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Research Article

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A class of $p_1(x, \cdot)$ & $p_2(x, \cdot)$-fractional Kirchhoff-type problem with variable $s(x, \cdot)$-order and without the Ambrosetti-Rabinowitz condition in $\mathbb{R}^N$

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Abstract: In this article, we study a class of Kirchhoff-type equation driven by the variable $s(x, \cdot)$-order fractional $p_1(x, \cdot)$ & $p_2(x, \cdot)$-Laplacian. With the help of three different critical point theories, we obtain the existence and multiplicity of solutions in an appropriate space of functions. The main difficulties and innovations are the Kirchhoff functions with double Laplace operators in the whole space $\mathbb{R}^N$. Moreover, the approach is variational, but we do not impose any Ambrosetti-Rabinowitz condition for the nonlinear term.

Keywords: Kirchhoff-type equation, fractional $p_1(x, \cdot)$ & $p_2(x, \cdot)$-Laplacian, variable $s(x, \cdot)$-order, abstract critical point theory

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1 Introduction

In this article, we study the existence and multiplicity of solutions for the following Kirchhoff-type equation:

$$\sum_{i=1}^{2} M_i\left(\iint_{\mathbb{R}^{2N}} \frac{|\eta(x)-\eta(y)|^{p_i(x,y)}}{p_i(x,y)|x-y|^{N+s(x,y)p_i(x,y)}}\mathrm{d}x\mathrm{d}y\right)(-\Delta)^{s(x,\cdot)}_{p_i(x,\cdot)}\eta + \sum_{i=1}^{2}|\eta|^{\overline{p}_i(x)-2}\eta = \lambda f(x,\eta), \tag{1.1}$$

for all $x \in \mathbb{R}^N$. M_i $(i = 1, 2)$ are continuous Kirchhoff-type functions in $\mathbb{R}^N$, λ is a real positive parameter, and the nonlinearity f is a Carathéodory function, whose hypothesis will be introduced later. $(-\Delta)^{s(x,\cdot)}_{p_i(x,\cdot)}$ are called fractional $p_i(x, \cdot)$-Laplacian with variable $s(x, \cdot)$-order, given $p_i(x, \cdot) : \mathbb{R}^{2N} \to (1, +\infty)$ $(i = 1, 2)$ and $s(x, \cdot) : \mathbb{R}^{2N} \to (0, 1)$ with $N > s(x, y)p_i(x, y)$ for all $(x, y) \in \mathbb{R}^N \times \mathbb{R}^N$, which can be defined as follows:

$$(-\Delta)^{s(x,\cdot)}_{p_i(x,\cdot)}\eta(x) \coloneqq \text{P.V.} \int_{\mathbb{R}^N} \frac{|\eta(x)-\eta(y)|^{p_i(x,y)-2}(\eta(x)-\eta(y))}{|x-y|^{N+s(x,y)p_i(x,y)}}\mathrm{d}y, \tag{1.2}$$

for all x in $\mathbb{R}^N$, $\eta \in C_0^\infty(\mathbb{R}^N)$ and P.V. stands for the Cauchy principal value. Especially, when $s(x, \cdot) \equiv s$ and $p_i(x, \cdot) \equiv p_i$, $(-\Delta)^{s(x,\cdot)}_{p_i(x,\cdot)}$ in (1.1) reduces to the fractional p-Laplace operator $(-\Delta)^s_p$, e.g., see [1] involving the

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fractional p-Laplacian problem without the Ambrosetti-Rabinowitz (A-R) condition and see [2] on the fractional p & q-Laplacian problem with critical Sobolev-hardy exponents.

Throughout this article, we assume that $p_i(x,y)$ $(i=1,2)$ and $s(x,y)$ are continuous functions and the hypotheses we impose on $p_i(x,y)$ and $s(x,y)$ are as follows:

(P): $p_i(x,y)$ are symmetric functions, i.e., $p_i(x,y)=p_i(y,x), 1<p_i^- \coloneqq \inf_{(x,y)\in\mathbb{R}^N\times\mathbb{R}^N} p_i(x,y) \le p_i^+ \coloneqq \sup_{(x,y)\in\mathbb{R}^N\times\mathbb{R}^N} p_i(x,y) < +\infty$. We denote $p_{\max}(x,y)=\max\{p_1(x,y),p_2(x,y)\}$, $p_{\min}(x,y)=\min\{p_1(x,y),p_2(x,y)\}$, and $\overline{p_i}(x)=p_i(x,x)$.

(S): $s(x,y)$ is a symmetric function, i.e., $s(x,y)=s(y,x)$, $0<s^- \coloneqq \inf_{(x,y)\in\mathbb{R}^N\times\mathbb{R}^N} s(x,y) \le s^+ \coloneqq \sup_{(x,y)\in\mathbb{R}^N\times\mathbb{R}^N} s(x,y)<1$, and $\bar{s}(x)=s(x,x)$.

Kirchhoff in [3] introduced the following model, which came to be known as the Kirchhof-type equation:

$$\rho\frac{\partial^2\eta(x)}{\partial t^2}-\left(\frac{p_0}{h}+\frac{E}{2L}\int_0^L\left|\frac{\partial\eta(x)}{\partial t}\right|^2 \mathrm{d}x\right)\frac{\partial^2\eta(x)}{\partial x^2}=0, \tag{1.3}$$

where parameters ρ, p_0, h, E, and L, with some specific physical meaning, are real positive constants. Particularly, Equation (1.3) is nonlocal fractional problem that contains a nonlocal coefficient $\frac{p_0}{h}+\frac{E}{2L}\int_0^L\left|\frac{\partial\eta(x)}{\partial t}\right|^2 \mathrm{d}x$ and can be used to model some physical systems in concrete real-world application, such as anomalous diffusion, ultra-relativistic of quantum mechanics, and water waves. Since then the literature on Kirchhof-type equations and Kirchhoff-type systems are quite large, and here we just list a few, e.g., see [4–7] for further details.

We assume that $M_i:\mathbb{R}_0^+\to\mathbb{R}^+$ $(i=1,2)$ are continuous functions, which the following conditions are satisfied:

$(\mathcal{M}_1)$: There are some positive constants $\vartheta_i\in[1,p^*_{s(x,\cdot)}/p^+_{\max})$ $(i=1,2)$ and $\vartheta=\max\{\vartheta_1,\vartheta_2\}$ such that

$$tM_i(t)\le\vartheta_i\widetilde{M}_i(t),\quad \text{for any } t\in\mathbb{R}_0^+,\quad \text{where } \widetilde{M}_i(t)=\int_0^t M_i(\tau)\mathrm{d}\tau.$$

$(\mathcal{M}_2)$: There are some positive constants $m_i=m_i(\tau)>0$ $(i=1,2)$ for all $\tau>0$ such that

$$M_i(t)\ge m_i,\quad \text{for any } t>\tau.$$

$(\mathcal{M}_3)$: $M_i(t)$ is a decreasing function.

The evolution of the Laplace operator has been progressively deepened and has taken many forms so far. Many mathematical scholars have been devoted to the integer Laplace operators, fractional Laplace operators, and variable-order fractional Laplace operators. For some important results of these operators, we recommend the readers to refer to previous studies in [8–19] and literature cited therein.

In the framework of variable exponents involving fractional $p(x,\cdot)$-Laplace operator with variable $s(x,\cdot)$-order, there have been some papers on this topic involving both with and without a Kirchhoff coefficient, for instance, see [20–25]. Especially, an embedding theorem for variable exponential Sobolev spaces was first proved in [26]. In addition, with the help of variational methods, Zuo et al. in [27] studied a class of fractional $p(x,\cdot)$-Kirchhoff-type problem with the presence of a single Laplace operator in the whole space $\mathbb{R}^N$.

Problem (1.1) comes from the following system:

$$\eta_t=\operatorname{div}[D\eta\nabla\eta]+c(x,\eta), \tag{1.4}$$

where $D\eta=|\nabla\eta|^{p-2}+|\nabla\eta|^{q-2}$. System (1.4) had a wide range of applications in the field of physics and related sciences and had been paid much attention, for example, see [28–32].

Since both p & q-Laplacian that is not homogeneous are involved, it is more difficult to get the corresponding estimates to compare to the case $p = q > 1$; therefore, we do need more careful analysis. The case on the whole space $\mathbb{R}^N$ was studied in [33], and He and Li used the constraint minimization to study the subcritical growth problem:

$$\begin{cases} -\operatorname{div}|\nabla\eta|^{p-2}\nabla\eta - \operatorname{div}|\nabla\eta|^{q-2}\nabla v + m|\eta|^{p-2}\eta + n|\eta|^{q-2}\eta = f(x,\eta(x)), & x \in \mathbb{R}^N, \\ \eta(x) \in W^{1,p}(\mathbb{R}^N) \bigcap W^{1,q}(\mathbb{R}^N), \end{cases}$$

where $m, n > 0$, $N \geq 3$, and $1 < q < p < N, f(x,\eta)/\eta^{p-1}$ tend to a positive constant l as $v \to \infty$.

Chaves et al. in [34] analyzed the existence of weak solution in $D^{1,p}(\mathbb{R}^N) \bigcap D^{1,q}(\mathbb{R}^N)$ for the equation involving weight functions as follows:

$$\begin{cases} -\Delta_p\eta - \Delta_q\eta + a(x)|\eta|^{p-2}\eta + b(x)|\eta|^{q-2}\eta = f(x,\eta), & x \in \mathbb{R}^N, \\ \eta(x) \in W^{1,p}(\mathbb{R}^N) \bigcap W^{1,q}(\mathbb{R}^N), \end{cases} \tag{1.5}$$

where $1 < q < p < q^\star := \frac{Nq}{N-q}$, $p < N$. They proved that problem (1.5) possessed at least one weak solution even if the nonlinear term f did not satisfy the (A-R) condition.

There are few papers to consider the $p(x,\cdot)$ & $q(x,\cdot)$-Laplacian problem. The case is on the bounded domain Ω, and Zuo et al. in [35] investigated a kind of the Choquard-type problems without a Kirchhoff coefficient:

$$\begin{cases} (-\Delta)^{s(x,\cdot)}_{p(x,\cdot)}\eta + (-\Delta)^{s(x,\cdot)}_{q(x,\cdot)}\eta = \lambda|\eta|^{\beta(x)-2}\eta + \left(\displaystyle\int_\Omega \frac{G(y,\eta(y))}{|x-y|^{\mu(x,y)}}\mathrm{d}y\right)g(x,\eta(x)) + k(x), & x \in \Omega, \\ \eta(x) = 0, \quad x \in \mathbb{R}^N\backslash\Omega, \end{cases} \tag{1.6}$$

where the operators $(-\Delta)^{s(x,\cdot)}_{p(x,\cdot)}$ & $(-\Delta)^{s(x,\cdot)}_{q(x,\cdot)}$ are two fractional Laplace operators with variable order $s(x,\cdot) : \mathbb{R}^{2N} \to (0,1)$ and different variable exponents $p(x,\cdot), q(x,\cdot) : \mathbb{R}^{2N} \to (1,\infty)$. The results of problem (1.6) are different from the single fractional Laplace operator.

While combining the $p(x,\cdot)$ & $q(x,\cdot)$-Laplacian with Kirchhoff coefficients, Zhang in [36] devoted to the study of the following equations:

$$\begin{cases} \displaystyle\sum_{i=1}^{2} M_i\left(\int_{\Omega\times\Omega} \frac{|\eta(x)-\eta(y)|^{p_i(x,y)}}{p_i(x,y)|x-y|^{N+sp_i(x,y)}}\mathrm{d}x\mathrm{d}y\right)\left(-\Delta_{p_i(x)}\right)^s\eta = f(x,\eta), & x \in \Omega, \\ \eta = 0, \quad x \in \mathbb{R}^N\backslash\Omega, \end{cases} \tag{1.7}$$

where M_i $(i = 1, 2)$ is a model of Kirchhoff coefficient on the bounded domain Ω. and $(-\Delta_{p_i(x)})^s$ is fractional Laplace operators with a constant order. On the basis of variational method and critical point theory, he proved the existence of solutions for problems (1.7) in an appropriate space of functions.

In the famous paper [37], Ambrosetti and Rabinowitz introduced the well-known (A-R) condition on the nonlinearity, that is, there exist some positive constants μ_0 such that

$$0 < \mu_0 F(x,\eta) \leq f(x,\eta)\eta, \quad \text{for all } (x,\eta) \in \mathbb{R}^N \times \mathbb{R},$$

where $F(x,\eta) = \int_0^\eta f(x,t)\mathrm{d}t$. As is known, the (A-R) condition plays a very important role in the application of the variational method, which is widely used to guarantee that the Palais-Smale sequences are bounded and the function I_λ has a mountain pass geometry. However, some interesting nonlinearities do not satisfy the (A-R) condition, and an example of such a function (see [38]) is expressed as follows:

$$f(\eta) = \begin{cases} |\eta|^{p_0-2}\eta - \left(\dfrac{p_0-1}{p_0}\right)|\eta|^{r_0-2}\eta, & |\eta| \leq 1, \\ |\eta|^{p_0-2}\eta\log\left(\dfrac{1}{p_0} + |\eta|\right), & |\eta| > 1, \end{cases}$$

where $1 < p_0 < r_0 < p^* = \frac{Np_0}{N-p_0}$. Indeed, this function does not satisfy

$$F(x,\eta) \geq d_1|\eta|^{\mu_0} - d_2, \quad \text{for all } (x,\eta) \in \mathbb{R}^N \times \mathbb{R},$$

where $d_1, d_2 > 0$. Hence, many researchers pay attention to find the new reasonable conditions instead of the (A-R) condition, see [24,39] and the references therein.

As far as we know, there is no result for Kirchhoff-type equation involving double fractional $p_1(x,\cdot)$ & $p_2(x,\cdot)$-Laplace operators with a variable $s(x,\cdot)$-order without the (A-R) condition in the whole space $\mathbb{R}^N$. Therefore, motivated by the previous and aforementioned cited works, we will investigate the existence and multiplicity of solutions for this kind of equation, which is different from the work of [15,34,35] that the equations of these problems involve the fractional p-Laplace operator with a constant order or do not contain Kirchhoff terms, and more general than (1.7), which authors considered a local version of the fractional operator, that is, with an integral set in Ω and not in the whole space $\mathbb{R}^N$. Our study extends previous results in some ways.

Throughout this article, C_j $(j = 1, 2, \dots, N)$ denote distinct positive constants, and $i = 1$ and 2. For any real-valued function H defined on a domain $\mathfrak{D}$, we denote:

$$C_+(\mathfrak{D}) := \{H \in C(\mathfrak{D}, \mathbb{R}) : \quad 1 < H^- := \inf_{\mathfrak{D}} H \leq H \leq H^+ := \sup_{\mathfrak{D}} H < +\infty\}.$$

The function $a_i : \mathbb{R}^N \to \mathbb{R}$ are continuous functions, which satisfy the following conditions:

($\mathcal{A}_1$): $a, a_i \in L^{h(x)}(\mathbb{R}^N)$ such that $a(x), a_i(x) \geq 0$, where $h \in C_+(\mathbb{R}^N)$.

($\mathcal{A}_2$): $a, a_i \in C(\mathbb{R}^N \times \mathbb{R})$ such that $a(x), a_i(x) \geq 0$ for all $x \in \mathbb{R}^N$ and $a, a_i \neq 0$.

The nonlinearity $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, satisfying:

($\mathcal{F}_1$): Let $p_i, q \in C_+(\mathfrak{D})$ and $1 < p_{\max}^+ < q^- \leq q(x) \leq q^+ < \vartheta < p_s^*(x)$, and there exist $a_1(x)$ and $a_2(x)$, given by ($\mathcal{A}_1$) such that

$$|f(x,t)| \leq a_1(x) + a_2(x)|t|^{q(x)-1} \text{ for all } (x,t) \in \mathbb{R}^N \times \mathbb{R}.$$

($\mathcal{F}_2$): $\lim_{|t|\to\infty} \frac{F(x,t)}{|t|^{\vartheta p_{\max}^+}} = +\infty$, uniformly for all $x \in \mathbb{R}^N$, where $F(x,t) = \int_0^t f(x,s)\mathrm{d}s > 0$.

($\mathcal{F}_3$): There exists $\beta(x) \in L^\infty(\mathbb{R}^N)_+$ such that $\limsup_{|t|\to 0} \frac{p_{\max}^+ F(x,t)}{|t|^{p_{\max}^+}} \leq \beta(x)$, uniformly for all $x \in \mathbb{R}^N$.

($\mathcal{F}_4$): There exists a constant $\tau \geq 1$ such that $\tau\varrho(x,t) \geq \varrho(x,\iota t)$ for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$ and $\iota \in [0,1]$, where $\varrho(x,t) := f(x,t)t - p_{\max}^+ F(x,t)$;

($\mathcal{F}_5$): $f(x,-t) = -f(x,t)$ for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$.

The paper is organized into five sections. Aside from Sections 1, 2 presents the main results, Section 3 presents some preliminary notions and results about fractional Lebesgue spaces and Sobolev spaces, Section 4 proves the compactness condition of Cerami sequence and Theorems 2.1–2.3, and Section 5 presents a conclusion.

2 The main results

We need to present the corresponding definition and variational framework before stating our main results.

Definition 1. We say that $\eta \in X$ is a weak solution of problem (1.1), if

$$\sum_{i=1}^{2} M_i(\delta_{p_i}(\eta)) \times \langle \delta'_{p_i}(\eta), \varphi\rangle + \sum_{i=1}^{2} \int_{\mathbb{R}} |\eta|^{\overline{p}_i(x)-2}\eta\varphi \mathrm{d}x = \lambda \int_{\mathbb{R}} f(x,\eta)\varphi \mathrm{d}x,$$

for any $\varphi \in X$, where X will be introduced in Section 2 and

$$\delta_{p_i}(\eta) = \iint\limits_{\mathbb{R}^2} \frac{|\eta(x) - \eta(y)|^{p_i(x,y)}}{p_i(x,y)|x-y|^{N+s(x,y)p_i(x,y)}} \mathrm{d}x\mathrm{d}y,$$

$$\langle \delta'_{p_i}(\eta), \varphi \rangle = \iint\limits_{\mathbb{R}^2} \frac{|\eta(x) - \eta(y)|^{p_i(x,y)-2}(\eta(x) - \eta(y))(\varphi(x) - \varphi(y))}{|x-y|^{N+s(x,y)p_i(x,y)}} \mathrm{d}x\mathrm{d}y.$$

The problem (1.1) has a variational form with the function $I_\lambda : X \to \mathbb{R}$, defined as follows:

$$I_\lambda(\eta) := \sum_{i=1}^{2} \widetilde{M_i}(\delta_{p_i}(\eta)) + \sum_{i=1}^{2} \int\limits_{\mathbb{R}} \frac{1}{\overline{p}_i(x)} |\eta|^{\overline{p}_i(x)} \mathrm{d}x - \lambda \int\limits_{\mathbb{R}} F(x,\eta)\mathrm{d}x, \tag{2.1}$$

for all $\eta \in X$ and $\widetilde{M_i}$ given in $(\mathcal{M}_1)$. Moreover, the function I_λ is well defined on the Sobolev spaces X and belongs to the class $C^1(X, \mathbb{R})$, which the argument is similar to [21], and

$$\langle I'_\lambda(\eta), \varphi \rangle := \sum_{i=1}^{2} M_i(\delta_{p_i}(\eta)) \times \langle \delta'_{p_i}(\eta), \varphi \rangle + \sum_{i=1}^{2} \int\limits_{\mathbb{R}} |\eta|^{\overline{p}_i(x)-2} \eta\varphi \mathrm{d}x - \lambda \int\limits_{\mathbb{R}} f(x,\eta)\varphi \mathrm{d}x, \tag{2.2}$$

for any $\eta, \varphi \in X$. Thus, under our assumptions, the existence and multiplicity of solutions for problem (1.1) is equivalent to the existence of critical points for the function I_λ.

Now, we are ready to state three results of this paper.

Theorem 2.1. *Assume that* $(\mathcal{S})$, $(\mathcal{P})$, $(\mathcal{M}_1)$–$(\mathcal{M}_3)$, $(\mathcal{A}_1)$–$(\mathcal{A}_2)$, *and* $(\mathcal{F}_1)$–$(\mathcal{F}_4)$ *are satisfied. Then, for any* $\lambda > 0$, *the problem* (1.1) *admits at least one nontrivial solution* η_0 *in* X.

Theorem 2.2. *Assume that* $(\mathcal{S})$, $(\mathcal{P})$, $(\mathcal{M}_1)$–$(\mathcal{M}_3)$, $(\mathcal{A}_1)$–$(\mathcal{A}_2)$, *and* $(\mathcal{F}_1)$–$(\mathcal{F}_4)$ *are satisfied. Then, there exists* $\lambda^* > 0$, *for any* $\lambda \in (0, \lambda^*)$, *such that problem* (1.1) *possesses at least two distinct nontrivial solutions* η_1, η_2 *in* X.

Theorem 2.3. *Assume that* $(\mathcal{S})$, $(\mathcal{P})$, $(\mathcal{M}_1)$–$(\mathcal{M}_3)$, $(\mathcal{A}_1)$–$(\mathcal{A}_2)$, *and* $(\mathcal{F}_1)$–$(\mathcal{F}_5)$ *are satisfied. Then, for any* $\lambda > 0$, *the problem* (1.1) *has infinitely many nontrivial solutions in* X.

Remark 2.1. The main idea to overcome these difficulties lies on the $p_1(x, \cdot)$ & $p_1(x, \cdot)$-Laplace operators developed in [27,36], recently. Under the weaker conditions for the nonlinear term, the existence of at least one nontrivial solution (Theorem 2.1) was proved by applying the mountain pass theorem [40], and then, we obtain at least two distinct solutions (Theorem 2.2) and infinitely many solutions (Theorem 2.3) by using the generalized abstract critical point theorem [41] and fountain theorem [40], respectively.

Remark 2.2. Our work is different from the previous papers [2,15,33,35] in the sense because of Kirchhoff terms and the presence of the more complicated $p_1(x, \cdot)$ & $p_1(x, \cdot)$-Laplace operators, which makes our analysis more complicated. The work of this paper is to be of great importance in the development of the $p_1(x, \cdot)$ & $p_1(x, \cdot)$-Laplace operators theory.

3 Preliminary results

3.1 Variable exponents Lebesgue spaces

In this section, we briefly review some basic knowledge, lemmas, and propositions of generalized variable exponents Lebesgue spaces.

Let $\vartheta(x) \in C_+(\mathbb{R}^N)$, and we define the variable exponents Lebesgue spaces as follows:

$$L^{\vartheta(x)}(\mathbb{R}^N) := \left\{\eta : \eta \text{ is a measurable and } \int_{\mathbb{R}^N} |\eta|^{\vartheta(x)} \mathrm{d}x < \infty\right\},$$

provided with the Luxemburg norm

$$\|\eta\|_{\vartheta(x)} := \inf\left\{\chi > 0 : \int_{\mathbb{R}^N} \left|\frac{\eta}{\chi}\right|^{\vartheta(x)} \mathrm{d}x \le 1\right\}.$$

Then, $(L^{\vartheta(x)}(\mathbb{R}^N), \|\cdot\|_{\vartheta(x)})$ is a separable and reflexive Banach spaces, which is called generalized Lebesgue spaces, see [17,42].

Lemma 3.1. (See [42]) *Let $\vartheta(x)$ be the conjugate exponent of $\widetilde{\vartheta}(x) \in C_+(\mathbb{R}^N)$, that is, $\frac{1}{\vartheta(x)} + \frac{1}{\widetilde{\vartheta}(x)} = 1$, for all $x \in \mathbb{R}^N$. Suppose that $\eta \in L^{\vartheta(x)}(\mathbb{R}^N)$ and $u \in L^{\widetilde{\vartheta}(x)}(\mathbb{R}^N)$, then*

$$\left|\int_{\mathbb{R}^N} \eta u \mathrm{d}x\right| \le \left(\frac{1}{\vartheta^-} + \frac{1}{\widetilde{\vartheta}^-}\right)\|\eta\|_{\vartheta(x)}\|u\|_{\widetilde{\vartheta}(x)} \le 2\|\eta\|_{\vartheta(x)}\|u\|_{\widetilde{\vartheta}(x)}.$$

Proposition 3.1. (See [43]) *The modular of $L^{\vartheta(x)}(\mathbb{R}^N)$, which is the mapping $\rho_{\vartheta(x)} : L^{\vartheta(x)}(\mathbb{R}^N) \to \mathbb{R}$, is defined by*

$$\rho_{\vartheta(x)}(\eta) := \int_{\mathbb{R}^N} |\eta|^{\vartheta(x)} \mathrm{d}x.$$

Suppose that $\eta_n, \eta \in L^{\vartheta(x)}(\mathbb{R}^N)$, then the following properties hold

(1) $\|\eta\|_{\vartheta(x)} > 1 \Rightarrow \|\eta\|_{\vartheta(x)}^{\vartheta^-} \le \rho_{\vartheta(x)}(\eta) \le \|\eta\|_{\vartheta(x)}^{\vartheta^+}$,

(2) $\|\eta\|_{\vartheta(x)} < 1 \Rightarrow \|\eta\|_{\vartheta(x)}^{\vartheta^+} \le \rho_{\vartheta(x)}(\eta) \le \|\eta\|_{\vartheta(x)}^{\vartheta^-}$,

(3) $\|\eta\|_{\vartheta(x)} < 1$ (*resp.* $=1, >1$) $\Leftrightarrow \rho_{\vartheta(x)}(\eta) < 1$ (*resp.* $= 1, >1$),

(4) $\|\eta_n\|_{\vartheta(x)} \to 0$ (*resp.* $\to +\infty$) $\Leftrightarrow \rho_{\vartheta(x)}(\eta_n) \to 0$ (*resp.* $\to +\infty$),

(5) $\lim_{n\to\infty}|\eta_n - \eta|_{\vartheta(x)} = 0 \Leftrightarrow \lim_{n\to\infty}\rho_{\vartheta(x)}(\eta_n - \eta) = 0$.

Lemma 3.2. (See [44]) *Suppose that $|\eta|^{\vartheta(x)} \in L^{\beta_1(x)/\vartheta(x)}(\mathbb{R}^N)$, where $\vartheta(x), \beta_1(x) \in C_+(\mathbb{R}^N)$, and $\vartheta(x) \le \beta_1(x)$ for all $x \in \mathbb{R}^N$, then $\eta \in L^{\beta_1(x)}(\mathbb{R}^N)$, and there exists a number $\overline{\vartheta} \in [\vartheta^-, \vartheta^+]$ such that*

$$\||\eta|^{\vartheta(x)}\|_{\beta_1(x)/\vartheta(x)} = (\|\eta\|_{\beta_1(x)})^{\overline{\vartheta}}.$$

3.2 Variable-order fractional Sobolev spaces

From now on, we recall some important lemmas and properties about fractional Sobolev spaces with variable $s(x,\cdot)$-order.

Let $p(x,\cdot) \in C_+(\mathbb{R}^{2N})$ and define the Gagliardo seminorm by

$$[\eta]_{s(x,\cdot),p(x,\cdot)} := \inf\left\{\chi > 0 : \iint_{\mathbb{R}^{2N}} \frac{|\eta(x) - \eta(y)|^{p(x,y)}}{\chi^{p(x,y)}|x - y|^{N+p(x,y)s(x,y)}} \mathrm{d}x\mathrm{d}y < 1\right\},$$

and we consider the following variable $s(x,\cdot)$-order fractional Sobolev spaces with variable exponents

$$W = W^{s(x,\cdot),p(x,\cdot)}(\mathbb{R}^N) := \{\eta \in L^{\overline{p}(\cdot)}(\mathbb{R}^N) : \eta \text{ is a measurable and } [\eta]_{s(x,\cdot),p(x,\cdot)} < \infty\},$$

endowed with the norm

$$\|\eta\|_W := \|\eta\|_{\overline{p}(\cdot)} + [\eta]_{s(x,\cdot),p(x,\cdot)}.$$

Then, $(W, \|\cdot\|_W)$ is a separable and reflexive Banach spaces, see [26].

Proposition 3.2. (See [20,44]) *Define the modular function* $\rho^{s(x,\cdot)}_{p(x,\cdot)} : W \to \mathbb{R}$ *by*

$$\rho^{s(x,\cdot)}_{p(x,\cdot)}(\eta) = \iint_{\mathbb{R}^{2N}} \frac{|\eta(x) - \eta(y)|^{p(x,y)}}{|x-y|^{N+p(x,y)s(x,y)}} \mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}^N} |\eta|^{\overline{p}(x)} \mathrm{d}x.$$

Suppose that $\eta_n, \eta \in W$, *then the following properties hold*

(1) $\|\eta\|_W < 1$ $\quad$ (*resp.* $= 1, > 1$) $\Leftrightarrow \rho^{s(x,\cdot)}_{p(x,\cdot)}(\eta) < 1$ (*resp.* $= 1, > 1$),

(2) $\|\eta\|_W < 1 \Rightarrow \|\eta\|_W^{p^+} \le \rho^{s(x,\cdot)}_{p(x,\cdot)}(\eta) \le \|\eta\|_W^{p^-}$,

(3) $\|\eta\|_W > 1 \Rightarrow \|\eta\|_W^{p^-} \le \rho^{s(x,\cdot)}_{p(x,\cdot)}(\eta) \le \|\eta\|_W^{p^+}$,

(4) $\lim_{n\to\infty}\|\eta_n\|_W = 0$ (*resp.* $\to +\infty$) $\Leftrightarrow \lim_{n\to\infty}\rho^{s(x,\cdot)}_{p(x,\cdot)}(\eta_n) = 0$ $\quad$ (*resp.* $\to +\infty$),

(5) $\lim_{n\to\infty}\|\eta_n - \eta\|_W = 0 \Leftrightarrow \lim_{n\to\infty}\rho^{s(x,\cdot)}_{p(x,\cdot)}(\eta_n - \eta) = 0$.

Lemma 3.3. (See [26]) *Assume that* $s(x,\cdot)$, $p(x,\cdot)$ *fulfill* $(\mathcal{S})$, $(\mathcal{P})$ *with* $N > p(x,y)s(x,y)$ *for any* $(x,y) \in \overline{\Omega} \times \overline{\Omega}$. Set $\vartheta(x) \in C_+(\overline{\Omega})$ *fulfill*

$$1 < \vartheta^- = \min_{x\in\overline{\Omega}} \vartheta(x) \le \vartheta(x) < p^*_{s(x,\cdot)}(x) = \frac{N\overline{p}(x)}{N - \overline{p}(x)\overline{s}(x)}, \quad \textit{for any } x \in \overline{\Omega},$$

where $\overline{p}(x) = p(x,x)$ *and* $\overline{s}(x) = s(x,x)$. *Then, there exists* $C_\vartheta = C_\vartheta(N, s, p, \vartheta, \Omega) > 0$ *such that*

$$\|\eta\|_{\vartheta(\cdot)} \le C_\vartheta \|\eta\|_W,$$

for any $\eta \in W$. *Moreover, the embedding* $W \hookrightarrow L^{\vartheta(\cdot)}(\Omega)$ *is compact.*

Lemma 3.4. (See [26]) *Assume that* $s(x,\cdot)$, $p(x,\cdot)$ *fulfill* $(\mathcal{S})$, $(\mathcal{P})$ *with* $N > p(x,y)s(x,y)$ *for any* $(x,y) \in \mathbb{R}^N \times \mathbb{R}^N$, *and suppose that* $h \in C_+(\mathbb{R}^N)$ *is a uniformly continuous such that* $\overline{p}(x) \le h(x) < p^*_s(x)$ *for* $x \in \mathbb{R}^N$. *Then, the embedding* $X \hookrightarrow L^{h(\cdot)}(\mathbb{R}^N)$ *is continuous.*

3.3 $L^{\vartheta(x)}_{a(x)}(\mathbb{R}^N)$ spaces

We assume that $\vartheta(x) \in C_+(\mathbb{R}^N)$ and $a(x)$ satisfying $(\mathcal{A}_2)$, and consider the following spaces

$$L^{\vartheta(x)}_{a(x)}(\mathbb{R}^N) := \left\{\eta : \mathbb{R}^N \to \mathbb{R} \text{ is a measurable and } \int_{\mathbb{R}^N} a(x)|\eta(x)|^{\vartheta(x)}\mathrm{d}x < \infty\right\},$$

with the norm

$$\|\eta\|_{L^{\vartheta(x)}_{a(x)}(\mathbb{R}^N)} = \|\eta\|_{\vartheta,a(x)} := \inf\left\{\gamma > 0 : \int_{\mathbb{R}^N} a(x)\left|\frac{\eta(x)}{\gamma}\right|^{\vartheta(x)} \mathrm{d}x \le 1\right\}.$$

Obviously, $(L^{\vartheta(x)}_{a(x)}(\mathbb{R}^N), \|\cdot\|_{\vartheta,a(x)})$ is a uniformly convex Banach spaces and hence reflexive (see [27,43]). As the following lemma states, the norm $\|\eta\|_{\vartheta,a(x)}$ is connected to a semimodular $\varrho_{\vartheta,a(x)} = \int_{\mathbb{R}^N} a(x)|\eta(x)|^{\vartheta(x)}\mathrm{d}x$.

Lemma 3.5. (See [45]) *Suppose that $\eta_n \in L^{\vartheta(x)}_{a(x)}(\mathbb{R}^N)$, then the following result holds*

$$\lim_{n\to\infty} \|\eta_n\|_{\vartheta,a(x)} = 0 \Leftrightarrow \lim_{n\to\infty} \varrho_{\vartheta,a(x)}(\eta_n) = 0.$$

Lemma 3.6. (See [27]) *Let $s(x,\cdot)$ and $p(x,\cdot)$ satisfy $(\mathcal{S})$, $(\mathcal{P})$. Let $\vartheta(x) \in C_+(\mathbb{R}^N)$ with $1 < \vartheta^- \le \vartheta(x) \le \vartheta^+ < p_s^*(x)$ for all $x \in \mathbb{R}^N$. Suppose that $(\mathcal{A}_1)$ holds with h fulfilling*

$$\overline{p}(x) \le \psi(x) = \frac{h(x)\vartheta(x)}{h(x)-1} \le p_s^*(x) \text{ for all } x \in \mathbb{R}^N.$$

Then, the embedding $W \hookrightarrow L^{\vartheta(x)}_{a(x)}(\mathbb{R}^N)$ is continuous. Furthermore, if $\psi^+ < p_s^*(x)$ for all $x \in \mathbb{R}^N$, and then $W \hookrightarrow L^{\vartheta(x)}_{a(x)}(\mathbb{R}^N)$ is compact.

Lemma 3.7. (See [27]) *Let $s(x,\cdot)$ and $p(x,\cdot)$ satisfy $(\mathcal{S})$, $(\mathcal{P})$. Suppose that $\vartheta(x) \in C_+(\mathbb{R}^N)$ and $(\mathcal{A}_1)$ hold. Then, for any $\eta \in W$, there exist two positive constants $\overline{\vartheta} \in [\vartheta^-, \vartheta^+]$ and $C_{\vartheta,a(x)}$ such that*

$$\varrho_{\vartheta,a(x)}(\eta) \le C_{\vartheta,a(x)}\|\eta\|^{\overline{\vartheta}}.$$

Later on, we consider the following spaces:

$$X_i := \left\{\eta \in W = W^{s(x,\cdot),p_i(x,\cdot)}(\mathbb{R}^N) : \int_{\mathbb{R}^N} |\eta(x)|^{\overline{p}_i(x)}\mathrm{d}x < \infty\right\},$$

and this space endowed with the norm

$$\|\eta\|_{X_i} := \inf\left\{\gamma > 0 : \rho_{p_i(x,\cdot)}\left(\frac{\eta}{\gamma}\right) \le 1\right\},$$

where the function $\rho_{p_i(x,\cdot)} : X_i \to \mathbb{R}$ defined by

$$\rho_{p_i(x,\cdot)}(\eta) = \iint_{\mathbb{R}^{2N}} \frac{|\eta(x)-\eta(y)|^{p_i(x,y)}}{|x-y|^{N+p_i(x,y)s(x,y)}}\mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}^N} |\eta|^{\overline{p}_i(x)}\mathrm{d}x.$$

Obviously, the Banach spaces $(X_i, \|\cdot\|_{X_i})$ is uniformly convex and hence reflexive. Similar to Proposition 3.2, we obtain the following lemma.

Proposition 3.3. *Suppose that $\eta_n, \eta \in X_i$, then the following properties hold*
(1) $\|\eta\|_{X_i} < 1$ (*resp.* $= 1, >1$) $\Leftrightarrow \rho_{p_i(x,\cdot)}(\eta) < 1$ (*resp.* $= 1, > 1$),
(2) $\|\eta\|_{X_i} < 1 \Rightarrow \|\eta\|_{X_i}^{p_i^+} \le \rho_{p_i(x,\cdot)}(\eta) \le \|\eta\|_{X_i}^{p_i^-}$,
(3) $\|\eta\|_{X_i} > 1 \Rightarrow \|\eta\|_{X_i}^{p_i^-} \le \rho_{p_i(x,\cdot)}(\eta) \le \|\eta\|_{X_i}^{p_i^+}$,
(4) $\lim_{n\to\infty}\|\eta_n\|_{X_i} = 0$ (*resp.* $\to +\infty$) $\Leftrightarrow \lim_{n\to\infty}\rho_{p_i(x,\cdot)}(\eta_n) = 0$ (*resp.* $\to +\infty$),
(5) $\lim_{n\to\infty}\|\eta_n - \eta\|_{X_i} = 0 \Leftrightarrow \lim_{n\to\infty}\rho_{p_i(x,\cdot)}(\eta_n - \eta) = 0$.

Let $X = X_1 \cap X_2$, which is a separable and reflexive Banach space endowed with the norm

$$\|\eta\|_X = \|\eta\|_{X_1} + \|\eta\|_{X_2}.$$

To simplify the presentation, we will denote the norm of X by $\|\cdot\|$ instead of $\|\cdot\|_X$. X^* denotes the dual space of X.

We note that the embedding $X \hookrightarrow L^{\vartheta(x)}(\mathbb{R}^N)$ is no longer compact, which makes it difficult to verify the Cerami condition. The following embedding result provides a new tool to overcome this difficulty.

Lemma 3.8. *Let $s(x,\cdot)$ and $p(x,\cdot)$ satisfy $(\mathcal{S})$, $(\mathcal{P})$. Let $\vartheta(x) \in C_+(\mathbb{R}^N)$ with $1 < \vartheta^- \le \vartheta(x) \le \vartheta^+ < p_s^*(x)$ for all $x \in \mathbb{R}^N$. Suppose that $(\mathcal{A}_1)$ holds with h fulfilling*

$$\overline{p}(x) \le \psi(x) = \frac{h(x)\vartheta(x)}{h(x)-1} \le p_s^*(x) \text{ for all } x \in \mathbb{R}^N.$$

Then, the embedding $X \hookrightarrow L_{a(x)}^{\vartheta(x)}(\mathbb{R}^N)$ is continuous. Furthermore, if $\psi^+ < p_s^*(x)$ for all $x \in \mathbb{R}^N$, and then, $X \hookrightarrow L_{a(x)}^{\vartheta(x)}(\mathbb{R}^N)$ is compact.

Proof. For any $\psi(x) \in C^+(\mathbb{R}^N)$ satisfies $\psi^+ < p_s^*(x)$ for all $x \in \mathbb{R}^N$, and form Lemma 3.6, we obtain that $X_1 \hookrightarrow L_{a(x)}^{\vartheta(x)}(\mathbb{R}^N)$ and $X_2 \hookrightarrow L_{a(x)}^{\vartheta(x)}(\mathbb{R}^N)$, and therefore, we get $X \hookrightarrow L_{a(x)}^{\vartheta(x)}(\mathbb{R}^N)$, where the imbedding is continuous and compact. □

The function $\mathcal{L}(\eta) : X \to \mathbb{R}$ is defined as follows:

$$\mathcal{L}(\eta) = \mathcal{L}_{p_1(x,\cdot)}(\eta) + \mathcal{L}_{p_2(x,\cdot)}(\eta),$$

where

$$\mathcal{L}_{p_i(x,\cdot)}(\eta) = \iint_{\mathbb{R}^{2N}} \frac{1}{p_i(x,y)} \frac{|\eta(x)-\eta(y)|^{p_i(x,y)}}{|x-y|^{N+p_i(x,y)s(x,y)}} \mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}^N} \frac{1}{\overline{p}_i(x)} |\eta(x)|^{\overline{p}_i(x)} \mathrm{d}x.$$

Proposition 3.4. *Let $s(x,\cdot)$ and $p_i(x,\cdot)$ satisfy $\mathcal{S}$ and $\mathcal{P}$. We consider the following function $\mathcal{L}'_{p_i(x,\cdot)} : X_i \to X_i^*$, with X_i^* the dual space of X_i, such that*

$$\langle \mathcal{L}'_{p_i(x,\cdot)}(\eta), \phi \rangle = \iint_{\mathbb{R}^{2N}} \frac{|\eta(x)-\eta(y)|^{p_i(x,y)-2}(\eta(x)-\eta(y))(\phi(x)-\phi(y))}{|x-y|^{N+p_i(x,y)s(x,y)}} \mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}^N} |\eta(x)|^{\overline{p}_i(x)-2}\eta(x)\phi(x)\mathrm{d}x,$$

for any $\eta, \phi \in X_i$. Then:

(i) *$\mathcal{L}'_{p_i(x,\cdot)}$ is a bounded and strictly monotone operator;*

(ii) *$\mathcal{L}'_{p_i(x,\cdot)}$ is a mapping of type (S_+), that is, if $\eta_n \rightharpoonup \eta$ in X_i and $\limsup_{n\to\infty}\langle \mathcal{L}'_{p_i(x,\cdot)}(\eta_n) - \mathcal{L}'_{p_i(x,\cdot)}(\eta), \eta_n - \eta\rangle \le 0$, then $\eta_n \to \eta$ in X_i;*

(iii) *$\mathcal{L}'_{p_i(x,\cdot)} : X_i \to X_i^*$ is a homeomorphism.*

Proof. The proof of this proposition can be given arguing similar to Lemma 2.6 in [44] and Lemma 4.2 in [46] by combining with Lemma 3.6, which we omit here. □

Similar to Proposition 3.4, we have the following result.

Proposition 3.5. *Let $s(x,\cdot)$ and $p_i(x,\cdot)$ satisfy $\mathcal{S}$ and $\mathcal{P}$. Then:*

(i) *$\mathcal{L}'$ is a bounded and strictly monotone operator;*

(ii) *$\mathcal{L}'$ is a mapping of type (S_+), that is, if $\eta_n \rightharpoonup \eta$ in X and $\limsup_{n\to\infty}\langle \mathcal{L}'(\eta_n) - \mathcal{L}'(\eta), \eta_n - \eta\rangle \le 0$, then $\eta_n \to \eta$ in X;*

(iii) *$\mathcal{L}' : X \to X^*$ is a homeomorphism.*

Proof. Here, the proof is similar to [36], and we give a brief proof process for completeness.

(i) Obviously, $\mathcal{L}'$ is continuous and bounded. According to Proposition 3.4, since $\mathcal{L}'_{p_i(x,\cdot)}$ is a strictly monotone operator, $\mathcal{L}' = \mathcal{L}'_{p_1(x,\cdot)} + \mathcal{L}'_{p_2(x,\cdot)}$ is strictly a monotone operator. Consequently, $\mathcal{L}'$ is a bounded and strictly a monotone operator.

(ii) From (i) and if $\eta_n \rightharpoonup \eta$ in X and $\limsup_{n\to\infty}\langle \mathcal{L}'(\eta_n) - \mathcal{L}'(\eta), \eta_n - \eta\rangle \le 0$, as a consequence,

$$\lim_{n\to\infty} \langle \mathcal{L}'_{p_1(x,\cdot)}(\eta_n) - \mathcal{L}'_{p_1(x,\cdot)}(\eta) + \mathcal{L}'_{p_2(x,\cdot)}(\eta_n) - \mathcal{L}'_{p_2(x,\cdot)}(\eta), \eta_n - \eta\rangle = 0,$$

and it follows that as $n \to \infty$

$$\langle \mathcal{L}'_{p_1(x,\cdot)}(\eta_n) - \mathcal{L}'_{p_1(x,\cdot)}(\eta), \eta_n - \eta \rangle \to 0 \tag{3.1}$$

$$\text{and} \quad \langle \mathcal{L}'_{p_2(x,\cdot)}(\eta_n) - \mathcal{L}'_{p_2(x,\cdot)}(\eta), \eta_n - \eta \rangle \to 0. \tag{3.2}$$

Since $\eta_n \to \eta$ in $L^{\overline{p}_1(x)}(\mathbb{R}^N)$, which implies that

$$|u_n|^{\overline{p}_1(x)-2}u_n \to |u|^{\overline{p}_1(x)-2}u \ \text{ in } \ L^{\frac{\overline{p}_1(x)}{\overline{p}_1(x)-1}}(\mathbb{R}^N),$$

and according to the Hölder's inequality, we obtain

$$\int_{\mathbb{R}^N} \left[|\eta_n(x)|^{\overline{p}_1(x)-2}\eta_n(x) - |\eta(x)|^{\overline{p}_1(x)-2}\eta(x)\right](\eta_n - \eta)\mathrm{d}x \to 0 \quad \text{as} \quad n \to \infty. \tag{3.3}$$

To simplify the presentation, we define the following functions:

$$\begin{aligned} &\xi_n(x,y) := \eta_n(x) - \eta_n(y), \\ &\xi(x,y) := \eta(x) - \eta(y), \\ &V_n(x) := \eta_n(x) - \eta(x). \end{aligned} \tag{3.4}$$

It follows that $V_n(x) - V_n(y) = \xi_n(x,y) - \xi(x,y)$. Combining with (3.1) and (3.4) and as $n \to \infty$, we deduce that

$$\begin{aligned} &\langle \mathcal{L}'_{p_1(x,\cdot)}(\eta_n) - \mathcal{L}'_{p_1(x,\cdot)}(\eta), \eta_n - \eta \rangle \\ &= \langle \mathcal{L}'_{p_1(x,\cdot)}(\eta_n) - \mathcal{L}'_{p_1(x,\cdot)}(\eta), V_n \rangle \\ &= \iint_{\mathbb{R}^{2N}} \frac{|\eta_n(x) - \eta_n(y)|^{p_1(x,y)-2}(\eta_n(x) - \eta_n(y))(V_n(x) - V_n(y))}{|x-y|^{N+p_1(x,y)s(x,y)}}\mathrm{d}x\mathrm{d}y \\ &\quad - \iint_{\mathbb{R}^{2N}} \frac{|\eta(x) - \eta(y)|^{p_1(x,y)-2}(\eta(x) - \eta(y))(V_n(x) - V_n(y))}{|x-y|^{N+p_1(x,y)s(x,y)}}\mathrm{d}x\mathrm{d}y \\ &\quad + \int_{\mathbb{R}^N} \left[|\eta_n(x)|^{\overline{p}_1(x)-2}\eta_n(x) - |\eta(x)|^{\overline{p}_1(x)-2}\eta(x)\right](V_n(x) - V_n(y))\mathrm{d}x \\ &= \iint_{\mathbb{R}^{2N}} \frac{[|\xi_n(x,y)|^{p_1(x,y)-2}\xi_n(x,y) - |\xi(x,y)|^{p_1(x,y)-2}\xi(x,y)](V_n(x) - V_n(y))}{|x-y|^{N+p_1(x,y)s(x,y)}}\mathrm{d}x\mathrm{d}y + o(1). \end{aligned} \tag{3.5}$$

From the well-known Simon inequality, when $p_1(x,y) \geq 2$ and as $n \to \infty$, we have the following estimate:

$$\begin{aligned} &\iint_{\mathbb{R}^{2N}} \frac{|\xi_n(x,y) - \xi(x,y)|^{p_1(x,y)}}{|x-y|^{N+p_1(x,y)s(x,y)}}\mathrm{d}x\mathrm{d}y \\ &\leq C_1 \iint_{\mathbb{R}^{2N}} \frac{[|\xi_n(x,y)|^{p_1(x,y)-2}\xi_n(x,y) - |\xi(x,y)|^{p_1(x,y)-2}\xi(x,y)](\xi_n(x,y) - \xi_n(x,y))}{|x-y|^{N+p_1(x,y)s(x,y)}}\mathrm{d}x\mathrm{d}y + o(1) \\ &\leq C_1 \iint_{\mathbb{R}^{2N}} \frac{[|\xi_n(x)|^{p_1(x,y)-2}\xi_n(x,y) - |\xi(x,y)|^{p_1(x,y)-2}\xi(x,y)](V_n(x) - V_n(y))}{|x-y|^{N+p_1(x,y)s(x,y)}}\mathrm{d}x\mathrm{d}y + o(1) \\ &= C_2 \langle \mathcal{L}'_{p_1(x,\cdot)}(\eta_n) - \mathcal{L}'_{p_1(x,\cdot)}(\eta), V_n \rangle \\ &= C_2 \langle \mathcal{L}'_{p_1(x,\cdot)}(\eta_n) - \mathcal{L}'_{p_1(x,\cdot)}(\eta), \eta_n - \eta \rangle. \end{aligned} \tag{3.6}$$

Taking into account (3.1), (3.3), (3.5), and (3.6), we obtain

$$\rho_{p_1(x,\cdot)}(\eta_n - \eta) \to 0 \quad \text{as } n \to \infty. \tag{3.7}$$

When $1 < p_1(x,y) < 2$ and $n \to \infty$, we have the following estimate:

$$\begin{aligned}
&\int_{\mathbb{R}^{2N}} \frac{|\xi_n(x,y) - \xi(x,y)|^{p_1(x,y)}}{|x-y|^{N+p_1(x,y)s(x,y)}} \mathrm{d}x\mathrm{d}y \\
&\le C_3 \iint_{\mathbb{R}^{2N}} \frac{1}{|x-y|^{N+p_1(x,y)s(x,y)}} [(|\xi_n(x,y)|^{p_1(x,y)-2}\xi_n(x,y) - |\xi(x,y)|^{p_1(x,y)-2}\xi(x,y)) \\
&\quad \times (\xi_n(x,y) - \xi(x,y)) + o(1)]^{\frac{p_1(x,y)}{2}} [|\xi_n(x,y)|^{p_1(x,y)} + |\xi(x,y)|^{p_1(x,y)}]^{\frac{2-p_1(x,y)}{2}} \mathrm{d}x\mathrm{d}y \\
&\le C_4 \iint_{\mathbb{R}^{2N}} \left[\frac{(|\xi_n(x,y)|^{p_1(x,y)-2}\xi_n(x,y) - |\xi(x,y)|^{p_1(x,y)-2}\xi(x,y))}{|x-y|^{N+p_1(x,y)s(x,y)}} (\xi_n(x,y) - \xi(x,y)) + o(1) \right]^{\frac{p_1(x,y)}{2}} \\
&\quad \times \left[\left(\frac{|\xi_n(x,y)|^{p_1(x,y)}}{|x-y|^{N+p_1(x,y)s(x,y)}} \right)^{\frac{2-p_1(x,y)}{2}} + \left(\frac{|\xi(x,y)|^{p_1(x,y)}}{|x-y|^{N+p_1(x,y)s(x,y)}} \right)^{\frac{2-p_1(x,y)}{2}} \right] \mathrm{d}x\mathrm{d}y.
\end{aligned} \tag{3.8}$$

To simplify the presentation, we define the functions $\mathfrak{f}_1, \mathfrak{f}_2, \mathfrak{f}_3, :\mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ as follows:

$$\mathfrak{f}_1 := \frac{(|\xi_n(x)|^{p_1(x,y)-2}\xi_n(x,y) - |\xi(x,y)|^{p_1(x,y)-2}\xi(x,y))}{|x-y|^{N+p_1(x,y)s(x,y)}} (\xi_n(x,y) - \xi(x,y))$$

and

$$\mathfrak{f}_2 := \frac{|\xi_n(x)|^{p_1(x,y)}}{|x-y|^{N+p_1(x,y)s(x,y)}}, \quad \mathfrak{f}_3 := \frac{|\xi(x)|^{p_1(x,y)}}{|x-y|^{N+p_1(x,y)s(x,y)}}.$$

Then, from (3.8), Hölder inequality, the Proposition 3.3, we derive that

$$\begin{aligned}
&\int_{\mathbb{R}^{2N}} \frac{|\xi_n(x,y) - \xi(x,y)|^{p_1(x,y)}}{|x-y|^{N+p_1(x,y)s(x,y)}} \mathrm{d}x\mathrm{d}y \\
&\le C_5 \iint_{\mathbb{R}^{2N}} (\mathfrak{f}_1 + o(1))^{\frac{p_1(x,y)}{2}} \left(\mathfrak{f}_2^{\frac{2-p_1(x,y)}{2}} + \mathfrak{f}_3^{\frac{2-p_1(x,y)}{2}} \right) \mathrm{d}x\mathrm{d}y \\
&\le C_5 \left(\|(\mathfrak{f}_1 + o(1))\|_{L^1(\mathbb{R}^{2N})}^{\frac{p_1^-}{2}} + \|(\mathfrak{f}_1 + o(1))\|_{L^1(\mathbb{R}^{2N})}^{\frac{p_1^+}{2}} \right) \left(\|\mathfrak{f}_2\|_{L^1(\mathbb{R}^{2N})}^{\frac{2-p_1^-}{2}} + \|\mathfrak{f}_2\|_{L^1(\mathbb{R}^{2N})}^{\frac{2-p_1^+}{2}} + \|\mathfrak{f}_3\|_{L^1(\mathbb{R}^{2N})}^{\frac{2-p_1^-}{2}} + \|\mathfrak{f}_3\|_{L^1(\mathbb{R}^{2N})}^{\frac{2-p_1^+}{2}} \right).
\end{aligned} \tag{3.9}$$

Since η_n and η are bounded in X_1, from Proposition 3.3, $\|\mathfrak{f}_2\|_{L^1(\mathbb{R}^{2N})}$ and $\|\mathfrak{f}_3\|_{L^1(\mathbb{R}^{2N})}$ are bounded. Therefore, combining with (3.1), (3.3), (3.5), and (3.9), we have

$$\rho_{p_1(x,\cdot)}(\eta_n - \eta) \to 0 \text{ as } n \to \infty. \tag{3.10}$$

Consequently, (3.7), (3.10), and Proposition 3.3 imply that

$$\|\eta_n - \eta\|_{X_1} \to 0 \text{ as } n \to \infty.$$

Similarly,

$$\|\eta_n - \eta\|_{X_2} \to 0 \text{ as } n \to \infty.$$

So, we get that

$$\|\eta_n - \eta\| \to 0 \text{ as } n \to \infty,$$

that is, if $\eta_n \rightharpoonup \eta$ in X and $\limsup_{n\to\infty} \langle \mathcal{L}'(\eta_n) - \mathcal{L}'(\eta), \eta_n - \eta \rangle \le 0$, then $\eta_n \to \eta$ in X.

(iii) Since $\mathcal{L}(\eta)$ is strictly a monotone operator in X, $\mathcal{L}(\eta)$ is an injection. From Proposition 3.3, we obtain

$$\frac{\langle \mathcal{L}'(\eta), \eta\rangle}{\|\eta\|} = \frac{\langle \mathcal{L}'_{p_1(x,\cdot)}(\eta), \eta\rangle + \langle \mathcal{L}'_{p_2(x,\cdot)}(\eta), \eta\rangle}{\|\eta\|}$$
$$= \frac{\iint_{\mathbb{R}^{2N}} \frac{|\eta(x)-\eta(y)|^{p_1(x,y)}}{|x-y|^{N+p_1(x,y)s(x,y)}}\mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}^N} |\eta(x)|^{\overline{p}_1(x)}\mathrm{d}x}{\|\eta\|} + \frac{\iint_{\mathbb{R}^{2N}} \frac{|\eta(x)-\eta(y)|^{p_2(x,y)}}{|x-y|^{N+p_2(x,y)s(x,y)}}\mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}^N} |\eta(x)|^{\overline{p}_2(x)}\mathrm{d}x}{\|\eta\|}$$
$$= \frac{\min\{\|\eta\|_{X_1}^{p_1^-}, \|\eta\|_{X_1}^{p_1^+}\} + \min\{\|\eta\|_{X_2}^{p_2^-}, \|\eta\|_{X_2}^{p_2^+}\}}{\|\eta\|}.$$

It means that

$$\lim_{\|\eta\|\to\infty} \frac{\langle \mathcal{L}'(\eta), \eta\rangle}{\|\eta\|} = \infty.$$

Therefore, $\mathcal{L}'(\eta)$ is coercive operator, thanks to the Minty-Browder theorem (see [47], Theorem 26A), $\mathcal{L}'(\eta)$ is a surjection. Due to its monotonicity, $\mathcal{L}'(\eta)$ is an injection. So, $(\mathcal{L}'(\eta))^{-1}$ exists. Consequently, the continuity of $(\mathcal{L}'(\eta))^{-1}$ is sufficient to ensure that $\mathcal{L}'(\eta)$ is a homeomorphism.

If $\{\mathfrak{g}_n\}_{n\in\mathbb{N}} \subset X^*$, then $\mathfrak{g}_n \to \mathfrak{g}$ in X^*. we assume that there are $\eta_n, \eta \in X$ such that

$$\eta_n = (\mathcal{L}'(\eta))^{-1}(\mathfrak{g}_n), \text{ and } \eta = (\mathcal{L}'(\eta))^{-1}(\mathfrak{g}).$$

In view of the coercivity of $\mathcal{L}'(\eta)$, we conclude that $\{\eta_n\}_{n\in\mathbb{N}}$is bounded in X. Then, up to subsequence $\eta_n \rightharpoonup \eta$ in X, which implies

$$\lim_{n\to\infty} \langle \mathcal{L}'(\eta_n) - \mathcal{L}'(\eta), \eta_n - \eta\rangle = \lim_{n\to\infty} \langle \mathfrak{g}_n - \mathfrak{g}, \eta_n - \eta\rangle = 0.$$

Since $\mathcal{L}'$ is of (S_+)-type operator, we have $\eta_n \to \eta$ in X. □

4 The proof of the main results

4.1 Compactness condition

Let X be a Banach space and $I_\lambda \in C^1(X, \mathbb{R})$. We review that $\{\eta_n\}_{n\in\mathbb{N}} \subset X$ is a Cerami sequence, if

$$|I_\lambda(\eta_n)| \le c, \quad (1 + \|\eta_n\|)I'_\lambda(\eta_n) \to 0 \text{ in } X^* \text{ as } n \to \infty, \tag{4.1}$$

and we say that a function I_λ satisfies the Cerami condition at the level $c \in \mathbb{R}$, and I_λ has a strongly convergent subsequence in X.

Remark 4.1. As being known, the Cerami condition is weaker than the Palais-Smale compactness condition. Thus, if a function, I_λ satisfies both Cerami condition and mountain pass geometry, abstract critical point theory, and fountain theorem, then I_λ has a critical point in X.

Lemma 4.1. *Assume that $(\mathcal{S})$, $(\mathcal{P})$, $(\mathcal{M}_1)$–$(\mathcal{M}_3)$, $(\mathcal{A}_1)$–$(\mathcal{A}_2)$, $(\mathcal{F}_2)$, and $(\mathcal{F}_4)$ are satisfied. Then, for any $\lambda > 0$, the sequence $\{\eta_n\}_{n\in\mathbb{N}}$ is bounded in X.*

Proof. Let $\lambda > 0$, and $\{\eta_n\}_{n\in\mathbb{N}} \subset X$ is a Cerami sequence (4.1) associated with I_λ, which implies

$$|I_\lambda(\eta_n)| \le c, \tag{4.2}$$

for some positive constant c, which does not depend on n, and

$$(1 + \|\eta_n\|)I'_\lambda(\eta_n) \to 0 \text{ in } X^* \text{ as } n \to \infty. \tag{4.3}$$

In view of (4.3), there exists $\kappa_n \to 0$ such that

$$|\langle I'_\lambda(\eta_n), \xi\rangle| \le \frac{\kappa_n\|\xi\|}{1 + \|\eta_n\|}, \quad \text{for all } \xi \in X \text{ and } n \in \mathbb{N}. \tag{4.4}$$

By choosing $\xi = \eta_n$, we obtain

$$-\langle I'_\lambda(\eta_n), \eta_n\rangle \le -\sum_{i=1}^{2} m_i\langle\delta'_{p_i}(\eta_n), \eta_n\rangle - \sum_{i=1}^{2}\int_{\mathbb{R}} |\eta_n|^{\overline{p}_i(x)}\mathrm{d}x + \lambda\int_{\mathbb{R}} f(x, \eta_n)\eta_n \mathrm{d}x \le \frac{\kappa_n\|\eta_n\|}{1 + \|\eta_n\|} \le \kappa_n \le C_6. \tag{4.5}$$

From now on, we show that the sequence $\{\eta_n\}_{n\in\mathbb{N}}$ is bounded in X by contrary arguments. Assume that

$$\|\eta_n\| \to \infty, \text{ as } n \to \infty. \tag{4.6}$$

We define a new sequence $\{\omega_n\}_{n\in\mathbb{N}}$ to be denoted by $\omega_n = \eta_n / \|\eta_n\|$, then $\{\omega_n\}_{n\in\mathbb{N}} \subset X$ and $\|\omega_n\| = 1$. By Lemma 3.8, there exists a subsequence, without loss of generality, still denoted by $\{\omega_n\}_{n\in\mathbb{N}}$, such that

$$\omega_n \rightharpoonup \omega \text{ weakly in } X, \quad \omega_n \to \omega \text{ strongly in } L^{\vartheta(x)}_{a(x)}(\mathbb{R}^N), \quad \omega_n \to \omega \text{ a.e. in } \mathbb{R}^N \tag{4.7}$$

for $\vartheta(x) \in (1, p_s^*(x))$ and $\omega \ge 0$.

If $\omega \ne 0$, the set $\Omega_+ := \{x \in \mathbb{R}^N : \omega(x) > 0\}$ has positive Lebesgue measure and $\eta_n(x) \to +\infty$ for all $x \in \Omega_+$. Therefore, on the basis of the hypothesis $(\mathcal{F}_2)$, we deduce that

$$\limsup_{n\to\infty}\frac{F(x, \eta_n)}{\|\eta_n\|^{\vartheta p^+_{\max}}} = \limsup_{n\to\infty}\frac{F(x, \eta_n)\omega_n^{\vartheta p^+_{\max}}}{|\eta_n|^{\vartheta p^+_{\max}}} = \infty, \text{ in } \Omega_+. \tag{4.8}$$

From Fatou's lemma, we obtain

$$\limsup_{n\to\infty}\int_{\mathbb{R}^N}\frac{F(x, \eta_n)}{\|\eta_n\|^{\vartheta p^+_{\max}}}\mathrm{d}x = \limsup_{n\to\infty}\int_{\mathbb{R}^N}\frac{F(x, \eta_n)\omega_n^{\vartheta p^+_{\max}}}{|\eta_n|^{\vartheta p^+_{\max}}}\mathrm{d}x = \infty, \text{ in } \Omega_+. \tag{4.9}$$

As a consequence of (4.2), we have

$$\int_{\mathbb{R}} F(x, \eta_n) \le \frac{1}{\lambda}\sum_{i=1}^{2}\widetilde{M}_i(\delta_{p_i}(\eta_n)) + \frac{1}{\lambda}\sum_{i=1}^{2}\int_{\mathbb{R}}\frac{1}{\overline{p}_i(x)}|\eta_n|^{\overline{p}_i(x)}\mathrm{d}x + \frac{C_7}{\lambda}. \tag{4.10}$$

and then, it follows by the condition $(\mathcal{M}_1)$ that

$$\int_{\mathbb{R}}\frac{F(x, \eta_n)}{\|\eta_n\|^{\vartheta p^+_{\max}}} \le \frac{1}{\lambda\|\eta_n\|^{\vartheta p^+_{\max}}}\sum_{i=1}^{2}\widetilde{M}_i(1)\delta_{p_i}^{\vartheta_i}(\eta_n) + \frac{1}{\lambda\|\eta_n\|^{\vartheta p^+_{\max}}}\sum_{i=1}^{2}\int_{\mathbb{R}}\frac{1}{\overline{p}_i(x)}|\eta_n|^{\overline{p}_i(x)}\mathrm{d}x + \frac{C_7}{\lambda\|\eta_n\|^{\vartheta p^+_{\max}}}. \tag{4.11}$$

Hence,

$$\limsup_{n\to\infty}\int_{\mathbb{R}^N}\frac{F(x, \eta_n)}{\|\eta_n\|^{\vartheta p^+_{\max}}}\mathrm{d}x \le \frac{\max\{1, \widetilde{M}_1(1), \widetilde{M}_2(1)\}}{\lambda p_{\min}}, \tag{4.12}$$

and this contradicts (4.9).

We suppose that ω is the null function and again arrive at a contradiction. Since $I_\lambda(t\eta_n)$ is continuous function in $t \in [0, 1]$, we define the sequence $t_n \subset [0, 1]$ by

$$I_\lambda(t_n\eta_n) := \max_{0\le t\le 1} I_\lambda(t\eta_n). \tag{4.13}$$

Without loss of generality, we assume that $p_2(x,\cdot) < p_1(x,\cdot)$, and there exists a positive sequence $v_n := (2\mu)^{1/p_2^-}\omega_n = \frac{(2\mu)^{1/p_2^-}\eta_n}{\|\eta_n\|}$, where $\mu > \frac{1}{2}\left(\frac{p_1^+}{p_2^+}\right)^{\frac{p_2^-}{p_1^+-p_2^-}}$. On the basis of the continuity of the Nemytskii operator, we have that $F(x, v_n) \to 0$ in $L^1(\mathbb{R}^N)$ due to $v_n \to 0$ in $L^{\vartheta(x)}(\mathbb{R}^N)$ as $n \to \infty$. Therefore,

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} F(x, v_n) = 0. \tag{4.14}$$

According to $\|\eta_n\| \to \infty$ as $n \to \infty$, there exists n_0 large enough such that $\frac{(2\mu)^{1/p_2^-}}{\|\eta_n\|} \in (0, 1)$ for all $n \ge n_0$. Thus, from (4.14) and the conditions $(\mathcal{M}_1)$ and $(\mathcal{M}_2)$, we obtain

$$\begin{aligned}
I_\lambda(t_n\eta_n) &\ge I_\lambda(v_n)\\
&\ge \sum_{i=1}^{2}\frac{m_i}{\vartheta_i}(\delta_{p_i}(v_n)) + \sum_{i=1}^{2}\int_{\mathbb{R}}\frac{1}{\overline{p}_i(x)}|v_n|^{\overline{p}_i(x)}\mathrm{d}x - \lambda\int_{\mathbb{R}}F(x, v_n)\mathrm{d}x\\
&\ge \frac{\min\{1, m_1, m_2\}}{\vartheta p_{\max}^+}\sum_{i=1}^{2}\left(\iint_{\mathbb{R}^{2N}}\frac{|v_n(x)-v_n(y)|^{p_i(x,y)}}{|x-y|^{N+s(x,y)p_i(x,y)}}\mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}}|v_n|^{\overline{p}_i(x)}\mathrm{d}x\right) - \lambda\int_{\mathbb{R}}F(x, v_n)\mathrm{d}x\\
&\ge \frac{\min\{1, m_1, m_2\}}{\vartheta p_{\max}^+}\left(\frac{(2\mu)^{p_1^-/p_2^-}}{p_1^+}\|\omega_n\|_{X_1}^{p_1^+} + \frac{2\mu}{p_2^+}\|\omega_n\|_{X_2}^{p_2^+}\right) - \lambda\int_{\mathbb{R}}F(x, v_n)\mathrm{d}x\\
&\ge \frac{\mu\min\{1, m_1, m_2\}}{2^{p_1^+-2}\vartheta(p_{\max}^+)^2}(\|\omega_n\|_{X_1} + \|\omega_n\|_{X_2})^{p_1^+} - \lambda\int_{\mathbb{R}}F(x, v_n)\mathrm{d}x\\
&= \frac{\mu\min\{1, m_1, m_2\}}{2^{p_1^+-2}\vartheta(p_{\max}^+)^2} - \lambda\int_{\mathbb{R}}F(x, v_n)\mathrm{d}x,
\end{aligned} \tag{4.15}$$

where we have used that $\|\omega_n\|_{X_2} \le \|\omega_n\|_{X_1} + \|\omega_n\|_{X_2} = \|\omega_n\| = 1$ and also that $(\mathbf{a}+\mathbf{b})^p \le 2^{p-1}(\mathbf{a}^p + \mathbf{b}^p)$ for $\mathbf{a} > 0$ and $\mathbf{b} > 0$. From (4.14), we take $n_1 \ge n_0$ such that

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} F(x, v_n) < \frac{\mu\min\{1, m_1, m_2\}}{2^{p_1^+-1}\vartheta(p_{\max}^+)^2}, \quad \text{for all } n \ge n_1. \tag{4.16}$$

which joint with (4.15), we obtain

$$I_\lambda(t_n\eta_n) > \frac{\mu\min\{1, m_1, m_2\}}{2^{p_1^+-1}\vartheta(p_{\max}^+)^2}, \quad \text{for all } n \ge n_1. \tag{4.17}$$

Due to μ being arbitrary, we have the following conclusion:

$$I_\lambda(t_n\eta_n) = \infty, \quad \text{for all } n \ge n_1. \tag{4.18}$$

Since $0 \le t_n\eta_n \le \eta_n$ and the hypothesis $(\mathcal{F}_4)$ yields

$$\int_{\mathbb{R}^N}\varrho(x, t_n\eta_n)\mathrm{d}x \le \int_{\mathbb{R}^N}\tau\varrho(x, \eta_n)\mathrm{d}x, \quad \text{for all } n \ge n_1. \tag{4.19}$$

By passing to new subsequence, if necessary, we can assume that $0 < t_n < 1$ for all $n \ge n_2 \ge n_1$. Indeed, (4.18) combined with (4.2) implies that $t_n \ne 1$ and the fact that $I_\lambda(0) = 0$ implies that $t_n \ne 0$. Thus,

$$\begin{aligned}
0 &= t_n\frac{\mathrm{d}}{\mathrm{d}t}I_\lambda(t\eta_n)|_{t=t_n} = \langle I_\lambda'(t_n\eta_n), t_n\eta_n\rangle\\
&= \sum_{i=1}^{2}M_i(\delta_{p_i}(t_n\eta_n)) \times \langle\delta_{p_i}'(t_n\eta_n), t_n\eta_n\rangle + \sum_{i=1}^{2}\int_{\mathbb{R}}|t_n\eta_n|^{\overline{p}_i(x)}\mathrm{d}x - \lambda\int_{\mathbb{R}}f(x, t_n\eta_n)t_n\eta_n\mathrm{d}x.
\end{aligned} \tag{4.20}$$

Hence, for all sufficiently large n, we have

$$\begin{aligned}
\frac{1}{\tau}I_\lambda(t_n\eta_n) &:= \frac{1}{\tau}\sum_{i=1}^{2}\widetilde{M}_i(\delta_{p_i}(t_n\eta_n)) + \frac{1}{\tau}\sum_{i=1}^{2}\int_{\mathbb{R}}\frac{1}{\overline{p}_i(x)}|t_n\eta_n|^{\overline{p}_i(x)}\,\mathrm{d}x - \frac{\lambda}{\tau}\int_{\mathbb{R}}F(x,t_n\eta_n)\mathrm{d}x \\
&\quad - \frac{1}{\tau p^+_{\max}}\sum_{i=1}^{2}M_i(\delta_{p_i}(t_n\eta_n)) \times \langle \delta'_{p_i}(t_n\eta_n), t_n\eta_n\rangle - \frac{1}{\tau p^+_{\max}}\sum_{i=1}^{2}\int_{\mathbb{R}}|t_n\eta_n|^{\overline{p}_i(x)}\,\mathrm{d}x \\
&\quad + \frac{\lambda}{\tau p^+_{\max}}\int_{\mathbb{R}} f(x,t_n\eta_n)t_n\eta_n\mathrm{d}x + o(1) \\
&= \frac{1}{\tau}\sum_{i=1}^{2}\widetilde{M}_i(\delta_{p_i}(t_n\eta_n)) + \frac{1}{\tau}\sum_{i=1}^{2}\int_{\mathbb{R}}\frac{1}{\overline{p}_i(x)}|t_n\eta_n|^{\overline{p}_i(x)}\,\mathrm{d}x - \frac{1}{\tau p^+_{\max}}\sum_{i=1}^{2}M_i(\delta_{p_i}(t_n\eta_n)) \times \langle \delta'_{p_i}(t_n\eta_n), t_n\eta_n\rangle \\
&\quad - \frac{1}{\tau p^+_{\max}}\sum_{i=1}^{2}\int_{\mathbb{R}}|t_n\eta_n|^{\overline{p}_i(x)}\,\mathrm{d}x + \frac{\lambda}{\tau}\int_{\mathbb{R}}\left(\frac{1}{p^+_{\max}}f(x,t_n\eta_n)t_n\eta_n - F(x,t_n\eta_n)\right)\mathrm{d}x + o(1).
\end{aligned} \tag{4.21}$$

By combining with (4.19) and the hypothesis $(\mathcal{F}_4)$, we obtain

$$\int_{\mathbb{R}} f(x,t_n\eta_n)t_n\eta_n\mathrm{d}x = \int_{\mathbb{R}} p^+_{\max}F(x,t_n\eta_n)\mathrm{d}x + \int_{\mathbb{R}} \varrho(x,t_n\eta_n)\mathrm{d}x \le \int_{\mathbb{R}} p^+_{\max}F(x,t_n\eta_n)\mathrm{d}x + \int_{\mathbb{R}} \tau\varrho(x,\eta_n)\mathrm{d}x, \tag{4.22}$$

which joint with (4.21) and the condition $(\mathcal{M}_3)$, we obtain

$$\begin{aligned}
\frac{1}{\tau}I_\lambda(t_n\eta_n) &\le \frac{1}{\tau}\sum_{i=1}^{2}\widetilde{M}_i(\delta_{p_i}(t_n\eta_n)) + \frac{1}{\tau}\sum_{i=1}^{2}\int_{\mathbb{R}}\frac{1}{\overline{p}_i(x)}|t_n\eta_n|^{\overline{p}_i(x)}\,\mathrm{d}x + \frac{\lambda}{p^+_{\max}}\int_{\mathbb{R}}\varrho(x,\eta_n)\mathrm{d}x \\
&\quad - \frac{1}{\tau p^+_{\max}}\sum_{i=1}^{2}M_i(\delta_{p_i}(t_n\eta_n)) \times \langle \delta'_{p_i}(t_n\eta_n), t_n\eta_n\rangle - \frac{1}{\tau p^+_{\max}}\sum_{i=1}^{2}\int_{\mathbb{R}}|t_n\eta_n|^{\overline{p}_i(x)}\,\mathrm{d}x + o(1) \\
&\le \sum_{i=1}^{2}\widetilde{M}_i(\delta_{p_i}(\eta_n)) + \sum_{i=1}^{2}\int_{\mathbb{R}}\frac{1}{\overline{p}_i(x)}|\eta_n|^{\overline{p}_i(x)}\,\mathrm{d}x + \frac{\lambda}{p^+_{\max}}\int_{\mathbb{R}}\varrho(x,\eta_n)\mathrm{d}x \\
&\quad - \frac{1}{p^+_{\max}}\sum_{i=1}^{2}M_i(\delta_{p_i}(\eta_n)) \times \langle \delta'_{p_i}(\eta_n), \eta_n\rangle - \frac{1}{p^+_{\max}}\sum_{i=1}^{2}\int_{\mathbb{R}}|\eta_n|^{\overline{p}_i(x)}\,\mathrm{d}x + o(1) \\
&\le \sum_{i=1}^{2}\widetilde{M}_i(\delta_{p_i}(\eta_n)) + \sum_{i=1}^{2}\int_{\mathbb{R}}\frac{1}{\overline{p}_i(x)}|\eta_n|^{\overline{p}_i(x)}\,\mathrm{d}x + \lambda\int_{\mathbb{R}}\left(\frac{1}{p^+_{\max}}f(x,\eta_n)\eta_n - F(x,\eta_n)\right)\mathrm{d}x \\
&\quad - \frac{1}{p^+_{\max}}\sum_{i=1}^{2}M_i(\delta_{p_i}(\eta_n)) \times \langle \delta'_{p_i}(\eta_n), \eta_n\rangle - \frac{1}{p^+_{\max}}\sum_{i=1}^{2}\int_{\mathbb{R}}|\eta_n|^{\overline{p}_i(x)}\,\mathrm{d}x + o(1) \\
&\le C_8,
\end{aligned} \tag{4.23}$$

as $n \to \infty$, which contradicts (4.18). Therefore, we conclude that the sequence $\{\eta_n\}_{n\in\mathbb{N}}$ is bounded in X. ◻

Lemma 4.2. *Assume that $(\mathcal{S})$, $(\mathcal{P})$, $(\mathcal{M}_2)$–$(\mathcal{M}_3)$, $(\mathcal{A}_1)$–$(\mathcal{A}_2)$, and $(\mathcal{F}_1)$ are satisfied. If the sequence $\{\eta_n\}_{n\in\mathbb{N}} \subset X$ is a Cerami sequence of I_λ at the level $c \in \mathbb{R}$, then, for any $\lambda > 0$, $\{\eta_n\}_{n\in\mathbb{N}}$ has a strong convergent subsequence.*

Proof. We assume that $\{\eta_n\}_{n\in\mathbb{N}} \subset X$ be a Cerami sequence. From Lemma 4.1, the sequence η_n is bounded in X. It follows from Lemma 3.8, combined with the reflexivity of X, that there exists a subsequence, which is still expressed as $\{\eta_n\}_{n\in\mathbb{N}}$, such that

$$\eta_n \rightharpoonup \eta \text{ weakly in } X, \quad \eta_n \to \eta \text{ strongly in } L^{\vartheta(x)}_{a(x)}(\mathbb{R}^N), \quad \eta_n \to \eta \text{ a.e. in } \mathbb{R}^N, \tag{4.24}$$

for $\vartheta(x) \in (1, p^*_s(x))$. Since η_n is bounded in X and $I'_\lambda(\eta_n) \to 0$, we derive that

$$\langle I'_\lambda(\eta_n), \eta_n - \eta\rangle \to 0, \text{ as } n \to \infty,$$

and it follows that

$$
\begin{aligned}
o_n(1) = \langle I'_\lambda(\eta_n), \eta_n - \eta\rangle &= \sum_{i=1}^{2} M_i(\delta_{p_i}(\eta_n)) \times \langle \delta'_{p_i}(\eta_n), \eta_n - \eta\rangle + \sum_{i=1}^{2} \int_{\mathbb{R}} |\eta_n|^{\overline{p}_i(x)-1}(\eta_n - \eta)\mathrm{d}x \\
&\quad - \lambda \int_{\mathbb{R}} f(x, \eta_n)(\eta_n - \eta)\mathrm{d}x.
\end{aligned} \tag{4.25}
$$

Indeed, by using the fact that $\eta_n \to \eta$ strongly in $L^{g(x)}_{a(x)}(\mathbb{R}^N)$ together with Hölder's inequality and Lemma 3.8, we obtain

$$
\lim_{n\to\infty} \int_{\mathbb{R}} |\eta_n|^{\overline{p}_i(x)-1}(\eta_n - \eta)\mathrm{d}x = 0. \tag{4.26}
$$

According to the hypothesis $(\mathcal{A}_i)$ and using the hypothesis $(\mathcal{F}_1)$, we infer that

$$
\int_{\mathbb{R}} f(x, \eta_n)(\eta_n - \eta)\mathrm{d}x \le \int_{\mathbb{R}} a_1(x)(\eta_n - \eta)\mathrm{d}x + \int_{\mathbb{R}} a_2(x)|\eta_n|^{q(x)-1}(\eta_n - \eta)\mathrm{d}x \tag{4.27}
$$

and

$$
\int_{\mathbb{R}} a_2(x)|\eta_n|^{q(x)-1}(\eta_n - \eta)\mathrm{d}x \le 2^{q^+-1}\left(\int_{\mathbb{R}} a_2(x)|\eta_n - \eta|^{q(x)}\mathrm{d}x + \int_{\mathbb{R}} a_2(x)|\eta|^{q(x)-1}(\eta_n - \eta)\mathrm{d}x\right). \tag{4.28}
$$

By (4.24), Hölder inequality, and Lemma 3.5, we have

$$
\begin{aligned}
&\lim_{n\to\infty} \int_{\mathbb{R}} a_1(x)(\eta_n - \eta)\mathrm{d}x = 0, \\
&\lim_{n\to\infty} \int_{\mathbb{R}} a_2(x)|\eta_n - \eta|^{q(x)}\mathrm{d}x = 0, \\
&\lim_{n\to\infty} \int_{\mathbb{R}} a_2(x)|\eta|^{q(x)-1}(\eta_n - \eta)\mathrm{d}x = 0,
\end{aligned} \tag{4.29}
$$

which joint with (4.27), we obtain

$$
\lim_{n\to\infty} \int_{\mathbb{R}} f(x, \eta_n)(\eta_n - \eta)\mathrm{d}x = 0. \tag{4.30}
$$

Therefore, from (4.25), (4.26), and (4.30), we deduce that

$$
\lim_{n\to\infty} \langle I'_\lambda(\eta_n), \eta_n - \eta\rangle = \lim_{n\to\infty} \sum_{i=1}^{2} M_i(\delta_{p_i}(\eta_n)) \times \langle \delta'_{p_i}(\eta_n), \eta_n - \eta\rangle = 0. \tag{4.31}
$$

Since $\{\eta_n\}_{n\in\mathbb{N}}$ is bounded in X, passing to a subsequence, we suppose that

$$
\iint_{\mathbb{R}^2} \frac{|\eta_n(x) - \eta_n(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+s(x,y)p(x,y)}}\mathrm{d}x\mathrm{d}y \to t_0 \ge 0 \text{ as } n \to \infty.
$$

(i) If $t_0 = 0$, then η_n strongly converges to $\eta = 0$ in X.

(ii) If $t_0 > 0$, since the function M_i is continuous, we have

$$
\lim_{n\to\infty} \sum_{i=1}^{2} M_i(\delta_{p_i}(\eta_n)) = M_i(t_0) \ge 0. \tag{4.32}
$$

Therefore, from the hypothesis $(\mathcal{M}_2)$ and $(\mathcal{M}_3)$, for n large enough, there exist $C_9, C_{10} > 0$ such that

$$0 < C_9 \le M_i(\delta_{p_i}(\eta_n)) \le C_{10}, \tag{4.33}$$

which joint with (4.31), we obtain

$$\begin{aligned}\sum_{i=1}^{2}\lim_{n\to\infty}\langle\delta'_{p_i}(\eta_n), \eta_n-\eta\rangle &= \lim_{n\to\infty}\langle\delta'_{p_1}(\eta_n), \eta_n-\eta\rangle + \lim_{n\to\infty}\langle\delta'_{p_2}(\eta_n), \eta_n-\eta\rangle\\ &= \lim_{n\to\infty}\langle\delta'_{p_1}(\eta_n)+\delta'_{p_2}(\eta_n), \eta_n-\eta\rangle\\ &= \lim_{n\to\infty}\langle\mathcal{L}'(\eta_n), \eta_n-\eta\rangle\\ &= 0.\end{aligned} \tag{4.34}$$

Thus, according to (4.34) and Proposition 3.5, we finally achieve the strong convergence of $\eta_n \to v$ as $n \to \infty$ in X. I_λ satisfies the Cerami condition for all $c \in \mathbb{R}$. □

4.2 Proof of Theorem 2.1

In what follows, we prove Theorem 2.1 by applying the mountain pass theorem, see [40].

Lemma 4.3. *Assume that $(\mathcal{S})$, $(\mathcal{P})$, $(\mathcal{M}_1)$, $(\mathcal{M}_2)$, $(\mathcal{F}_1)$, and $(\mathcal{F}_3)$ are satisfied. Then, there exist positive constants ρ_0 and $\alpha_0 = \alpha_0(\rho_0)$ such that for all $\lambda > 0$ we have the function $I_\lambda \ge \alpha_0(\rho_0)$ for all $\eta \in X$ with $\|\eta\| = \rho_0$.*

Proof. From the hypotheses $(\mathcal{F}_1)$ and $(\mathcal{F}_3)$, for any $\varepsilon > 0$, we can verify that there exists $C_\varepsilon > 0$ such that

$$F(x,t) \le \frac{1}{p^+_{\max}}(\beta(x)+\varepsilon)|t|^{p^+_{\max}} + C_\varepsilon|t|^{q(x)}, \quad \text{for all } (x,t) \in \mathbb{R}^N \times \mathbb{R}. \tag{4.35}$$

Let $\lambda > 0$ and $\|\eta\| < 1$, and then, by using the conditions $(\mathcal{M}_1)$, $(\mathcal{M}_2)$, and (4.35), for any $\eta \in X$, we deduce that

$$\begin{aligned} I_\lambda(\eta) &\ge \frac{1}{\vartheta p^+_{\max}}\sum_{i=1}^{2} M_i(\delta_{p_i}(\eta)) \times \langle\delta'_{p_i}(\eta), \eta\rangle + \sum_{i=1}^{2}\int_{\mathbb{R}}\frac{1}{\overline{p}_i(x)}|\eta|^{\overline{p}_i(x)}\,dx - \frac{\lambda(l+\varepsilon)}{p^+_{\max}}\int_{\mathbb{R}}|\eta|^{p^+_{\max}}\,dx\\ &\quad - \lambda C_\varepsilon\int_{\mathbb{R}}|\eta|^{q(x)}\,dx\\ &\ge \frac{M_{\min}}{\vartheta p^+_{\max}}\sum_{i=1}^{2}\left(\int_{\Omega\times\Omega}\frac{|\eta(x)-\eta(y)|^{p_i(x,y)}}{|x-y|^{N+p_i(x,y)s(x,y)}}\,dx\,dy + \int_{\mathbb{R}}|\eta|^{\overline{p}_i(x)}\,dx\right) - \frac{\lambda(l+\varepsilon)}{p^+_{\max}}\int_{\mathbb{R}}|\eta|^{p^+_{\max}}\,dx\\ &\quad - \lambda C_\varepsilon\int_{\mathbb{R}}|\eta|^{q(x)}\,dx\\ &\ge \frac{M_{\min}}{\vartheta p^+_{\max}}(\|\eta\|^{p_1^+}_{X_1} + \|\eta\|^{p_2^+}_{X_2}) - \frac{\lambda(l+\varepsilon)}{p^+_{\max}}\int_{\mathbb{R}}|\eta|^{p^+_{\max}}\,dx - \lambda C_\varepsilon\int_{\mathbb{R}}|\eta|^{q(x)}\,dx, \end{aligned} \tag{4.36}$$

where $M_{\min} = \min\{1, m_1, m_2\}$ and $l = \|\beta(x)\|_{L^\infty(\mathbb{R}^N)_+}$. According to Lemma 3.8, there exist C_{11} and C_{12} such that

$$I_\lambda(\eta) \ge \frac{M_{\min}}{\vartheta p^+_{\max}}(\|\eta\|^{p_1^+}_{X_1} + \|\eta\|^{p_2^+}_{X_2}) - \frac{\lambda(l+\varepsilon)}{p^+_{\max}}C_{11}\|\eta\|^{p^+_{\max}} - \lambda C_\varepsilon C_{12}\|\eta\|^{q^-}. \tag{4.37}$$

without loss of generality, we suppose that $\|\eta\|_{X_1}^{p_1^+} \geq \frac{\|\eta\|}{2} \geq \|\eta\|_{X_2}^{p_2^+}$, and it follows that

$$I_\lambda(\eta) \geq \frac{M_{\min}}{2^{p_{\max}^+}\vartheta p_{\max}^+}\|\eta\|^{p_1^+} - \frac{\lambda(l+\varepsilon)}{p_{\max}^+}C_{11}\|\eta\|^{p_{\max}^+} - \lambda C_\varepsilon C_{12}\|\eta\|^{q^-}. \tag{4.38}$$

Since $q^- > p_{\max}^+ > p_1^+$ and choosing $\rho \in \left(0, \min\left\{1, 1/C_\vartheta, [M_{\min}/2^{p_{\max}^+}\lambda C_{11}\vartheta(l+\varepsilon)]^{\frac{1}{p_{\max}^+ - p_1^+}}\right\}\right)$, we obtain

$$I_\lambda(\eta) \geq \frac{M_{\min}}{2^{p_{\max}^+ + 1}\vartheta p_{\max}^+}\|\rho\|_{X_1}^{p_1^+} - \lambda C_\varepsilon C_{12}\|\rho\|^{q^-}. \tag{4.39}$$

Let us take

$$0 < \rho_0 < \min\left\{1, 1/C_\vartheta, \left[M_{\min}/2^{p_{\max}^+}\lambda C_{11}\vartheta(l+\varepsilon)\right]^{\frac{1}{p_{\max}^+ - p_1^+}}, \left[M_{\min}/2^{p_{\max}^+ + 1}\lambda C_{12}\vartheta p_{\max}^+ C_\varepsilon\right]^{\frac{1}{q^- - p_1^+}}\right\}, \tag{4.40}$$

combining with (4.39) implies that

$$I_\lambda(\eta) \geq \rho_0^{q^-}\left(\frac{M_{\min}}{2^{p_{\max}^+ + 2}\vartheta p_{\max}^+}\right) = \alpha(\rho_0). \tag{4.41}$$

Consequently, there exists a constant $\alpha_0 = \alpha_0(\rho_0) > 0$ such that the function $I_\lambda(\eta) \geq \alpha_0$ for all $\eta \in X$ with $\|\eta\| = \rho_0$. □

Lemma 4.4. *Assume that* $(\mathcal{S})$, $(\mathcal{P})$, $(\mathcal{M}_1)$–$(\mathcal{M}_2)$, *and* $(\mathcal{F}_1)$–$(\mathcal{F}_4)$ *are satisfied. Then, for all* $\lambda > 0$, *there exists* $\eta_0 \in X$ *with* $\|\eta_0\| > \rho_0$, *where* ρ_0 *is given in* Lemma 4.3, *such that* $I(\eta_0) < 0$ *for all* $t > 1$ *sufficient large.*

Proof. Let $\lambda > 0$, and it follows from the hypothesis $(\mathcal{F}_1)$ and $(\mathcal{F}_2)$ that for a positive constant M:

$$M > \left(\sum_{i=1}^{2}\frac{\widetilde{M}_i(1)}{(p_i^-)^{\vartheta_i}}\|\varphi_0\|_{X_i}^{\vartheta_i p_i^+} + \sum_{i=1}^{2}\frac{1}{\overline{p}_i^-}\|\varphi_0\|_{X_i}^{p_i^+}\right)\left(\lambda\int_{\mathbb{R}}|\varphi_0|^{\vartheta p_{\max}^+}\mathrm{d}x\right)^{-1}, \tag{4.42}$$

and there exists a corresponding positive constant C_M such that

$$F(x,t) \geq M|t|^{\vartheta p_{\max}^+} - C_M, \quad \text{uniformly for all } x \in \mathbb{R}^N. \tag{4.43}$$

By using the condition $(\mathcal{M}_1)$, we have

$$\widetilde{M}_i(t) \leq \widetilde{M}_i(1)t^\vartheta, \quad \text{for any } t \geq 1. \tag{4.44}$$

Let us take $\varphi_0 \in X\backslash\{0\}$, from (4.42)–(4.44), we obtain

$$\begin{aligned} I_\lambda(t\varphi_0) &\leq \sum_{i=1}^{2}\widetilde{M}_i(1)(\delta_{p_i}(t\varphi_0))^{\vartheta_i} + \sum_{i=1}^{2}\frac{1}{\overline{p}_i^-}\int_{\mathbb{R}}|t\varphi_0|^{\overline{p}_i(x)}\mathrm{d}x - \lambda M\int_{\mathbb{R}}|t\varphi_0|^{\vartheta p_{\max}^+}\mathrm{d}x + \lambda C_M\int_{\mathbb{R}}\mathrm{d}x \\ &\leq |t|^{\vartheta p_{\max}^+}\left(\sum_{i=1}^{2}\widetilde{M}_i(1)(\delta_{p_i}(\varphi_0))^{\vartheta_i} + \sum_{i=1}^{2}\frac{1}{\overline{p}_i^-}\int_{\mathbb{R}}|\varphi_0|^{\overline{p}_i(x)}\mathrm{d}x - \lambda M\int_{\mathbb{R}}|\varphi_0|^{\vartheta p_{\max}^+}\mathrm{d}x\right) + \lambda C_M\int_{\mathbb{R}}\mathrm{d}x \\ &\leq |t|^{\vartheta p_{\max}^+}\left(\sum_{i=1}^{2}\frac{\widetilde{M}_i(1)}{(p_i^-)^{\vartheta_i}}\|\varphi_0\|_{X_i}^{\vartheta_i p_i^+} + \sum_{i=1}^{2}\frac{1}{\overline{p}_i^-}\|\varphi_0\|_{X_i}^{p_i^+} - \lambda M\int_{\mathbb{R}}|\varphi_0|^{\vartheta p_{\max}^+}\mathrm{d}x\right) + \lambda C_M\int_{\mathbb{R}}\mathrm{d}x. \end{aligned} \tag{4.45}$$

By choosing M large enough, we deduce that $I(t\varphi_0) \to -\infty$ as $t \to \infty$. In consequence, there exists $\eta_0 := t\varphi_0 \in X$ such that $\|\eta_0\| > \rho_0$ and $I(\eta_0) < 0$ for all $t > 1$ sufficient large. □

Proof of Theorem 2.1. Let X be a real Banach space, according to Lemmas 4.1 and 4.2, and therefore, there exists a Cerami subsequence $\{\eta_n\}_{n\in\mathbb{N}} \subset X$, such that $\eta_n \to \eta_0$ in X as $n \to \infty$, and the function I_λ fulfills Cerami condition for any $\lambda > 0$. Moreover, $I_\lambda(0) = 0$ is obvious, from Lemmas 4.3 and 4.4, we know that all conditions of the mountain pass theorem are fulfilled. It follows that I_λ has at least one critical point η_0 such that $I_\lambda(\eta_0) \ge \alpha_0(\rho_0) > 0$, namely, problem (1.1) has at least one nontrivial weak solution in X. □

4.3 Proof of Theorem 2.2

The function I_λ, which is relevant to problem (1.1), is rewritten in the following form:

$$I_\lambda(\eta) := \mathcal{U}(\eta) - \lambda\mathcal{V}(\eta), \tag{4.46}$$

where

$$\begin{aligned}\mathcal{U}(\eta) &= \sum_{i=1}^{2}\widetilde{M}_i(\delta_{p_i}(\eta)) + \sum_{i=1}^{2}\int_{\mathbb{R}}\frac{1}{\overline{p}_i(x)}|\eta|^{\overline{p}_i(x)}\mathrm{d}x,\\ \mathcal{V}(\eta) &= \int_{\mathbb{R}}F(x,\eta)\mathrm{d}x.\end{aligned} \tag{4.47}$$

In this section, we prove Theorem 2.2 by using the following abstract critical point theorem, see [41].

Theorem 4.1. *Let X be a real Banach space and consider two locally Lipschitz continuous functions $\mathcal{U},\mathcal{V} : X \to R$. Assume that $\mathcal{U}$ is bounded from below and $\mathcal{U}(0) = \mathcal{V}(0) = 0$. Set $\zeta > 0$ be fixed, and it is supposed that for each*

$$\lambda \in \lambda_0 := \left(0, \zeta\left(\sup_{\eta\in\mathcal{U}^{-1}(-\infty,\zeta)}\mathcal{V}(\eta)\right)^{-1}\right), \tag{4.48}$$

the function $I_\lambda = \mathcal{U} - \lambda\mathcal{V}$ fulfills the Cerami condition for any $\lambda \in \lambda_0$ and is unbounded from below. Then, for any $\lambda \in \lambda_0$, the function I_λ has two distinct critical points.

Proof of Theorem 2.2. Obviously, $\mathcal{U}(0) = \mathcal{V}(0) = 0$ and $\mathcal{U}$ is bounded from below.

According to an argument similar to Lemma 4.4 of Theorem 2.1. Let $\eta \in X\backslash\{0\}$, from the hypotheses $(\mathcal{F}_1)$, $(\mathcal{F}_2)$, $(\mathcal{M}_1)$, and enough large $t > 1$, we have

$$\begin{aligned}I_\lambda(t\eta) &= \sum_{i=1}^{2}\widetilde{M}_i(\delta_{p_i}(t\eta)) + \sum_{i=1}^{2}\int_{\mathbb{R}}\frac{1}{\overline{p}_i(x)}|t\eta|^{\overline{p}_i(x)}\mathrm{d}x - \lambda\int_{\mathbb{R}}F(x,t\eta)\mathrm{d}x\\ &\le |t|^{\vartheta p^+_{\max}}\left(\sum_{i=1}^{2}\frac{\widetilde{M}_i(1)}{(p_i^-)^{\vartheta_i}}\|\eta\|_{X_i}^{\vartheta_i p_i^+} + \sum_{i=1}^{2}\frac{1}{\overline{p}_i^-}\|\eta\|^{p_i^+} - \lambda M\int_{\mathbb{R}}|\eta|^{\vartheta p^+_{\max}}\mathrm{d}x\right) + \lambda C_M\int_{\mathbb{R}}\mathrm{d}x,\end{aligned} \tag{4.49}$$

by choosing $M > \left(\sum_{i=1}^{2}\frac{\widetilde{M}_i(1)}{(p_i^-)^{\vartheta_i}}\|\eta\|_{X_i}^{\vartheta_i p_i^+} + \sum_{i=1}^{2}\frac{1}{\overline{p}_i^-}\|\eta\|_{X_i}^{p_i^+}\right)\left(\lambda\int_{\mathbb{R}}|\eta|^{\vartheta p^+_{\max}}\mathrm{d}x\right)^{-1}$ and M is large enough, we deduce that $I(t\eta) \to -\infty$ as $t \to \infty$. As a consequence, $I_\lambda = \mathcal{U} - \lambda\mathcal{V}$ is unbounded from below.

In view of the hypothesis $(\mathcal{F}_1)$ and Lemma 3.7, it follows that

$$\begin{aligned}\mathcal{V}(\eta) &= \int_{\mathbb{R}}F(x,\eta)\mathrm{d}x\\ &\le \int_{\mathbb{R}}a_1(x)|\eta|\mathrm{d}x + \frac{1}{q^-}\int_{\mathbb{R}}a_2(x)|\eta|^{q(x)}\mathrm{d}x\end{aligned} \tag{4.50}$$

$$
\begin{aligned}
&\leq C_{15}\max\{\|\eta\|, \|\eta\|^{q^+}, \|\eta\|^{q^-}\}\\
&\leq C_{15}\max\{\|\eta\|_{X_1} + \|\eta\|_{X_2}, 2^{q^+-1}\left(\|\eta\|_{X_1}^{q^+} + \|\eta\|_{X_2}^{q^+}\right), 2^{q^--1}\left(\|\eta\|_{X_1}^{q^-} + \|\eta\|_{X_2}^{q^-}\right)\},
\end{aligned}
$$

where we have used that $(\mathbf{a} + \mathbf{b})^p \leq 2^{p-1}(\mathbf{a}^p + \mathbf{b}^p)$ for $\mathbf{a} > 0$ and $\mathbf{b} > 0$.

On the other hand, using (4.2) and the conditions $(\mathcal{M}_1)$ and $(\mathcal{M}_2)$, Proposition 3.3, we deduce that

$$
\begin{aligned}
\mathcal{U}(\eta) &= \sum_{i=1}^{2}\widetilde{M_i}(\delta_{p_i}(\eta)) + \sum_{i=1}^{2}\int_{\mathbb{R}}\frac{1}{\overline{p_i}(x)}|\eta|^{\overline{p_i}(x)}\mathrm{d}x,\\
&\geq \frac{1}{\vartheta p_{\max}^+}\sum_{i=1}^{2}M_i(\delta_{p_i}(\eta)) \times \langle\delta_{p_i}'(\eta), \eta\rangle + \sum_{i=1}^{2}\int_{\mathbb{R}}\frac{1}{\overline{p_i}(x)}|\eta|^{\overline{p_i}(x)}\mathrm{d}x\\
&\geq \frac{M_{\min}}{\vartheta p_{\max}^+}\sum_{i=1}^{2}\left(\int_{\Omega\times\Omega}\frac{|\eta(x) - \eta(y)|^{p_i(x,y)}}{|x - y|^{N+p_i(x,y)s(x,y)}}\mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}}|\eta|^{\overline{p_i}(x)}\mathrm{d}x\right)\\
&= \frac{M_{\min}}{\vartheta p_{\max}^+}(\|\eta\|_{X_1}^{p_1(x,\cdot)} + \|\eta\|_{X_2}^{p_2(x,\cdot)}),
\end{aligned} \tag{4.51}
$$

where $M_{\min} = \min\{1, m_1, m_2\}$. Without the loss of generality, we suppose that $\|\eta\|_{X_1}^{p_1(x,\cdot)} \geq \frac{\|\eta\|}{2} \geq \|\eta\|_{X_2}^{p_2(x,\cdot)} > 1$. Let us take $\zeta = 1$, it follows for each $\eta \in \mathcal{U}^{-1}(-\infty, 1)$ that

$$
\begin{aligned}
\|\eta\|_{X_1} &\leq \max\left\{\left(\frac{\vartheta p_{\max}^+}{2M_{\min}}\mathcal{U}(\eta)\right)^{\frac{1}{p_1^+}}, \quad \left(\frac{\vartheta p_{\max}^+}{2M_{\min}}\mathcal{U}(\eta)\right)^{\frac{1}{p_1^-}}\right\}\\
&\leq \max\left\{\left(\frac{\vartheta p_{\max}^+}{2M_{\min}}\right)^{\frac{1}{p_1^+}}, \left(\frac{\vartheta p_{\max}^+}{2M_{\min}}\right)^{\frac{1}{p_1^-}}\right\}\\
&\leq \left(\frac{\vartheta p_{\max}^+}{2M_{\min}}\right)^{\frac{1}{p_1^-}}.
\end{aligned} \tag{4.52}
$$

Similar to the aforementioned discussion, considering $\|\eta\|_{X_2}^{p_2(x,\cdot)} \geq \frac{\|\eta\|}{2} \geq \|\eta\|_{X_1}^{p_1(x,\cdot)} > 1$, we have

$$
\begin{aligned}
\|\eta\|_{X_2} &\leq \max\left\{\left(\frac{\vartheta p_{\max}^+}{2M_{\min}}\mathcal{U}(\eta)\right)^{\frac{1}{p_2^+}}, \left(\frac{\vartheta p_{\max}^+}{2M_{\min}}\mathcal{U}(\eta)\right)^{\frac{1}{p_2^-}}\right\}\\
&\leq \max\left\{\left(\frac{\vartheta p_{\max}^+}{2M_{\min}}\right)^{\frac{1}{p_2^+}}, \left(\frac{\vartheta p_{\max}^+}{2M_{\min}}\right)^{\frac{1}{p_2^-}}\right\}\\
&\leq \left(\frac{\vartheta p_{\max}^+}{2M_{\min}}\right)^{\frac{1}{p_2^-}}.
\end{aligned} \tag{4.53}
$$

Let us denote

$$
\lambda^* = \left(2^{q^+-1}C_{15}\left(\left(\frac{\vartheta p_{\max}^+}{2M_{\min}}\right)^{\frac{q^+}{p_1^-}} + \left(\frac{\vartheta p_{\max}^+}{2M_{\min}}\right)^{\frac{q^+}{p_2^-}}\right)\right)^{-1}, \tag{4.54}
$$

which joint with (4.50), we have

$$
\sup_{\eta\in\mathcal{U}^{-1}(-\infty,1)}\mathcal{V}(\eta) \leq 2^{q^+-1}C_{15}\left(\left(\frac{\vartheta p_{\max}^+}{2M_{\min}}\right)^{\frac{q^+}{p_1^-}} + \left(\frac{\vartheta p_{\max}^+}{2M_{\min}}\right)^{\frac{q^+}{p_2^-}}\right) = \frac{1}{\lambda^*} < \frac{1}{\lambda}. \tag{4.55}
$$

According to Lemmas 4.1 and 4.2, there exists a Cerami subsequence $\{\eta_n\}_{n\in\mathbb{N}} \subset X$, such that $\eta_n \to \eta_1$ in X as $n \to \infty$, and the function I_λ fulfills the Cerami condition. Moreover, all conditions of Theorem 4.1 are satisfied. Therefore, for any $\lambda \in (0, \lambda^*) \subset \lambda_0$, problem (1.1) possesses at least two distinct nontrivial solutions η_1, η_2 in X. □

4.4 Proof of Theorem 2.3

The space X is a separable and reflexive real Banach space, and there exist $\{e_j\} \subset X$ and $\{e_j^*\} \subset X^*$ such that

$$X = \overline{\operatorname{span}\{e_j : j = 1, 2, \ldots\}},$$
$$X^* = \overline{\operatorname{span}\{e_j^* : j = 1, 2, \ldots\}}$$

and

$$\langle e_j^*, e_i \rangle = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

Let $X_j = \operatorname{span}\{e_j : j = 1, 2, \ldots\}$ and define

$$A_k = \oplus_{i=1}^{k} X_i = \operatorname{span}\{e_j : j = 1, 2, \ldots, k\},$$
$$B_k = \overline{\oplus_{i=k}^{\infty} X_i} = \overline{\operatorname{span}\{e_j : j = k+1, k+2, \ldots\}}.$$

Theorem 4.2. (Fountain theorem, see [40]) *Let X be a real Banach space and an even function $I_\lambda \in C^1(X, \mathbb{R})$ satisfies the Cerami condition for every $c > 0$, and that there is $k_0 > 0$, such that for every $k \geq k_0$, there exists $\rho_k > r_k > 0$, so that the following properties hold:*
(i) $a_k = \max\{I_\lambda(\eta) : \eta \in A_k, \|\eta\| = \rho_k\} \leq 0$;
(ii) $b_k = \inf\{I_\lambda(\eta) : \eta \in B_k, \|\eta\| = r_k\} \to +\infty$ *as* $k \to \infty$.

Then, I_λ has a sequence of critical points η_k such that $I_\lambda(\eta_k) \to +\infty$.

Lemma 4.5. (See [27]) *Assume that $(\mathcal{A}_1)$ and $(\mathcal{A}_2)$ are satisfied. Let $\zeta(x) \in C_+(\mathbb{R}^N)$, $\zeta(x) < p_s^*(x)$, for any $x \in \mathbb{R}^N$ and denote*

$$\beta_k = \sup_{u \in B_k, \|u\|_X = 1} \|\eta\|_{\zeta, a(x)},$$

and then, $\lim_{k\to\infty} \beta_k = 0$.

To prove Theorem 2.3, we will utilize the fountain theorem, we first need to prove two lemmas.

Lemma 4.6. *Under the conditions of* Theorem 2.3, *there exists $r_k > 0$ such that*

$$\inf_{u \in B_k, \|u\| = r_k} I_\lambda(\eta) > +\infty.$$

Proof. According to the hypothesis $(\mathcal{F}_1)$, we can verify that there exists C_{21}, C_{22} such that

$$F(x, t) \leq C_{21} a_1(x)|t| + C_{22} a_2(x)|t|^{q(x)}, \quad \text{for all } (x, t) \in \mathbb{R}^N \times \mathbb{R}. \tag{4.56}$$

Set $\lambda > 0$ and $\|\eta\| > 1$ for any $\eta \in B_k$. Then, using the conditions $(\mathcal{M}_1)$, $(\mathcal{M}_2)$ and Lemma 3.8, we deduce that

$$\begin{aligned} I_\lambda(\eta) &\geq \frac{1}{\vartheta p_{\max}^+} \sum_{i=1}^{2} M_i(\delta_{p_i}(\eta)) \times \langle \delta'_{p_i}(\eta), \eta \rangle + \sum_{i=1}^{2} \int_{\mathbb{R}} \frac{1}{\overline{p}_i(x)} |\eta|^{\overline{p}_i(x)} \mathrm{d}x - \lambda C_{21} \int_{\mathbb{R}} a_4(x)|\eta| \mathrm{d}x - \lambda C_{22} \int_{\mathbb{R}} a_3(x)|\eta|^{q(x)} \mathrm{d}x \\ &\geq \frac{M_{\min}}{\vartheta p_{\max}^+} \sum_{i=1}^{2} \left(\int_{\Omega\times\Omega} \frac{|\eta(x) - \eta(y)|^{p_i(x,y)}}{|x-y|^{N + p_i(x,y)s(x,y)}} \mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}} |\eta|^{\overline{p}_i(x)} \mathrm{d}x \right) - \lambda C_{21} \int_{\mathbb{R}} a_1(x)|\eta| \mathrm{d}x - \lambda C_{22} \int_{\mathbb{R}} a_2(x)|\eta|^{q(x)} \mathrm{d}x \\ &\geq \frac{M_{\min}}{\vartheta p_{\max}^+} (\|\eta\|_{X_1}^{p_1^-} + \|\eta\|_{X_2}^{p_2^-}) - \lambda C_{23}\|\eta\| - \lambda C_{24} \beta_k^{q^+} \|\eta\|^{q^+} \\ &\geq \frac{M_{\min}}{2^{p_{\max}^+ - 1} \vartheta p_{\max}^+} \|\eta\|^{p_{\min}^-} - \lambda C_{23}\|\eta\| - \lambda C_{24} \beta_k^{q^+} \|\eta\|^{q^+}, \end{aligned} \tag{4.57}$$

where $M_{\min} = \min\{1, m_1, m_2\}$, where β_k is defined as in Lemma 4.5. Choosing

$$r_k = (M_{\min})^{\frac{1}{q^+ - p_{\min}^-}} \left(2^{p_{\max}^+ - 1} \theta p_{\max}^+ \lambda C_{24} \beta_k^{q^+}\right)^{\frac{-1}{q^+ - p_{\min}^-}}.$$

It is easy to see that $r_k \to +\infty$ as $k \to +\infty$, thanks to Lemma 4.5 and the fact that $p_{\min}^- < p_{\max}^+ < q^+$. Thus, by the choice of $r_k \in Z_k$ with $\|u\| = r_k$ such that $\rho_k > r_k > 0$, we obtain

$$B_k = \inf_{u \in B_k, \|u\| = r_k} I(u) \geq \frac{M_{\min}}{2^{p_{\max}^+} \vartheta p_{\max}^+} r_k^{p_{\min}^-} - \lambda C_{23} r_k \to +\infty,$$

as $k \to +\infty$. □

Lemma 4.7. *Under the conditions of* Theorem 2.3, *then there exists* $\rho_k > 0$ *such that*

$$\max_{u \in A_k, \|u\| = \rho_k} I_\lambda(\eta) \leq 0.$$

Proof. Let $\lambda > 0$ and it follows from the hypotheses $(\mathcal{F}_1)$ and $(\mathcal{F}_2)$ that there exist corresponding positive constants C_{16}, C_{17}, C_{18} such that

$$F(x,t) \geq C_{16} |t|^{\vartheta p_{\max}^+} - C_{17} a_1(x)|t| - C_{18} a_2(x) |t|^{q(x)}, \quad \text{uniformly for all } x \in \mathbb{R}^N. \tag{4.58}$$

For any $\eta \in A_k$ with $\|\eta\| = 1$. Combining with $(\mathcal{M}_1)$ and (4.58), we obtain

$$\begin{aligned} I_\lambda(t\eta) &\leq \sum_{i=1}^{2} \widetilde{M}_i(1) (\delta_{p_i}(t\eta))^{\vartheta_i} + \sum_{i=1}^{2} \frac{1}{\overline{p}_i^-} \int_{\mathbb{R}} |t\eta|^{\overline{p}_i(x)} \mathrm{d}x \\ &\quad - \lambda C_{16} \int_{\mathbb{R}} |t\eta|^{\vartheta p_{\max}^+} \mathrm{d}x + \lambda C_{17} \int_{\mathbb{R}} a_1(x) |t\eta| \mathrm{d}x + \lambda C_{18} \int_{\mathbb{R}} a_2(x) |t\eta|^{q(x)} \mathrm{d}x \\ &\leq \left(\frac{2\max\{1, \widetilde{M}_1(1), \widetilde{M}_2(1)\}}{p_{\min}^-} \|\eta\|^{\vartheta p_{\max}^+} - \lambda C_{16} \int_{\mathbb{R}} |\eta|^{\vartheta p_{\max}^+} \mathrm{d}x \right) t^{\vartheta p_{\max}^+} \\ &\quad + \lambda C_{17} t \int_{\mathbb{R}} a_1(x) |\eta| \mathrm{d}x + \lambda C_{18} t^{q^+} \int_{\mathbb{R}} a_2(x) |\eta|^{q(x)} \mathrm{d}x. \end{aligned} \tag{4.59}$$

Since all norms are equivalent on the finite dimensional Banach space A_k, there exist some constants C_{19}, C_{20}, C_{21}, and C_{21} such that

$$\frac{\max\{1, \widetilde{M}_1(1), \widetilde{M}_2(1)\}}{p_{\min}^-} \|\eta\|^{\vartheta p_{\max}^+} \leq \frac{C_{19}}{4} \lambda C_{16} \int_{\mathbb{R}} |\eta|^{\vartheta p_{\max}^+} \mathrm{d}x,$$

$$\int_{\mathbb{R}} |\eta|^{\vartheta p_{\max}^+} \mathrm{d}x \geq C_{20} \|\eta\|, \qquad \int_{\mathbb{R}} a_1(x) |\eta| \mathrm{d}x \leq C_{21} \|\eta\|, \qquad \int_{\mathbb{R}} a_2(x) |\eta|^{q(x)} \mathrm{d}x \leq C_{22} \|\eta\|,$$

which combine with (4.59), we have

$$I_\lambda(t\eta) \leq -\frac{C_{19} C_{20}}{2} \lambda C_{16} t^{\vartheta p_{\max}^+} + \lambda C_{17} C_{21} t + \lambda C_{18} C_{22} t^{q^+}. \tag{4.60}$$

Since $\vartheta p_{\max}^+ > q^+$, we obtain that $I_\lambda(t\eta) \to -\infty$ as $t \to +\infty$. Therefore, there exists $t_0 > r_k > 0$ large enough such that $I_\lambda(t_0 \eta) \leq 0$, and thus, let us take $\rho_k = t_0$, and we conclude that

$$a_k = \max_{u \in A_k, \|u\| = \rho_k} I(\eta) \leq 0.$$ □

Proof of Theorem 2.3. Let X be a real Banach space, from Lemmas 4.1 and 4.2; therefore, there exists a Cerami subsequence $\{\eta_n\}_{n \in \mathbb{N}} \subset X$, such that $\eta_n \to \eta_1$ in X as $n \to \infty$, and the function I_λ fulfills the Cerami condition. Moreover, Lemmas 4.6, 4.7, and $I_\lambda(0) = 0$ imply that I satisfies all conditions of Theorem 4.2. Consequently, for any $\lambda > 0$, problem (1.1) has infinitely many nontrivial weak solutions in X. □

5 Conclusion

In this article, we study a kind of Kirchhoff-type problem in the whole space $\mathbb{R}^N$. Under some reasonable assumptions of f and with the help of variational and critical point theory, we get at least one nontrivial solution, two distinct nontrivial solutions, and infinitely many nontrivial solutions in an appropriate space of functions without the Ambrosetti-Rabinowitz condition. Several recent results of the pieces of literature are extended and improved.

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