3}\varphi_j^p\varphi_i^q\mathrm{d}t, \quad d_{ij}^{pq} = \varphi_j^p(1)\varphi_i^q(1),$$

where $p, q = 0, 1$.

Next, we will derive the matrix form of the discrete variational scheme (3.2).

Case 1. When $m = 0$, let

$$u_{0h} = u_0^0\varphi_0^0 + u_n^1\varphi_n^1 + \sum_{i=1}^{n-1}(u_i^0\varphi_i^0 + u_i^1\varphi_i^1). \tag{5.1}$$

Plugging the expression (5.1) in (3.2) and taking v_{0h} through all the basis functions in $S_h(0)$, we derive that

$$(A_0 + B_0 + D_0)U^0 = \lambda_{0h}RD_0U^0, \tag{5.2}$$

where

$$A_0 = \begin{pmatrix} (a_{ij}^{00}) & (a_{ij}^{10}) \\ (a_{ij}^{01}) & (a_{ij}^{11}) \end{pmatrix}, \quad B_0 = \begin{pmatrix} (b_{ij}^{00}) & (b_{ij}^{10}) \\ (b_{ij}^{01}) & (b_{ij}^{11}) \end{pmatrix}, \quad D_0 = \begin{pmatrix} (d_{ij}^{00}) & (d_{ij}^{10}) \\ (d_{ij}^{01}) & (d_{ij}^{11}) \end{pmatrix},$$
$$U^0 = (u_0^0, \dots, u_{N-1}^0, u_1^1, \dots, u_n^1)^T.$$

Similarly, when $m = 1$, let

$$u_{1h} = u_0^1\varphi_0^1 + u_n^1\varphi_n^1 + \sum_{i=1}^{n-1}(u_i^0\varphi_i^0 + u_i^1\varphi_i^1). \tag{5.3}$$

Plugging the expression (5.3) in (3.2) and taking v_{1h} through all the basis functions in $S_h(1)$, we obtain

$$[A_1 + (1+m)D_1]U^1 = \lambda_{1h}RD_1U^1, \tag{5.4}$$

where

$$A_1 = \begin{pmatrix} (\tilde{a}_{ij}^{00}) & (\tilde{a}_{ij}^{10}) \\ (\tilde{a}_{ij}^{01}) & (\tilde{a}_{ij}^{11}) \end{pmatrix}, \quad D_1 = \begin{pmatrix} (\tilde{d}_{ij}^{00}) & (\tilde{d}_{ij}^{10}) \\ (\tilde{d}_{ij}^{01}) & (\tilde{d}_{ij}^{11}) \end{pmatrix},$$
$$\tilde{a}_{ij}^{pq} = \int_I (t+1)L_1\varphi_j^pL_1\varphi_i^q\mathrm{d}t, \quad \tilde{d}_{ij}^{pq} = \varphi_j^p(1)\varphi_i^q(1),$$
$$U^1 = (u_1^0 \dots, u_{n-1}^0, u_0^1, \dots, u_n^1)^T.$$

When $m \geq 2$, let

$$u_{2h} = u_n^1\varphi_n^1 + \sum_{i=1}^{n-1}(u_i^0\varphi_i^0 + u_i^1\varphi_i^1). \tag{5.5}$$

Plugging the expression (5.5) in (3.2) and taking v_{mh} through all the basis functions in $S_h(m)$, we gain

$$[A_m + (2m^2 + 1)B_m + m^2(m^2 - 4)C_m + D_m]U^m = \lambda_{mh}RD_mU^m, \tag{5.6}$$

where

$$A_m = \begin{pmatrix} (a_{ij}^{00}) & (a_{ij}^{10}) \\ (a_{ij}^{01}) & (a_{ij}^{11}) \end{pmatrix}, \quad B_m = \begin{pmatrix} (b_{ij}^{00}) & (b_{ij}^{10}) \\ (b_{ij}^{01}) & (b_{ij}^{11}) \end{pmatrix}, \quad C_m = \begin{pmatrix} (c_{ij}^{00}) & (c_{ij}^{10}) \\ (c_{ij}^{01}) & (c_{ij}^{11}) \end{pmatrix}, \quad D_m = \begin{pmatrix} (d_{ij}^{00}) & (d_{ij}^{10}) \\ (d_{ij}^{01}) & (d_{ij}^{11}) \end{pmatrix} \tag{5.7}$$

$$U^m = (u_1^0, \ldots, u_{n-1}^0, u_1^1, \ldots, u_n^1)^T.$$

Note that we know from the properties of cubic hermit interpolation basis function that the stiff matrices and mass matrices in (5.4)–(5.6) are all sparse. Thus, they can be efficiently solved.

6 Numerical experiments

In order to show the accuracy and convergence of the proposed algorithm, we will carry out a series of numerical tests. We operate our programs in MATLAB 2016b.

Example 1. We take $R = 1$ and $m = 0, 1, 2, 3$. The eigenvalues for different m and h are listed in Table 1.

We know from Table 1 that the eigenvalues achieve at least six-digit accuracy with $h \le \frac{1}{32}$ for $m = 0, 1, 2, 3$. In order to further show the convergence of the algorithm, we choose the numerical solutions of $h = \frac{1}{64}$ as reference solutions, and the error figures of the approximate eigenvalues $\lambda_{mh}(m = 0, 1, 2, 3)$ with different h are presented in Figure 1. We observe from Figure 1 that the numerical eigenvalues are also convergent.

Example 2. We take $R = 2$ and $m = 0, 1, 2, 3$. The eigenvalues for different m and h are listed in Table 2.

Similarly, we observe from Table 2 that the eigenvalues have at least six-digit accuracy with $h \le \frac{1}{32}$ for $m = 0, 1, 2, 3$. We still choose the numerical solutions of $h = \frac{1}{64}$ as reference solutions, the error figures of the approximate eigenvalues $\lambda_{mh}(m = 0, 1, 2, 3)$ with different h are listed in Figure 2. We see from Figure 2 that the approximation eigenvalues are also convergent. Besides, in order to show the convergence rate of our algorithm more intuitively, we also plot the error figures in semilog scale in Figures 3 and 4.

Next, we shall provide a numerical example for some larger Fourier norm m.

Example 3. We take $R = 1$ and $m = 4, 5, 6, 7$. The eigenvalues for different m and h are listed in Table 3.

Likewise, we observe from Table 3 that the eigenvalues have at least five-digit accuracy with $h \le \frac{1}{64}$ for $m = 4, 5, 6, 7$. We choose the numerical solutions of $h = \frac{1}{128}$ as reference solutions, and the error figures of

Table 1: Eigenvalues for $m = 0, 1, 2, 3$ and different h

h	λ_{0h}	λ_{1h}	λ_{2h}	λ_{3h}
1/8	2.000000024461073	4.000000073005518	6.000025208583129	8.000307091454056
1/16	2.000000015899493	4.000000072449404	6.000001539514262	8.000019101762424
1/32	2.000000006492949	4.000000084945128	6.000000095601314	8.000001192525161
1/64	2.000000002660813	4.000000091047035	6.000000005945074	8.000000074519297

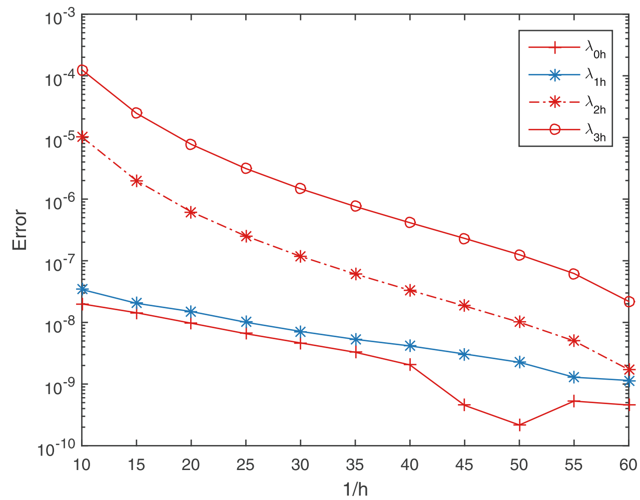

Figure 1: Errors between numerical solutions and the reference solution for $R = 1$.

Table 2: Eigenvalues for $m = 0, 1, 2, 3$ and different h

h	λ_{0h}	λ_{1h}	λ_{2h}	λ_{3h}
1/8	1.000000012230537	2.000000036502759	3.000012604291565	4.000153545727028
1/16	1.000000007949747	2.000000036224702	3.000000769757131	4.000009550881212
1/32	1.000000003246474	2.000000042472564	3.000000047800657	4.000000596262581
1/64	1.000000001330406	2.000000045523517	3.000000002972537	4.000000037259649

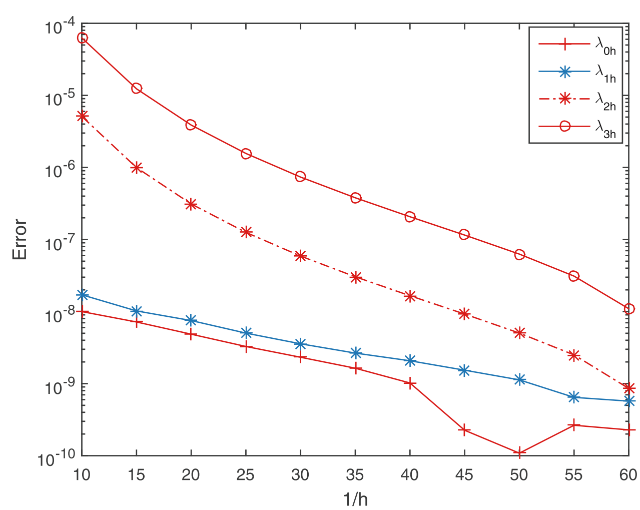

Figure 2: Errors between numerical solutions and the reference solution for $R = 2$.

the approximate eigenvalues $\lambda_{mh}(m = 4, 5, 6, 7)$ with different h are listed in Figure 5. We see from Figure 5 that the approximation eigenvalues are also convergent.

Introducing the usual definition of convergence order:

$$c(h) = \log_2\left(\frac{|\lambda_h - \lambda_{\frac{h}{2}}|}{|\lambda_{\frac{h}{2}} - \lambda_{\frac{h}{4}}|}\right). \tag{6.1}$$

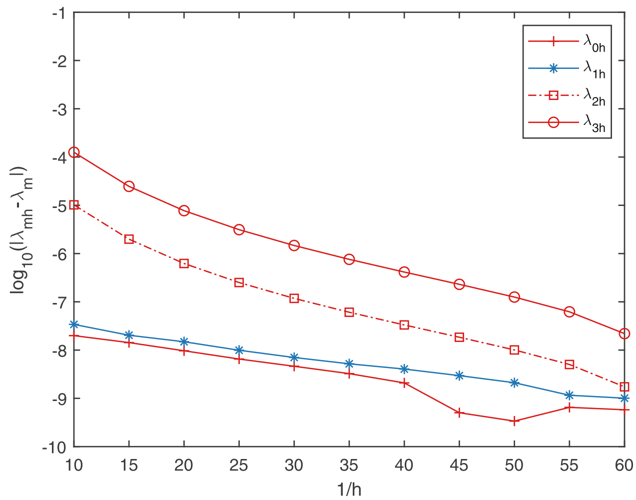

Figure 3: Error curves in semilog scale between the numerical solution and the reference solution for $R = 1$.

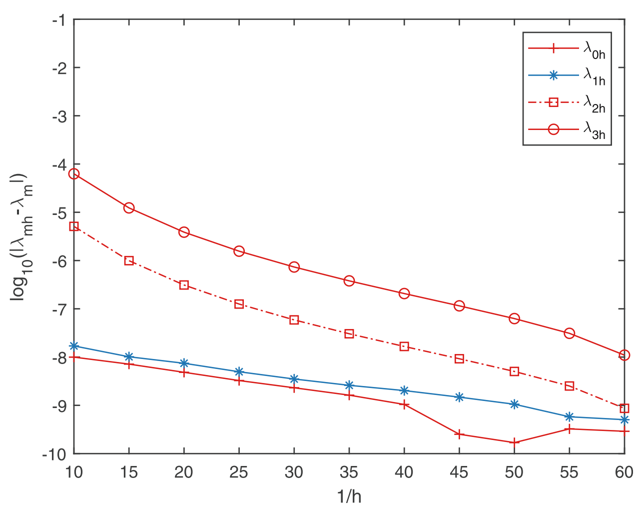

Figure 4: Error curves in semilog scale between the numerical solution and the reference solution for $R = 2$.

Table 3: Eigenvalues for $m = 4, 5, 6, 7$ and different h

h	λ_{4h}	λ_{5h}	λ_{6h}	λ_{7h}
1/4	10.0238272394670	12.0756338414226	14.1876937235392	16.3931316663901
1/8	10.0014906003953	12.0048786723652	14.0126111745548	16.0277527229022
1/16	10.0000931108253	12.0003074765839	14.0008034680874	16.0017927950984
1/32	10.0000058180331	12.0000192583640	14.0000504622624	16.0001129998648
1/64	10.0000003635851	12.0000012042859	14.0000031577541	16.0000070775063
1/128	10.0000000229184	12.0000000756600	14.0000001974885	16.0000004427145

For brevity, we shall use formula (6.1) to calculate the convergence order of the approximation eigenvalues of $m = 4, 5, 6, 7$ in the unit disk and list them in Table 4. We observe from Table 4 that the convergence order is about 4.

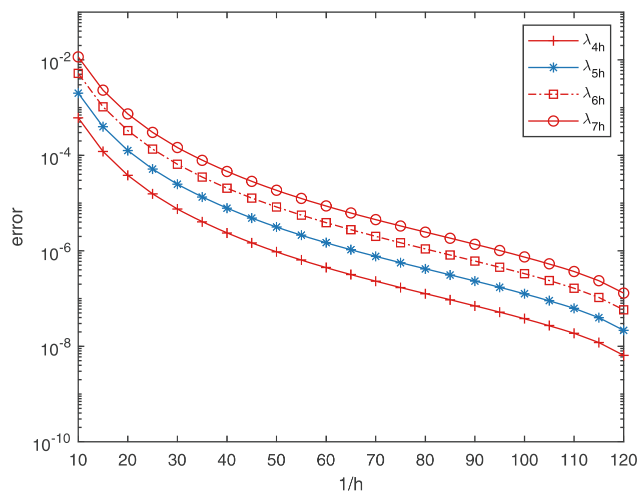

Figure 5: Errors between numerical solutions and the reference solution for $m = 4, 5, 6, 7$ and $R = 1$.

Table 4: Convergence order $c(h)$ for the eigenvalues with $m = 4, 5, 6, 7$

h	λ_{4h}	λ_{5h}	λ_{6h}	λ_{7h}
1/2	3.9924	3.8459	3.6960	3.5770
1/4	3.9985	3.9522	3.8902	3.8150
1/8	4.0008	3.9873	3.9709	3.9499
1/16	4.0003	3.9968	3.9926	3.9872
1/32	4.0010	3.9997	3.9982	3.9968

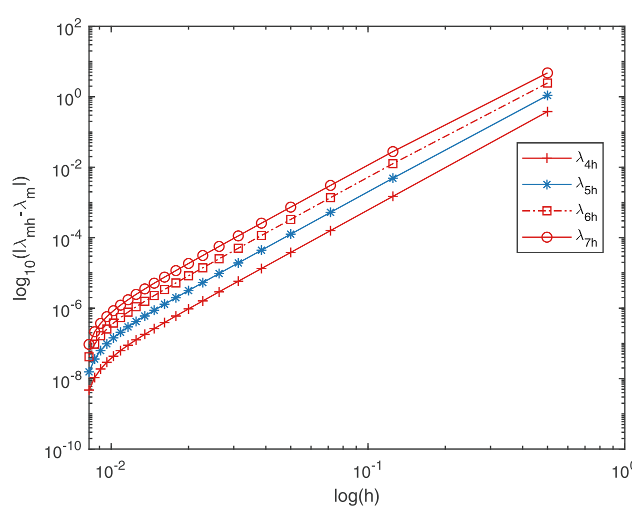

Figure 6: The eigenvalue errors in log-log scale between numerical solutions and the reference solution for $m = 4, 5, 6, 7$ and $R = 1$.

Finally, we plot the eigenvalue error in log-log scale to describe the algebra convergence rate in Figure 6.

7 Conclusion

We present in this article a novel finite element method based on a dimension reduction scheme for a fourth-order Steklov eigenvalue problem in a circular domain. The main advantage of this method is that the original problem is transformed into a series of one-dimensional problems which can be solved in parallel. Then, by introducing the polar conditions and the weighted Sobolev space, we prove the error estimates of approximation eigenvalues by using the minimax principle. The method developed in this article can be applied to more complex problems or more general polar geometric domains which will be the subject of our future endeavors.

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Author contributions: H. Zhang analyzed most of the data and wrote the initial draft of the article. J. Zhang contributed to refining the ideas, carrying out additional analyses, and finalizing this article. Z. X. Liu contributed the central idea.

Conflict of interest: The authors declare that we have no conflict of interest.

Data availability statement: Some or all data, models, or code generated or used during the study are available from the corresponding author by request.

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