Research Article

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Extensions of Gronwall-Bellman type integral inequalities with two independent variables

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Abstract: In this paper, we establish several kinds of integral inequalities in two independent variables, which improve well-known versions of Gronwall-Bellman inequalities and extend them to fractional integral form. By using these inequalities, we can provide explicit bounds on unknown functions. The integral inequalities play an important role in the qualitative theory of differential and integral equations and partial differential equations.

Keywords: fractional integral inequalities, two independent variables, differential and integral equations, partial differential equations

MSC 2020: 6D10, 34A40, 35A23

1 Introduction

It is well known that Gronwall's inequality is an important tool in the quantitative and qualitative analysis of solutions to differential and integral equations. For example, it has been used to study the boundedness, existence, uniqueness, and stability of solutions of differential-integral equations (cf. [1–6]). Gronwall's original result [7] appeared in 1919, and Bellman [2] proved the integral version of Gronwall's inequality in 1943. Since then, many researchers have spent a lot of effort studying more general Gronwall-type integral inequalities with a single variable and discussed their applications to ordinary differential equations.

Let us recall the standard Gronwall inequality, which can be found in [5,8].

Theorem 1.1. *Let $u(t) \in L_+^\infty[0, T]$ satisfy*

$$u(t) \le u_0(t) + \int_0^t f(s)u(s)\mathrm{d}s, \quad a.e.\ t \in [0, T], \tag{1.1}$$

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where u_0 *is nonnegative and nondecreasing, and* $f \in L^1_+[0, T]$. *Then,*

$$u(t) \le u_0(t) \exp\left(\int_0^t f(s) \mathrm{d}s \right), \quad a.e.\ t \in [0, T]. \tag{1.2}$$

Some Gronwall inequalities for fractional order are proved in [9,10]. Alzabut-Abdeljawad [11] proved a discrete fractional version of the generalized Gronwall inequality. For convenience, we state the following version of a fractional Gronwall inequality:

Theorem 1.2. [10] *Suppose* $\beta > 0$, $a(t)$ *is a nonnegative function locally integrable on* $0 \le t < T$ (*some* $T < +\infty$) *and* $g(t)$ *is a nonnegative, nondecreasing continuous function defined on* $0 \le t < T$, $g(t) \le M$ (*constant*), *and suppose* $u(t)$ *is nonnegative and locally integrable on* $0 \le t < T$ *with*

$$u(t) \le a(t) + g(t) \int_0^t (t - s)^{\beta - 1} u(s) \mathrm{d}s, \quad a.e.\ t \in [0, T). \tag{1.3}$$

Then,

$$u(t) \le a(t) + \int_0^t \left[\sum_{n=1}^{\infty} \frac{(g(t)\Gamma(\beta))^n}{\Gamma(n\beta)} (t - s)^{n\beta - 1} a(s) \right] \mathrm{d}s, \quad a.e.\ t \in [0, T). \tag{1.4}$$

We may also find various applications of integer and fractional Gronwall-Bellman type of inequalities to study the qualitative properties of solutions to differential and integral equations of fractional order in [12–18].

Furthermore, many authors extended one variable Gronwall-Bellman type integral inequalities to two or more independent variables (cf. [19–30]). Especially papers [20–24] proved some results about integral inequalities of Gronwall-Bellman type with two independent variables and presented some definite applications of their results to the boundedness, uniqueness, and continuous dependence of the solutions of some nonlinear hyperbolic partial integrodifferential equations. Recently, Boudeliou [31] considered Gronwall-type inequalities with two independent variables and applied his new theoretic results to obtain the boundedness of solutions of some integral equations successfully. For more recent developments of Gronwall-type inequalities with two independent variables, we refer the readers to [32–38] and the references therein. For example, Khan discussed several new integral inequalities of two independent variables, and one of the interesting inequalities is:

Theorem 1.3. [33] *Let* $\phi(x, y)$, $A(x, y)$, $B(x, y)$, *and* $H(x, y)$ *be real-valued nonnegative, nondecreasing continuous functions, defined* $x, y \in \mathbf{R}_+$, $c > 0$. *If*

$$\phi(x, y) \le c + \int_0^x A(s, y)\phi(s, y) \mathrm{d}s + \int_0^y B(x, t)\phi(x, t) \mathrm{d}t + \int_0^x \int_0^y H(s, t)\phi(s, t) \mathrm{d}t \mathrm{d}s, \tag{1.5}$$

for all $x, y \in \mathbf{R}_+$. *Then,*

$$\phi(x, y) \le c Q(x, y) E(x, y) \exp\left[\int_0^x \int_0^y H(s, t) Q(s, t) E(s, t) \mathrm{d}t \mathrm{d}s \right], \tag{1.6}$$

where

$$Q(x, y) = \exp\left[\int_0^y B(x, t) E(x, t) \mathrm{d}t \right]$$

and

$$E(x,y)=\exp\left[\int_0^x A(s,y)\mathrm{d}s\right].$$

In this paper, we extend and generalize the main results in [10,33] and show several new types of Gronwall-Bellman inequalities, which arises from a class of integral equations with a mixture of integer-order and fractional-order integrals. The results can be used to study the boundedness of solutions of several special kinds of integral equations.

2 Main results

In this section, we shall show several new inequalities, which are more general than (1.5)–(1.6). For conveniences, we set

$$M(x,y)=c+\int_0^x\int_0^y h(s,t)u^q(s,t)\mathrm{d}t\mathrm{d}s,$$

$$A(x,y)=1+\int_0^x (x-s)^{\alpha-1}\frac{u(s,y)}{M(s,y)}\,\mathrm{d}s+\int_0^y b(x,t)\frac{u(x,t)}{M(x,t)}\mathrm{d}t.$$

Our first result is the following.

Theorem 2.1. *Let $u(x,y)$, $b(x,y)$, $h(x,y)$, and $A(x,y)$ be real-valued nonnegative, nondecreasing continuous functions, and $c>0$, $0<q<1$, $\alpha>0$. If*

$$u(x,y)\le c+\int_0^x (x-s)^{\alpha-1}u(s,y)\mathrm{d}s+\int_0^y b(x,t)u(x,t)\mathrm{d}t+\int_0^x\int_0^y h(s,t)u^q(s,t)\mathrm{d}t\mathrm{d}s \tag{2.1}$$

for all $x,y\in\mathbf{R}_+$. Then,

$$u(x,y)\le W(x,y)E_\alpha(\Gamma(\alpha)x^\alpha)\left[c^{1-q}+(1-q)\int_0^x\int_0^y h(s,t)(W(s,t)E_\alpha(\Gamma(\alpha)s^\alpha))^q\mathrm{d}t\mathrm{d}s\right]^{\frac{1}{1-q}},$$

where

$$W(x,y)=\exp\left[\int_0^y b(x,t)E_\alpha(\Gamma(\alpha)x^\alpha)\mathrm{d}t\right],$$

and $E_\alpha(Z)=\sum_{k=0}^{\infty}\frac{Z^k}{\Gamma(k\alpha+1)}$ stands for the Mittag-Leffler Function (cf. [39]).

Proof. Let us set

$$M(x,y)=c+\int_0^x\int_0^y h(s,t)u^q(s,t)\mathrm{d}t\mathrm{d}s, \tag{2.2}$$

and then, the inequality (2.1) becomes

$$u(x,y)\le M(x,y)+\int_0^x (x-s)^{\alpha-1}u(s,y)\mathrm{d}s+\int_0^y b(x,t)u(x,t)\mathrm{d}t.$$

Since $M(x,y)$ is positive, nondecreasing continuous function, it is easy to show that

$$\frac{u(x,y)}{M(x,y)} \le 1 + \int_0^x (x-s)^{\alpha-1}\frac{u(s,y)}{M(s,y)}\mathrm{d}s + \int_0^y b(x,t)\frac{u(x,t)}{M(x,t)}\mathrm{d}t.$$

Set

$$A(x,y) = 1 + \int_0^x (x-s)^{\alpha-1}\frac{u(s,y)}{M(s,y)}\mathrm{d}s + \int_0^y b(x,t)\frac{u(x,t)}{M(x,t)}\mathrm{d}t.$$

So, one has

$$\frac{u(x,y)}{M(x,y)} \le A(x,y),$$
$$A(x,y) \le 1 + \int_0^x (x-s)^{\alpha-1}A(s,y)\mathrm{d}s + \int_0^y b(x,t)A(x,t)\mathrm{d}t. \tag{2.3}$$

Define

$$B(x,y) = 1 + \int_0^y b(x,t)A(x,t)\mathrm{d}t. \tag{2.4}$$

Therefore, we have

$$A(x,y) \le B(x,y) + \int_0^x (x-s)^{\alpha-1}A(s,y)\mathrm{d}s. \tag{2.5}$$

Since $b(x,y), A(x,y)$ are nonnegative and nondecreasing by the assumptions, $B(x,y)$ is positive, nondecreasing continuous function. We infer

$$\frac{A(x,y)}{B(x,y)} \le 1 + \int_0^x (x-s)^{\alpha-1}\frac{A(s,y)}{B(s,y)}\mathrm{d}s.$$

Set

$$C(x,y) = 1 + \int_0^x (x-s)^{\alpha-1}\frac{A(s,y)}{B(s,y)}\mathrm{d}s.$$

Then, we obtain

$$\frac{A(x,y)}{B(x,y)} \le C(x,y), \tag{2.6}$$

$$C(x,y) \le 1 + \int_0^x (x-s)^{\alpha-1}C(s,y)\mathrm{d}s. \tag{2.7}$$

Define an integral operator:

$$\mathbf{B}\phi(x,y) = \int_0^x (x-s)^{\alpha-1}\phi(s,y)\mathrm{d}s.$$

Then, formula (2.7) implies that

$$C(x,y) \le 1 + \mathbf{B}C(x,y), \quad C(x,y) \le \sum_{k=0}^{n-1}\mathbf{B}^k 1 + \mathbf{B}^n C(x,y).$$

In the following, we prove that

$$\mathbf{B}^n C(x,y) \le \frac{(\Gamma(\alpha))^n}{\Gamma(n\alpha)} \int_0^x (x-s)^{n\alpha-1} C(s,y)\mathrm{d}s, \tag{2.8}$$

$$\mathbf{B}^n C(x,y) \to 0 \quad \text{as } n \to +\infty \text{ for each } x, y \in \mathbf{R}_+.$$

By the use of induction, we easily obtain that relation (2.8) is true for $n = 1$. Next, assume that it is also true for $n = k$. If $n = k+1$, one has from inductive hypothesis

$$\mathbf{B}^{k+1} C(x,y) = \mathbf{B}\mathbf{B}^k C(x,y) \le \int_0^x (x-s)^{\alpha-1} \left[\frac{(\Gamma(\alpha))^k}{\Gamma(k\alpha)} \int_0^s (s-\tau)^{k\alpha-1} C(\tau,y)\mathrm{d}\tau \right] \mathrm{d}s.$$

By exchanging integration order, we have

$$\mathbf{B}^{k+1} C(x,y) \le \frac{(\Gamma(\alpha))^k}{\Gamma(k\alpha)} \int_0^x \left[\int_\tau^x (x-s)^{\alpha-1} (s-\tau)^{k\alpha-1} \mathrm{d}s \right] C(\tau,y)\mathrm{d}\tau.$$

In virtue of the properties of the beta functions (see [39, p. 6]) and the variable substitution $s = \tau + u(x-\tau)$, we easily obtain

$$\begin{aligned} \mathbf{B}^{k+1} C(x,y) &\le \frac{(\Gamma(\alpha))^k}{\Gamma(k\alpha)} \int_0^x \left[\int_0^1 (1-u)^{\alpha-1} (u)^{k\alpha-1} \mathrm{d}u \right] (x-\tau)^{(k+1)\alpha-1} C(\tau,y)\mathrm{d}\tau \\ &= \frac{(\Gamma(\alpha))^k}{\Gamma(k\alpha)} \int_0^x \frac{\Gamma(\alpha)\Gamma(k\alpha)}{\Gamma((k+1)\alpha)} (x-\tau)^{(k+1)\alpha-1} C(\tau,y)\mathrm{d}\tau \\ &= \frac{(\Gamma(\alpha))^{k+1}}{\Gamma((k+1)\alpha)} \int_0^x (x-\tau)^{(k+1)\alpha-1} C(\tau,y)\mathrm{d}\tau, \end{aligned}$$

which proves that the inequality (2.8) holds for $n = k+1$. And

$$0 \le \mathbf{B}^n C(x,y) \le \frac{(\Gamma(\alpha))^n}{\Gamma(n\alpha)} \int_0^x (x-s)^{n\alpha-1} C(s,y)\mathrm{d}s \to 0 \quad \text{as } n \to +\infty \text{ for all } x, y \in \mathbf{R}_+.$$

Therefore, we readily obtain

$$\begin{aligned} C(x,y) &\le \sum_{k=0}^{\infty} \mathbf{B}^k 1 \le \sum_{k=0}^{\infty} \frac{(\Gamma(\alpha))^k}{\Gamma(k\alpha)} \int_0^x (x-s)^{k\alpha-1} 1\mathrm{d}s \\ &= \sum_{k=0}^{\infty} \frac{(\Gamma(\alpha))^k}{\Gamma(k\alpha)} \frac{x^{k\alpha}}{k\alpha} = \sum_{k=0}^{\infty} \frac{(\Gamma(\alpha)x^\alpha)^k}{\Gamma(k\alpha+1)} = E_\alpha(\Gamma(\alpha)x^\alpha). \end{aligned}$$

From the relation (2.6), one has

$$A(x,y) \le B(x,y)C(x,y) \le B(x,y)E_\alpha(\Gamma(\alpha)x^\alpha). \tag{2.9}$$

Taking the partial derivative with respect to y on both sides of (2.4), we obtain

$$B_y(x,y) = b(x,y)A(x,y). \tag{2.10}$$

By substituting (2.9) in (2.10), we have

$$\frac{B_y(x,y)}{B(x,y)} \le b(x,y)E_\alpha(\Gamma(\alpha)x^\alpha).$$

Integrating both sides of aforementioned inequality with respect to y from 0 to y, we obtain

$$B(x,y) \le \exp\left[\int_0^y b(x,t)E_\alpha(\Gamma(\alpha)x^\alpha)\mathrm{d}t\right] = W(x,y).$$

Hence, we obtain from (2.9)

$$A(x,y) \le W(x,y)E_\alpha(\Gamma(\alpha)x^\alpha).$$

Furthermore, we have from (2.3)

$$u(x,y) \le A(x,y)M(x,y) \le W(x,y)E_\alpha(\Gamma(\alpha)x^\alpha)M(x,y). \tag{2.11}$$

Taking the partial derivatives with respect to x and y on both sides of (2.2), respectively, we obtain

$$M_{xy}(x,y) = h(x,y)u^q(x,y). \tag{2.12}$$

From (2.11) and (2.12), we have

$$\begin{aligned}
&\frac{M_{xy}(x,y)}{M^q(x,y)} \le h(x,y)(W(x,y)E_\alpha(\Gamma(\alpha)x^\alpha))^q \\
\Rightarrow\ &\frac{M_{xy}(x,y)M(x,y)}{M^{q+1}(x,y)} - \frac{qM_x(x,y)M_y(x,y)}{M^{q+1}(x,y)} \le h(x,y)(W(x,y)E_\alpha(\Gamma(\alpha)x^\alpha))^q \\
\Rightarrow\ &\frac{\partial}{\partial y}\left[\frac{M_x(x,y)}{M^q(x,y)}\right] \le h(x,y)(W(x,y)E_\alpha(\Gamma(\alpha)x^\alpha))^q.
\end{aligned}$$

Then, integrating both sides of aforementioned inequality, first, integrating y from 0 to y, and then integrating x from 0 to x, we obtain

$$M(x,y) \le \left[c^{1-q} + (1-q)\int_0^x\int_0^y h(s,t)(W(s,t)E_\alpha(\Gamma(\alpha)s^\alpha))^q\mathrm{d}t\mathrm{d}s\right]^{\frac{1}{1-q}}. \tag{2.13}$$

By substituting (2.13) in (2.11), we obtain

$$u(x,y) \le W(x,y)E_\alpha(\Gamma(\alpha)x^\alpha)\left[c^{1-q} + (1-q)\int_0^x\int_0^y h(s,t)(W(s,t)E_\alpha(\Gamma(\alpha)s^\alpha))^q\mathrm{d}t\mathrm{d}s\right]^{\frac{1}{1-q}},$$

which completes our proof. □

Theorem 2.2. *Under the assumptions in* Theorem 2.1, *but* $q = 1$, *if the following inequality holds*

$$u(x,y) \le c + \int_0^x (x-s)^{\alpha-1}u(s,y)\mathrm{d}s + \int_0^y b(x,t)u(x,t)\mathrm{d}t + \int_0^x\int_0^y h(s,t)u(s,t)\mathrm{d}t\mathrm{d}s$$

for all $x, y \in \mathbf{R}_+$. *Then,*

$$u(x,y) \le cW(x,y)E_\alpha(\Gamma(\alpha)x^\alpha)\exp\left[\int_0^x\int_0^y h(s,t)W(s,t)E_\alpha(\Gamma(\alpha)s^\alpha)\mathrm{d}t\mathrm{d}s\right],$$

where $W(x,y)$ *is defined in* Theorem 2.1.

Proof. Similar to the proof of Theorem 2.1, we set

$$M(x,y) = c + \int_0^x\int_0^y h(s,t)u(s,t)\mathrm{d}t\mathrm{d}s.$$

By using the same steps as in (2.2)–(2.11), we easily have

$$u(x,y) \le W(x,y)E_\alpha(\Gamma(\alpha)x^\alpha)M(x,y). \tag{2.14}$$

Due to $M_{xy}(x,y) = h(x,y)u(x,y)$ and the inequality (2.14), we obtain

$$\begin{aligned}
&\frac{M_{xy}(x,y)}{M(x,y)} \le h(x,y)W(x,y)E_\alpha(\Gamma(\alpha)x^\alpha)\\
\Rightarrow\ &\frac{M_{xy}(x,y)M(x,y)}{M^2(x,y)} - \frac{M_x(x,y)M_y(x,y)}{M^2(x,y)} \le h(x,y)W(x,y)E_\alpha(\Gamma(\alpha)x^\alpha)\\
\Rightarrow\ &\frac{\partial}{\partial y}\left[\frac{M_x(x,y)}{M(x,y)}\right] \le h(x,y)W(x,y)E_\alpha(\Gamma(\alpha)x^\alpha).
\end{aligned}$$

Integrating y from 0 to y, and then integrating x from 0 to x on both sides of the aforementioned inequality, we obtain

$$M(x,y) \le c\exp\left[\int_0^x\int_0^y h(s,t)W(s,t)E_\alpha(\Gamma(\alpha)s^\alpha)\mathrm{d}t\mathrm{d}s\right]. \tag{2.15}$$

Then, substituting (2.15) into (2.14), we obtain

$$u(x,y) \le cW(x,y)E_\alpha(\Gamma(\alpha)x^\alpha)\exp\left[\int_0^x\int_0^y h(s,t)W(s,t)E_\alpha(\Gamma(\alpha)s^\alpha)\mathrm{d}t\mathrm{d}s\right].$$ □

Theorem 2.3. *Under the assumptions in* Theorem 2.1 *and* $0 < p < 1$, *if the following inequality holds*

$$u(x,y) \le c + \int_0^x (x-s)^{\alpha-1}u(s,y)\mathrm{d}s + \int_0^y b(x,t)u^p(x,t)\mathrm{d}t + \int_0^x\int_0^y h(s,t)u^q(s,t)\mathrm{d}t\mathrm{d}s, \tag{2.16}$$

for all $x, y \in \mathbf{R}_+$. *Then,*

$$u(x,y) \le E_\alpha(\Gamma(\alpha)x^\alpha)W_1(x,y)\left[c^{1-q} + (1-q)\int_0^x\int_0^y h(s,t)(E_\alpha(\Gamma(\alpha)s^\alpha)W_1(s,t))^q\mathrm{d}t\mathrm{d}s\right]^{\frac{1}{1-q}},$$

where

$$W_1(x,y) = \left[1 + (1-p)c^{p-1}\int_0^y b(x,t)(E_\alpha(\Gamma(\alpha)x^\alpha))^p\mathrm{d}t\right]^{\frac{1}{1-p}}.$$

Proof. Define

$$M(x,y) = c + \int_0^x\int_0^y h(s,t)u^q(s,t)\mathrm{d}t\mathrm{d}s. \tag{2.17}$$

By substituting (2.17) into (2.16), we obtain

$$u(x,y) \le M(x,y) + \int_0^x (x-s)^{\alpha-1}u(s,y)\mathrm{d}s + \int_0^y b(x,t)u^p(x,t)\mathrm{d}t.$$

Since $M(x,y)$ is positive and nondecreasing, one has

$$\frac{u(x,y)}{M(x,y)} \le 1 + \int_0^x (x-s)^{\alpha-1}\frac{u(s,y)}{M(s,y)}\mathrm{d}s + \int_0^y b(x,t)\frac{u^p(x,t)}{M(x,t)}\mathrm{d}t.$$

Set

$$A(x,y)=1+\int_0^x (x-s)^{\alpha-1}\frac{u(s,y)}{M(s,y)}\mathrm{d}s+\int_0^y b(x,t)\frac{u^p(x,t)}{M(x,t)}\mathrm{d}t.$$

So, we obtain

$$\frac{u(x,y)}{M(x,y)}\le A(x,y). \tag{2.18}$$

Hence,

$$A(x,y)\le 1+\int_0^x (x-s)^{\alpha-1}A(s,y)\mathrm{d}s+\int_0^y b(x,t)A^p(x,t)M^{p-1}(x,t)\mathrm{d}t.$$

By the definition of the function $M(x,y)$ (see (2.17)) and $0<p<1$, it is easy to know that

$$c^{p-1}\ge M^{p-1}(x,y).$$

Therefore, one has

$$A(x,y)\le 1+\int_0^x (x-s)^{\alpha-1}A(s,y)\mathrm{d}s+c^{p-1}\int_0^y b(x,t)A^p(x,t)\mathrm{d}t.$$

Set

$$B(x,y)=1+c^{p-1}\int_0^y b(x,t)A^p(x,t)\mathrm{d}t. \tag{2.19}$$

Then, we obtain

$$A(x,y)\le B(x,y)+\int_0^x (x-s)^{\alpha-1}A(s,y)\mathrm{d}s.$$

Similarly, using the same steps from (2.5)–(2.9), we obtain

$$A(x,y)\le B(x,y)E_\alpha(\Gamma(\alpha)x^\alpha). \tag{2.20}$$

Now by taking the partial derivative with respect to y both sides of (2.19), we obtain

$$B_y(x,y)=c^{p-1}b(x,y)A^p(x,y).$$

From (2.20), we have

$$B_y(x,y)\le c^{p-1}b(x,y)(B(x,y)E_\alpha(\Gamma(\alpha)x^\alpha))^p.$$

Hence,

$$\frac{B_y(x,y)}{B^p(x,y)}\le c^{p-1}b(x,y)(E_\alpha(\Gamma(\alpha)x^\alpha))^p,$$

which implies that

$$B(x,y)\le\left[1+(1-p)c^{p-1}\int_0^y b(x,t)(E_\alpha(\Gamma(\alpha)x^\alpha))^p\,\mathrm{d}t\right]^{\frac{1}{1-p}}=W_1(x,y).$$

Therefore, we have from (2.20)

$$A(x,y)\le E_\alpha(\Gamma(\alpha)x^\alpha)W_1(x,y).$$

Thanks to (2.18), we also have

$$u(x,y) \le M(x,y)E_\alpha(\Gamma(\alpha)x^\alpha)W_1(x,y). \tag{2.21}$$

Taking the partial derivatives with respect to x and y on both sides of (2.17), respectively, we obtain

$$M_{xy}(x,y) = h(x,y)u^q(x,y).$$

By using (2.21), we have

$$\frac{M_{xy}(x,y)}{M^q(x,y)} \le h(x,y)(E_\alpha(\Gamma(\alpha)x^\alpha)W_1(x,y))^q.$$

Then using the similar steps from (2.12) to (2.13) in Theorem 2.1, we obtain

$$M(x,y) \le \left[c^{1-q} + (1-q)\int_0^x\int_0^y h(s,t)(E_\alpha(\Gamma(\alpha)s^\alpha)W_1(s,t))^q \mathrm{d}t\mathrm{d}s\right]^{\frac{1}{1-q}},$$

which implies from (2.21)

$$u(x,y) \le E_\alpha(\Gamma(\alpha)x^\alpha)W_1(x,y)\left[c^{1-q} + (1-q)\int_0^x\int_0^y h(s,t)(E_\alpha(\Gamma(\alpha)s^\alpha)W_1(s,t))^q \mathrm{d}t\mathrm{d}s\right]^{\frac{1}{1-q}}.$$

This completes the proof. □

Theorem 2.4. *Under the same assumptions in* Theorem 2.3, *but* $q = 1$, *if the following inequality holds,*

$$u(x,y) \le c + \int_0^x (x-s)^{\alpha-1}u(s,y)\mathrm{d}s + \int_0^y b(x,t)u^p(x,t)\mathrm{d}t + \int_0^x\int_0^y h(s,t)u(s,t)\mathrm{d}t\mathrm{d}s$$

for all $x, y \in \mathbf{R}_+$. *Then,*

$$u(x,y) \le cE_\alpha(\Gamma(\alpha)x^\alpha)W_1(x,y)\exp\left[\int_0^x\int_0^y h(s,t)E_\alpha(\Gamma(\alpha)s^\alpha)W_1(s,t)\mathrm{d}t\mathrm{d}s\right],$$

where $W_1(x,y)$ *is defined in* Theorem 2.3.

Proof. Define

$$M(x,y) = c + \int_0^x\int_0^y h(s,t)u(s,t)\mathrm{d}t\mathrm{d}s. \tag{2.22}$$

Taking the partial derivatives with respect to x and y on both sides of (2.22), respectively, we obtain

$$M_{xy}(x,y) = h(x,y)u(x,y).$$

By using the similar steps of (2.17)–(2.21), we have

$$u(x,y) \le W_1(x,y)E_\alpha(\Gamma(\alpha)x^\alpha)M(x,y). \tag{2.23}$$

Therefore, one easily obtain

$$\frac{M_{xy}(x,y)}{M(x,y)} \le h(x,y)W_1(x,y)E_\alpha(\Gamma(\alpha)x^\alpha).$$

By using the similar discuss of the proof in Theorem 2.2, we easily obtain

$$M(x,y)\le c\exp\left[\int_0^x\int_0^y h(s,t)W_1(s,t)E_\alpha(\Gamma(\alpha)s^\alpha)\mathrm{d}t\mathrm{d}s\right].$$

Then, the inequality (2.23) reduces to

$$u(x,y)\le cW_1(x,y)E_\alpha(\Gamma(\alpha)x^\alpha)\exp\left[\int_0^x\int_0^y h(s,t)W_1(s,t)E_\alpha(\Gamma(\alpha)s^\alpha)\mathrm{d}t\mathrm{d}s\right].$$

The proof is complete. □

Theorem 2.5. *Let $u(x,y)$ and $h(x,y)$ be real-valued nonnegative, nondecreasing continuous functions, and $c>0$, $0<q<1$, $\alpha>0$, $\beta>0$. If*

$$u(x,y)\le c+\int_0^x(x-s)^{\alpha-1}u(s,y)\mathrm{d}s+\int_0^y(y-t)^{\beta-1}u(x,t)\mathrm{d}t+\int_0^x\int_0^y h(s,t)u^q(s,t)\mathrm{d}t\mathrm{d}s, \tag{2.24}$$

for all $x,y\in\mathbf{R}_+$. Then,

$$u(x,y)\le W_2(x,y)\left[c^{1-q}+(1-q)\int_0^x\int_0^y h(s,t)W_2^q(s,t)\mathrm{d}t\mathrm{d}s\right]^{\frac{1}{1-q}},$$

where

$$W_2(x,y)=E_\alpha(\Gamma(\alpha)x^\alpha)E_\beta(E_\alpha(\Gamma(\alpha)x^\alpha)\Gamma(\beta)y^\beta).$$

Proof. Define

$$M(x,y)=c+\int_0^x\int_0^y h(s,t)u^q(s,t)\mathrm{d}t\mathrm{d}s. \tag{2.25}$$

Then, (2.24) reduces to

$$u(x,y)\le M(x,y)+\int_0^x(x-s)^{\alpha-1}u(s,y)\mathrm{d}s+\int_0^y(y-t)^{\beta-1}u(x,t)\mathrm{d}t.$$

Obviously, $M(x,y)$ is positive, nondecreasing continuous function. So one has

$$\frac{u(x,y)}{M(x,y)}\le 1+\int_0^x(x-s)^{\alpha-1}\frac{u(s,y)}{M(s,y)}\mathrm{d}s+\int_0^y(y-t)^{\beta-1}\frac{u(x,t)}{M(x,t)}\mathrm{d}t.$$

Set

$$A(x,y)=1+\int_0^x(x-s)^{\alpha-1}\frac{u(s,y)}{M(s,y)}\mathrm{d}s+\int_0^y(y-t)^{\beta-1}\frac{u(x,t)}{M(x,t)}\mathrm{d}t.$$

We have

$$\frac{u(x,y)}{M(x,y)}\le A(x,y),$$
$$A(x,y)\le 1+\int_0^x(x-s)^{\alpha-1}A(s,y)\mathrm{d}s+\int_0^y(y-t)^{\beta-1}A(x,t)\mathrm{d}t.$$

Let

$$B(x,y) = 1 + \int_0^y (y-t)^{\beta-1} A(x,t)\mathrm{d}t, \tag{2.26}$$

which implies

$$A(x,y) \le B(x,y) + \int_0^x (x-s)^{\alpha-1} A(s,y)\mathrm{d}s.$$

Recalling $B(x,y)$ is positive, nondecreasing continuous function, we obtain

$$\frac{A(x,y)}{B(x,y)} \le 1 + \int_0^x (x-s)^{\alpha-1} \frac{A(s,y)}{B(s,y)}\mathrm{d}s.$$

Set

$$C(x,y) = 1 + \int_0^x (x-s)^{\alpha-1} \frac{A(s,y)}{B(s,y)}\mathrm{d}s.$$

Therefore, we obtain

$$\begin{aligned} \frac{A(x,y)}{B(x,y)} &\le C(x,y), \\ C(x,y) &\le 1 + \int_0^x (x-s)^{\alpha-1} C(s,y)\mathrm{d}s. \end{aligned}$$

By using to the same steps from (2.7) to (2.9), we have

$$A(x,y) \le B(x,y) E_\alpha(\Gamma(\alpha)x^\alpha). \tag{2.27}$$

By combining (2.26) and (2.27), we obtain

$$\begin{aligned} A(x,y) &\le E_\alpha(\Gamma(\alpha)x^\alpha)\left[1 + \int_0^y (y-t)^{\beta-1} A(x,t)\mathrm{d}t\right] \\ &\le E_\alpha(\Gamma(\alpha)x^\alpha) + E_\alpha(\Gamma(\alpha)x^\alpha)\int_0^y (y-t)^{\beta-1} A(x,t)\mathrm{d}t. \end{aligned} \tag{2.28}$$

Let $\mathbf{B}\phi(x,y) = E_\alpha(\Gamma(\alpha)x^\alpha)\int_0^y (y-t)^{\beta-1}\phi(x,t)\mathrm{d}t$. Then, formula (2.28) implies that

$$\begin{aligned} A(x,y) &\le E_\alpha(\Gamma(\alpha)x^\alpha) + \mathbf{B}A(x,y), \\ A(x,y) &\le \sum_{k=0}^{n-1} \mathbf{B}^k E_\alpha(\Gamma(\alpha)x^\alpha) + \mathbf{B}^n A(x,y). \end{aligned}$$

By using the similar steps in (2.8) and (2.9), we obtain

$$\begin{aligned} &\mathbf{B}^n A(x,y) \le \frac{(E_\alpha(\Gamma(\alpha)x^\alpha)\Gamma(\beta))^n}{\Gamma(n\beta)} \int_0^y (y-t)^{n\beta-1} A(x,t)\mathrm{d}t, \\ &\mathbf{B}^n A(x,y) \to 0 \quad \text{as } n \to +\infty \text{ for each } x, y \in \mathbf{R}_+. \end{aligned}$$

Then,

$$A(x,y) \le \sum_{k=0}^{\infty} \mathbf{B}^k E_\alpha(\Gamma(\alpha)x^\alpha) \le E_\alpha(\Gamma(\alpha)x^\alpha)E_\beta(E_\alpha(\Gamma(\alpha)x^\alpha)\Gamma(\beta)y^\beta) = W_2(x,y).$$

Therefore, we have

$$u(x,y) \le A(x,y)M(x,y) \le W_2(x,y)M(x,y). \tag{2.29}$$

Taking the partial derivatives with respect to x and y on both sides of (2.25), respectively, we also have

$$M_{xy}(x,y) = h(x,y)u^q(x,y),$$

which implies from (2.29)

$$\frac{M_{xy}(x,y)}{M^q(x,y)} \le h(x,y)W_2^q(x,y).$$

By using the similar steps of (2.12)–(2.13), one has

$$M(x,y) \le \left[c^{1-q} + (1-q)\int_0^x\int_0^y h(s,t)W_2^q(s,t)\mathrm{d}t\mathrm{d}s\right]^{\frac{1}{1-q}},$$

and hence,

$$u(x,y) \le W_2(x,y)\left[c^{1-q} + (1-q)\int_0^x\int_0^y h(s,t)W_2^q(s,t)\mathrm{d}t\mathrm{d}s\right]^{\frac{1}{1-q}}.$$

The proof is complete. □

Corollary 2.1. *Under the assumptions in* Theorem 2.5, *but* $q = 1$, *if the following inequality holds*

$$u(x,y) \le c + \int_0^x (x-s)^{\alpha-1}u(s,y)\mathrm{d}s + \int_0^y (y-t)^{\beta-1}u(x,t)\mathrm{d}t + \int_0^x\int_0^y h(s,t)u(s,t)\mathrm{d}t\mathrm{d}s,$$

for all $x, y \in \mathbf{R}_+$. *Then,*

$$u(x,y) \le cW_2(x,y)\exp\left[\int_0^x\int_0^y h(s,t)W_2(s,t)\mathrm{d}t\mathrm{d}s\right], \tag{2.30}$$

where $W_2(x,y)$ *is defined in* Theorem 2.5.

By using the proof of Theorem 2.5, we can easily obtain the conclusion (2.30), and we omit the details here.

3 Applications

In this section, two applications are presented to demonstrate the applicability of the main results.

For conveniences, we denote by $C(\mathbf{R}_+ \times \mathbf{R}_+ \times \mathbf{R}, \mathbf{R})$ the set of all continuous functions from $\mathbf{R}_+ \times \mathbf{R}_+ \times \mathbf{R}$ into $\mathbf{R}$. We first apply our Theorem 2.5 to study the boundedness of solutions for the following integral equation:

$$u(x,y) = u_0(x,y) + \int_0^x A(x,s,u(s,y))ds + \int_0^y B(y,t,u(x,t))dt + \int_0^x\int_0^y C(s,t,u(s,t))dtds, \quad \forall x,y \in \mathbf{R}_+, \tag{3.1}$$

where $u_0 \in C(\mathbf{R}_+ \times \mathbf{R}_+)$, $A, B, C \in C(\mathbf{R}_+ \times \mathbf{R}_+ \times \mathbf{R}, \mathbf{R})$. We will show the following:

Theorem 3.1. *Assume that the functions u_0, A, B, and C* in (3.1) *satisfy the following conditions,*

$$|u_0| \le c, \tag{3.2}$$

$$|A(x,s,u)| \le (x-s)^{\alpha-1}|u|, \tag{3.3}$$

$$|B(y,t,u)| \le (y-t)^{\beta-1}|u|, \tag{3.4}$$

$$|C(s,t,u)| \le h(s,t)|u|^q, \tag{3.5}$$

where $c > 0$, q, α, β, and h satisfy the same conditions as those in Theorem 2.5. *If $u(x,y)$ is a solution of equation* (3.1), *then*

$$|u(x,y)| \le W_2(x,y)\left[c^{1-q} + (1-q)\int_0^x\int_0^y h(s,t)W_2^q(s,t)dtds\right]^{\frac{1}{1-q}}, \tag{3.6}$$

for all $x, y \in \mathbf{R}_+$, where W_2 is defined as in Theorem 2.5.

Proof. By using the conditions (3.2)–(3.5) into (3.1), we obtain

$$\begin{aligned}|u(x,y)| \le c &+ \int_0^x (x-s)^{\alpha-1}|u(s,y)|ds + \int_0^y (y-t)^{\beta-1}|u(x,t)|dt \\ &+ \int_0^x\int_0^y h(s,t)|u(s,t)|^q dtds \quad \forall x,y \in \mathbf{R}_+.\end{aligned} \tag{3.7}$$

By applying Theorem 2.5 to (3.7), we can obtain the desired result (3.6). □

In the sequel, we shall illustrate that our Theorem 2.1 can be applied to study the boundedness of solutions of a class of partial differential equations in two independent variables.

Consider the problem of the form

$$u_{xy}(x,y) = f(y)u_x(x,y) + F(x,y,u(x,y)), \quad \forall x,y \in \mathbf{R}_+, \tag{3.8}$$

$$u(x,0) = a_1(x), \quad u(0,y) = a_2(y), \quad a_1(0) = a_2(0) = 0, \quad \forall x,y \in \mathbf{R}_+, \tag{3.9}$$

where $f \in C(\mathbf{R}_+, \mathbf{R})$ and $F \in C(\mathbf{R}_+ \times \mathbf{R}_+ \times \mathbf{R}, \mathbf{R})$.

We use our result to study the boundedness of the solution of the aforementioned initial boundary value problem.

Theorem 3.2. *Assume that $f \in C(\mathbf{R}_+, \mathbf{R}) \cap L^1(\mathbf{R}_+, \mathbf{R})$, $F \in C(\mathbf{R}_+ \times \mathbf{R}_+ \times \mathbf{R}, \mathbf{R})$, $a_1, a_2 \in L^\infty(\mathbf{R}_+, \mathbf{R})$ and*

$$|F(s,t,u)| \le h(s,t)|u|^q. \tag{3.10}$$

If f and h are nonnegative, nondecreasing continuous functions and $0 < q < 1$, $\alpha > 0$, $\beta > 0$, *then*

$$u(x,y) \le W(x,y)E_\alpha(\Gamma(\alpha)x^\alpha)\left[c^{1-q} + (1-q)\int_0^x\int_0^y h(s,t)(W(s,t)E_\alpha(\Gamma(\alpha)s^\alpha))^q dtds\right]^{\frac{1}{1-q}}, \tag{3.11}$$

for all $x, y \in \mathbf{R}_+$, where W is defined as in Theorem 2.1.

Proof. If the boundary value problem (3.8) with (3.9) has a solution $u(x, y)$, then u satisfies the following integral equation:

$$u(x,y) = a_1(x) + a_2(y) - \int_0^y f(t)a_2(t)\mathrm{d}t + \int_0^y f(t)u(x,t)\mathrm{d}t + \int_0^x\int_0^y F(s,t,u(s,t))\mathrm{d}t\mathrm{d}s, \quad \text{for all } x, y \in \mathbf{R}_+.$$

By using the assumptions and (3.10), we have

$$\begin{aligned}|u(x,y)| &\le c + \int_0^y f(t)|u(x,t)|\mathrm{d}t + \int_0^x\int_0^y h(s,t)|u(s,t)|^q\mathrm{d}t\mathrm{d}s\\ &\le c + \int_0^x (x-s)^{\alpha-1}|u(s,y)|\mathrm{d}s + \int_0^y f(t)|u(x,t)|\mathrm{d}t + \int_0^x\int_0^y h(s,t)|u(s,t)|^q\mathrm{d}t\mathrm{d}s, \quad \forall x, y \in \mathbf{R}_+,\end{aligned}$$

where $c := |a_1(x)| + |a_2(y)| + \int_0^\infty |f(t)a_2(t)|\mathrm{d}t$. Then, Theorem 2.1 implies the desired result (3.11). □

4 Conclusion

In this paper, we establish several kinds of integral inequalities in two independent variables, which improve well-known versions of Gronwall-Bellman inequalities and extend them to fractional integral form. Although, we only give two simple examples in the article as applications, our theoretical results can be widely used, cf. [11,33] and the references therein.

There are many issues worthy of further study in the article. On the one hand, the exponent power q of the unknown function $u(s, t)$ in our theorems, we only discuss two cases, $0 < q < 1$ and $q = 1$. The proofs of the theorems for the two cases are not mutually included. We do not know what will happen if the exponent $q > 1$. On the other hand, apart from studying the fractional integral inequality for two independent variables, we can extend our results with two independent variables to those of n independent variables. Studying those problems will be useful for us to further study the theory of fractional calculus equations in the future.

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