Research Article

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The equivalent condition of G-asymptotic tracking property and G-Lipschitz tracking property

https://doi.org/10.1515/math-2022-0026
received September 13, 2021; accepted February 16, 2022

Abstract: In this paper, we introduce the concepts of G-Lipschitz tracking property and G-asymptotic tracking property in metric G-space and obtain the equivalent conditions of G-asymptotic tracking property in metric G-space. In addition, we prove that the self-map f has the G-Lipschitz tracking property if and only if the shift map σ has the $\overline{G}$-Lipschitz tracking property in the inverse limit space under the topological group action. These results generalize the corresponding results in [Proc. Amer. Math. Soc. **115** (1992), 573–580].

Keywords: metric G-space, topological group, inverse limit space, G-Lipschitz tracking property, G-asymptotic tracking property

MSC 2020: 37B99

1 Introduction

The tracking property has an important application in topological dynamical systems. In recent years, more and more scholars pay attention to it, and the relevant research results are shown in [1–17]. Liang and Li [1] proved that the self-map f has the tracking property if and only if the shift map σ has the tracking property in the inverse limit space. Ji et al. [2] proved that the shift map has the Lipschitz shadowing property if and only if the self-map has the Lipschitz shadowing property in the inverse limit space. Wang and Zeng [3] gave the relationship between average tracking property and $\underline{q}$-average tracking property. Wu [4] proved that the self-map f has the $\overline{d}$-tracking property if and only if the shift map σ has the $\overline{d}$-tracking property in the inverse limit space.

The map f has G-asymptotic tracking property if for each $\varepsilon > 0$ there exists $\delta > 0$ such that for any (G, δ)-pseudo orbit $\{x_i\}_{i\geq 0}$ of f, and there exists a point $y \in Y$ and $l \geq 0$ such that the sequence $\{x_i\}_{i=l}^{\infty}$ is (G, ε) shadowed by point y. We obtained the equivalent condition of the G-asymptotic tracking property in metric G-space.

The map f has G-Lipschitz tracking property if there exists positive constant L and δ_0 such that for any $0 < \delta < \delta_0$ and any (G, δ)-pseudo orbit $\{x_i\}_{i\geq 0}$ of f, there exists a point $x \in X$ such that the sequence $\{x_i\}_{i\geq 0}$ is $(G, L\delta)$ shadowed by point x (see [18]). We proved that the map f has the G-Lipschitz tracking property if and only if the shift map σ has the $\overline{G}$-Lipschitz tracking property. The main results are as follows in this paper.

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Theorem 1.1. *Let (X, d) be a compact metric G-space, the map $f : X \to X$ be an equivalent map and the metric d be invariant to the topological group G, where G is exchangeable. Then, the map f has the G-asymptotic tracking property if and only if for any $\varepsilon > 0$, there exists $0 < \delta < \varepsilon$ and $l \geq 0$ such that if $\{x_k\}_{k=0}^{\infty}$ is (G, δ)-pseudo orbit of the map f, then there exists a point y in X such that $\{f^l(x_k)\}_{k=0}^{\infty}$ is (G, ε) shadowed by point y.*

Theorem 1.2. *Let (X, d) be a compact metric G-space, $(X_f, \overline{G}, \overline{d}, \sigma)$ be the inverse limit space of (X, G, d, f) and the map $f : X \longrightarrow X$ be an equivalent surjection. If the map f is an Lipschitz map with Lipschitz constant L, then we have that the map f has the G-Lipschitz tracking property if and only if the shift map σ has the $\overline{G}$-Lipschitz tracking property.*

2 The equivalent condition of G-asymptotic tracking property

In this section, we present some concepts that may be used in the following. The concept of metric G-space and equivariant map can be found in [17].

Definition 2.1. [19] Let (X, d) be a metric space and f be a continuous map from X to X. The map f is called to be uniformly continuous if for any $\varepsilon > 0$ there exists $0 < \delta < \varepsilon$ such that $d(x, y) < \delta$ implies $d(f(x), f(y)) < \varepsilon$ for all $x, y \in X$.

Definition 2.2. [20] Let (X, d) be a metric G-space. The metric d is said to be invariant to topological group G provided that $d(gx, gy) = d(x, y)$ for all $x, y \in X$ and $g \in G$.

Definition 2.3. Let (X, d) be a metric G-space and f be a continuous map from X to X. The map f has G-asymptotic tracking property if for each $\varepsilon > 0$ there exists $\delta > 0$ such that for any (G, δ)-pseudo orbit $\{x_i\}_{i\geq 0}$ of f, there exists a point $y \in Y$ and $l \geq 0$ such that the sequence $\{x_i\}_{i=l}^{\infty}$ is (G, ε) shadowed by point y.

Lemma 2.4. *Let (X, d) be a compact metric G-space, the map $f : X \to X$ be an equivalent map, the metric d be invariant to the topological group G, where G is exchangeable and $m > 0$. Then, for any $\varepsilon > 0$, there exists $0 < \delta < \varepsilon$ such that if for any $k \geq 0$, there exists $g_k \in G$ such that $d(g_k f(x_k), x_{k+1}) < \delta$, then we have $d(g_{k+m-1}g_{k+m-2}g_{k+m-3} \cdots g_{k+1}g_k f^m(x_k), x_{m+k}) < \varepsilon$.*

Proof. By continuity of the map f, for any $\varepsilon > 0$ and $0 \leq i < m$, there exists $0 < \delta < \varepsilon$ such that $d(x, y) < \delta$ implies

$$d(f^i(x), f^i(y)) < \frac{\varepsilon}{m}. \tag{1}$$

Suppose that for any $k > 0$, there exists $g_k \in G$ such that

$$d(g_k f(x_k), x_{k+1}) < \delta.$$

According to the equivalent definition of the map f and (1), for any $k > 0$ and $0 \leq i < m$, it follows that

$$d(g_k f^{i+1}(x_k), f^i(x_{k+1})) < \frac{\varepsilon}{m}.$$

Then,

$$d(g_k f^m(x_k), f^{m-1}(x_{k+1})) < \frac{\varepsilon}{m}.$$

$$d(g_{k+1} f^{m-1}(x_{k+1}), f^{m-2}(x_{k+2})) < \frac{\varepsilon}{m}.$$

$$d(g_{k+2}f^{m-2}(x_{k+2}), f^{m-3}(x_{k+3})) < \frac{\varepsilon}{m}.$$
$$\cdots\cdots$$
$$d(g_{k+m-2}f^{2}(x_{k+m-2}), f(x_{k+m-1})) < \frac{\varepsilon}{m}.$$
$$d(g_{k+m-1}f(x_{k+m-1}), x_{k+m}) < \frac{\varepsilon}{m}.$$

Since the metric d is invariant to the topological group G and G is exchangeable, we have

$$d(g_k g_{k+1} g_{k+2}\cdots g_{k+m-1}f^{m}(x_k), g_{k+1}g_{k+2}\cdots g_{k+m-1}f^{m-1}(x_{k+1})) < \frac{\varepsilon}{m}.$$
$$d(g_{k+1}g_{k+2}g_{k+3}\cdots g_{k+m-1}f^{m-1}(x_{k+1}), g_{k+2}g_{k+3}\cdots g_{k+m-1}f^{m-2}(x_{k+2})) < \frac{\varepsilon}{m}.$$
$$d(g_{k+2}g_{k+3}\cdots g_{k+m-1}f^{m-2}(x_{k+2}), g_{k+3}\cdots g_{k+m-1}f^{m-3}(x_{k+3})) < \frac{\varepsilon}{m}.$$
$$\cdots\cdots$$
$$d(g_{k+m-2}g_{k+m-1}f^{2}(x_{k+m-2}), g_{k+m-1}f(x_{k+m-1})) < \frac{\varepsilon}{m}.$$
$$d(g_{k+m-1}f(x_{k+m-1}), x_{k+m}) < \frac{\varepsilon}{m}.$$

Therefore,

$$\begin{aligned}
&d(g_k g_{k+1} g_{k+2}\cdots g_{k+m-1}f^{m}(x_k), x_{m+k})\\
&\quad < d(g_k g_{k+1} g_{k+2}\cdots g_{k+m-1}f^{m}(x_k), g_{k+1}g_{k+2}\cdots g_{k+m-1}f^{m-1}(x_{k+1}))\\
&\qquad + d(g_{k+1}g_{k+2}g_{k+3}\cdots g_{k+m-1}f^{m-1}(x_{k+1}), g_{k+2}g_{k+3}\cdots g_{k+m-1}f^{m-2}(x_{k+2}))\\
&\qquad + d(g_{k+2}g_{k+3}\cdots g_{k+m-1}f^{m-2}(x_{k+2}), g_{k+3}\cdots g_{k+m-1}f^{m-3}(x_{k+3}))\\
&\qquad + \cdots\cdots + d(g_{k+m-2}g_{k+m-1}f^{2}(x_{k+m-2}), g_{k+m-1}f(x_{k+m-1})) + d(g_{k+m-1}f(x_{k+m-1}), x_{k+m})\\
&\quad < \frac{\varepsilon}{m} + \frac{\varepsilon}{m} + \frac{\varepsilon}{m} + \cdots\cdots + \frac{\varepsilon}{m} = \varepsilon.
\end{aligned}$$
□

Theorem 2.5. *Let (X, d) be a compact metric G-space, the map $f : X \to X$ be an equivalent map and the metric d be invariant to the topological group G where G is exchangeable. Then, the map f has the G-asymptotic tracking property if and only if for any $\varepsilon > 0$ there exists $0 < \delta < \varepsilon$ and $l \geq 0$ such that if $\{x_k\}_{k=0}^{\infty}$ is (G, δ)-pseudo orbit of the map f, then there exists a point y in X such that $\{f^l(x_k)\}_{k=0}^{\infty}$ is (G, ε) shadowed by point y.*

Proof. (*Necessity*) Suppose that the map f has the G-asymptotic tracking property. Then, for each $\varepsilon > 0$, there exists $0 < \tau < \frac{\varepsilon}{2}$ such that for any (G, τ)-pseudo orbit $\{x_k\}_{k\geq 0}$ of f, there exists a point $y \in X$ and $l \geq 0$ such that the sequence $\{x_k\}_{k=l}^{\infty}$ is $(G, \frac{\varepsilon}{2})$ shadowed by point y. If $l = 0$, the results are obvious. Now we assume $l > 0$. Since the map f is uniformly continuous, for given $\frac{\varepsilon}{2l} > 0$ and any $0 \leq i < l$, there exists $0 < \delta < \min\left\{\frac{\varepsilon}{2l}, \tau\right\}$ such that $d(x, y) < \delta$ implies

$$d(f^i(x), f^i(y)) < \frac{\varepsilon}{2l}. \tag{2}$$

Let $\{x_k\}_{k=0}^{\infty}$ be (G, δ)-pseudo orbit of the map f. Then, for any $k \geq 0$, there exists $t_k \in G$ such that

$$d(t_k f(x_k), x_{k+1}) < \delta.$$

By (2) and the equivalent definition of the map f, for any $k \geq 0$ and $0 \leq i < l$, we have that

$$d(t_k f^{i+1}(x_k), f^i(x_{k+1})) < \frac{\varepsilon}{2l}. \tag{3}$$

Noting that the metric d is invariant to the topological group G, where G is exchangeable and (3), we have

$$d(t_kt_{k+1}t_{k+2}\cdots t_{k+l-2}t_{k+l-1}f^l(x_k), t_{k+1}t_{k+2}\cdots t_{k+l-2}t_{k+l-1}f^{l-1}(x_{k+1})) < \frac{\varepsilon}{2l},$$

$$d(t_{k+1}t_{k+2}t_{k+3}\cdots t_{k+l-2}t_{k+l-1}f^{l-1}(x_{k+1}), t_{k+2}t_{k+3}\cdots t_{k+l-2}t_{k+l-1}f^{l-2}(x_{k+2})) < \frac{\varepsilon}{2l},$$

$$d(t_{k+2}t_{k+3}\cdots t_{k+l-2}t_{k+l-1}f^{l-2}(x_{k+2}), t_{k+3}\cdots t_{k+l-2}t_{k+l-1}f^{l-3}(x_{k+3})) < \frac{\varepsilon}{2l},$$

$$\cdots\cdots$$

$$d(t_{k+l-2}t_{k+l-1}f^2(x_{k+l-2}), t_{k+l-1}f(x_{k+l-1})) < \frac{\varepsilon}{2l},$$

$$d(t_{k+l-1}f(x_{k+l-1}), x_{k+l}) < \frac{\varepsilon}{2l},$$

and thus,

$$\begin{aligned}
&d(t_kt_{k+1}t_{k+2}\cdots t_{k+l-2}t_{k+l-1}f^l(x_k), x_{k+l})\\
&\quad < d(t_kt_{k+1}t_{k+2}\cdots t_{k+l-2}t_{k+l-1}f^l(x_k), t_{k+1}t_{k+2}\cdots t_{k+l-2}t_{k+l-1}f^{l-1}(x_{k+1}))\\
&\qquad + d(t_{k+1}t_{k+2}t_{k+3}\cdots t_{k+l-2}t_{k+l-1}f^{l-1}(x_{k+1}), t_{k+2}t_{k+3}\cdots t_{k+l-2}t_{k+l-1}f^{l-2}(x_{k+2}))\\
&\qquad + d(t_{k+2}t_{k+3}\cdots t_{k+l-2}t_{k+l-1}f^{l-2}(x_{k+2}), t_{k+3}\cdots t_{k+l-2}t_{k+l-1}f^{l-3}(x_{k+3}))\\
&\qquad +\cdots\cdots+ d(t_{k+l-2}t_{k+l-1}f^2(x_{k+l-2}), t_{k+l-1}f(x_{k+l-1})) + d(t_{k+l-1}f(x_{k+l-1}), x_{k+l})\\
&\quad < \frac{\varepsilon}{2l} + \frac{\varepsilon}{2l} + \frac{\varepsilon}{2l} + \cdots\cdots + \frac{\varepsilon}{2l} = \frac{\varepsilon}{2}.
\end{aligned}$$

So for any $k \geq 0$, we have

$$d(t_kt_{k+1}t_{k+2}\cdots t_{k+l-2}t_{k+l-1}f^l(x_k), x_{k+l}) < \frac{\varepsilon}{2}. \tag{4}$$

Since the map f has the G-asymptotic tracking property, for any $k \geq 0$, there exists $g_k \in G$ and $y \in Y$ such that

$$d(f^k(y), g_kx_{k+l}) < \frac{\varepsilon}{2}.$$

Since the metric d is invariant to the topological group G, then

$$d(g_k^{-1}f^k(y), x_{k+l}) < \frac{\varepsilon}{2}. \tag{5}$$

By (4) and (5), for any $k \geq 0$, we obtain

$$d(t_kt_{k+1}t_{k+2}\cdots t_{k+l-2}t_{k+l-1}f^l(x_k), g_k^{-1}f^k(y)) < d(t_kt_{k+1}t_{k+2}\cdots t_{k+l-2}t_{k+l-1}f^l(x_k), x_{k+l}) + d(x_{k+l}, g_k^{-1}f^k(y)) < \varepsilon.$$

Together with the fact that the metric d is invariant to the topological group G again, it follows that

$$d(f^k(y), g_kt_kt_{k+1}t_{k+2}\cdots t_{k+l-2}t_{k+l-1}f^l(x_k)) < \varepsilon.$$

Hence, the sequence $\{f^l(x_k)\}_{k=0}^{\infty}$ is (G, ε) shadowed by point y.

(*Sufficiency*) Suppose that for any $\varepsilon > 0$ there exists $0 < \tau < \varepsilon$ and $l \geq 0$ such that if $\{x_k\}_{k=0}^{\infty}$ is (G, τ)-pseudo orbit of the map f, then there exists a point z in X such that $\{f^l(x_k)\}_{k=0}^{\infty}$ is $\left(G, \frac{\varepsilon}{2}\right)$ shadowed by point z. If $l = 0$, the results are obvious. Now we assume $l > 0$. Since the map f is uniformly continuous, for given $\varepsilon > 0$ and any $0 \leq i < l$, there exists $0 < \delta_0 < \tau$ such that $d(x, y) < \delta$ implies

$$d(f^i(x), f^i(y)) < \frac{\varepsilon}{2}.$$

Let $\{x_k\}_{k=0}^{\infty}$ be (G, δ_0)-pseudo orbit of the map f. Then, there exists $z \in X$ such that for any $k \geq 0$ there exists $p_k \in G$ such that

$$d\left(f^k(z), p_k f^l(x_k)\right) < \frac{\varepsilon}{2}. \tag{6}$$

In addition, for any $k \geq 0$, there exists $s_k \in G$ such that

$$d(s_k f(x_k), x_{k+1}) < \delta_0.$$

By Lemma 2.9, for any $k \geq 0$, we have that

$$d(s_k s_{k+1} s_{k+2} \cdots s_{k+l-2} s_{k+l-1} f^l(x_k), x_{k+l}) < \frac{\varepsilon}{2}. \tag{7}$$

Since the metric d is invariant to the topological group G and equations (6) and (7), we have

$$d\left(s_k s_{k+1} s_{k+2} \cdots s_{k+l-2} s_{k+l-1} f^k(z), s_k s_{k+1} s_{k+2} \cdots s_{k+l-2} s_{k+l-1} p_k f^l(x_k)\right) < \frac{\varepsilon}{2}$$

and

$$d(p_k s_k s_{k+1} s_{k+2} \cdots s_{k+l-2} s_{k+l-1} f(x_k), p_k x_{k+l}) < \frac{\varepsilon}{2}.$$

Since G is exchangeable, we have that

$$\begin{aligned}
& d(s_k s_{k+1} s_{k+2} \cdots s_{k+l-2} s_{k+l-1} f^k(z), p_k x_{k+l}) \\
& \quad < d(s_k s_{k+1} s_{k+2} \cdots s_{k+l-2} s_{k+l-1} f^k(z), s_k s_{k+1} s_{k+2} \cdots s_{k+l-2} s_{k+l-1} p_k f^l(x_k)) \\
& \qquad + d(s_k s_{k+1} s_{k+2} \cdots s_{k+l-2} s_{k+l-1} p_k f^l(x_k), p_k x_{k+l}) \\
& \quad = d(s_k s_{k+1} s_{k+2} \cdots s_{k+l-2} s_{k+l-1} f^k(z), s_k s_{k+1} s_{k+2} \cdots s_{k+l-2} s_{k+l-1} p_k f^l(x_k)) \\
& \qquad + d(p_k s_k s_{k+1} s_{k+2} \cdots s_{k+l-2} s_{k+l-1} f^l(x_k), p_k x_{k+1}) \\
& \quad < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} < \varepsilon.
\end{aligned}$$

Hence, the map f has the G-asymptotic tracking property. Thus, we complete the proof. □

3 *G*-Lipschitz tracking property

The concept of the inverse limit spaces in this section under group action can be found in [21].

Definition 3.1. [5] Let(X, d) be a metric space and $f : X \to X$ be a continuous map.The map f is said to be an Lipschitz map if there exists a positive constant $L > 0$ such that $d(f(x), f(y)) \leq Ld(x, y)$ for all $x, y \in X$.

Definition 3.2. [18] Let(X, d) be a metric G-space and $f : X \to X$ be a continuous map. The map f has G-Lipschitz tracking property if there exists positive constant L and δ_0 such that for any $0 < \delta < \delta_0$ and (G, δ)-pseudo orbit $\{x_i\}_{i\geq 0}$ of f there exists a point z in X such that the sequence $\{x_i\}_{i\geq 0}$ is $(G, L\delta)$ shadowed by point z.

Now, we give the proof of Theorem 3.3.

Theorem 3.3. *Let (X, d) be a compact metric G-space, $(X_f, \overline{G}, \overline{d}, \sigma)$ be the inverse limit space of (X, G, d, f) and the map $f : X \longrightarrow X$ be an equivalent surjection. If the map f is an Lipschitz map with Lipschitz constant L, then we have that the map f has the G-Lipschitz tracking property if and only if the shift map σ has the $\overline{G}$-Lipschitz tracking property.*

Proof. (*Necessity*) Suppose that the map f has the G-Lipschitz tracking property. Then, there exists positive constant L_1 and ε_1 such that for any $0 < \varepsilon < \varepsilon_1$ and (G, ε)-pseudo orbit $\{x_i\}_{i=0}^{\infty}$ of the map f, there exists a point $x \in X$ such that $\{x_i\}_{i=0}^{\infty}$ is $(G, L_1\varepsilon)$ shadowed by x. By the compactness of X, let $M = \operatorname{diam}(X)$. For given $\varepsilon > 0$, choose a positive constant $m > 0$ such that $\frac{M}{2^m} < \varepsilon$. Let

$$L_2 = L_m + \frac{L_{m-1}}{2} + \frac{L_{m-2}}{2^2} + \cdots + \frac{L}{2^{m-1}} + \frac{1}{2^m},$$

$$L_3 = L_1 L_2 2^m,$$

$$\varepsilon_2 = \frac{\varepsilon_1}{2^m}.$$

For any $0 < \eta < \varepsilon_2$, let $\{\overline{y}_k\}_{k=0}^{\infty} = (y_k^0, y_k^1, y_k^2 \cdots)$ be (G, η)-pseudo orbit of the shift map σ in X_f. It is obvious that for each $k \geq 0$, there exists $\overline{g}_k = (g_k, g_k, g_k \cdots) \in \overline{G}$ such that

$$\overline{d}(\overline{g}_k \sigma(\overline{y}_k), \overline{y}_{k+1}) < \eta.$$

From the definition of the metric $\overline{d}$, for every $k \geq 0$, it follows that

$$d(g_k f(y_k^m), y_{k+1}^m) < 2^m \eta < \varepsilon_1.$$

Thus, $\{y_k^m\}_{k=0}^{\infty}$ is $(G, 2^m\eta)$-pseudo orbit of the map f. By the G-Lipschitz tracking property of the map f in X, there exists a point y in X such that for every $k \geq 0$ there exists $t_k \in G$ such that

$$d(f^k(y), t_k y_k^m) < L_1 2^m \eta. \tag{8}$$

Since the map f is onto, we write

$$\overline{y} = (f^m(y), f^{m-1}(y), \cdots, f(y), y, \cdots) \in X_f.$$

$$\overline{t}_k = (t_k, t_k, t_k \cdots) \in \overline{G}.$$

By the definition of the equivalent map f, for any $k \geq 0$, we have

$$\begin{aligned}\overline{d}(\sigma^k(\overline{y}), \overline{t}_k \overline{y}_k) &< \sum_{i=0}^{i=m} \frac{d(f^{k+m-i}(y), t_k y_k^i)}{2^i} + \sum_{i=m+1}^{\infty} \frac{M}{2^i} \\ &< \sum_{i=0}^{i=m} \frac{d(f^{m-i}(f^k(y)), f^{m-i}(t_k y_k^m))}{2^i} + \frac{M}{2^m} \\ &< \sum_{i=0}^{i=m} \frac{d(f^{m-i}(f^k(y)), f^{m-i}(t_k y_k^m))}{2^i} + \varepsilon \\ &\leq \sum_{i=0}^{i=m} \frac{d(f^{m-i}(f^k(y)), f^{m-i}(t_k y_k^m))}{2^i}.\end{aligned}$$

According to (8) and the definition of Lipschitz map f, we obtain

$$\sum_{i=0}^{i=m} \frac{d(f^{m-i}(f^k(y)), f^{m-i}(t_k y_k^m))}{2^i} < \sum_{i=0}^{i=m} \frac{L^{m-i} d(f^k(y), t_k y_k^m)}{2^i} < \sum_{i=0}^{i=m} \frac{L^{m-i} L_1 2^m \eta}{2^i} = L_1 L_2 2^m \eta.$$

Then, for any $k \geq 0$, it follows that

$$\overline{d}(\sigma^k(\overline{y}), \overline{t}_k \overline{y}_k) < L_3 \eta.$$

Therefore, the shift map σ has the G-Lipschitz tracking property.

(*Sufficiency*) Next, we suppose that the shift map σ has the $\overline{G}$-Lipschitz tracking property. Then, there exists positive constant L_4 and ε_3 such that for any $0 < \delta' < \varepsilon_3$ and $(\overline{G}, \delta')$-pseudo orbit $\{\overline{z}_k\}_{k=0}^{\infty}$ of the shift map σ there exists a point $\overline{z} \in X_f$ such that $\{\overline{z}_k\}_{k=0}^{\infty}$ is $(\overline{G}, L_4\delta')$ shadowed by point $\overline{z}$. For given $\delta' > 0$, choose $n > 0$ such that $\frac{M}{2^n} < \delta'$. Then, we write

$$L_5 = L_n + \frac{L_{n-1}}{2} \frac{L_{n-2}}{2^2} + \cdots + \frac{L}{2^{n-1}} + \frac{1}{2^n},$$

$$L_6 = 2^n L_4 L_5,$$

$$\varepsilon_4 = \frac{\varepsilon_3}{L_5}.$$

For any $0 < \delta < \varepsilon_4$, suppose that $\{x_k\}_{k=0}^{\infty}$ is (G, δ)-pseudo orbit of the map f. It is obvious that for each $k \geq 0$, there exists a point $p_k \in G$ such that

$$d(p_k f(x_k), x_{k+1}) < \delta. \tag{9}$$

Since the map f is onto, we write

$$\overline{x}_k = (f^n(x_k), f^{n-1}(x_k), \cdots, f(x_k), x_k, \cdots) \in X_f.$$

$$\overline{p}_k = (p_k, p_k, p_k \cdots) \in \overline{G}.$$

Then, for any $k \geq 0$, we have that

$$\begin{aligned}
\overline{d}(\overline{p}_k \sigma(\overline{x}_k), \overline{x}_{k+1}) &< \sum_{i=0}^{i=n} \frac{d(p_k f^{n+1-i}(x_k), f^{n-i}(x_{k+1}))}{2^i} + \sum_{i=n+1}^{\infty} \frac{M}{2^i} \\
&< \sum_{i=0}^{i=n} \frac{d(p_k f^{n+1-i}(x_k), f^{n-i}(x_{k+1}))}{2^i} + \frac{M}{2^n} \\
&< \sum_{i=0}^{i=n} \frac{d(p_k f^{n+1-i}(x_k), f^{n-i}(x_{k+1}))}{2^i} + \delta' \\
&\leq \sum_{i=0}^{i=n} \frac{d(p_k f^{n+1-i}(x_k), f^{n-i}(x_{k+1}))}{2^i}.
\end{aligned}$$

According to (9), the definition of Lipschitz map f and the definition of equivalent map f, we obtain

$$\sum_{i=0}^{i=n} \frac{d(p_k f^{n+1-i}(x_k), f^{n-i}(x_{k+1}))}{2^i} \leq \sum_{i=0}^{i=n} \frac{L^{n-i} d(p_k f(x_k), x_{k+1})}{2^i} \leq \sum_{i=0}^{i=n} \frac{L^{n-i}\delta}{2^i} = L^5\delta < \varepsilon_3.$$

So for any $k \geq 0$, we have

$$\overline{d}(\overline{p}_k \sigma(\overline{x}_k), \overline{x}_{k+1}) < L^5\delta < \varepsilon_3.$$

Hence, $\{\overline{x}_k\}_{k=0}^{\infty}$ is $(\overline{G}, L^5\delta)$-pseudo orbit of the shift map σ. By the G-Lipschitz tracking property of the shift map σ, there exists a point $\overline{z} = (z_0, z_1, z_2, \cdots z_n \cdots) \in X_f$ such that for every $k \geq 0$ there exists $\overline{s}_k = (s_k, s_k, s_k, \cdots) \in \overline{G}$ with $s_k \in G$ such that

$$\overline{d}(\sigma^k(\overline{z}), \overline{s}_k \overline{x}_k) < L^4 L^5 \delta.$$

From the definition of the metric $\overline{d}$, it follows that

$$d(f^k(z_n), s_k x_k) < 2^n L^4 L^5 \delta.$$

Thus,

$$d(f^k(z_n), s_k x_k) < L^6 \delta.$$

So the map f has the G-Lipschitz tracking property. □

4 Conclusion

In this paper, we studied dynamical properties of G-Lipschitz tracking property and G-asymptotic tracking property. It was obtained that the equivalent conditions of G-asymptotic tracking property in metric G-space. In addition, it was proved that the self-map f has the G-Lipschitz tracking property if and only if the shift map σ has the $\overline{G}$-Lipschitz tracking property in the inverse limit space under topological group action. These results generalize the corresponding results in [Proc. Amer. Math. Soc. **115** (1992), 573–580].

Acknowledgements: This work was partially supported by the NSF of Guangxi Province (2020JJA110021) and construction project of Wuzhou University of China (2020B007).

Conflict of interest: The author states no conflict of interest.

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