Research Article

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General complex L_p projection bodies and complex L_p mixed projection bodies

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Abstract: Abardia and Bernig proposed the notions of complex projection body and complex mixed projection body. In this paper, we introduce the concepts of the general complex L_p projection body and complex L_p mixed projection body. Furthermore, we establish the Brunn-Minkowski-type inequalities for the general complex L_p projection bodies and the Aleksandrov-Fenchel-type inequalities for the general complex L_p mixed projection bodies.

Keywords: the general complex L_p mixed projection body, Brunn-Minkowski-type inequalities, Aleksandrov-Fenchel-type inequalities

MSC 2020: 52A20, 52A40

1 Introduction

The classical Brunn-Minkowski theory appeared at the turn of the nineteenth into the twentieth century. One of the core concepts that Minkowski introduced within the Brunn-Minkowski theory is the projection body. There are important inequalities involving the volume of the projection body and its polar, such as Petty projection inequality [1] and Zhang projection inequality [2].

In the early 1960s, Firey [3] introduced the Firey-Minkowski L_p-addition of a convex body. In the mid-1990s, it was shown in [4] and [5] that a study of the volume of the Firey-Minkowski L_p combinations leads to the L_p Brunn-Minkowski theory. This theory has expanded rapidly. An early achievement of the L_p Brunn-Minkowski theory was the discovery of L_p projection body, introduced by Lutwak et al. [6]. Since then, Ludwig [7,8] extended the projection body to an entire class that can be called the general L_p projection body.

A mixed projection body was introduced in the classic volume of Bonnesen-Fenchel [9]. In [10] and [11], Lutwak considered the volume of the mixed projection body and established the classical mixed volume inequalities, such as Aleksandrov-Fenchel inequalities and Brunn-Minkowski inequalities.

Let us mention that the projection bodies described above are all real. The theory of the real projection body continues to be a very active field now. For additional information and some results on real projection body see, e.g., [8,12–17]. However, some classical concepts of convex geometry in real vector space were extended to complex cases, such as complex difference body [18], complex intersection body [19], complex centroid body [20,21], and complex projection body [22–25].

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In this paper, we mainly study the projection body in complex vector space. Let $\mathcal{K}(\mathbb{C}^n)$ be the set of convex body (nonempty compact convex subsets) in complex vector space $\mathbb{C}^n$. For the set of the convex body containing the origin in their interiors and the set of an origin-symmetric convex body in $\mathbb{C}^n$, we write $\mathcal{K}_o(\mathbb{C}^n)$ and $\mathcal{K}_{os}(\mathbb{C}^n)$, respectively. Let $V(K)$ denote the complex volume of K, B the complex unit ball, and S^{2n-1} the complex unit sphere.

In 2011, the complex L_p projection body $\Pi_C K$ was defined by Abardia and Bernig [22]: For $K \in \mathcal{K}(\mathbb{C}^n)$ and $C \in \mathcal{K}(\mathbb{C})$,

$$h(\Pi_C K, u) = nV_1(K, C \cdot u) = \frac{1}{2} \int\limits_{S^{2n-1}} h(C \cdot u, v) \mathrm{d}S_K(v) \tag{1.1}$$

for every $u \in S^{2n-1}$, where $C \cdot u := \{cu : c \in C\}$. They also defined the mixed complex projection body as

$$h(\Pi_C(K_1, K_2, \ldots, K_{2n-1}), u) = nV(K_1, K_2, \ldots, K_{2n-1}, C \cdot u). \tag{1.2}$$

Very recently, the concept of asymmetric complex L_p projection body $\Pi^+_{p,C} K$ was introduced in [24]. First, a convex body $C \in \mathcal{K}(\mathbb{C})$ is called an asymmetric L_p zonoid if there exists a finite even Borel measure $\mu_{p,C}(v)$ on S^{n-1} such that

$$h_C(u)^p = \int\limits_{S^{n-1}} (\mathcal{R}[cu \cdot v])^p_+ \mathrm{d}\mu_{p,C}(c).$$

Based on the fact that $h_{C \cdot u}(v) = h_C(u \cdot v)$ and the sesquilinearity of the Hermitian inner product in $\mathbb{C}^n$, we obtain

$$h_{C \cdot u}(v)^p = \int\limits_{S^{n-1}} (\mathcal{R}[c \cdot (u \cdot v)])^p_+ \mathrm{d}\mu_{p,C}(c) = \int\limits_{S^{n-1}} (\mathcal{R}[cu \cdot v])^p_+ \mathrm{d}\mu_{p,C}(c)$$

for all $u, v \in S^{n-1}$. Then, if $p \geq 1$, $K \in \mathcal{K}_o(\mathbb{C}^n)$, and $C \in \mathcal{K}(\mathbb{C})$ is an asymmetric L_p zonoid, the asymmetric complex L_p projection body $\Pi^+_{p,C} K$ is

$$h(\Pi^+_{p,C} K, u)^p = 2nV_p(K, C \cdot u) = \int\limits_{S^{2n-1}} \int\limits_{S^{n-1}} (\mathcal{R}[cu \cdot v])^p_+ \mathrm{d}\mu_{p,C}(c) \mathrm{d}S_{p,K}(v) \tag{1.3}$$

for all $u, v \in S^{2n-1}$, where $S_{p,K}(v)$ is the L_p surface area measure of K on S^{2n-1}.

Motivated by the works of Abardia and Bernig [22], Haberl [20], and Liu and Wang [24], we introduce more general definitions of complex L_p projection body and complex L_p mixed projection body.

Definition 1.1. If $p \geq 1$, $K \in \mathcal{K}_o(\mathbb{C}^n)$, and $C \in \mathcal{K}(\mathbb{C})$ is an asymmetric L_p zonoid, the general complex L_p projection body $\Pi^\lambda_{p,C} K$ is defined by

$$h(\Pi^\lambda_{p,C} K, u)^p = f_1(\lambda) h(\Pi^+_{p,C} K, u)^p + f_2(\lambda) h(\Pi^-_{p,C} K, u)^p \tag{1.4}$$

for every $\lambda \in [-1, 1]$, where

$$f_1(\lambda) = \frac{(1+\lambda)^p}{(1+\lambda)^p + (1-\lambda)^p}, \quad f_2(\lambda) = \frac{(1-\lambda)^p}{(1+\lambda)^p + (1-\lambda)^p},$$

and $f_1(\lambda) + f_2(\lambda) = 1$.

We use $\Pi^{\lambda,*}_{p,C} K$ to denote the polar body $\Pi^\lambda_{p,C} K$. The normalization is chosen such that $\Pi^\lambda_{p,C} B = B$ and $\Pi^0_{p,C} K = \Pi_{p,C} K$. If $\lambda = 1$ in (1.4), then $\Pi^1_{p,C} K = \Pi^+_{p,C} K$. In addition, set $\Pi^-_{p,C} K = \Pi^+_{p,C}(-K)$. By the definitions of $\Pi^\pm_{p,C} K$ and $\Pi^\lambda_{p,C} K$, we obtain

$$\Pi^\lambda_{p,C} K = \frac{(1+\lambda)^p}{(1+\lambda)^p + (1-\lambda)^p} \cdot \Pi^+_{p,C} K +_p \frac{(1-\lambda)^p}{(1+\lambda)^p + (1-\lambda)^p} \cdot \Pi^-_{p,C} K, \tag{1.5}$$

and the complex L_p projection body is defined as

$$\Pi_{p,C}K = \frac{1}{2} \cdot \Pi^{+}_{p,C}K +_p \frac{1}{2} \cdot \Pi^{-}_{p,C}K. \tag{1.6}$$

It is clear that if $p = 1$ in (1.6), $\Pi_{p,C}K$ is $\Pi_C K$ defined in (1.1).

Definition 1.2. If $p \geq 1$, $K_1, K_2, \ldots, K_{2n-1} \in \mathcal{K}_o(\mathbb{C}^n)$, and $C \in \mathcal{K}(\mathbb{C})$ is an asymmetric L_p zonoid, the general complex L_p mixed projection body $\Pi^{\lambda}_{p,C}(K_1, K_2, \ldots, K_{2n-1})$ is defined by

$$h(\Pi^{\lambda}_{p,C}(K_1, K_2, \ldots, K_{2n-1}), u)^p = f_1(\lambda)h(\Pi^{+}_{p,C}(K_1, K_2, \ldots, K_{2n-1}), u)^p + f_2(\lambda)h(\Pi^{-}_{p,C}(K_1, K_2, \ldots, K_{2n-1}), u)^p \tag{1.7}$$

for every $\lambda \in [-1, 1]$.

Moreover, if $K_{2n-i} = \cdots = K_{2n-1} = B$ and $\mathbf{M} := (K_1, K_2, \ldots, K_{2n-1-i})$, then for $0 \leq i \leq 2n-2$, $\Pi^{\lambda}_{p,C}(K_1, K_2, \ldots, K_{2n-1})$ is written as $\Pi^{\lambda}_{p,i,C}(\mathbf{M})$. If $K_1 = \cdots = K_{2n-1-i} = K$, we simply write $\Pi^{\lambda}_{p,i,C}(\mathbf{M})$ as $\Pi^{\lambda}_{p,i,C}(K)$ that is called the ith L_p mixed complex projection body of K. If $i = 0$, we write $\Pi^{\lambda}_{p,0,C}K$ as $\Pi^{\lambda}_{p,C}K$.

Before stating our main results, let us introduce the L_p Blaschke combination. For $2n \neq p \geq 1$ and $K, L \in \mathcal{K}_{os}(\mathbb{C}^n)$, the L_p Blaschke combination $K \#_p L \in \mathcal{K}_{os}(\mathbb{C}^n)$ is defined by Lutwak [4] as

$$\mathrm{d}S_p(K \#_p L, \cdot) := \mathrm{d}S_p(K, \cdot) + \mathrm{d}S_p(L, \cdot), \tag{1.8}$$

where $S_p(K, \cdot)$ denotes the L_p surface area measure of K on S^{2n-1}. If $p = 1$ and $K, L \in \mathcal{K}(\mathbb{C}^n)$, it is a classical Blaschke combination.

Our main results can be stated as the following Theorems 1.1–1.4 and among them, Theorems 1.1–1.2 are the Brunn-Minkowski-type inequalities for the general complex L_p projection bodies. Theorems 1.3–1.4 are the Aleksandrov-Fenchel-type inequalities for the general complex L_p mixed projection bodies.

Theorem 1.1. *If $p \geq 1$, $K, L \in \mathcal{K}_{os}(\mathbb{C}^n)$, and $C \in \mathcal{K}(\mathbb{C})$ is an asymmetric L_p zonoid, then*

$$V(\Pi^{\lambda}_{p,C}(K \#_p L))^{\frac{p}{2n}} \geq V(\Pi^{\lambda}_{p,C}K)^{\frac{p}{2n}} + V(\Pi^{\lambda}_{p,C}L)^{\frac{p}{2n}} \tag{1.9}$$

with equality if and only if K and L are real dilates.

Theorem 1.2. *If $p \geq 1$, $K, L \in \mathcal{K}_{os}(\mathbb{C}^n)$, and $C \in \mathcal{K}(\mathbb{C})$ is an asymmetric L_p zonoid, then*

$$V(\Pi^{\lambda,*}_{p,C}(K \#_p L))^{-\frac{p}{2n}} \geq V(\Pi^{\lambda,*}_{p,C}K)^{-\frac{p}{2n}} + V(\Pi^{\lambda,*}_{p,C}L)^{-\frac{p}{2n}} \tag{1.10}$$

with equality if and only if K and L are real dilates.

Theorem 1.3. *If $p \geq 1$, $K_1, K_2, \ldots, K_{2n-1} \in \mathcal{K}_o(\mathbb{C}^n)$, and $C \in \mathcal{K}(\mathbb{C})$ is an asymmetric L_p zonoid, then*

$$V(\Pi^{\lambda}_{p,C}(K_1, K_2, \ldots, K_{2n-1}))^{\frac{pr}{2n}} \geq (2n)^{1-p} \prod_{j=1}^{2n-1} V(K_1, \ldots, K_{2n-1}, K_j)^{\frac{r(1-p)}{2n-1}} \prod_{j=1}^{r} V(\Pi_C(K_j[r], K_{r+1}, \ldots, K_{2n-1}))^{\frac{p}{2n}}. \tag{1.11}$$

If K_j is an ellipsoid centered at the origin or an Hermitian ellipsoid, the equality holds.

Theorem 1.4. *If $K_1, K_2, \ldots, K_{2n-1-i} \in \mathcal{K}_o(\mathbb{C}^n)$ and $C \in \mathcal{K}(\mathbb{C})$ is an asymmetric L_p zonoid. Let $\mathbf{M} := (K_1, K_2, \ldots, K_{2n-1-i})$. If $p \geq 1$, $i = 0, 1, \ldots, 2n-2$, then*

$$V(\Pi^{\lambda}_{p,i,C}\mathbf{M})^{\frac{p}{2n}} \geq (2n)^{1-p} \prod_{j=1}^{2n-1-i} W_i(\mathbf{M}, K_j)^{\frac{1-p}{2n-1-i}} \prod_{j=1}^{r} V(\Pi_{i,C}(K_j[r], K_{r+1}, \ldots, K_{2n-1-i}))^{\frac{p}{2nr}}. \tag{1.12}$$

If K_j is an ellipsoid centered at the origin or an Hermitian ellipsoid, the equality holds.

Remark 1.1. Note that the cases of $C = [0, 1]$ of Theorems 1.1–1.4 are the Brunn-Minkowski-type inequalities [26] and the Aleksandrov-Fenchel-type inequalities for the general L_p projection bodies [27].

If C is just a point, then $\Pi^{\lambda}_{p,C}K = \{0\}$ for every $K \in \mathcal{K}_o(\mathbb{C}^n)$ and every $\lambda \in [-1, 1]$. We assume that dim $C > 0$ throughout this paper.

2 Preliminaries

2.1 Support function, radial function, and polar of convex body

For a complex number $c \in \mathbb{C}$, we write $\bar{c}$ for its complex conjugate and $|c|$ for its norm. If $\phi \in \mathbb{C}^{m\times n}$, then ϕ^* denotes the conjugate transpose of ϕ. We denote by "$\cdot$" the standard Hermitian inner product on $\mathbb{C}^n$ which is conjugate linear in first argument. Koldobsky et al. [19] identified $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ using the standard mapping:

$$\iota(c) = (\mathcal{R}[c_1], \dots, \mathcal{R}[c_n], \zeta[c_1], \dots, \zeta[c_n]), \quad c \in \mathbb{C}^n,$$

where $\mathcal{R}$ and ζ are the real and imaginary parts, respectively. Note that $\mathcal{R}[x \cdot y] = \iota x \cdot \iota y$ for all $x, y \in \mathbb{C}^n$, where the inner product on the right hand is the standard Euclidean inner product on $\mathbb{R}^{2n}$.

We collect complex reformulations of well-known results from convex geometry. These complex version can be directly deduced from their real counterparts by an appropriate application of ι. For more details refer to the books in [28,29].

A convex body $K \in \mathcal{K}(\mathbb{C}^n)$ is determined by its support function $h_K : \mathbb{C}^n \to \mathbb{R}$, where

$$h_K(x) = \max\{\mathcal{R}[x \cdot y] : y \in K\}.$$

For every Borel set $\omega \subset S^{2n-1}$, the complex surface area measure S_K of $K \in \mathcal{K}(\mathbb{C}^n)$ is defined by

$$S_K(\omega) = \mathcal{H}^{2n-1}(\iota\{x \in K : \exists u \in \omega, \mathcal{R}[x \cdot u] = h_K(u)\}),$$

where $\mathcal{H}^{2n-1}$ stands for $(2n-1)$-dimensional Hausdorff measure on $\mathbb{R}^{2n}$. In addition, the complex surface area measures are translation invariant and $S_{cK}(\omega) = S_K(\bar{c}\omega)$ for all $c \in S^{n-1}$ and each Borel set $\omega \in S^{2n-1}$.

K is an Hermitian ellipsoid if $K = \{x \in \mathbb{C}^n : x \cdot \phi x \le 1\} + t$ for a positive definite Hermitian matrix $\phi \in GL(n, \mathbb{C})$ and a $t \in \mathbb{C}^n$. Note that if K is an Hermitian ellipsoid if and only if $K = \psi B + t$ for some $\psi \in GL(n, \mathbb{C})$ and a $t \in \mathbb{C}^n$ (see [20]).

If K is a compact star-shaped (about the origin) in $\mathbb{C}^n$, its radial function, $\rho_K(x) = \rho(K, x) : \mathbb{C}^n\backslash\{0\} \to [0, \infty)$ is given by

$$\rho_K(x) = \max\{\lambda \ge 0 : \lambda x \in K\}.$$

If $\rho_K(x)$ is positive and continuous, K will be called a star body. For the set of star body containing the origin in their interiors, we write $\mathcal{S}_o(\mathbb{C}^n)$. Moreover, if $K \in \mathcal{K}_o(\mathbb{C}^n)$, then $K^* \in \mathcal{K}_o(\mathbb{C}^n)$. On $\mathbb{C}^n\backslash\{0\}$, the support function and radial function of the polar body K^* can be given, respectively, by

$$h_{K^*} = \frac{1}{\rho_K}, \quad \rho_{K^*} = \frac{1}{h_K}.$$

2.2 The L_p mixed quermassintegrals

For $K \in \mathcal{K}_o(\mathbb{C}^n)$, $i = 1, 2, \dots, 2n-1$, the quermassintegrals $W_i(K)$ of K are defined by (see [29])

$$W_i(K) = \frac{1}{2n} \int_{S^{2n-1}} h(K, u) \mathrm{d}S_i(K, u), \tag{2.1}$$

where $S_i(K, u)$ is the mixed surface area of K.

If $p \geq 1$, $K, L \in \mathcal{K}_o(\mathbb{C}^n)$, and $\alpha, \beta \geq 0$ that are not both zero, the L_p Minkowski combination $\alpha \cdot K +_p \beta \cdot L$ is defined by

$$h^p_{\alpha\cdot K+_p\beta\cdot L} = \alpha h^p_K + \beta h^p_L.$$

The extension of the Brunn-Minkowski inequality (see [4]) is as follows: If $K, L \in \mathcal{K}_o(\mathbb{C}^n)$, $p > 1$, then

$$W_i(K +_p L)^{\frac{p}{2n-i}} \geq W_i(K)^{\frac{p}{2n-i}} + W_i(L)^{\frac{p}{2n-i}} \tag{2.2}$$

with equality if and only if K and L are dilates. In particular, if $p = 1$, $i = 0$, the inequality is the Brunn-Minkowski inequality.

If $p \geq 1$, $i = 1, 2, \ldots, 2n-1$, and $K \in \mathcal{K}_o(\mathbb{C}^n)$, there exists a positive Borel measure $S_{p,i}(K, \cdot)$ on S^{2n-1}, such that the L_p mixed quermassintegral $W_{p,i}(K, L)$ has the following integral representation (see [4]):

$$W_{p,i}(K, L) = \frac{1}{2n}\int_{S^{2n-1}} h(L, u)^p \mathrm{d}S_{p,i}(K, u) \tag{2.3}$$

for all $L \in \mathcal{K}_o(\mathbb{C}^n)$. It turns out that the measure $S_{p,i}(K, \cdot)$ on S^{2n-1} is absolutely continuous with respect to $S_i(K, \cdot)$ and has the Radon-Nikodym derivative $\frac{\mathrm{d}S_{p,i}(K,\cdot)}{\mathrm{d}S_i(K,\cdot)} = h_K(\cdot)^{1-p}$. Obviously, $S_{p,0}(K, \cdot) = S_p(K, \cdot)$.

In view of the L_p Minkowski inequality in $\mathbb{R}^n$ by Lutwak [4], there is an L_p Minkowski inequality about the L_p mixed quermassintegrals in $\mathbb{C}^n$. That is, if $K, L \in \mathcal{K}_o(\mathbb{C}^n)$, $p \geq 1$, and $0 \leq i < 2n$, then

$$W_{p,i}(K, L)^{2n-i} \geq W_i(K)^{2n-i-p} W_i(L)^p \tag{2.4}$$

with equality for $p = 1$ if and only if K and L are real homothetic, while for $p > 1$ if and only if K and L are real dilates. In particular, if $i = 0$ in (2.4), then

$$V_p(K, L)^{2n} \geq V(K)^{2n-p} V(L)^p \tag{2.5}$$

with equality if and only if K and L are real dilates.

2.3 The dual L_p mixed quermassintegrals

For $K \in \mathcal{S}_o(\mathbb{C}^n)$ and $i = 1, 2, \ldots, 2n-1$, the dual quermassintegrals $\widetilde{W}_i(K)$ are defined by (see [30])

$$\widetilde{W}_i(K) = \frac{1}{2n}\int_{S^{2n-1}} \rho(K, u)^{2n-i} \mathrm{d}S(u), \tag{2.6}$$

where $S(u)$ stands for the push forward with respect to ι^{-1} of $\mathcal{H}^{2n-1}$ on the $(2n-1)$-dimensional Euclidean unit sphere.

If $p \geq 1$, $K, L \in \mathcal{S}_o(\mathbb{C}^n)$, and $\alpha, \beta \geq 0$ that are not both zero, the L_p harmonic radial combination $\alpha \cdot K \tilde{+}_p \beta \cdot L$ is defined by

$$\rho^{-p}_{\alpha\cdot K\tilde{+}_p\beta\cdot L} = \alpha\rho^{-p}_K + \beta\rho^{-p}_L.$$

For $p \geq 1$, $i = 1, 2, \ldots, 2n-1$, and $K, L \in \mathcal{S}_o(\mathbb{C}^n)$, the dual L_p mixed quermassintegral $\widetilde{W}_{p,i}(K, L)$ has the following integral representation (see [31]):

$$\widetilde{W}_{-p,i}(K, L) = \frac{1}{2n}\int_{S^{2n-1}} \rho(K, u)^{2n+p-i}\rho(L, u)^{-p} \mathrm{d}S(u). \tag{2.7}$$

The dual L_p Minkowski inequality in $\mathbb{C}^n$ can be stated as follows: If $p \geq 1$, $K, L \in \mathcal{S}_o(\mathbb{C}^n)$, and $0 \leq i < 2n$, then

$$\widetilde{W}_{-p,i}(K, L)^{2n-i} \geq \widetilde{W}_i(K)^{2n+p-i} \widetilde{W}_i(L)^{-p} \tag{2.8}$$

with equality if and only if K and L are real dilates.

2.4 The general complex L_p projection body and complex L_p mixed projection body

Since the integral representations of the general complex L_p projection body and complex L_p mixed projection body need to be used in Section 3, we will present their integral representations in this part.

First of all, by combining (1.3) and (1.4), we obtain the integral representation of the general complex L_p projection body $\Pi^{\lambda}_{p,C}K$ as

$$h(\Pi^{\lambda}_{p,C}K, u)^p = \int\limits_{S^{2n-1}}\int\limits_{S^{n-1}} f_1(\lambda)(\mathcal{R}[cu\cdot v])_+^p \mathrm{d}\mu_{p,C}(c)\mathrm{d}S_p(K, v) + \int\limits_{S^{2n-1}}\int\limits_{S^{n-1}} f_2(\lambda)(\mathcal{R}[cu\cdot v])_-^p \mathrm{d}\mu_{p,C}(c)\mathrm{d}S_p(K, v) \tag{2.9}$$

for all $u \in S^{2n-1}$ and every $\lambda \in [-1, 1]$.

In order to give the integral representation of $\Pi^{\lambda}_{p,C}(K_1, K_2, \ldots, K_{2n-1})$, we introduce the following definition.

Definition 2.1. For $p \geq 0$ and $K_1, K_2, \ldots, K_{2n-1} \in \mathcal{K}_o(\mathbb{C}^n)$, we define Borel measure $S_p(K_1, K_2, \ldots, K_{2n-1}, \cdot)$ on S^{2n-1} as

$$S_p(K_1, K_2, \ldots, K_{2n-1}; \omega) = \int\limits_{\omega} (h(K_1, u)h(K_2, u)\ldots h(K_{2n-1}, u))^{\frac{1-p}{2n-1}} \mathrm{d}S(K_1, K_2, \ldots, K_{2n-1}; u) \tag{2.10}$$

for each Borel $\omega \subset S^{2n-1}$.

Let us mention that $S_p(K_1, K_2, \ldots, K_{2n-1}; \cdot)$ on S^{2n-1} is called the general complex L_p mixed surface area measure of $K_1, K_2, \ldots, K_{2n-1}$ and has the Radon-Nikodym derivative

$$\frac{\mathrm{d}S_p(K_1, K_2, \ldots, K_{2n-1}; \cdot)}{\mathrm{d}S(K_1, K_2, \ldots, K_{2n-1}; \cdot)} = (h(K_1, \cdot)h(K_2, \cdot)\ldots h(K_{2n-1}, \cdot))^{\frac{1-p}{2n-1}}. \tag{2.11}$$

If $K_1 = K_2 = \cdots = K_{2n-1} = K$, then $S_p(K, K, \ldots, K, \cdot) = S_p(K, \cdot)$. In particular, if $K_1, K_2, \ldots, K_{2n-1-i} \in \mathcal{K}_o(\mathbb{C}^n)$, $K_{2n-i} = \cdots = K_{2n-1} = B$, then for $i = 0, 1, \ldots, 2n-2$, we obtain

$$\frac{\mathrm{d}S_{p,i}(K_1, K_2, \ldots, K_{2n-1-i}; \cdot)}{\mathrm{d}S_i(K_1, K_2, \ldots, K_{2n-1-i}; \cdot)} = (h(K_1, \cdot)h(K_2, \cdot)\ldots h(K_{2n-1-i}, \cdot))^{\frac{1-p}{2n-1-i}}. \tag{2.12}$$

Next, with respect to (1.7) and (2.10), we have the following integral representation of the general complex L_p mixed projection body $\Pi^{\lambda}_{p,C}(K_1, K_2, \ldots, K_{2n-1})$ as

$$\begin{aligned} h(\Pi^{\lambda}_{p,C}(K_1, K_2, \ldots, K_{2n-1}), u)^p = &\int\limits_{S^{2n-1}}\int\limits_{S^{n-1}} f_1(\lambda)(\mathcal{R}[cu\cdot v])_+^p \mathrm{d}S_p(K_1, K_2, \ldots, K_{2n-1}; v) \\ &+ \int\limits_{S^{2n-1}}\int\limits_{S^{n-1}} f_2(\lambda)(\mathcal{R}[cu\cdot v])_+^p \mathrm{d}S_p(K_1, K_2, \ldots, K_{2n-1}; v) \end{aligned} \tag{2.13}$$

for all $u, v \in S^{2n-1}$ and every $\lambda \in [-1, 1]$.

3 Proofs of main results

In this section, we give the proofs of Theorems 1.1–1.4. First, the proof of Theorem 1.1 needs the following Lemma 3.1.

Lemma 3.1. *Let $K \in \mathcal{K}_o(\mathbb{C}^n)$ and $C \in \mathcal{K}(\mathbb{C})$ be an asymmetric L_p zonoid. If $p \geq 1$ and $0 \leq i < 2n$, then for all $Q \in \mathcal{K}_o(\mathbb{C}^n)$, we have*

$$W_{p,i}(Q, \Pi^{\lambda}_{p,C}K) = V_p\big(K, \Pi^{\lambda}_{p,i,\bar{C}}Q\big). \tag{3.1}$$

Proof. From (2.3), (1.4), and the conjugate linear of Hermitian inner product, we know that for all $Q \in \mathcal{K}_o(\mathbb{C}^n)$,

$$\begin{aligned}
W_{p,i}(Q, \Pi^\lambda_{p,C}K) &= \frac{1}{2n}\int_{S^{2n-1}} h(\Pi^\lambda_{p,C}K, u)^p \mathrm{d}S_{p,i}(Q, u)\\
&= \frac{1}{2n}\int_{S^{2n-1}}\int_{S^{2n-1}}\int_{S^{n-1}} f_1(\lambda)(\mathcal{R}[cu\cdot v])_+^p \mathrm{d}\mu_{p,C}(c)\mathrm{d}S_p(K, v)\mathrm{d}S_{p,i}(Q, u)\\
&\quad + \frac{1}{2n}\int_{S^{2n-1}}\int_{S^{2n-1}}\int_{S^{n-1}} f_2(\lambda)(\mathcal{R}[cu\cdot v])_-^p \mathrm{d}\mu_{p,C}(c)\mathrm{d}S_p(K, v)\mathrm{d}S_{p,i}(Q, u)\\
&= \frac{1}{2n}\int_{S^{2n-1}}\int_{S^{2n-1}}\int_{S^{n-1}} f_1(\lambda)(\mathcal{R}[u\cdot \bar{c}v])_+^p \mathrm{d}\mu_{p,C}(c)\mathrm{d}S_{p,i}(Q, u)S_p(K, v)\\
&\quad + \frac{1}{2n}\int_{S^{2n-1}}\int_{S^{2n-1}}\int_{S^{n-1}} f_2(\lambda)(\mathcal{R}[u\cdot \bar{c}v])_-^p \mathrm{d}\mu_{p,C}(c)\mathrm{d}S_{p,i}(Q, u)S_p(K, v)\\
&= \frac{1}{2n}\int_{S^{2n-1}}\int_{S^{2n-1}} f_1(\lambda)h(\bar{C}u, v)^p \mathrm{d}S_{p,i}(Q, u)\mathrm{d}S_p(K, v)\\
&\quad + \frac{1}{2n}\int_{S^{2n-1}}\int_{S^{2n-1}} f_2(\lambda)h(-\bar{C}u, v)^p \mathrm{d}S_{p,i}(Q, u)\mathrm{d}S_p(K, v)\\
&= \frac{1}{2n}\int_{S^{2n-1}} h\left(\Pi^\lambda_{p,i,\bar{C}}Q, v\right)^p \mathrm{d}S_p(K, v)\\
&= V_p\left(K, \Pi^\lambda_{p,i,\bar{C}}Q\right),
\end{aligned}$$

where $\bar{C}$ is the conjugate of C and then we conclude the proof. □

Now, we are in a position to prove Theorem 1.1.

Proof of Theorem 1.1. Since $K, L \in \mathcal{K}_{os}(\mathbb{C}^n)$, then for $N \in \mathcal{K}_{os}(\mathbb{C}^n)$ and by (1.8), we have

$$V_p(K\#_pL, N) = \frac{1}{2n}\int_{S^{2n-1}} h(N, u)^p \mathrm{d}S_p(K\#_pL) = V_p(K, N) + V_p(L, N). \tag{3.2}$$

According to Lemma 3.1, for all $Q \in \mathcal{K}_{os}(\mathbb{C}^n)$, one deduces

$$\begin{aligned}
W_{p,i}(Q, \Pi^\lambda_{p,C}(K\#_pL)) &= V_p\left(K\#_pL, \Pi^\lambda_{p,\bar{C}}Q\right)\\
&= V_p\left(K, \Pi^\lambda_{p,i,\bar{C}}Q\right) + V_p\left(L, \Pi^\lambda_{p,i,\bar{C}}Q\right)\\
&= W_{p,i}(Q, \Pi^\lambda_{p,C}K) + W_{p,i}(Q, \Pi^\lambda_{p,C}L).
\end{aligned}$$

By (2.4), we have

$$W_{p,i}(Q, \Pi^\lambda_{p,C}(K\#_pL)) \geq W_i(Q)^{\frac{2n-p-i}{2n-i}}\left[W_i(\Pi^\lambda_{p,C}K)^{\frac{p}{2n-i}} + W_i(\Pi^\lambda_{p,C}L)^{\frac{p}{2n-i}}\right] \tag{3.3}$$

with equality if and only if Q, K, and L are real dilates.

Taking $Q = \Pi^\lambda_{p,C}(K\#_pL)$ in (3.3), we obtain

$$W_i(\Pi^\lambda_{p,C}(K\#_pL))^{\frac{p}{2n-i}} \geq W_i(\Pi^\lambda_{p,C}K)^{\frac{p}{2n-i}} + W_i(\Pi^\lambda_{p,C}L)^{\frac{p}{2n-i}} \tag{3.4}$$

with equality if and only K and L are real dilates. □

The following lemma provides a connection of $\Pi^{\lambda,*}_{p,C}$ and $M^{\lambda}_{p,i,\bar{C}}$ in terms of the dual L_p mixed quermassintergral and their mixed volume. We need the lemma to prove Theorem 1.2.

Lemma 3.2. *Let $K \in \mathcal{K}_o(\mathbb{C}^n)$ and $C \in \mathcal{K}(\mathbb{C})$ be an asymmetric L_p zonoid. If $p \geq 1$ and $0 \leq i < 2n$, then for all $Q \in \mathcal{S}_o(\mathbb{C}^n)$, we have*

$$\widetilde{W}_{-p,i}(Q, \Pi^{\lambda,*}_{p,C}K) = \frac{2n+p}{2} V_p\big(K, M^{\lambda}_{p,i,\bar{C}}Q\big), \tag{3.5}$$

where $M^{\lambda}_{p,C}Q$ is the general complex L_p moment body [24]. *If $K_{2n-i} = \cdots = K_{2n-1} = B$, we write $M^{\lambda}_{p,i,C}(Q, B)$ as $M^{\lambda}_{p,i,C}Q$.*

Proof. By (2.7), (2.9), and the conjugate linear of Hermitian inner product, we have

$$\begin{aligned}
V_p\big(K, M^{\lambda}_{p,i,\bar{C}}Q\big) &= \frac{1}{2n}\int_{S^{2n-1}} h\left(M^{\lambda}_{p,i,\bar{C}}Q, u\right)^p \mathrm{d}S_p(K,u)\\
&= \frac{2}{2n(2n+p)}\int_{S^{2n-1}}\int_{S^{2n-1}} f_1(\lambda)h(\bar{C}u, v)^p\rho(Q,v)^{2n+p-i}\mathrm{d}S(v)\mathrm{d}S_p(K,u)\\
&\quad + \frac{2}{2n(2n+p)}\int_{S^{2n-1}}\int_{S^{2n-1}} f_2(\lambda)h(-\bar{C}u, v)^p\rho(Q,v)^{2n+p-i}\mathrm{d}S(v)\mathrm{d}S_p(K,u)\\
&= \frac{2}{2n(2n+p)}\int_{S^{2n-1}}\int_{S^{2n-1}} f_1(\lambda)h(Cu, v)^p\rho(Q,v)^{2n+p-i}\mathrm{d}S_p(K,u)\mathrm{d}S(v)\\
&\quad + \frac{2}{2n(2n+p)}\int_{S^{2n-1}}\int_{S^{2n-1}} f_2(\lambda)h(-Cu, v)^p\rho(Q,v)^{2n+p-i}\mathrm{d}S(v)\mathrm{d}S_p(K,u)\\
&= \frac{2}{2n(2n+p)}\int_{S^{2n-1}} h(\Pi^{\lambda}_{p,C}K, v)^p\rho(Q,v)^{2n+p-i}\mathrm{d}S(v)\\
&= \frac{2}{2n(2n+p)}\int_{S^{2n-1}} \rho(\Pi^{\lambda,*}_{p,C}K, v)^{-p}\rho(Q,v)^{2n+p-i}\mathrm{d}S(v)\\
&= \frac{2}{2n+p}\widetilde{W}_{-p,i}(Q, \Pi^{\lambda,*}_{p,C}K),
\end{aligned}$$

which ends the proof of Lemma 3.2. □

Proof of Theorem 1.2. From (2.8), (3.2), and Lemma 3.2, we have for all $Q \in \mathcal{S}_o(\mathbb{C}^n)$

$$\begin{aligned}
\widetilde{W}_{-p,i}(Q, \Pi^{\lambda,*}_{p,C}(K\#_pL)) &= \frac{2n+p}{2}V_p\big(K\#_pL, M^{\lambda}_{p,i,\bar{C}}Q\big)\\
&= \frac{2n+p}{2}[V_p\big(K, M^{\lambda}_{p,i,\bar{C}}Q\big) + V_p\big(L, M^{\lambda}_{p,i,\bar{C}}Q\big)]\\
&= \widetilde{W}_{-p,i}(Q, \Pi^{\lambda,*}_{p,C}K) + \widetilde{W}_{-p,i}(Q, \Pi^{\lambda,*}_{p,C}L)\\
&\geq \widetilde{W}_i(Q)^{\frac{2n+p-i}{2n-i}}\left[\widetilde{W}_i(\Pi^{\lambda,*}_{p,C}K)^{-\frac{p}{2n-i}} + \widetilde{W}_i(\Pi^{\lambda,*}_{p,C}L)^{-\frac{p}{2n-i}}\right]
\end{aligned} \tag{3.6}$$

with equality if and only if Q, K, and L are real dilates.

Taking $Q = \Pi^{\lambda,*}_{p,C}(K\#_pL)$ in (3.6), it yields that

$$\widetilde{W}_i(\Pi^{\lambda,*}_{p,C}(K\#_pL))^{-\frac{p}{2n-i}} \geq \widetilde{W}_i(\Pi^{\lambda,*}_{p,C}K)^{-\frac{p}{2n-i}} + \widetilde{W}_i(\Pi^{\lambda,*}_{p,C}L)^{-\frac{p}{2n-i}} \tag{3.7}$$

with equality if and only if K and L are real dilates. Finally, let $i = 0$ in (3.7), we conclude the proof of Theorem 1.2. □

From now on, we pay attention to prove Theorems 1.3 and 1.4.

Proof of Theorem 1.3. From (1.3), (2.11), and the Hölder's integral inequality [32], one has

$$\begin{aligned}
h(\Pi^{+}_{p,C}(K_1, K_2, \dots, K_{2n-1}), u)^p &= \int_{S^{2n-1}} h(C \cdot u, v)^p \mathrm{d}S_p(K_1, K_2, \dots, K_{2n-1}; v) \\
&= \int_{S^{2n-1}} h(C \cdot u, v)^p (h(K_1, u) \cdots h(K_{2n-1}, u))^{\frac{1-p}{2n-1}} \mathrm{d}S(K_1, K_2, \dots, K_{2n-1}; v) \\
&\geq \left(\int_{S^{2n-1}} h(C \cdot u, v) \mathrm{d}S(K_1, K_2, \dots, K_{2n-1}; v) \right)^p (2n)^{1-p} \prod_{j=1}^{2n-1} \\
&\quad V(K_1, K_2, \dots, K_{2n-1}, K_j)^{\frac{1-p}{2n-1}} \\
&= h(\Pi_C(K_1, K_2, \dots, K_{2n-1}), u)^p (2n)^{1-p} \prod_{j=1}^{2n-1} V(K_1, K_2, \dots, K_{2n-1}, K_j)^{\frac{1-p}{2n-1}}.
\end{aligned} \tag{3.8}$$

According to the equality condition of Hölder's integral inequality, the equality holds if and only if K_j and $C \cdot u$ are dilates.

For each $Q \in \mathcal{K}_o(\mathbb{C}^n)$, integrating both sides of (3.8) for $\mathrm{d}S_{p,i}(Q, u)$ in $u \in S^{2n-1}$, we obtain

$$W_{p,i}(Q, \Pi^{+}_{p,C}(K_1, K_2, \dots, K_{2n-1})) \geq (2n)^{1-p} \prod_{j=1}^{2n-1} V(K_1, K_2, \dots, K_{2n-1}, K_j)^{\frac{1-p}{2n-1}} W_{p,i}(Q, \Pi_C(K_1, K_2, \dots, K_{2n-1})).$$

Taking $Q = \Pi^{+}_{p,C}(K_1, K_2, \dots, K_{2n-1})$ and by (2.4), we have

$$W_i(\Pi^{+}_{p,C}(K_1, K_2, \dots, K_{2n-1}))^{\frac{p}{2n-i}} \geq (2n)^{1-p} \prod_{j=1}^{2n-1} V(K_1, K_2, \dots, K_{2n-1}, K_j)^{\frac{1-p}{2n-1}} W_i(\Pi_C(K_1, K_2, \dots, K_{2n-1}))^{\frac{p}{2n-i}} \tag{3.9}$$

with equality if and only if $\Pi^{+}_{p,C}(K_1, K_2, \dots, K_{2n-1}), \Pi_C(K_1, K_2, \dots, K_{2n-1})$ are real dilates. But the equality condition in inequality (3.8) reveals that this happens precisely if K_j and $C \cdot u$ are dilates.

By the extension of the Brunn-Minkowski inequality (2.2), we obtain

$$W_i(\Pi^{\lambda}_{p,C}(K_1, K_2, \dots, K_{2n-1}))^{\frac{p}{2n-i}} \geq W_i(\Pi^{\pm}_{p,C}(K_1, K_2, \dots, K_{2n-1}))^{\frac{p}{2n-i}} \tag{3.10}$$

with equality if and only if $\Pi^{+}_{p,C}(K_1, K_2, \dots, K_{2n-1})$, $\Pi^{-}_{p,C}(K_1, K_2, \dots, K_{2n-1})$ are real dilates which is only possible if $\Pi^{-}_{p,C}(K_1, K_2, \dots, K_{2n-1}) = \Pi^{+}_{p,C}(K_1, K_2, \dots, K_{2n-1})$. It means that if $\Pi^{-}_{p,C}(K_1, K_2, \dots, K_{2n-1}) \neq \Pi^{+}_{p,C}(K_1, K_2, \dots, K_{2n-1})$, the inequality is strict unless $\lambda = -1, 1$, or 0.

Combining (3.9) and (3.10), we obtain

$$W_i(\Pi^{\lambda}_{p,C}(K_1, K_2, \dots, K_{2n-1}))^{\frac{p}{2n-i}} \geq (2n)^{1-p} \prod_{j=1}^{2n-1} V(K_1, K_2, \dots, K_{2n-1}, K_j)^{\frac{1-p}{2n-1}} W_i(\Pi_C(K_1, K_2, \dots, K_{2n-1}))^{\frac{p}{2n-i}}. \tag{3.11}$$

If $p = 1$, the Aleksandrov-Fenchel-type inequality (see [22]) is

$$V(\Pi_C(K_1, K_2, \dots, K_{2n-1}))^r \geq \prod_{j=1}^{r} V(\Pi_C(K_j[r], K_{r+1}, \dots, K_{2n-1})).$$

After that, let $i = 0$ in (3.11) and combine with the case of $p = 1$, we obtain

$$V(\Pi^{\lambda}_{p,C}(K_1, K_2, \dots, K_{2n-1}))^{\frac{p}{2n}} \geq (2n)^{1-p} \prod_{j=1}^{2n-1} V(K_1, \dots, K_{2n-1}, K_j)^{\frac{1-p}{2n-1}} \prod_{j=1}^{r} V(\Pi_C(K_j[r], K_{r+1}, \dots, K_{2n-1}))^{\frac{p}{2nr}}. \tag{3.12}$$

Let us turn toward the equality condition. If K_j is an ellipsoid centered at the origin, then $K_j = \phi B$ for $\phi \in GL(n, \mathbb{C})$. From $\Pi^{\lambda}_{p,C}(\phi B) = |\det \phi|^{\frac{2}{p}}\phi^{-*}\Pi^{\lambda}_{p,C}B$ and the fact that $\Pi^{\lambda}_{p,C}B = aB, \Pi_C B = bB$ $a, b > 0$ (see [20,24]), we have

$$\begin{aligned} V(\Pi^{\lambda}_{p,C}(K_1, K_2, \ldots, K_{2n-1}))^{\frac{p}{2n}} &= V(\Pi^{\lambda}_{p,C}(\phi B))^{\frac{p}{2n}} \\ &= V(|\det \phi|^{\frac{2}{p}}\phi^{-*}\Pi^{\lambda}_{p,C}B)^{\frac{p}{2n}} \\ &= |\det \phi|^{2} V(\phi^{-*}\Pi^{\lambda}_{p,C}B)^{\frac{p}{2n}} \\ &= |\det \phi|^{2}|\det \phi^{-*}|^{\frac{p}{n}} V(aB)^{\frac{p}{2n}} \\ &= |\det \phi|^{2}|\det \phi^{-*}|^{\frac{p}{n}} a^{p} V(B)^{\frac{p}{2n}}. \end{aligned} \tag{3.13}$$

Similarly, we also obtain

$$V(\Pi_C(K_1, K_2, \ldots, K_{2n-1}))^{\frac{p}{2n}} = V(\Pi_C(\phi B))^{\frac{p}{2n}} = |\det \phi|^{2p}|\det \phi^{-*}|^{\frac{p}{n}} b^{p} V(B)^{\frac{p}{2n}}, \tag{3.14}$$

$$(2n)^{1-p}\prod_{j=1}^{2n-1} V(K_1, \ldots, K_{2n-1}, K_j)^{\frac{1-p}{2n-1}} = (2n)^{1-p}V(\phi B)^{1-p} = (2n)^{1-p}|\det \phi|^{2(1-p)}V(B)^{1-p} = c|\det \phi|^{2(1-p)}, \tag{3.15}$$

where $c = \left(\int_{S^{2n-1}} h(B, u)\mathrm{d}S(B, u)\right)^{1-p}$ is a constant.

From (3.14) and (3.15), we have

$$(2n)^{1-p}\prod_{j=1}^{2n-1} V(K_1, \ldots, K_{2n-1}, K_j)^{\frac{1-p}{2n-1}}\prod_{j=1}^{r} V(\Pi_C(K_j[r], K_{r+1}, \ldots, K_{2n-1}))^{\frac{p}{2nr}} = c|\det \phi|^{2}|\det \phi^{-*}|^{\frac{p}{n}} b^{p} V(B)^{\frac{p}{2n}}. \tag{3.16}$$

Comparing equations (3.13) and (3.16), we know that $a = c^{\frac{1}{p}}b$ is possible, which means that if K_j is an ellipsoid centered at the origin, the equality holds in (3.12).

If K_j is an Hermitian ellipsoid, there exists a positive Hermitian matrix $\phi \in GL(n, \mathbb{C})$ and a vector $t \in \mathbb{C}^n$ such that $K_j = \phi B + t$. The definition of $\Pi^{\lambda}_{p,C}$ reveals that $\Pi^{\lambda}_{p,C}$ is translation invariant. Hence, if K_j is an Hermitian ellipsoid, equality also holds in (3.12). □

The case of $\lambda = 0$ of Theorem 1.3 is the following Aleksandrov-Fenchel-type inequality for the complex L_p mixed projection bodies.

Corollary 3.1. *If $p > 1$, $K_1, K_2, \ldots, K_{2n-1} \in \mathcal{K}_o(\mathbb{C}^n)$, and $C \in \mathcal{K}(\mathbb{C})$ is an asymmetric L_p zonoid, then*

$$V(\Pi_{p,C}(K_1, K_2, \ldots, K_{2n-1}))^{\frac{p}{2n}} \geq (2n)^{1-p}\prod_{j=1}^{2n-1} V(K_1, \ldots, K_{2n-1}, K_j)^{\frac{1-p}{2n-1}}\prod_{j=1}^{r} V(\Pi_C(K_j[r], K_{r+1}, \ldots, K_{2n-1}))^{\frac{p}{2nr}}.$$

For $p = 1$, the inequality is Aleksandrov-Fenchel-type inequality for mixed projection bodies [22].

In particular, the case of $C = [0, 1]$ and $\lambda = 0$ of Theorem 1.3 is Aleksandrov-Fenchel-type inequality for the general L_p mixed projection bodies [27]. We are now ready to prove Theorem 1.4. Note that Theorem 1.3 reduces to Theorem 1.4 by setting $K_{2n-i} = \cdots = K_{2n-1} = B$.

Proof of Theorem 1.4. Recall that $\mathbf{M} := (K_1, K_2, \ldots, K_{2n-1-i})$. From (2.12), (2.13), and the Hölder's integral inequality, we have

$$h(\Pi^{\lambda}_{p,i,C}\mathbf{M}, u)^{p} \geq (2n)^{1-p}\prod_{j=1}^{2n-1-i} W_i(\mathbf{M}, K_j)^{\frac{1-p}{2n-1-i}} h(\Pi_{i,C}\mathbf{M}, u)^{p}. \tag{3.17}$$

For all $Q \in \mathcal{K}_o(\mathbb{C}^n)$, we integrate both sides of (3.17) for $\mathrm{d}S_p(Q, u)$ and obtain

$$V_p(Q, \Pi^{\lambda}_{p,i,C}\mathbf{M}) \geq (2n)^{1-p}V_p(Q, \Pi_{i,C}\mathbf{M})\prod_{j=1}^{2n-1-i} W_i(\mathbf{M}, K_j)^{\frac{1-p}{2n-1-i}}.$$

Taking $Q = \Pi^{\lambda}_{p,i,C}\mathbf{M}$ and using (2.5), we obtain

$$V(\Pi^{\lambda}_{p,i,C}\mathbf{M})^{\frac{p}{2n}} \geq (2n)^{1-p}\prod_{j=1}^{r} V(\Pi_{i,C}(K_j[r], K_{r+1}, \ldots, K_{2n-1-i}))^{\frac{p}{2nr}} \prod_{j=1}^{2n-1-i} W_i(\mathbf{M}, K_j)^{\frac{1-p}{2n-1-i}}.$$

If K_j is an ellipsoid centered at the origin or an Hermitian ellipsoid, then the equality holds. □

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